WORD EQUATIONS I: PAIRS AND THEIR MAKANIN-RAZBOROV DIAGRAMS

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This paper is the first in a sequence on the structure of sets of solutions to systems of equations over a free semigroup. To describe the structure, we present a Makanin-Razborov diagram that encodes the set of solutions to such system of equations. In the sequel we show how this diagram, and the tools that are used in constructing it, can be applied to analyze fragments of the first order theory of free semigroups.

A free semigroup can be viewed as the most basic formal language. This connection, and the analogy with Tarski's problem on the first order theories of free groups, led W. Quine to study the first order theory of a free semigroup. In 1946 Quine proved that arithmetic can be defined in a free semigroup. Hence, by Godels incompleteness theorems, the theory is not axiomatizable, nor decidable [Qu]. Later, smaller and smaller fragments of the theory have been shown to be undecidable, including sentences with only two quantifiers, by Durnev ([Du1],[Du2]), Marchenkov [Mar], and others.

On the positive side, the Diophantine theory of a free semigroup has been shown to be decidable by G. S. Makanin [Ma1]. Makanin presented an algorithm that decides if a given system of equations over a free semigroup has a solution. Several years later Makanin was able to modify his algorithm to decide if a given system of equations over a free group has a solution [Ma2].

In 1987, A. A. Razborov managed to use Makanins techniques and gave a combinatorial description of the set of solutions to a system of equations (variety) over a free group. This description was further developed by O. Kharlampovich and A. Myasnikov [Kh-My], and a more geometric approach was given by the author in [Se1]. The description of varieties over a free group is the starting point to a structure theory that finally led to quantifier elimination ([Se3],[Se4]), and to the solution of Tarski's problem on the first order theories of free groups [Se5].

The search for a description of the set of solutions to a system of equations over free semigroups has a long history (see [Di]). For equations with one variable, and then for equations with 3 variables, the structure of the set of solutions over free semigroup was achieved before the analogous structures over free groups (see [Khm]). In these cases the structure that was found was purely combinatorial.

Since the mid 1980's and in particular after Razborov's thesis, there have been quite a few attempts to study sets of solutions over a free semigroup, usually for particular families of systems, that are often either with small number of variables,

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[2]Partially supported by an Israel academy of sciences fellowship.

or that are of rather particular type, mostly connected to quadratic equations over a free group (e.g. [Ma3],[Ma4],[Ma5],[Ly],[Di] and their references).

In 2013 an algorithm to enumerate and encode the set of solutions to a general system of equations over a free semigroup was found (by A. Jez). This algorithm that enumerates the solutions efficiently is based on variants of the Ziv-Lempel algorithm from information theory ([Je],[DJP]).

In this paper we present a geometric approach to study varieties over a free semigroup. We use the combinatorial techniques that were introduced by Makanin in proving the decidability of the Diophantine theory [Ma1], and we combine them with geometric techniques that were used in the construction of the JSJ decomposition of finitely presented groups ([Se7],[Ri-Se]), and with techniques that appear in the solution to Tarski's problem.

Unfortunately, even though our approach is based on the construction of the JSJ decomposition for groups, we were not able to find an analogue of the JSJ for studying varieties over free semigroups (we were able to get such analogue in some special cases). However, we are able to find an analogue of Razborov's work over free groups, and associate what we call a Makanin-Razborov diagram with each variety over a free semigroup.

The MR diagram that we construct is not canonical, but it encodes all the points in a given variety over a free semigroup. Furthermore, given a path in the diagram, that we call a *resolution*, there exists a sequence of points in the associated variety that factor through the resolution, and such that the sequence converges in the Gromov-Hausdorff topology to an object from which the resolution can be reconstructed. Such sequences are viewed as generic points in the variety (a replacement for test sequences that exist over a free group), and are used in the sequel to prove model theoretic results, that include generalizations of Merzlyakov's theorem (over varieties) for free semigroups (see [Me] and [Se2]).

The MR diagram over semigroups, its properties and the way it is constructed, suggest that very basic questions about words (in formal languages) are connected to concepts, objects and tools from low dimensional topology. These include the JSJ decomposition, the geometry and topology of surfaces, the dynamics of automorphisms of surfaces and of free groups, and finally the dynamics of actions of groups on real trees. We expect that this combination of techniques and structure can be modified to describe sets of solutions to systems of equations over free objects in other (associative and non-commutative) algebraic categories, and we plan to continue in these directions.

Given a system of equations Σ over a free semigroup:

$$\begin{aligned} u_1(x_1,\ldots,x_n) &= v_1(x_1,\ldots,x_n) \\ &\vdots \\ u_s(x_1,\ldots,x_n) &= v_s(x_1,\ldots,x_n) \end{aligned}$$

we naturally associate with it a f.p. semigroup:

$$S(\Sigma) \;=\; \langle\, x_1,\ldots,x_n \,|\, u_1=v_1,\ldots,u_s=v_s \,\rangle$$

The set of solutions of Σ over a free semigroup, $FS_k =< a_1,\ldots,a_k >$, is in bijection with the set of semigroup homomorphisms: $\{h : S(\Sigma) \to FS_k\}$. Hence, studying the variety of solutions to Σ is equivalent to studying the structure of the set of homomorphisms from $S(\Sigma)$ to the free semigroup FS_k.

With a f.g. semigroup S we associate a group $G(S)$, that is obtained by forcing all the elements in S to have inverses, or alternatively, by looking at a presentation of S as a semigroup as if it is a presentation of a group. Naturally, S is mapped into $G(S)$, but in general it is not embedded in it. Let $\hat{S}$ be the (semigroup) image of S in $G(S)$.

The free semigroup FS_k embeds in the free group F_k in a standard way. A simple observation shows that every semigroup homomorphism from S to FS_k extends uniquely to a group homomorphism from $G(S)$ to F_k (see section 1). Hence, one can replace the set of semigroup homomorphisms from S to the free semigroup FS_k, with the set of pair homomorphisms: $\{\eta : (\hat{S}, G(S)) \to (FS_k, F_k)\}$. i.e., those group homomorphisms from $G(S)$ to F_k that map $\hat{S}$ into the standard free semigroup FS_k.

Our whole approach is based on studying the structure of these pair homomorphisms. An immediate application of the techniques that were used over free groups, shows that the set of pair homomorphisms from $(\hat{S}, G(S))$ to (FS_k, F_k) is canonically a finite union of the sets of pair homomorphisms from pairs of the form (S_i, L_i) to (FS_k, F_k), where for each i, L_i is a limit group, S_i is a semigroup that generates L_i as a group, and (S_i, L_i) is a (limit) quotient of the pair $(\hat{S}, G(S))$.

A free semigroup, and the semigroups that we need to consider, have usually very few automorphisms. The ability to replace a semigroup with a pair in which the ambient group is a limit group, enables one to work with a large group of automorphisms (of the limit group), that usually do not preserve the embedded semigroup. In section 2 we describe an analogue of the shortening procedure for homomorphisms of pairs. Technically, the shortening procedure for homomorphisms of pairs is much more involved than its analogue for homomorphisms of groups.

In section 3 we describe a construction of a JSJ decomposition for a pair (S, L), where L is a limit group, and S generates it as a group. Unfortunately, the construction applies only in special cases. Note that free products exist in the categories of groups, of semigroups, and of pairs. However, it may be that a pair (S, L), where L is a limit group, is freely indecomposable as a pair, but the limit group L is freely decomposable. This simple fact implies that any attempt to borrow concepts from the JSJ theory and from the construction of the MR diagram over free groups must be further refined. In section 6 we describe the construction of the MR diagram for pairs in the freely indecomposable case (theorem 6.8). In section 7 we finally describe the construction of the MR diagram for pairs in the general case (theorems 7.17 and 7.18).

As over free (and hyperbolic) groups, there are pairs, (S_i, L_i), associated with the nodes of the MR diagram, together with their associated decompositions and modular groups. However, unlike the case of groups, the abelian decompositions that are associated with the pairs that appear in the nodes of the diagram, need to recall not only the algebraic structure of the group in question, but rather dynamical properties and the associated modular groups. Hence, we need to extend the classes of vertex groups that we borrow from the JSJ theory of groups. In particular, the abelian decompositions that appear along the MR diagram for pairs contain a new type of vertex groups that we call *Levitt*. These Levitt vertex groups are free factors of the ambient limit groups that are connected to other vertex groups by edges with trivial stabilizers, and each such vertex group contributes the group of its automorphisms (that are automorphisms of free groups that are not necessarily geometric) to the modular group of the pair with which it is associated.

In the MR diagram over free groups, when we follow a path (resolution) in the diagram, the groups that are associated with the nodes along the path, form a finite sequence of successive proper quotients (maximal shortening quotients as they appear in [Se1]). In the MR diagram over free semigroups that we construct this is not true. When we go over a resolution in the diagram, the pairs (and groups) that appear along the nodes of the diagram are quotients, but not necessarily proper quotients.

The resolutions in the MR diagram over free groups end with free groups of various ranks. In the MR diagram over free semigroups, resolutions end with marked graphs with directed edges, that have free groups as their fundamental groups. Group homomorphisms from a given free group into the coefficient free group are easily understood using substitutions, and they are in bijection with a product set of the coefficient group. In the case of semigroups, the set of homomorphisms that are associated with a terminal node are also obtained by substitutions, and these sets can be viewed as natural projections of affine spaces, i.e., natural projections of product sets of the coefficient semigroup.

Finally I would like to thank Eliyahu Rips who encouraged me to proceed along this long project, that somehow involved repeated mistakes and misconceptions.

§1. Maximal Pairs

In a similar way to the study of equations over groups [Ra1], with a finite system of equations Φ over a free semigroup $FS_k = < a_1, \dots, a_k >$ it is natural to associate a f.p. semigroup $S(\Phi)$. If the system Φ is defined by the coefficients $a_1, \dots, a_k$, the unknowns $x_1, \dots, x_n$ and the equations:

$$u_1(a_1, \dots, a_k, x_1, \dots, x_n) = v_1(a_1, \dots, a_k, x_1, \dots, x_n)$$
$$\vdots$$
$$u_s(a_1, \dots, a_k, x_1, \dots, x_n) = v_s(a_1, \dots, a_k, x_1, \dots, x_n)$$

we set the associated f.p. semigroup $S(\Phi)$ to be:

$$S(\Phi) \ = < \ a_1, \dots, a_k, x_1, \dots, x_n \,|\, u_1 = v_1, \dots, u_s = v_s \ >$$

Clearly, every solution of the system Φ corresponds to a homomorphism (of semigroups) $h : S(\Phi) \to FS_k$ for which $h(a_i) = a_i$, and every such homomorphism corresponds to a solution of the system Φ. Therefore, the study of sets of solutions to systems of equations in a free semigroup is equivalent to the study of all homomorphisms from a fixed f.p. semigroup S into a free semigroup FS_k, for which a given prescribed set of elements in S is mapped to a fixed basis of the free semigroup FS_k.

Hence, as in studying sets of solutions to systems of equations over a free or a hyperbolic group [Se1], to study sets of solutions to systems of equations over a free semigroup, we fix a f.p. (or even a f.g.) semigroup S, and study the structure of its set of homomorphisms into a free semigroup, FS_k, that we denote, $Hom(S, FS_k)$.

Given a f.g. semigroup, S, we can naturally associate a group with it. Given a presentation of S as a semigroup, we set the f.g. group $Gr(S)$ to be the group with the presentation of S interpreted as a presentation of a group. Clearly, the semigroup S is naturally mapped into the group, $Gr(S)$, and the image of S in $Gr(S)$ generates $Gr(S)$. We set $\eta_S : S \to Gr(S)$ to be this natural homomorphism of semigroups.
The free semigroup, FS_k, naturally embeds into a free group, F_k. By the construction of the group, $Gr(S)$, every homomorphism of semigroups, $h : S \to FS_k$, extends to a homomorphism of groups, $h_G : Gr(S) \to F_k$, so that: $h = h_G \circ \eta_S$.

Our goal is to study the structure of the set of homomorphisms (of semigroups), $Hom(S, FS_k)$. By construction, every homomorphism (of semigroups), $h : S \to FS_k$, extends to a homomorphism (of groups), $h_G : Gr(S) \to F_k$. Therefore, the study of the structure of $Hom(S, FS_k)$, is equivalent to the study of the structure of the collection of homomorphisms of groups, $Hom(Gr(S), F_k)$, that restrict to homomorphisms of (the semigroup) S into the free semigroup (the *positive cone*), FS_k.

By the techniques of section 5 in [Se1], with any given collection of homomorphisms of a f.g. group into a free group, we can associate its Zariski closure, and with the Zariski closure one can associate canonically a finite collection of limit groups, that are all (maximal) limit quotients of the given f.g. group, so that every homomorphism from the given collection factors through at least one of the quotient maps from the given f.g. group into the (finitely many) limit quotients. By (canonically) associating a finite collection of maximal limit quotients with the set of homomorphisms, $Hom(Gr(S), F_k)$, that restrict to (semigroup) homomorphisms from S to FS_k, we get the following basic theorem, which is the basis for our approach to study the structure of $Hom(S, FS_k)$.

Theorem 1.1. *Let S be a f.g. semigroup, and let $Gr(S)$ be the f.g. group that is associated with S, by interpreting a semigroup presentation of S, as a presentation of a group. Let $\eta_S : S \to Gr(S)$ be the natural semigroup homomorphism, and note that $\eta_S(S)$ generates $Gr(S)$ as a group.*

There exists a finite canonical collection of (limit) pairs, $(S_1, L_1), \dots, (S_m, L_m)$, where the L_i's are limit quotients of $Gr(S)$, and the semigroups, S_i, are quotients of the semigroup S that generate the limit groups L_i as groups, with the following properties:

1. *for each index i, $1 \leq i \leq m$, there exists a (canonical) quotient map of pairs, $\tau_i : (S, Gr(S)) \to (S_i, L_i)$.*
2. *by construction, every homomorphism of semigroups, $h : S \to FS_k$, extends to a map of pairs, $h_P : (S, Gr(S)) \to (FS_k, F_k)$. For each such homomorphism of pairs, there exists an index i, $1 \leq i \leq m$, and a homomorphism of pairs, $u_h : (S_i, L_i) \to (FS_k, F_k)$, for which: $h_P = u_h \circ \tau_i$.*

Proof: Identical to the proof of theorem 7.2 in [Se1].

□

In theorem 7.2 in [Se1], it is shown that with each f.g. group it is possible to associate a canonical finite collection of limit groups, so that each homomorphism from the f.g. group into a free group factors through at least one of the finitely many limit quotients. Theorem 1.1 is the analogue of that theorem for semigroups.

It reduces (canonically) the study of the structure of the set of semigroup homomorphisms from a given semigroup to a free semigroup, $Hom(S, FS_k)$, to the study of the structure of homomorphisms of finitely many pairs $\{(S_i, L_i)\}$ into (FS_k, F_k), where the L_i's are limit groups, and the S_i's are subsemigroups of the L_i's that generate the limit groups L_i's as groups.

§2. Positive Cones, their embeddings in Standard Cones, and Shortenings

To analyze homomorphisms of a f.g. semigroup into the free semigroup we analyzed sequences of homomorphisms of pairs (S, G) into the standard pair (FS_k, F_k). In studying such homomorphisms of pairs, the subsemigroup S is viewed as the positive cone in the ambient group G. To get a structure theory for the entire collection of pair homomorphisms we are led to replacing the given cone with a standard cone. The structure of a standard cone depends on the structure of the ambient group, and more specifically on the structure of the tree that is obtained as a limit of a sequence of homomorphisms, and the dynamics of the action of the ambient group on that tree.

In general, a standard cone can not be embedded into the ambient group G, but rather into an extension of G. Still, a standard cone is essential in applying the shortening argument for semigroups, for any attempt to construct JSJ decompositions, for separating free factors, and in general for constructing a Makanin-Razborov diagram that is associated with a pair (S, L).

Let (S, L) be a pair of a limit group, L, and its subsemigroup, S, that generates it as a group. Let $\{h_n : (S, L) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on a real tree Y. In this section we show how to extend the subsemigroup S to a bigger semigroup, by adding to it standard generating sets of the various components of the real tree Y. We further make sure that the (finitely many) generators of the original subsemigroup S can be expressed as positive words in the standard generating sets that are associated with the components of the limit tree Y.

The standard generating sets that we add depend on the dynamics of the action of the limit group L on the limit tree Y. We start with the rather basic case, of an axial action of a free abelian group on a line.

Theorem 2.1. *Let (S, A) be a pair of a free abelian group group A and a subsemigroup S that generates A. Let $\{h_n\}$ be a sequence of pair homomorphisms of (S, A) into (FS_k, F_k) that converges into a faithful axial action of A on a line Y. Let A_0 be the direct summand of A that acts trivially on Y, and suppose that: $rk(A) - rk(A_0) = \ell$.*

Then there exists a positive collection of generators, $a_1, \dots, a_\ell$, in $A \setminus A_0$ so that:

(1) $A = A_0 + < a_1 > + \ldots + < a_\ell >$.
(2) *there exists some index n_0, such that for every $n > n_0$: $h_n(a_1), \dots, h_n(a_\ell) \in FS_k$.*
(1) *(3) for each of the generators $s_1, \dots, s_r$ of the semigroup S, s_j can be expressed as a positive word in $a_1, \dots, a_\ell$ modulo an element in A_0. i.e., for every j, $1 \leq j \leq r$:*

$$s_j = a_1^{m_1^j} \ldots a_\ell^{m_\ell^j} a_0(j)$$

where $m_i^j \geq 0$ and $a_0(j) \in A_0$.

(4) *Wlog we can assume that for every j, $1 \leq j \leq r$, and every $n > n_0$, $h_n(a_0(j)) \in FS_k$.*

Proof:

Let $a_1, \dots, a_\ell$ be elements in A, for which $A = A_0 + < a_1 > + \dots + < a_\ell >$, and so that the elements $a_1, \dots, a_\ell$ translate along the axis of A (the real tree Y) in a positive direction. Since the action of A on its axis is an axial action, a_j translates a positive distance α_j, and the real numbers α_j, $1 \leq j \leq \ell$, are independent over the rationals.

Each of the fixed set of generators of the semigroup S, $s_1, \dots, s_r$, can be written as a word in the elements $a_1, \dots, A_\ell$ times an element in A_0. We modify the elements $a_1, \dots, a_\ell$ iteratively, so that the elements $s_1, \dots, s_r$ can be represented as positive words in the modified set of generators. If an element s_j is already a positive word in the generators, then the modifications of the generators that we perform, will change the positive word to another positive word in the new generators.

Suppose that after rearranging the order of the given generators, s, one of the elements, $s_1, \dots, s_f$, is represented by the word:

$$s \;=\; \Sigma_{j=1}^{m} k_j a_j \;-\; \Sigma_{j=m+1}^{\ell} t_j a_j$$

Wlog we may assume that t_{m+1} is one of the maximal elements from the set $t_{m+1}, \dots, t_\ell$. To prove the theorem we show that after finitely many modifications of the set of generators, s can be represented by another word in a new set of positive generators, so that the absolute value of the negative coefficients is still bounded by t_{m+1}, and the number of appearances of the coefficient $-t_{m+1}$ is strictly smaller than the number of appearances of it in the original word.

Suppose that one of the elements a_j, $j = 1, \dots, m$, satisfies $a_j > a_{m+1}$. In that case we replace the (positive) generator a_j with a positive generator $\hat{a}_j = a_j - a_{m+1}$. With respect to the new set of generators the coefficients k_j are unchanged, $-t_{m+1}$ is replaced by $-t_{m+1} + k_j$ and t_j, $j = m+2, \dots, \ell$ are unchanged. In particular, all the new negative coefficients are bounded below by the previous $-t_{m+1}$, and the number of negative coefficients that are equal to $-t_{m+1}$ is reduced by 1.

Suppose that all the elements a_j, $j = 1, \dots, m$, satisfy $a_j < a_{m+1}$. We start by replacing the generator a_{m+1} with $\hat{a}_{m+1} = a_{m+1} - a_1$. This replaces k_1 by $\hat{k}_1 = k_1 - t_{m+1}$ and keeps all the other coefficients unchanged. If $\hat{k}_j = 0$ we have reduced the number of generators that participate in the word that represents the element s. If $\hat{k}_j > 0$ we do the following:

(1) If there exists an index j, $j = 1, \dots, m$, for which $a_j > \hat{a}_{m+1}$, we replace the generator a_j with $a_j - a_{m+1}$, and hence replace the coefficient $-t_{m+1}$ with $-t_{m+1} + k_j$. This reduces the number of negative coefficients that are equal to the previous $-t_{m+1}$ by 1.

(2) Suppose that for all j, $j = 1, \dots, m$, $a_j < \hat{a}_{m+1}$. In that case we replace $\hat{a}_{m+1}$ by $\hat{a}_{m+1} - a_1$, and hence replace $\hat{k}_1$ by $\hat{k}_1 - t_{m+1}$.

In that case after finitely many steps either the coefficient of the first generator becomes 0 or negative, or we reduced the number of negative coefficients that are the biggest in their absolute value (i.e., equal to $-t_{m+1}$).

If $\hat{k}_1 < 0$ then we reduced the number of generators with positive coefficients in the representation of the element s. Hence, we repeat the same steps where the elements that have positive coefficients are only $a_2, \dots, a_m$. Therefore, after finitely many steps the number of generators with negative coefficient that are the biggest in their absolute value is strictly smaller. Finally, after finitely many steps we obtain a new set of generators for which the element s is represented by a positive word. Continuing to the other elements $s_1, \dots, s_r$, we get a generating set for which all these elements are represented by positive words.

□

Theorem 2.1 constructs a positive basis in case the ambient limit group is abelian and its limit action is axial. The next case that we consider is the case of a pair (S, L), in which the ambient limit group L is a surface group, and L acts freely on the limit (real) tree. In this case of an IET action of the surface group L, we were not able to prove the existence of a positive standard cone. Instead we prove a weaker statement that enables one to apply shortening arguments in the sequel (shortening arguments are essential in the constructions of the JSJ decompositions, and the Makanin-Razborov diagrams in the sequel).

Theorem 2.2. *Let Q be a (closed) surface group, and let (S, Q) be a pair where S is a f.g. subsemigroup that generates Q as a group. Let $\{h_n\}$ be a sequence of pair homomorphisms of (S, Q) into (FS_k, F_k) that converges into a free (IET) action of Q on a real tree Y.*

Then there exists a sequence of automorphisms $\{\varphi_\ell \in ACT(Q)\}_{\ell=1}^{\infty}$ with the following properties:

1. *for each index ℓ, there exists $n_\ell > 0$, so that for every $n > n_\ell$, and all the generators of the subsemigroup S, $s_1, \dots, s_r$, $h_n \circ \varphi_\ell(s_j) \in FS_k$. (note that the automorphisms φ_ℓ do not preserve the subsemigroup S in general, but the image of S under the twisted homomorphisms $h_n \circ \varphi_\ell$ remains in the standard positive cone in F_k).*
2. *for every $n > n_\ell$ and every j, $1 \le j \le r$:*

$$\frac{d_T(h_n \circ \varphi_\ell(s_j), id)}{\max_{1 \le j \le r} d_T(h_n(s_j), id.)} < \frac{1}{\ell}$$

   where T is a Cayley graph of the coefficient free group F_k with respect to a fixed set of generators.

Proof: The generators $s_1, \dots, s_r$ of the subsemigroup S generate the surface group Q. The action of Q on the limit tree Y is an IET action, which is in particular a geometric action. Hence, there exists a finite subtree R of Y, such that if one applies the Rips machine to the action of the pseudogroup generated by the restrictions of the actions of the generators $s_1, \dots, s_r$ to the finite tree R, the output is a standard IET pseudogroup. i.e., the output is supported on a union of finitely many intervals that are all part of one positively oriented interval, and these finitely many intervals are divided into finitely many subintervals, a permutation of these subintervals, and a standard set of generators of Q that map each subinterval to its image (dual) under the permutation. Note that the generators of Q that are associated with the IET presentation, are obtained by gluing the finitely many intervals on which the IET is supported into a (finite) tree, and then the generators of Q are the elements that map a subinterval into its dual. Hence, the generators of Q that are associated with the IET transformation need not be positively oriented in case the IET is supported on more than one connected interval. This is in contrast with the positive orientation of the original generators $s_1, \dots, s_r$, and of each of the subintervals that define the IET transformation.

For presentation purposes we first assume that the endpoints of the intervals in the interval exchange (that is obtained by the Rips machine from the action of Q on Y) belong to a single orbit under Q (i.e., that the generators of the interval exchange act transitively on the endpoints of the intervals in the IET), and that the base point in Y (i.e., the point in Y which is a limit of the identity element in the Cayley graph of F_k) belongs to that orbit.

The action of the surface group Q on the tree Y is reduced to an interval exchange transformation that is based on positively oriented intervals $I_1, \dots, I_g$. By our assumption, the points that are at the beginning and at the end of the subintervals that define the interval exchange are all in the orbit of the basepoint in Y. Let $q_0, \dots, q_f \in I$ be the points that are at the beginnings and at the ends of the subintervals that define the IET transformation on the intervals $I_1, \dots, I_g$. We look at those indices i, $1 \leq i \leq f$, for which the segment $[q_{i-1}, q_i]$ is supported on one of the segments $I_1, \dots, I_g$. For each such index i, there is a unique element in Q that maps the subinterval $[q_{i-1}, q_i]$ to a (positively oriented) subinterval $[y_0, y_i]$, where y_0 is the basepoint in Y. For each i, $1 \leq i \leq f$, we set $v_i \in Q$ to be the element that maps y_0 to y_i.

By construction, for large enough index n, $h_n(v_i) \in FS_k$, for every i, $1 \leq i \leq f$. The elements in Q, that restrict to the elements that are associated with the interval exchange transformation, i.e., the elements that map subintervals to their companions (after gluing the intervals $I_1, \dots, I_g$ and obtaining a finite tree), generate the surface group Q. Each of these elements is in the subgroup that is generated by $v_1, \dots, v_f$. Hence, $v_1, \dots, v_f$ generate Q.

For each j, $1 \leq j \leq r$, the segment $[y_0, s_j(y_0)]$ is positively oriented, and s_j can be presented as a word in the elements $v_1, \dots, v_f$, $s_j = w_j(v_1, \dots, v_f)$. The words w_j need not be positive words in the elements $v_1, \dots, v_f$. For each index j, $w_j(v_1, \dots, v_f)$ represents a finite path in the tree Y, that starts at y_0 and ends at $s_j(y_0)$. We denote this path p_{w_j}. Since the words w_j need not be positive words, p_{w_j} need not be positively oriented.

Each path p_{w_j} is supported on a finite subtree of the real tree Y that we denote T_{w_j}. We view T_{w_j} as a combinatorial tree and not as a metric tree. To T_{w_j} we add a finite collection of vertices:

(1) a vertex for the base point of Y (the initial point of p_{w_j}), and for the point $s_j(y_0)$ (the terminal point of p_{w_j}).
(2) a vertex for each root and each branching point in T_{w_j}.
(3) each segment in T_{w_j}, being a subsegment of Y, can be divided into finitely many segments with either positive or negative orientation. If there exists an edge in T_{w_j} which is not oriented (i.e., the edge can be cut into subsegments with both positive and negative orientations), then the word w_j can be strictly shortened as a word in $v_1, \dots, v_f$, and still represents $s_j \in Q$. Therefore, we can assume that every edge in T_{w_j} is oriented.
(3) p_{w_j} starts at the base point in Y, and ends in $s_j(y_0)$. While moving along p_{w_j}, at certain segments of the path the distance to the base point $y_0 \in Y$

increases, and at other segments it decreases. At each point which is the boundary of such (increasing or decreasing) segments along p_{w_j} (i.e., points in which the distance to y_0 changes from increasing to decreasing or vice versa), we add a vertex to T_{w_j}.

In the sequel we denote the paths p_{w_j} that are associated with the words w_j on T_{w_j}.

At this point we apply a sequence of Dehn twists on the generators of the interval exchange (which is the output of the Rips machine from the action of the surface group Q on the real tree Y). We perform the sequence of Dehn twists from the positive side of the interval on which the IET is based. Hence, the sequence of IETs that we obtain are supported on a decreasing sequence of subintervals of the original interval I that supports the original IET. All these supporting subintervals share the endpoint y_0 which is the base point in Y. Since the action of Q on Y was assumed to be a minimal IET action, the lengths of the supporting intervals has to approach 0.

With each of the obtained IETs we associate a positive generating set for a subsemigroup of Q, in a similar way to the association of $v_1, \dots, v_f$ with the original IET. We denote the corresponding set of generators $v_1^t, \dots, v_f^t$ (note that the number of generators of the corresponding semigroups can only decrease, so by omitting finitely many of them, we can assume that their number is fixed).

Lemma 2.3. *For every index $t \geq 1$, each of the elements $v_1, \dots, v_f$ is contained in the subsemigroup that is generated by: $v_1^t, \dots, v_f^t$.*

Proof: By the definition of a Dehn twists that is performed from the positive side of the supporting interval, for every index $t \geq 1$, each of the elements $v_1^{t+1}, \dots, v_f^{t+1}$ can be written as a (concrete) positive word in the elements $v_1^t, \dots, v_f^t$. Hence, the claim of the lemma follows by induction.

□

By lemma 2.3 each of the elements $v_1, \dots, v_f$ can be written as a positive word in the elements $v_1^t, \dots, v_f^t$). By substituting these positive words in the words w_j we obtain presentations of the elements s_j, $1 \leq j \leq r$, in terms of the generating sets $v_1^t, \dots, v_f^t$:

$$s_j \,=\, w_j(v_1, \dots, v_f) \,=\, w_j^t(v_1^t, \dots, v_f^t).$$

Since the elements $v_1, \dots, v_f$ and $v_1^t, \dots, v_f^t$ are positively oriented, and the elements v_j are presented as positive words in the generators $v_1^t, \dots, v_f^t$, the paths that are associated with the words w_j^t in the tree Y, that we denote $p_{w_j^t}$, are identical to the paths p_{w_j}, that are associated with the words w_j.

While shortening the lengths of the elements $v_1^t, \dots, v_f^t$, the ratios between these lengths may not be bounded. To prove theorem 2.2 we first need to replace the elements $v_1^t, \dots, v_f^t$ by elements that are not longer than them, and for which the ratios between their lengths are universally bounded. We start by proving the existence of such generators assuming a global bound on the periodicity of the images of the given set of generators of the semigroup S, $s_1, \dots, s_r$, under the sequence of homomorphisms $\{h_n\}$. In the sequel we prove a general version of the proposition omitting the bounded periodicity assumption.

Proposition 2.4. *Suppose that there exists an integer c_p, such that the periodicity* *of the elements $h_n(s_1),\dots,h_n(s_r)$ is bounded by c_p for all integers n. i.e., for every n, and every j, $1\le j\le r$, $h_n(s_j)$ can not be written as a word of the form: $s_j=w_1\alpha^{c_p+1}w_2$, where $w_1,w_2,\alpha\in FS_k$, and α is not the empty word.*

For every $t\ge 1$, the elements $v_1^t,\dots,v_f^t\in Q$ can be replaced by elements $u_1^t,\dots,u_{g_t}^t\in Q$, with the following properties:

1. *$f\le g_t\le e(f)$.*
2. *for every index t, the semigroup that is generated by $u_1^t,\dots,u_{g_t}^t$ in Q contains the semigroup that is generated by $v_1^t,\dots,v_f^t$ in Q.*
3. *for every $t\ge 1$, and for large enough index n, $h_n(u_i^t)\in FS_k$ for every index i, $1\le i\le g_t$,*
4. *the tuples $u_1^t,\dots,u_{g_t}^t$ belong to finitely many isomorphism classes (under the action of $Aut(Q)$).*
5. *because of (2) each of the elements $v_1^t,\dots,v_f^t$ can be represented as a positive word in the elements $u_1^t,\dots,u_{g_t}^t$. Substituting these words in the words w_j^t, for each $t\ge 1$ and each j, $1\le j\le r$, $s_j=z_j^t(u_1^t,\dots,u_{g_t}^t)$. With each word $z_j^t(u_1^t,\dots,u_{g_t}^t)$ we can associate a path, $p_{z_j^t}$ in the tree Y, and this path is identical to the path $p_{w_j^t}$, that is identical to p_{w_j}.*
6. *there exist positive constants d_1,d_2 that depend only on f and c_p, such that for every $t\ge 1$, there exist a sequence of indices (that depend on t) $1\le i_1<i_2<\ldots<i_{b(t)}\le g_t$, such that for every $1\le m_1<m_2\le b(t)$:*

   $$d_1\cdot length(u_{i_{m_1}}^t)\;\le\; length(u_{i_{m_2}}^t)\;\le\; d_2\cdot length(u_{i_{m_1}}^t).$$

   Furthermore, for every index i, $1\le i\le g_t$ for which $i\ne i_m$, $m=1,\dots,b(t)$:

   $$10g_t\cdot d_2\cdot length(u_i^t)\;\le\; length(u_{i_1}^t)$$

7. *For each index t, in the words z_j^t, $1\le j\le r$, in a distance bounded by $g(t)$ (that is bounded by the function $e(f)$), either before or after the occurrence of an element u_i^t, for $i\ne i_1,\dots,i_{b(t)}$, appears one of the elements $u_{i_m}^t$, $1\le m\le b(t)$.*

Proof: To get a new set of elements that satisfy the conclusion of the proposition we need to modify the standard sequence of Dehn twists that we used to get the elements $v_1^t,\dots,v_f^t$ (which are Dehn twists that are performed on the pair of bases that are adjacent to the vertex at the positive end of the interval on which the interval exchange transformation is supported).

We start by dividing the generators $v_1^t,\dots,v_f^t$ into finitely many sets according to their length. We order the elements from the longest to the shortest. We place a separator between consecutive sets whenever there is a pair of consecutive elements (ordered according to length) that satisfy:

$$length(v_i^t)\ge c_1(f,c_p)\cdot length(v_{i+1}^t)$$

where $c_1(f,c_p)=4fc_p$.

Clearly the elements v_i^t are divided into at most f sets. If there are no separators, we set $g_t=f$, and $u_i^t=v_i^t$, $i=1,\dots,f$. Suppose that there is a separator. In that case our goal is to construct a procedure that will iteratively reduce the number of separators. We call the elements that are in the first (longest) set *long* and the elements in all the other (shorter) sets *short*.

At this point we modify the standard Dehn twists that are performed on an interval exchange basis, so that the performed Dehn twists do not change short bases (short elements can be modified in a controlled way). Let I be the interval that supports the IET transformation, and let pv be the vertex at the positive end of I. On the given IET transformation we perform the following operations:

(1) Suppose that the two bases that are adjacent to pv are long, i.e., each of them contains a long element. Let b_1 be the longer base that is adjacent to pv. Suppose that the other end of b_1 is not covered by a long base. In this case we cut b_1 at the end of the last long base that completely overlaps with b_1 and perform Makanin's *entire transformation* over the base b_1 (see [Ca-Ka] for Makanin's entire transformation). The length of what is left from b_1 after the entire transformation is at most f times the length of the longest short element. For the rest of this part, before we change the set of long elements, this part of b_1 is declared to be a *short* base. In this case we reduced the number of pairs of long bases by 1 and added a pair of short bases.

By performing an entire transformation over the base b_1 we transfered several long bases (and elements), and several short bases (and elements) that now replace the base $\hat{b}_1$ that was previously paired with the base b_1. In the original interval exchange, that is based on the entire interval I, every point is covered exactly twice. Hence, the subinterval that supports the base $\hat{b}_1$ supports another sequence of (possibly) long and short bases (and elements). The endpoints of these bases is now used together with the endpoints of bases that were transfered using the entire transformation to define the generators of a new semigroup, that contains the semigroup that is associated with the original IET that is supported on I.

The generators of the semigroup that is associated with the new IET contains some generators of the previous IET (that is based on the entire interval I), and new generators that are located between endpoints of bases that were transformed over the base b_1 and bases that overlapped with $\hat{b}_1$ in the IET that we started with. We divide the new elements according to the following rules.

Suppose that a previously long element (generator) is cut into finitely many new elements by the new endpoints of bases. If the length of all the fractions of the previously long element are at least $c_1(f, c_p)$ times the maximal length of a short element we consider all the new fractions to be *long*. Otherwise, the length of at least one of the fractions is at least $\frac{c_1(f,c_p)}{3}$ times the maximal length of a short element. If there are fractions that are of length at least $c_1(f, c_p)$ times the maximal length of a short element we declare them to be *long* and the rest to be *secondary short* elements. If there are no such fractions, we declare the fractions of length at least $\frac{c_1(f,c_p)}{3}$ times the length of the maximal short element to be *short* and the rest to be *secondary short*.

Suppose that a previously short element (generator) is cut into finitely many new elements by the new endpoints of bases. The length of at least one of the fractions is at least $\frac{1}{f}$ times the previous length of that short element. We declare the fraction with maximal length *short* and all the other fractions *secondary short*.

(2) Suppose that the other end (the beginning) of b_1 is covered by another long base. In this case we perform Makanin's entire transformation over the carrying base b_1, and continue as in part (1). If what left from b_1 after the entire transformation is a long element we continue to the next step, and what left from b_1 remains a long element. If what left is bounded by $c_1(f, c_p)$ times the length of the maximal short element it is set to be a *secondary short element*, and the corresponding base to be *short*. In this last case the number of pairs of long bases is reduced by 1 and the number of pairs of short bases is increased by 1.

(3) Suppose that the bases that are adjacent to pv are a short and a long base. Suppose that the length of the long base that is adjacent to pv, that we denote b_1, is bigger than the sum of the lengths of the short bases that are adjacent to pv and the first long base that is adjacent to them. In that case we perform Makanin's entire transformation precisely as in cases (1) and (2).

(4) Suppose that a long and a short base are adjacent to pv, and that the length of the long base that is adjacent to pv, that we denote b_1, is smaller than the sum of the short bases that are adjacent to pv and the first long base that is adjacent to these short bases. We set b_2 to be the first long base that is adjacent to the short bases that are adjacent to pv.

If b_1 and b_2 are not a pair of bases we do the following. We first transfer the short bases that are adjacent to pv using the base b_1, and then perform an entire transformation over the base b_2 precisely as we did in cases (1) and (2).

Suppose that b_1 and b_2 are a pair of bases. This implies that b_1 (and b_2) contains a large periodic word, which is a power of the chain of short bases that are adjacent to pv. We assumed that the periodicity of the words $h_n(s_j)$ is bounded by c_p for all n, and all j, $1 \leq j \leq r$. Hence, the length of b_1 (and b_2) is bounded by $f \cdot (c_p + 1)$ times the maximal length of a small element. This implies that there can not be a separator between the length of an element that is supported by b_1 or b_2 and the collection of small elements. Therefore, in this case b_1 (and b_2) can not be long bases, a contradiction, so in case (4) b_1 and b_2 can not be a pair of bases (assuming bounded periodicity).

(5) Suppose that two short bases are adjacent to pv. Let b_1 and b_2 be the two long bases that are adjacent to the two sequences of short bases that are adjacent to pv. b_1 and b_2 can not be a pair of bases because of our bounded periodicity assumption, as we argued in part (4). Suppose that b_1 is longer than b_2, and that the sequence of short bases that are adjacent to pv and b_1 is longer than the sequence that is adjacent to pv and b_2. In that case, we first cut b_2 (and the base that is paired with it) into two bases at the endpoint of b_1. We declare the part of b_2 that overlaps with short bases before it overlaps with b_1 to be *short*. Then we perform an entire transformation along b_1 precisely as we did in cases (1) and (2). In this case we either reduce the number of long pairs of bases (if the beginning of b_1 overlaps with a short base), or we added a short base and left behind at least two short bases, so the number of short bases that participate in

the next iterations in this part (before we change the set of long bases) is reduced by at least 1.

(6) With the notation and the assumptions of part (5), suppose that the sequence of short bases that are adjacent to pv and b_1 is shorter than the sequence that is adjacent to pv and b_2. In that case, we first cut b_1 (and the base that is paired with it) into two bases at the endpoint of b_2. We declare the part of b_2 that overlaps with short bases before it overlaps with b_1 to be short. Then we perform an entire transformation along the longer between the remaining of b_1 and b_2, precisely as we did in cases (1) and (2). As in part (5), in this case we either reduce the number of long pairs of bases, or we added a short base and left behind at least two short bases, so the number of short bases is reduced by at least 1.

In step (1) the number of pairs of long bases is reduced by at least one, and the number of short bases is increased by at most the reduction in the number of long bases. In step (2) there may be no change in the number of bases, and if there is a change the number of short bases is increased by at most the reduction in the number of long bases. The outcome of steps (3) and (4) in terms of the number of bases is identical to that of steps (1) and (2). The outcome of steps (5) and (6) on the number of bases is similar to that of steps (1) and (2) with an additional increase of the number of short bases by 1, but an additional reduction of the number of active short bases (i.e., the number of bases that take part in the next steps of the iterative procedure) by at least 1.

Hence, when we run this procedure until there are no long elements. i.e., until all the elements are either short or secondary short, the number of bases can grow to at most $2f$, and therefore the number of elements can grow to at most $4f$.

Along the procedure whenever a long base is cut into a finite collection of (only) short and secondary short elements, then the short elements have a length which is bounded below by $\frac{c_1(f,c_p)}{2f}$ times the maximal length of a previous short element. When a short base is cut into a collection of short and secondary short elements, then the length of the short (and not necessarily the secondary short) fractions is bounded below by $\frac{1}{4f}$ times the length of the original short base. Furthermore, the obtained short fractions can be effected only by steps (5) and (6) before the set of long elements is changed, and these steps do not effect these short fractions.

When the procedure reached the state in which there are no long elements, we change the place of the separators between the elements. Once again we divide the new collection of elements into finitely many sets according to their length. We order the short and secondary short elements from the longest to the shortest. We place a separator between consecutive sets whenever there is a pair of consecutive elements (ordered according to length) that satisfy:

$$length(\hat{v}_i^t) \geq c_2(f,c_p) \cdot length(\hat{v}_{i+1}^t)$$

where $c_2(f,c_p) = c_p(4f)^2$.

The new elements $\hat{v}_i^t$ may be divided (by the separators) into a larger number of sets than the previous separated sets. However, the number of such sets that contain at least one short element (and not only secondary short elements) is at most the previous number of (separated) sets minus 1, and in particular is bounded by $f-1$. Also, by the construction of the procedure, the (separated) set that contains the longest elements must contain a short element and not only secondary short elements.

If there is only one separated set that contains a short element (which must be the separated set of longest elements), we reached a terminal state of the iterative procedure. Otherwise, if there is more than one such set, we declare a new collection of long elements. We declare the set of long elements to be all the elements in the separated set with the longest elements, and the elements in all the next (according to length of elements) separated sets, until the next separating set that contains a short element (where this last separated set is excluded).

We continue itartively. After each round of iterations the number of separated sets that contain short elements reduces by at least one, so after at most f rounds we end up with a unique set, the one that contains the longest elements, that contains short elements. All the other separated sets contain only secondary short elements.

Since the number of bases can multiply by at most 4 in each iteration, the number of bases when the procedure terminates is bounded above by $4^f \cdot f$. In particular, the number of elements in the set that contains the longest elements (the only set that contains short and not only secondary short elements) is bounded by $4^f \cdot f$. The ratios between consecutive elements in this set is bounded above by $c_p \cdot (4^f \cdot f)^f$. Hence, the ratio between the longest and the shortest element in this set is bounded above by $\{c_p \cdot (4^f \cdot f)^f\}^{(4^f \cdot f)}$.

We set the elements that the iterative procedure ends up with, to be the generators $u_1^t, \dots, u_{g(t)}^t$. $g(t)$, the number of elements is bounded above by $e(f) = 4^f \cdot f$. We set the elements in the separated set with the longest elements, to be the elements $u_1^t, \dots, u_{b(t)}^t$, and the ratios between their lengths is bounded above by $\{c_p \cdot (4^f \cdot f)^f\}^{(4^f \cdot f)}$. The elements that are not in the separated set with the longest elements, are all secondary short elements. By construction, in representing the original elements v_i^t, in a distance bounded by $g(t)$, hence by $e(f)$, to the appearance of a secondary short element appears one of the short elements, so one of the elements, $u_{i_1}^t, \dots, u_{i_{b(t)}}^t$. Therefore, part (6) of the proposition holds.

□

The next proposition is the key in our proof of theorem 2.2. Since we proved proposition 2.4 under a bounded periodicity assumption, we state and prove it under the same (bounded periodicity) assumption (and generalize it in the sequel). It implies that the words z_j^t (in part (5) of the statement of proposition 2.4) can be replaced by words $\hat{z}_j^t$ with uniformly bounded cancellation.

Proposition 2.5. *Suppose that there exists an integer c_p, such that the periodicity of the elements $h_n(s_1), \dots, h_n(s_r)$ is bounded by c_p for all integers n.*

With the notation of proposition 2.4, there exists a constant $C > 0$, so that for every index $t \geq 1$ the words z_j^t, $1 \leq j \leq r$, can be replaced by words: $\hat{z}_j^t$ with the following properties:

1. *As elements in Q: $\hat{z}_j^t(u_1^t, \dots, u_{g_t}^t) = z_j^t(u_1^t, \dots, u_{g_t}^t)$.*
2. *$\hat{z}_j^t$ is obtained from z_j^t by eliminating distinct pairs of subwords. Each pair of eliminated subwords corresponds to two subpaths of the path $p_{z_j^t}$ that lie over the same segment in the tree T_{w_j}, where the two subpaths have opposite orientations.*
3. *With the word $\hat{z}_j^t(u_1^t, \dots, u_{g_t}^t)$ we can naturally associate a path in the tree*

Y, that we denote, $p_{\hat{z}^t_j}$. The path $p_{\hat{z}^t_j}$ can be naturally divided into subsegments according to the appearances of the subwords u^t_i in the word $\hat{z}^t_j$.

Let $DB_{\hat{z}^t_j}$ be the number of such subsegments that are associated with subwords u^t_i in $p_{\hat{z}^t_j}$, that at least part of them is covered more than once by the path $p_{\hat{z}^t_j}$. Then for every $t \geq 1$ and every j, $1 \leq j \leq r$, $DB_{\hat{z}^t_j} \leq C$.

Proof: Suppose that such a constant C does not exist. Then for every positive integer m, there exists an index j_m, $1 \leq j_m \leq r$, and an index $t_m > 1$, so that for every possible choice of words $\hat{z}^{t_m}_{j_m}$ that satisfy parts (1) and (2), part (3) is false for the constant $C = m$.

By passing to a subsequence, we may assume that j_m is fixed, and we denote it j. We can further assume that for the subsequence the integer g_{t_m}, that counts the number of generators of the semigroup that is constructed in step t_m according to the procedure that is described in proposition 2.4 is fixed and we denote it g. We may also assume that the integer $b(t_m)$ and the sequences of indices $i_1, \ldots, i_{b(t_m)}$, that are associated with the sets of generators of the semigroups that are constructed according to proposition 2.4 along the subsequence $\{t_m\}$ are fixed, and we denote them b and $i_1, \ldots, i_b$.

The tree T_{w_j} has finitely many edges. By traveling along the path p_{w_j} we pair subsegments of the path that cover the same edge in T_{w_j} with opposite orientations. We start from the basepoint in T_{w_j}. Given an edge in T_{w_j} that is covered more than once by the path p_{w_j}, we pair the first and the second subsegments of p_{w_j} that pass through this edge. Note that since T_{w_j} is a tree, these two subsegments have to be of opposite orientations. If p_{w_j} passes more than 3 times through an edge in T_{w_j}, we further pair the 3rd and 4th subsegments of p_{w_j} that pass through such an edge and so on.

The number of pairs of subsegments of p_{w_j} that we obtained depends only on the original tree T_{w_j}, and the path p_{w_j}. The paths $p_{z^t_j}$ are the same as the path p_{w_j} (as paths in the real tree Y) for every index t. Our goal is to show that there exists a subsequence of the given sequence, so that for every m in the subsequence, the subwords of $z^{t_m}_j$ that are associated with subsegments of the path $p_{z^{t_m}_j}$ that were paired together, can be replaced by eliminating distinct pair of subwords (part (2) in the statement of the proposition), to subwords of uniformly bounded word length, and so that by eliminating these distinct pairs the element $z^{t_m}_j$ is replaced by an element $\hat{z}^{t_m}_j$ that represents the same element in the surface group Q and satisfies part (3) of the proposition for some constant C.

Let e be an edge in T_{w_j}, and let p_1 and p_2 be subpaths of p_{w_j} that were paired together and are supported on e. Let $p^{t_m}_1$ and $p^{t_m}_2$ be the corresponding subpaths of $p_{z^{t_m}_j}$. By construction, the word lengths of the paths $p^{t_m}_1$ and $p^{t_m}_2$ is not bounded.

For each index m, we set sc_m to be the maximal length of the elements $u^{t_m}_{i_1}, \ldots, u^{t_m}_{i_b}$ (note that sc_m approaches 0 when m grows to infinity). We set Y_m to be the real tree Y equipped with the metric that is obtained from the metric on Y by dividing it by sc_m. Q acts isometrically on Y, so it naturally acts isometrically on Y_m. By proposition 2.4 the length of each of the elements $u^{t_m}_{i_1}, \ldots, u^{t_m}_{i_b}$ is bounded above by 1, and below by d_1 (the constant d_1 is defined in part (6) of proposition 2.4).

From the actions of Q on the trees Y_m it is possible to extract a subsequence (still denoted $\{t_m\}$) that converges into a (non-trivial) action of Q on some real tree Y_∞. Q acts faithfully on Y_∞ and the length of each of the elements $u_{i_1},\dots,u_{i_b}$ on Y_∞ is bounded below by d_1 and above by 1. Since the action of Q on Y_∞ is faithful, and Q is a surface group, the action of Q on Y_∞ contains only IET and discrete components. Since we assumed a global bound on the periodicity of the images: $h_n(s_j)$ for every n, and every j, $1 \leq j \leq r$, and the elements $s_1,\dots,s_r$ generate Q, the action of Q on the limit tree Y_∞ contains no discrete components. In particular the stabilizer of every non-degenerate segment in Y_∞ has to be trivial.

Let e be an edge in T_{w_j} with subpaths p_1 and p_2 that were paired together, and let $p_1^{t_m}$ and $p_2^{t_m}$ be the corresponding subpaths of $p_{\hat{z}_j^{t_m}}$. As the lengths of the elements $u_{i_1}^{t_m},\dots,u_{i_b}^{t_m}$ (when acting on the tree Y) approaches 0 with m, the combinatorial lengths of the paths $p_1^{t_m}$ and $p_2^{t_m}$ grows to ∞ with m.

Given the appearance of a generator $u_{i_s}^{t_m}$, $s = 1,\dots,b$ in $p_1^{t_m}$, and the appearance of a generator $(u_{i_{\hat{s}}}^{t_m})^{-1}$ in $p_2^{t_m}$ that overlap in a non-degenerate segment, we define their *dual position* to be the subsegment in which they overlap, and its image in the segments that are associated with both $u_{i_s}^{t_m}$ and $u_{i_{\hat{s}}}^{t_m}$. Although the combinatorial lengths of the paths $p_1^{t_m}$ and $p_2^{t_m}$ grows to infinity with m, the number of possible dual positions between the appearances of the various generators remain bounded.

Lemma 2.6. *With the assumptions of proposition 2.5 (in particular, the bounded periodicity of the images $h_n(s_j)$), there exists a global bound R, such that for every index m (from the chosen convergent subsequence), the number of dual positions of overlapping subwords $u_{i_s}^{t_m}$ in $p_1^{t_m}$ and $(u_{i_{\hat{s}}}^{t_m})^{-1}$ in $p_2^{t_m}$ $(s,\hat{s} = 1,\dots,b)$ is bounded by R.*

Proof: Suppose that the number of dual positions is not universally bounded. Then there exists a subsequence (still denoted $\{t_m\}$), pair of indices i_s and $i_{\hat{s}}$, $s,\hat{s} = 1,\dots,b$, such that the number of dual positions of $u_{i_s}^{t_m}$ in $p_1^{t_m}$ and $(u_{i_{\hat{s}}}^{t_m})^{-1}$ in $p_2^{t_m}$ is bigger than m.

In that case, a simple piegon hole argument implies that in the action of $u_{i_s}^{t_m}$ on Y_m, there exists a subinterval $J_m \subset [y_m, u_{i_s}^{t_m}(y_m)]$ (where y_m is the base point of Y_m) so that:

(i) the length of J_m satisfies: $1 \geq length(J_m) > \epsilon_0 > 0$ for every m.
(ii) there exists some non-trivial element $q_m \in Q$, for which $q_m(J_m)$ overlaps with J_m in an interval of length that is at least $(1-\delta_m)\cdot length(J_m)$, and the sequence δ_m approaches 0 with m.

Parts (i) and (ii) clearly imply that the periodicity of the elements $h_n(s_j)$ can not be globally bounded, a contradiction to the assumptions of propositions 2.4 and 2.5.

□

Lemma 2.6 proves that in the chosen subsequence of words, $z_j^{t_m}$, the number of possible dual positions between the appearances of the possible pairs of generators, $u_{i_s}^{t_m}$ and $(u_{i_{\hat{s}}}^{t_m})^{-1}$, in overlapping paths, $p_1^{t_m}$ and $p_2^{t_m}$, remain bounded. Given a word $z_j^{t_m}$, its associated path $p_{z_j^{t_m}}$, and two of its overlapping subpaths $p_1^{t_m}$ and $p_2^{t_m}$, we use the bound on the number of dual positions to replace $z_j^{t_m}$ with a shorter word $\tilde{z}_j^{t_m}$, so that:

$$z_j^{t_m}(u_1^{t_m},\dots,u_{g_{t_m}}^{t_m}) = \tilde{z}_j^{t_m}(u_1^{t_m},\dots,u_{g_{t_m}}^{t_m}).$$

When two appearances of a pair $u_{i_s}^{t_m}$ in $p_1^{t_m}$ and $(u_{i_{\hat{s}}}^{t_m})^{-1}$ in $p_2^{t_m}$ belong to the same dual position, we can trim the paths $p_1^{t_m}$ and $p_2^{t_m}$, by erasing the (identical) subpaths between these two appearances from $p_1^{t_m}$ and $p_2^{t_m}$. We set $\tilde{z}_j^{t_m}$ to be the word that is obtained from $z_j^{t_m}$ after erasing the corresponding identical subpaths. We further repeat this erasing for all the appearances of repeating pairs in the same dual position in all the (paired) overlapping subpaths $p_1^{t_m}$ and $p_2^{t_m}$ along the path $p_{z_j^{t_m}}$. We denote the words that are obtained after this erasing, $\hat{z}_j^{t_m}$. Since by lemma 2.6 there are at most R dual positions for every pair of generators, and there are at most $(2f)^2$ pairs of generators, overlapping subpaths in the path $p_{\hat{z}_j^{t_m}}$ have combinatorial length bounded by $R \cdot (2f)^2$. As the number of overlapping subpaths in $p_{\hat{z}_j^{t_m}}$ is bounded by the number of overlapping subpaths in $p_{z_j^{t_m}}$, which is identical to the number of overlapping subpaths in p_{w_j}, the total combinatorial length of the overlapping subpaths in $p_{\hat{z}_j^{t_m}}$ is universally bounded. Therefore, the words $\hat{z}_j^{t_m}$ satisfy the conclusions of proposition 2.5.

□

By part (3) of proposition 2.4 the tuples $u_1^{t_m}, \dots, u_{g_{t_m}}^{t_m}$ belong to finitely many isomorphism classes. By proposition 2.5 the total combinatorial lengths of the overlapping subpaths in the paths $p_{\hat{z}_j^{t_m}}$ are universally bounded. Hence, we can pass to a further subsequence (that we still denote $\{t_m\}$) for which the isomorphism class of the tuples $u_1^{t_m}, \dots, u_{g_{t_m}}^{t_m}$ is identical. In particular $g_{t_m} = g$ is fixed, and the overlapping subpaths in the paths $p_{\hat{z}_j^{t_m}}$ represent the same words in the generators $u_1^{t_m}, \dots, u_{g_{t_m}}^{t_m}$ (in correspondence).

For a fixed index t_{m_0} and each index t_m from this subsequence we set:

$$\hat{s}_j^{t_m} = \hat{z}_j^{t_{m_0}}(u_1^{t_m}, \dots, u_g^{t_m}).$$

Since the tuples $u_1^{t_m}, \dots, u_j^{t_m}$ belong to the same isomorphism class, the tuples $\hat{s}_1^{t_m}, \dots, \hat{s}_r^{t_m}$ belong to the same isomorphism class as the tuple $s_1, \dots, s_r$ as $s_j = \hat{s}_j^{t_{m_0}}$. Since the overlapping subpaths in the paths $p_{\hat{z}_j^{t_m}}$ represent the same words in the generators $u_1^{t_m}, \dots, u_g^{t_m}$ the paths $[y_0, \hat{s}_j^{t_m}]$ are all positively oriented. Since the lengths of the elements $u_i^{t_m}$ approaches 0 when m grows to ∞, so are the lengths of the elements $\hat{s}_j^{t_m}$. Therefore, if we denote the automorphism that maps the tuple $s_1, \dots, s_r$ to $\hat{s}_1^{t_m}, \dots, \hat{s}_r^{t_m}$ by φ_m, then a subsequence of the automorphisms $\{varphi_m\}$ that we denote $\{\varphi_\ell\}$ satisfy the conclusions of theorem 2.2.

So far we proved theorem 2.2 in case in the reduction of the action of the surface group Q on the real tree Y to an interval exchange transformation on a (positive) interval, all the endpoints of the intervals that define the IET transformation belong to the same orbit under the action of Q, and the periodicity of the images: $h_n(s_j)$ is globally bounded.

Suppose that the periodicity of the images $h_n(s_j)$, for every n and j, $1 \leq j \leq r$, is globally bounded, and not all the endpoints of the intervals belong to the same orbit. First, by possibly cutting one of the intervals into two intervals, we can always assume that one of the endpoints is in the orbit of the base point y_0 of the real tree Y, so we may assume that the interval on which the intervals exchange is supported is a positively oriented interval, that starts in the base point y_0 (as we did in case all the endpoints were in the same orbit).

With the interval exchange (on a positively oriented) interval that starts at the base point y_0, we associate the elements $v_1, \dots, v_f$, that map an endpoint of a subinterval to an endpoint of the next subinterval. Since the endpoints of the subintervals belong to more than one orbit, the elements $v_1, \dots, v_f$ belong in a nontrivial free product $\hat{Q} = \tilde{Q} * < e_1, \dots, e_c >$ where $\tilde{Q}$ is a surface group that contains Q as a subgroup of finite index (i.e., $\tilde{Q}$ is associated with a surface that finitely covers the surface that is associated with Q), and $< e_1, \dots, e_c >$ is a free group. The statements and the proofs of lemma 2.3 and proposition 2.4, that do not use any properties of the group that is generated by $v_1, \dots, v_f$ remain valid in the general case of more than one orbit of endpoints of subintervals. To prove proposition 2.5 we applied lemma 2.6. The proof of lemma 2.6 can be modified in a straightforward way to include the more general case of more than one orbit of endpoints of segments. This concludes the proof of theorem 2.2 under the bounded periodicity assumption.

To prove theorem 2.2 omitting the bounded periodicity assumption, we use the same strategy of proof, but modify the statements and the arguments to include long periodic subwords, or in the limit, to include non-degenerate segments with non-trivial stabilizers (that are contained in the discrete or simplicial part of the limit action). We start with a generalization of proposition 2.4.

Recall that we started the proof of theorem 2.2 with an iterative process of Dehn twists. $v_1^t, \dots, v_f^t$ are the (positive) elements between consecutive initial and final points of the bases that generate the IET after t Dehn twists iterations. The aim of proposition 2.4 was to replace these generators by a new (possibly larger set of) generators so that the ratios between their lengths is globally bounded.

Proposition 2.7. *For every $t \geq 1$, the elements $v_1^t, \dots, v_f^t \in Q$ can be replaced by elements $u_1^t, \dots, u_{g_t}^t \in Q$ that satisfy properties (1)-(5) in proposition 2.4. Properties (6) and (7) in proposition 2.4 are replaced by the following properties:*

(6) *there exist a real number $d_2 > 1$ and a subset of indices (that depend on t) $1 \leq i_1 < i_2 < \ldots < i_{b(t)} \leq g_t$, such that for every index i, $1 \leq i \leq g_t$ for which $i \neq i_m$, $m = 1, \dots, b(t)$:*

$$10 g_t \cdot d_2 \cdot length(u_i^t) \leq length(u_{i_1}^t)$$

(7) *there exists an integer $\ell(t)$, $0 \leq \ell(t) \leq b(t)$, and a positive real number d_1 such that for every $\ell(t) + 1 \leq m_1 < m_2 \leq b(t)$:*

$$d_1 \cdot length(u_{i_{m_1}}^t) \leq length(u_{i_{m_2}}^t) \leq d_2 \cdot length(u_{i_{m_1}}^t)$$

For every $m_1 \leq \ell(t)$ and $\ell(t) + 1 \leq m_2 \leq b(t)$:

$$d_1 \cdot length(u_{i_{m_2}}^t) \leq length(u_{i_{m_1}}^t)$$

(8) *For each t, and every index m, $1 \leq m \leq \ell(t)$, there exist distinct indices $1 \leq j_1, \dots, j_{e_m(t)} \leq g(t)$ (that depend on t) that do not belong to the set $i_1, \dots, i_{b(t)}$, such that: $w_m = u_{j_1}^t \dots u_{j_{e_m(t)}}^t$, and $u_m^t = \alpha w_m^{p_m}$ where α is a suffix of w_m.*

(9) *for each index t, in the words z_j^t, $1 \leq j \leq r$, in a bounded distance (where the bound depends only on f) either before or after the occurrence of an element u_i^t, for $i \neq i_1, \dots, i_{b(t)}$, appears one of the elements $u_{i_m}^t$, $1 \leq m \leq b(t)$.*

Proof: We use the same iterative procedure that was used to prove proposition 2.4. We divide the elements $v_1^t, \ldots, v_f^t$ into finitely many sets according to their length. We order the elements from the longest to the shortest. We place a separator between consecutive sets whenever there is a pair of consecutive elements (ordered according to length) that satisfy: $length(v_i^t) \geq c_1(f) \cdot length(v_{i+1}^t)$ where $c_1(f) = 4f$. Note that the constant $c_1(f)$ in the current proposition depends only on the number of generators f, and not on the periodicity bound, that we didn't assume exists.

We use the same iterative procedure that we used in the proof of proposition 2.4, and we refer to its various cases and notions as it appears in the proof of proposition 2.4. In cases (1)-(3) of this procedure we proceed precisely as in proposition 2.4 (the bounded periodicity case).

In case (4) of the procedure, we assumed that a long and a short base are adjacent to pv, the terminal point of the interval that supports the IET, and that the length of the long base that is adjacent to pv, that we denote b_1, is smaller than the sum of the short bases that are adjacent to pv and the first long base that is adjacent to these short bases. We set b_2 to be the first long base that is adjacent to the short bases that are adjacent to pv.

If b_1 and b_2 are not a pair of bases we do what we did in proposition 2.4 (essentially what we did in cases (1) and (2)). Suppose that b_1 and b_2 are a pair of bases. In this case b_1 and b_2 are a pair of long bases, such that b_1 is obtained by a shift of b_2 by a short word. Hence, b_1 (and b_2) can be presented as a short word times a high power of another short word, precisely as described in case (8). In this case we leave the bases b_1 and b_2 as they are, and continue analyzing the rest of the IET transformation, i.e., the bases that are supported on the interval which the complement of the union of b_1 and b_2. The number of pairs of long bases is reduced by 1. The number of active short bases, i.e., the number of short bases that participates in the rest of the procedure is reduced by at least 1.

In case (5) let b_1 and b_2 be the two long bases that are adjacent to the two sequences of short bases that are adjacent to pv. If b_1 and b_2 is not a pair of bases we do what we did in the proof of proposition 2.4. If b_1 and b_2 are a pair of bases we do what we did in case (4) when b_1 and b_2 are a pair of bases. Once again the number of pairs of long bases is reduced by 1, and the number of active short bases is reduced by at least 2.

In case (6) we act precsiely as we did in the proof of proposition 2.4. As in proposition 2.4, when we run this procedure until there are no long elements. i.e., until all the elements are either short, secondary short, and a product of a short and a high power of short elements (part (8) of the proposition), the number of bases can grow to at most $2f$, and therefore the number of elements can grow to at most $4f$.

When the procedure reached the state in which there are no long elements that do not fall into the description in part (8) of the proposition, we change the place of the separators between the elements. Once again we divide the new collection of elements into finitely many sets according to their length. We order the short and secondary short elements from the longest to the shortest (note that the elements that satisfy part (8) of the proposition, the semi-periodic elements, are not contained in the set that we order). We place a separator between consecutive sets whenever there is a pair of consecutive elements (ordered according to length) that satisfy:

$$length(\hat{v}_i^t) \geq c_2(f) \cdot length(\hat{v}_{i+1}^t)$$

where $c_2(f) = (4f)^2$.

If there is only one separated set that contains a short element, we reached a terminal state of the iterative procedure. In that case if there is a semi-periodic element (an element that satisfy part (8)), and its period contains a small element, we further perform Dehn twist and shorten the semi-periodic element, so that it becomes small as well. If its period does not contain small elements (only secondary small), we do not change it. Note that such a semi-periodic element can be much larger than the small elements.

Otherwise, if there is more than one such set, we declare a new collection of long elements. We declare the set of long elements to be all the elements that satisfy part (8) of the proposition (i.e., previously long semiperiodic elements), and the elements in the separated set with the longest short elements, and the elements in all the next (according to length of elements) separated sets, until the next separating set that contains a short element (where this last separated set is excluded).

We continue itartively. After each round of iterations the number of separated sets that contain short elements reduces by at least one, so after at most f rounds we end up with a unique separated set that contains short elements. All the other separated sets contain only secondary short elements.

The conclusions of the proposition follow from the termination of the iterative procedure, precisely as in the proof of proposition 2.4.

□

Proposition 2.7 replaces proposition 2.4 in the general case (i.e., when there is no periodicity assumption). To obtain the same conclusions as in proposition 2.5, we further modify the tuples, $u_1^t, \dots, u_{g_t}^t$.

Proposition 2.8. *With the notation of proposition 2.4, it is possible to further modify the tuple of elements $u_1^t, \dots, u_{g_t}^t$, by performing Dehn twists on some of the semi-periodic elements (the elements $u_1^t, \dots, u_{\ell(t)}^t$ that satisfy part (8) in proposition 2.7), so that there exists a constant $C > 0$, for which for the modified tuples, that we still denote: $u_1^t, \dots, u_{g_t}^t$, for every index $t \geq 1$ the words z_j^t, $1 \leq j \leq r$, can be replaced by words: $\hat{z}_j^t$ that satisfy properties (1)-(3) in proposition 2.5.*

Proof: Suppose that such a constant C does not exist (for any possible application of Dehn twists on the semiperiodic elements in the tuples: $u_1^t, \dots, u_{g_t}^t$). Then for every positive integer m, there exists an index $t_m > 1$, so that for every possible choice of (applications of) Dehn twists to the semiperiodic elements in the tuple, $u_1^{t_m}, \dots, u_{g_{t_m}}^{t_m}$, at least one of the words $\hat{z}_j^{t_m}$ that satisfy parts (1) and (2) (in proposition 2.5), part (3) is false for the constant $C = m$.

By passing to a subsequence, we may assume that for the subsequence the integer g_{t_m}, that counts the number of generators of the semigroup that is constructed in step t_m according to the procedure that is described in proposition 2.7 is fixed and we denote it g. We may also assume that the integers $\ell(t_m)$ and $b(t_m)$ and the sequences of indices $i_1, \dots, i_{b(t_m)}$, that are associated with the sets of generators of the semigroups that are constructed according to proposition 2.7 along the subsequence $\{t_m\}$ are fixed, and we denote them ℓ, b and $i_1, \dots, i_b$.

For each index t_m, we denote by $length_m$, the minimal length of a long element (i.e., the elements $u_1^{t_m},\ldots,u_b^{t_m}$. For each semi-periodic element $u_1^{t_m},\ldots,u_\ell^{t_m}$ we denote the length of its period by $lper_m^i$.

For each index m, and every i, $1 \leq i \leq \ell$, we look at the ratios: $\frac{lper_m^i}{length_m}$. We can pass to a subsequence of the indices m, for which (up to a change of order of indices): $0 < \epsilon < \frac{lper_m^i}{length_m}$ for some positive $\epsilon > 0$, and $i = 1,\ldots,\ell'$. And for every i, $\ell' < i \leq \ell$, the ratios $\frac{lper_m^i}{length_m}$ approaches 0. We perform Dehn twists along the semiperiodic elements $u_1^{t_m},\ldots,u_{\ell'}^{t_m}$ (from the subsequence of indices m), so that all these semiperiodic elements have lengths bounded by a constant times the length of a long element. These elements will be treated as long elements and not as semiperiodic elements in the sequel.

First, suppose that $\ell' = \ell$, i.e., that there exists an $\epsilon > 0$, such that for every i, $1 \leq i \leq \ell$, $0 < \epsilon < \frac{lper_m^i}{length_m}$. In that case, after applying Dehn twists to the semiperiodic elements, all the elements $u_1^{t_m},\ldots,u_g^{t_m}$ are either long or secondary short. By the argument that was used to prove proposition 2.5, either:

(i) the number of dual positions of the different elements is globally bounded (for the entire subsequence $\{t_m\}$).

(ii) there exists a subsequence (still denoted $\{t_m\}$), and a fixed positive word in a positive number of long elements and possibly some secondary short elements, which is a periodic word, and ratio between the length of the period and the length of the element that is represented by the positive word approaches 0.

Part (ii) implies that either the surface group Q contains a free abelian group of rank at least 2, or that Q is freely decomposable, and we get a contradiction. Hence, in case $\ell = \ell'$ there is a global bound on the number of dual positions of the different elements for the entire subsequence $\{t_m\}$, and the conclusion of the proposition follows by the same argument that was used to prove proposition 2.5. The same argument remains valid if $\ell' < \ell$ but the lengths of the semiperiodic elements $u_1^{t_m},\ldots,u_\ell^{t_m}$ can be bounded by a constant times the length of a long element.

Suppose that there exists a subsequence of indices (still denoted $\{t_m\}$), for which along a paired subpaths p_1 and p_2, at least one of the appearances of a semiperiodic element $u_i^{t_m}$, $\ell < i \leq \ell$ (along p_1 or p_2), overlaps with an unbounded number of elements (along p_2 or p_1 in correspondence). In that case, like part (ii) in the case in which the lengths of the semiperiodic elements are bounded by a (global) constant times the length of a long element, we get:

(ii') there exists a subsequence (still denoted $\{t_m\}$), and a fixed positive word in a positive number of either long elements or semiperiodic elements and possibly some secondary short elements, which is a periodic word, and ratio between the length of the period and the length of the element that is represented by the positive word approaches 0.

Hence, either the surface group Q contains a free abelian group of rank at least 2, or Q is freely decomposable, and we get a contradiction. Therefore, there exists a global bound on the number of elements that overlaps with a semiperiodic element that appears along a paired subpaths p_1 and p_2. In this last case, once again either part (i) or part (ii') holds, and if part (ii') holds we get a contradiction.

Finally, in all the cases we obtained a global bound on the number of dual positions of the different elements, so the proposition follows by the same argument that was used to prove proposition 2.5.

□

Given propositions 2.7 and 2.8, that generalize propositions 2.4 and 2.5, the rest of the proof of theorem 2.2 follows precisely as in the bounded periodicity case.

□

So far we studied axial and IET components in a limit action of pair on a real tree. The next theorem constructs a positive free basis in case the ambient group is free (non-abelian), and its action on the limiting tree is free and is either discrete or non-geometric. The remaining case, which is the Levitt action of a free group will be analyzed in the sequel. The existence of such a positive free basis is crucial in our general approach to the structure of varieties over a free semigroup.

Theorem 2.9. *Let F be a free group, and let $\{h_n : (S,F) \to (FS_k, F_k)$ be a sequence of homomorphisms of pairs that converges into a free action of the limit pair (S,F) on a real tree Y. Suppose that the action of F on the limit tree Y does not contain any Levitt components.*

Then there exists a directed finite graph Θ with the following properties:

1. *Θ contains a base point, and the free group F is identified with the fundamental group of the graph Θ. With each (positively oriented) edge in Θ we associate a label, and each of the given set of generators of the subsemigroup S corresponds to a positive path in Θ, that starts and ends at the base point, and can be expressed by a positive word in the labels that are associated with its edges.*
2. *there exists an index n_0 such that for every index $n > n_0$ the homomorphisms $\{h_n\}$ are encoded by the graph Θ. i.e., each of these homomorphisms is obtained by substituting elements from the free semigroup FS_k to the elements that are associated with the various positive edges in Θ (these substitutions define a homomorphism from the fundamental group F of Θ to the coefficient free group F_k, and this homomorphism is precisely the homomorphism from the subsequence).*

Proof: If the action of F on the limit tree Y is discrete then the conclusion of the theorem is immediate. Hence, we may assume that Y contains non-discrete parts and no Levitt components. In particular, the action is not geometric (Levitt (or thin) and non-geometric actions on a real tree are defined in [Be-Fe]).

Suppose that the action of F on Y is free and contains only discrete and non-geometric components in the sense of Bestvina-Feighn (i.e., there are no Levitt components). In that case every resolution (in the sense of [Be-Fe]) of the action of F on the real tree Y is discrete.

Suppose that $F \times T \to T$ is such a resolution. In particular, the action of F on T is free and discrete, and there exists an F-equivariant map from T onto Y. By the construction of a resolution (see [Be-Fe]), we can assume that the resolution was constructed so that the (finitely many) segments that connect between the base point in T and the images (in T) of the base point under the action of the given (finite) set of generators of the semigroup S are embedded isometrically into the limit tree Y. Since the finite union of the orbits of these segments in both trees T and Y cover these trees, and each of these segments is positively oriented, the (equivariant) orientation of segments in Y lifts to an equivariant orientation of segments in T.

The action of F on T is free and discrete. Hence, by Bass-Serre theory it is possible to associate with this action a finite graph of groups with a basepoint. We denote this graph Θ. Since the action is free, Θ has trivial vertex and edge stabilizers, and its fundamental group is identified with F. Furthermore, the segments in T are oriented equivariantly, and every edge in Θ is contained in (an orbit of) a segment that is associated with one of the given generators of the semigroup S, hence, the orientation of segments in T gives an orientation of the edges in Θ. Finally, since the segments that connect the base point to the images of the base point under the action of the given set of generators of S in T are all positively oriented, the loops in Θ that correspond to elements in the semigroup S are all positively oriented, so Θ satisfies the properties that are listed in part (1).

With each positive segment in the graph Θ we associate a label. Every substitution of values from the free semigroup FS_k to these labels gives a homomorphism of pairs from (S, L) to (FS_k, F_k). Since the action of F on T resolves the action of F on Y, and the action of F on Y is discrete and free, given a sequence of homomorphisms that converges into the action of F on the tree Y, there is an index n_0, for which for every $n > n_0$ positively oriented segments in Θ are mapped to elements in FS_k. Therefore, these homomorphisms are obtained from Θ by substituting these values to the labels that are associated with positively oriented edges, and we get part (2).

□

To conclude this section we need to analyze Levitt components, that play an essential role in our analysis of homomorphisms of pairs. The analysis of Levitt components that we use, is similar to our analysis of IET components, as it appears in the proof of theorem 2.2.

Theorem 2.10. *Let F be a f.g. free group, and let (S, F) be a pair where S is a f.g. subsemigroup that generates F as a group. Let $\{h_n\}$ be a sequence of pair homomorphisms of (S, F) into (FS_k, F_k) that converges into an indecomposable free action of F on a real tree Y of Levitt (thin) type (for the notion of an indecomposable action see definition 1.17 in [Gu]).*

Then the conclusions of theorem 2.2 (for the IET case) are valid. There exists a sequence of automorphisms $\{\varphi_\ell \in Aut(F)\}_{\ell=1}^{\infty}$ with the following properties:

1. *for each index ℓ, there exists $n_\ell > 0$, so that for every $n > n_\ell$, and all the generators of the subsemigroup S, $s_1, \dots, s_r$, $h_n \circ \varphi_\ell(s_j) \in FS_k$.*
2. *for every $n > n_\ell$:*

$$\frac{d_T(h_n \circ \varphi_\ell(s_j), id)}{\max_{1 \leq j \leq r} d_T(h_n(s_j), id.)} < \frac{1}{\ell}$$

where T is a Cayley graph of the coefficient free group F_k with respect to a fixed set of generators.

Proof: The argument that we use is an adaptation to the Levitt case of the argument in the IET case (theorem 2.2). Let $s_1, \dots, s_r$ be the generators of the subsemigroup S that generates the free group F that acts freely on the limit tree

Y, and the limit action is of Levitt type. Hence, there exists a finite subtree R of Y, such that if one applies the Rips machine to the action of the pseudogroup generated by the restrictions of the actions of the generators $s_1, \ldots, s_r$ to the finite tree R, the output is a Levitt type pseudogroup (see [Be-Fe]).

A Levitt type pseudogroup that is based on an oriented interval I, is generated by finitely many pairs of bases, where the union of the supports of these bases is a finite union of subintervals of I. Since it is a Levitt type pseudogroup, there are subintervals in I that are covered only once by the union of bases. Since every segment in the tree Y can be divided into finitely many oriented segments, and since a Levitt pseudogroup is mixing in the sense of [Mo], we can assume that the Levitt pseudogroup is supported on an oriented subinterval I of Y.

Levitt pseudo groups are analyzed in [Be-Fe], [GLP] and in [Ra]. Given a Levitt pseudogroup that is supported on some (subintervals of an) oriented interval I, there are some subintervals that are covered exactly once by the bases of the pseudogroup. Starting with the original f.g. pseudogroup one applies to it a sequence of moves (see [Be-Fe]). There are finitely many subintervals that are covered only once. In each move, one cuts such a subinterval that is contained in a base, hence, possibly cut the base that is supported on this subinterval (if the subinterval does not contain an endpoint of the base), and cuts a corresponding subinterval from the paired base.

Following section 7 in [Be-Fe], subintervals that are covered exactly once and removed along the process are divided into 3 classes:

(1) an isolated base, that is a base that its interior cuts no other bases. In this case the number of pairs of bases after removing the isolated base decreases by 1.
(2) a semi-isolated base, that is a subinterval that is covered only once, contained in a base and contains an endpoint of that base. Removing such a subinterval does not increase the number of bases.
(3) a subinterval that is covered only once and is contained in the interior of a base. In this case the number of pairs of bases after removing the subinterval increases by 1, and the number of connected components of subintervals that are covered by bases increases by at least 1.

In addition to removing subintervals that are covered exactly once, one performs the following operation:

(4) if there exists a subintervals that supports exactly two bases, and the support of both is the entire subinterval, one removes that subinterval and the bases that it supports. If the bases are paired, they are erased and the number of pairs of bases reduces by 1. If these bases are not paired, one further pairs the the two bases that were previously paired with the bases that were supported on that subinterval.

The Rips' machine applies moves of types (1) and (4) as long as possible (i.e., removes subintervals of type (1) and (4)). Then applies moves of type (2) as long as possible. If there are no more moves of types (1), (4) and (2), then one applies a move of type (3). Since we assumed that the action of the given group F on the tree Y is of Levitt type, and the given pseudogroup generates the action, moves of type (1) can not occur (since if a subinterval of type (1) exists, the action of the pseudogroup on the interior of that subinterval can not have a dense orbit). Furthermore, in all moves of type (4), the two bases that are supported on that subinterval can not be a pair of bases (by the same argument). By proposition 7.2

in [Be-Fe] moves of type (3) occur infinitely many times, i.e., a sequence of moves of type (4) and (2) has always a finite length. The Rips' machine does not give a priority to the order of the subintervals of type (3) that are being treated, as long as any such subinterval is treated after a finite time.

With the pseudogroup and the interval that supports it we can naturally associate a graph. The vertices in this graph are maximal subintervals that are covered at least once by bases of the pseudogroup. Each such maximal subinterval starts and ends with an endpoint of one of the bases that is not contained in the interior of another base. The edges in the graph are associated with the pairs of bases of the pseudogroup. For each pair of bases there is an edge that connects between a maximal subinterval that supports a base to a maximal subinterval (possibly the same one) that supports its paired base. Clearly the Euler characteristic of the graph that is associated with the initial Levitt pseudogroup is negative and is bounded below by $1-b$, where b is the number of paired bases.

The graph that is associated with a pseudogroup that is obtained from the original one after a sequence of moves of types (2)-(4) can have only a bigger (smaller in absolute value) Euler characteristic. After performing a further finite sequence of all the possible moves of types (2) and (4) (there are finitely many such by proposition 7.2 in [Be-Fe]), the valency of each vertex in the associated graph is at least 3, so at each step of the process in which all the moves of types (2) and (4) were performed, the number of generators of the obtained pseudogroup is universally bounded (in terms of the original Euler characteristic).

The sum of the lengths of the bases in the pseudogroups along the process that starts with a Levitt pseudogroup and applies moves of type (2)-(4) approaches 0 (see proposition 8.12 in [Be-Fe]). Since the number of bases is totally bounded, the (infinite) intersection of the unions of the subintervals that support these bases consists of finitely many points.

Lemma 2.11. *Let U be a f.g. Levitt pseudogroup that is based on some oriented interval I that generates a Levitt type action of a free group F on some real tree Y. Let $U_1,\dots$ be the pseudogroups that are obtained from U by applying the moves (2)-(4), where in each step we apply a single move of type (3) (if possible) and then all the possible moves of types (2) and (4). Then for some index t_0 and for all $t>t_0$:*

1. *the Euler characteristics of the graphs that are associated with the pseudogroups U_t are equal to the Euler characteristic of the graph that is associated with U_{t_0}.*
2. *there is no nontrivial word w in the generators of the pseudogroup U_t, that is defined on a non-degenerate subinterval of I, and acts trivially on that subinterval.*
3. *only moves of types (3) and (4) are applied in the process at each step t.*

Proof: For every index t, each of the the operations (2)-(4) do not reduce the absolute value of the Euler characteristic of the graph that is associated with the pseudogroup U_t, i.e., these operations can not increase the absolute value of the Euler characteristic. Hence, after finitely many steps the Euler characteristic stabilizes and part (1) follows.

Suppose that at some step t, there is a nontrivial word w in the generators of U_t, that is defined on some non-degenerate subsegment of I, and w fixes that subinterval pointwise. In that case there exists a point $q \in I$, that is contained in the interior of a subinterval J, $J \subset I$, such that the interval J has a periodic orbit under the action of the pseudogroup U_t. We can further assume that all the translates of J in that periodic orbit are disjoint.

Suppose that the disjoint translates of J in that periodic orbit are $J_0, \dots, J_\ell$. All these subintervals are covered by at least 2 bases from U_t, and for each index i there exists a pair of bases that map J_i to J_{i+1}, $i = 0, \dots, \ell - 1$, and J_ℓ to J_0. The subinterval J (or rather the subinterval to which it is going to move using moves of type (4)) will stay covered by at least two bases along the process, unless at some step $t_1 > t$, at least one of the subintervals in its orbit will be covered twice by a base and its paired base, and these two paired bases have the same support (i.e., the map from that base to its paired base is the identity map). Since the lengths of the bases approaches 0 when $t \to \infty$, such a step $t_1 > t$ must exist. In this case we can remove this pair of bases. Since the two paired bases that we remove can not be supported on a subinterval that does not support another base, the removal of this pair increases the Euler characteristic of the graph that is associated with the pseudogroup U_{t_1}. By part (1) such a reduction can occur only finitely many times and part (2) follows.

By parts (1) and (2) there is a step t_0 such that for every $t > t_0$ the Euler characteristic of the graph that is associated with the generating sets of the pseudogroups U_t is constant, and there is no non-degenerate subsegment of the interval I that is fixed pointwise by a non-trivial word w in the pseudogroup U_t. In particular, for all $t > t_0$ the pseudogroup U_t can not contain a pair of bases that are supported on the same subinterval of I (as otherwise this pair can be removed and increase the Euler characteristic, or leave a subinterval of I with trivial (simplicial) dynamics).

By the structure of the process, at step $t_0 + 1$ we start with U_{t_0} and apply to it all the possible moves of types (2) and (4) (there are finitely many such by proposition 7.2 in [Be-Fe]). At this point the endpoints of all the bases in the obtained pseudogroups are covered at least twice (otherwise a move of type (2) is still possible).

Now we apply move (3) along a subinterval $J_1 \subset I$ that is covered only once by a base b_1, and since it is a move of type (3), J is contained in the interior of the subinterval that supports the base b_1. The base b_1 is paired with a base b_2, so by move (3) we cut b_1 along J_1 and b_2 along a (paired) subinterval $J_2 \subset I$, and increase the number of paired bases by 1. We denoted the obtained pseudogroup U_{t_0+1}.

The subintervals J_1 and J_2 have to be disjoint. J_1 was covered exactly once before we erased the corresponding part of b_1, so every point in J_2 was covered by at least two bases before we erased the corresponding part of b_2, since otherwise there is a subinterval of J_2 on which the pseudogroup U_{t_0} act discretely, and a Levitt pseudogroup is mixing.

The endpoints of all the bases that are not supported by J_2 are covered at least twice, since they were covered twice before the move of type (3) was applied, and this move does not effect these endpoints. Suppose that there exists an endpoint of a base that is supported by J_2, and this endpoint is supported by a single base in the pseudogroup U_{t_0+1}. Since every point in the subinterval J_2 is covered at least once, in case there is a point in J_2 (including its endpoints) that is covered exactly once, there must exist a point in J_2 (including its endpoints) that is not contained in the interior of a base. Hence, the graph that is associated with U_{t_0+1} contains at least 2 new vertices of valency at least 2 in addition to the vertices that existed in the graph that was associated with the pseudogroup before the move of type (3). Since the number of paired bases was increased only by 1, the Euler characteristic of U_{t_0+1} is bigger by at least 1 from that of U_{t_0}, and we got a contradiction to part (1).

Therefore, all the endpoints of bases in U_{t_0+1} are covered by at least 2 bases. Moves of type (4) do not change this property, so no move of type (2) is required before a move of type (3) is applied. Hence, part (3) follows by a straightforward induction.

□

As was proved by Thurston for laminations on surfaces, and further generalized by Morgan-Shalen to codimension 1 laminations of a 3-manifold, the foliation of a band complex contains finitely many compact leaves up to isotopy.

Proposition 2.12. *Let X be a band complex (see section 5 in [Be-Fe] for the definition of a band complex). Then X contains finitely many isotopy classes of periodic leaves.*

Proof: See proposition 4.8 in [Be-Fe] and theorem 3.2 in [MS] for the analogous claim for codimension-1 laminations.

□

For a Levitt pseudogroup U, that is associated with a free action of a free group F on a real tree Y, proposition 2.12 gives presentations of the free group F in terms of the groups that are generated by the generators of the pseudogroup U and the pseudogroups U_t that are derived from it using the Rips machine.

Lemma 2.13. *Let U be a f.g. Levitt pseudogroup that is associated with a free action of a free group F on a real tree Y. Let $U_1, \dots$ be the pseudogroups that are derived from it using the Rips machine. Then:*

1. *for each index t, the group F that is generated by the generators of U_t can be presented as the quotient of a free group generated by generators that correspond to pairs of bases from U_t, divided by a normal subgroup that is (normally) generated by finitely many elements that are associated with the periodic leaves in the band complex that is associated with the pseudogroup U_t.*
2. *the number of distinct presentations of the free group F that are associated with the pseudogroups $U_1, \dots$ is finite.*

Proof: Part (1) is true for the initial pseudogroup, by construction. The moves of the Rips machine do not change that, as can be seen in section 6 in [Be-Fe].

By part (1), one can read a presentation for the free group F from the pseudogroup U_{t_0}. By proposition 2.12 the pseudogroup U_{t_0} has finitely many isotopy classes of periodic orbits. In particular, the length of a (simple) periodic orbit is bounded. By part (3) of proposition 2.11, the pseudogroups, $U_{t_0+1}, \dots$, are obtained from U_{t_0} by successive applications of moves of types (3) and (4). A move of type (4) can only reduce the length of a periodic orbit. A move of type (3) does not change the lengths of the periodic orbits. Therefore, the lengths of the periodic orbits in all the band complexes, $U_{t_0+1}, \dots$, are uniformly bounded.

By part (1) the periodic orbits determine the relations in the presentation of F that is associated with each of the pseudogroups, $U_{t_0+1}, \dots$. Since the number of generators in these pseudogroups are uniformly bounded, and the lengths of the periodic orbits is uniformly bounded, hence, the lengths of relations that appear in the presentations that are associated with these pseudogroups are uniformly bounded, the number of distinct presentations that can be read from the pseudogroups, $U_{t_0+1}, \dots$, is finite.

□

Propositions 2.11-2.13 enable one to modify the argument that was used in the proof of theorem 2.2 (the IET case), to prove theorem 2.10 (the Levitt case).

We started with the pseudogroup U, and applied the Rips machine to obtain the pseudogroups $U_1, \dots$. By proposition 2.11 starting from U_{t_0}, there is no non-degenerate subinterval in I that is stabilized pointwise by a non-trivial word in the pseudogroup U_t, $t > t_0$. Furthermore, only moves (3) and (4) are applied along the Rips machine at all the steps $t > t_0$.

Each of the pseudogroups U_t is supported on the oriented interval I, or rather on a finite union of maximal connected subintervals of I. As in the IET case, with the pseudogroup U_{t_0} we can associate a semigroup that its generators are elements in a group that contains the free group F, namely the elements between consecutive endpoints of bases that are supported on the same maximal connected subinterval of I. We denote the generators of this semigroup $v_1^{t_0}, \dots, v_{f_{t_0}}^{t_0}$.

In a similar way we can associate a set of generators of a semigroup with each of the pseudogroups U_t, $t > t_0$. We denote these sets of generators: $v_1^t, \dots, v_{f_t}^t$. The conclusion of lemma 2.3 is valid for the semigroups that are associated with U_t for every $t > t_0$.

Lemma 2.14. *For every pair of indices: $t_2 > t_1 \geq t_0$, each of the elements that generate the semigroup at step t_1: $v_1^{t_1}, \dots, v_{f_{t_1}}^{t_1}$ is contained in the semigroup that is constructed at step t_2 and is generated by: $v_1^{t_2}, \dots, v_{f_{t_2}}^{t_2}$.*

Proof: By proposition 2.11, for every $t > t_0$ the elements: $v_1^{t+1}, \dots, v_{f_{t+1}}^{t+1}$, are obtained from the elements: $v_1^t, \dots, v_{f_t}^t$ by a finite (possibly empty) sequence of moves of type (4) and a move of type (3). A move of type (4) does not change the semigroup that is generated by the elements: $v_1^t, \dots, v_{f_t}^t$. In a move of type (3) a subinterval J_1 is erased from a base b_1, that is covered exactly once along J_1, and a corresponding subinterval J_2 is erased from the base that is paired with b_1, that we denote b_2. Since each point in the subinterval J_2 has to be covered at least twice before it is erased from b_2, each of the elements: $v_1^t, \dots, v_{f_t}^t$ can be written as a positive word in the new elements: $v_1^{t+1}, \dots, v_{f_{t+1}}^{t+1}$ and the lemma follows.

□

Furthermore, for all $t > t_0$, the groups that are generated by the elements: $v_1^t, \dots, v_{f_t}^t$, that strictly contain the free group F, satisfy similar properties to the ones that are listed in lemma 2.13.

Lemma 2.15. *For every $t > t_0$, let V_t be the group that is generated by the elements: $v_1^t, \dots, v_{f_t}^t$. Then:*

1. *The groups V_t are all isomorphic and are all free.*
2. *From the pseudogroup U_t and the structure of its periodic orbits one can read a presentation of the free group V_t with generators: $v_1^t, \dots, v_{f_t}^t$. The presentations of the groups V_t that are are obtained in this way belong to*

finitely many classes, where the presentations in each class are similar. i.e., if t_1 and t_2 are in the same class, then the presentation that is associated with V_{t_1} is obtained from that associated with V_{t_2} by replacing $v_i^{t_2}$ with $v_i^{t_1}$, $i = 1, \dots, f_{t_1} = f_{t_2}$.

Proof: By lemma 2.14 for every $t > t_0$ and every $t_2 > t_1 \geq t_0$, $v_1^{t_1}, \dots, v_{f_{t_1}}^{t_1}$ can be written as positive words in $v_1^{t_2}, \dots, v_{f_{t_2}}^{t_2}$. In particular, $V_{t_1} < V_{t_2}$. By part (3) of proposition 2.11, for every $t > t_0$ one applies only moves of type (3) and (4). Moves of type (4) clearly do not change V_t. When one applies move (3) a subinterval J_1 is deleted from the interior of a base b_1, and a disjoint subinterval J_2 is deleted from a base b_2 that is paired with b_1. The interval J_1 is covered only once, where every point in the interval J_2 is covered by at least two bases before J_2 was erased from b_2.

Hence, from the two parts of the base b_1 that are left after deleting J_1 no new generators are added to the new generating set. The two parts of b_2 that are left after deleting J_2 can be presented as positive words in the generating set before the deletion. Therefore, all the new generators can be presented as words (not necessarily positive) in the previous generating set. So $V_{t+1} = V_t$.

The group F acts on a real tree Y, and the action is indecomposable (in the sense of [Gu]) and of Levitt type. With the action of F on Y once can associate finitely many orbits of branching points, and with each orbit of branching point finitely many orbits of germs. Each of the elements v_t^i starts at a germ of a branching point and ends at a germ of a branching point.

If all the starting and the ending branching points are in the same orbit under the action of F, then $V_t = F$ for every t, and V_t is free. Otherwise we divide the branching points in Y into finitely many orbits under the action of F. If all the elements, v_i^t, from the generating set of V_t starts and ends at branching points from the same orbit under the action of F, then all are elements in F and $V_t = F$. Hence, we look at those elements v_i^t that starts and ends at vertices from different orbits under the action of F. If v_i^t maps a germ of its starting point to a germ of its ending point, then v_i^t preserves the tree Y, i.e., it belongs to the stabilizer of the same indecomposable component under the action of V_t. By a result of P. Reynolds [Re], the subgroup that is generated by F and the elements that preserve the tree Y, that we denote $\hat{V}_t$, generate a free group that contains F as a finite index subgroup, Y is indecomposable under this action which is of Levitt type.

we look at the finite set of orbits of branching points of Y under the action of $\hat{V}_t$. Each of the generators v_t^i that is not contained in $\hat{V}_t$, map such an orbit to another orbit, and no germ of the staring orbit is mapped to a germ of the ending orbit by v_i^t. We order the orbits of branching points of Y under $\hat{V}_t$. We look at all pairs (j_1, j_2), $j_1 < j_2$, for which there is an element, v_i^t, that maps a branching point in orbit j_1 to a branching point in orbit j_2 or vice versa.

In [Gu] a graph of groups is associated with an action of a group on a real tree, that is obtained from the analysis of its indecomposable components. The group V_t admits an action on a real tree, in which the only indecomposable components are the orbits of the subtree Y, that is stabilized by the subgroup $\hat{V}_t$. The graph of groups that is associated with this action has one vertex stabilized by $\hat{V}_t$, and edges that are associated with the different pairs (j_1, j_2), $j_1 < j_2$, with which one can associate a generator v_i^t. Therefore, V_t is free, and we get part (1).

□

As in the proof of theorem 2.2 in the IET case, while shortening the lengths of the elements $v_1^t, \dots, v_{f_t}^t$ using the Rips machine (moves (2)-(4)), the ratios between the lengths of the elements $v_1^t, \dots, v_{f_t}^t$ may not be bounded. Hence, we need to replace the elements $v_1^t, \dots, v_{f_t}^t$ by elements that are not longer than them, and for which the ratios between their lengths are universally bounded. As in the IET case (proposition 2.4), We start by proving the existence of such generators assuming a global bound on the periodicity of the images of the given set of generators of the semigroup S, $s_1, \dots, s_r$, under the sequence of homomorphisms $\{h_n\}$, and then generalize the argument omitting the bounded periodicity assumption. Note that in order to find a replacement with a global bound on the ratios between elements, we may need to replace the original sequence $v_1^t, \dots, v_{f_t}^t$ by another sequence that satisfies the same properties (and in particular, all the claims 2.11-2.15).

Proposition 2.16 (cf. proposition 2.4). *Suppose that there exists an integer c_p, such that the periodicity of the elements $h_n(s_1), \dots, h_n(s_r)$ is bounded by c_p for all integers n.*

After possibly replacing the sequence of systems of generators, $v_1^t, \dots, v_{f_t}^t$, by a sequence of systems of generators that satisfy claims 2.11-2.15, that we still denote $v_1^t, \dots, v_{f_t}^t$, there exists a subsequence of indices, that for brevity we still denote t, such that for every index t (from the subsequence), the elements $v_1^t, \dots, v_{f_t}^t$ can be replaced by elements $u_1^t, \dots, u_g^t$, with the following properties:

(1) *for every index t (from the subsequence), the semigroup that is generated by $u_1^t, \dots, u_g^t$ contains the semigroup that is generated by $v_1^t, \dots, v_{f_t}^t$.*
(2) *for every t, and for large enough index n, $h_n(u_i^t) \in FS_k$ for every index i, $1 \leq i \leq g$.*
(3) *for every t it is possible to associate naturally a presentation with the generators: $u_1^t, \dots, u_g^t$. The presentations are similar for all t. i.e., for t_1, t_2 the presentations are identical if we replace $u_i^{t_1}$ with $u_i^{t_2}$, for $i = 1, \dots, g$.*
(4) *by part (1) for every t each of the elements $v_1^{t_0}, \dots, v_{f_{t_0}}^{t_0}$ can be represented as a positive word in the elements $u_1^t, \dots, u_g^t$. As in the IET case, each of the generators s_j, $j = 1, \dots, r$, can be written as a word, $s_j = w_j^{t_0}(v_1^{t_0}, \dots, v_{f_{t_0}}^{t_0}$. By lemma 2.14, for every t each of the elements $v_i^{t_0}$ can be written as a positive word in the elements, $v_1^t, \dots, v_{f(t)}^t$. Hence, by substituting these positive words instead of the elements, $v_i^{t_0}$, each of the elements s_j can be written as a word: $s_j = w_j^t(v_1^t, \dots, v_{f_t}^t$.*

   By part (1) each of the elements v_i^t can be presented as a positive word in the new generators: $u_1^t, \dots, u_g^t$. When we substitute these positive words in the words w_j^t, we get the words: $s_j = z_j^t(u_1^t, \dots, u_{g_t}^t)$.

   As in the IET case (part (5) in proposition 2.4), with each word, w_j^t and z_j^t, we can associate a path in the tree Y, $p_{w_j^t}$ and $p_{z_j^t}$. For a fixed j, $1 \leq j \leq r$, $p_{w_j^t}$ and $p_{z_j^t}$ are identical, and they are identical paths for all t.
(5) *there exist positive constants d_1, d_2, such that for every t, there exist a sequence of indices: $1 \leq i_1 < i_2 < \dots < i_b \leq g$, such that for every $1 \leq m_1 < m_2 \leq b$:*

$$d_1 \cdot length(u_{i_{m_1}}^t) \leq length(u_{i_{m_2}}^t) \leq d_2 \cdot length(u_{i_{m_1}}^t).$$

Furthermore, for every index i, $1 \le i \le g$ for which $i \ne i_m$, $m = 1, \dots, b$:

$$10g \cdot d_2 \cdot length(u_i^t) \le length(u_{i_1}^t)$$

(6) *For each index t, in the words z_j^t, $1 \le j \le r$, in a distance bounded by some constant c, either before or after the occurrence of an element u_i^t, for $i \ne i_1, \dots, i_b$, appears one of the elements $u_{i_m}^t$, $1 \le m \le b$.*

Proof: As in the IET case, to get the set of new generators $u_1^t, \dots, u_g^t$ we need to modify the Rips machine or the Makanin combinatorial algorithm. The Rips machine runs two processes. The first is a process that erases subsegments that are covered exactly once, and if that process terminates, the machine runs a second process that applies a sequence of entire transformations until there is a subsegment that is covered exactly once, or until every segment is covered exactly twice.

The pseudogroups U_t are all Levitt pseudogroups, hence, if one applies the Rips machine to them, the first process doesn't terminate. Since our aim is to construct generators with comparable lengths, we need to separate between long bases (or elements) and short ones. When we start with such separation, and modify accordingly the Rips machine, it may happen that even though we start with a Levitt pseudogroup, the first process terminates after finitely many steps (or doesn't apply at all), and then one applies the second process finitely many times.

Recall that by lemma 2.11, for $t > t_0$ no non-trivial word in the generators, $v_1^t, \dots, v_{f_t}^t$, fix pointwise a non-degenerate subinterval of the interval I. Furthermore, in applying the first process in the Rips machine to the pseudogroup U_t, only moves (3) and (4) of this process are applied.

As in the IET case, we start by dividing the generators $v_1^t, \dots, v_{f_t}^t$ into finitely many sets according to their length. We order the elements from the longest to the shortest. We place a separator between consecutive sets whenever there is a pair of consecutive elements (ordered according to length) that satisfy:

$$length(v_i^t) \ge c_1(f_t, c_p) \cdot length(v_{i+1}^t)$$

where $c_1(f_t, c_p) = 4f_t c_p$.

If there are no separators, we set $g_t = f_t$, and $u_i^t = v_i^t$, $i = 1, \dots, f_t$. Suppose that there is a separator. In that case we call the elements that are in the last (shortest) set *short* and the elements in all the other (longer) sets *long*.

The construction of the new set of generators starts with (possibly none) applications of a modification of the first process in the Rips machine. By lemma 2.11, for every $t > t_0$ the process that is used to construct the pseudogroups U_t, applies only moves of type (4) and (3). In each step of the modified process we first apply moves of type (4) as long as possible. Since each such move erases a pair of bases, one can apply only finitely many such moves. We continue a step by applying a modified move of type (3) if such modified move is possible.

Let b_1 be a base with a paired base b_2, and suppose that J_1 is a maximal subinterval of b_1 (J_1 is contained in the interior of b_1) that is covered only once. J_1 is one of the elements $v_1^t, \dots, v_{f_t}^t$. If it is not a long element, we do not perform a modified move of type (3) along J_1. If J_1 is long we perform move (3).

By the proof of lemma 2.11 every point in J_2 is covered by b_2 and at least one additional base. As in the proof of lemma 2.11, if there is a point in J_2 which is the endpoint of a base, and this endpoint is covered only once after erasing J_2 from b_2, the Euler characteristic of the graph that is associated with the new pseudogroup has increased by at least 1. In this case, we replace the system of generators $v_1^t, \dots, v_{f_t}^t$, by the new system of generators that is obtained after the modified move (3), and start the whole process again. The graphs that are associated with the new pseudogroups that we obtain have strictly bigger Euler characteristics than the previous ones, and they will satisfy all the properties that are listed in claims 2.11-2.15. Since their Euler characteristic is strictly bigger, and it has to be negative, such a replacement can occur only finitely many times. Hence, in the sequel we may assume that such an increase in the Euler characteristic does not happen along the modified process that we present.

After performing (the modified) move (3), we get a new set of elements. The endpoints of all the new bases are covered at least twice (so no move of type (2) is required as we argued in the proof of lemma 2.11). There are at most 4 new elements that have one endpoints at one of the two ends of J_2. A new element that is contained in a previous long element, and its length is smaller than the maximal length of (a previous) short element, is said to be *secondary short*. Every previous long element contains at least a single new element of length that is bounded below by f_t times the length of a maximal (previous) short element. There may be (at most two) short elements that were cut into two new elements. We just denote the cut points on these short elements, but continue to the next steps with the lengths of the previous short elements.

At this point we consider all the elements that are either short elements from the original pseudogroup, or new elements that are contained in previous long elements, and are not secondary short. We divide them into short and long elements precisely as we divided them before the first step. Note that every element that was short before the first step, and was perhaps cut into two new elements is still regarded short. Also, note that all the new short elements have lengths that are bounded below by the length of a maximal previous short element. Every previous long element either stayed long, or can be written as a positive word in the new generators, and this word contains at least a one new short or new long generator.

We repeat these steps iteratively. We first perform all the possible moves of type (4), and then perform a modified step of type (3). i.e., we perform move (3) only along a long element. As in lemma 2.11, since the endpoints of all the new bases are covered at least twice, no move of type (2) is required along the modified process. Afterwards we set some of the new elements (possibly none) to be secondary short, and redefine the elements that are long and short as we did in the first step. In considering lengths, we only denote new cutpoints on previously defined short elements, so the lengths of short elements can only increase along the process.

The modified process terminates after finitely many steps, since at each step we erase a long element, i.e., an element of length bounded below by $4 \cdot c_p \cdot f_t$ times the maximal length of the original short elements. The process terminates when there are no long elements that are covered exactly once. Long elements may be cut along the process, and at least one of the new elements that are obtained after the new cuts are either long or short, and in the last case, their length is bounded below by the maximal length of the original short elements, i.e., the short elements from the set of generators: $v_1^t, \dots, v_{f_t}^t$.

Short elements may be cut along the process, but the number of cuts is bounded by 3 times the number of the original long elements, which means that the number of cuts is bounded by $3 \cdot f_t$. Therefore, every short element is cut into at most $3 \cdot f_t$ new elements, so the length of at least one of them is bounded below by $\frac{1}{3 \cdot f_t}$ times the minimal length of an original short element, i.e., a short element from the set $v_1^t, \dots, v_{f_t}^t$. Therefore, every element from the set $v_1^t, \dots, v_{f_t}^t$ can be written as a positive word in the elements that are obtained after the first (modified) process, and there exists a constant $c_2(c_p, f_t)$, so that every subword of combinatorial length c_2 in these words, contains either a long element, or an element of length $\frac{1}{3 \cdot f_t}$ times the minimal length of a short element from the original set $v_1^t, \dots, v_{f_t}^t$.

Although after the first (modified) process there are no long elements that are covered exactly once, there may still remain long elements after that process. Hence, although we analyze pseudogroups of Levitt type, what is left to analyze are long elements, so that in the intervals that support them every point is covered at least twice. To analyze these long elements and shorten them we use a variation of the process that is used in the proof of proposition 2.4.

If there exists a subsequence of the indices t, for which there exists a constant $c_3 > 0$, so that the maximal length of a long element (after the first process) is bounded by c_3 times the maximal length of a short element, the conclusion of the theorem follows by the same argument that was used in the proof of proposition 2.4. Hence, in the sequel we will assume that there is no such subsequence.

We will denote the elements that are obtained after the first (modified) process, $\hat{v}_1^t, \dots, \hat{v}_{\hat{f}_t}^t$, and the pseudogroup that they generate by $\hat{U}_t$. By (the proof of) parts (2) of lemmas 2.13 and 2.15, with this infinite sets of generators, there are associated only finitely many presentations of the group F (lemma 2.13), and of the group that they generate (lemma 2.15). Hence, by passing to a subsequence we may assume that the presentations are all the same. Also note that since in the first process we only applied finitely many moves of types (3) and (4), lemma 2.14 remains valid for the semigroup that is associated with $\hat{U}_t$. i.e., the semigroup that is generated by the elements: $\hat{v}_1^t, \dots, \hat{v}_{\hat{f}_t}^t$ contains the semigroup that is generated by the elements: $v_1^t, \dots, v_{\hat{f}_t}^t$.

At this point we pass to a further subsequence, and divide the elements $\hat{v}_1^t, \dots, \hat{v}_{f_t}^t$ into finitely many equivalence classes. First, by passing to a subsequence, we may assume that f_t, the number of elements, is fixed and equals f. Moreover, we may assume that the short and secondary short elements (after changing the order) are the elements $\hat{v}_1^t, \dots, \hat{v}_{\ell_1}^t$.

In the lowest equivalence class we put all the short and the secondary short elements. Note that by construction, the ratios between the lengths of any two short elements is uniformly bounded (i.e., bounded by a constant independent of t), and the ratios between the lengths of secondary short elements and short elements are uniformly bounded from above.

We are left with long elements. The second equivalence class, that after change of order we may assume to include the elements $\hat{v}_{\ell_1+1}^t, \dots \hat{v}_{\ell_2}^t$. An element $\hat{v}_i^t$ is included in this class if the ratio between the lengths of $\hat{v}_i^t$ and the maximal length of a short element approaches infinity, the ratios between elements in this class are uniformly bounded, and the ratio between the length of an element in this class and long elements that are not in the class approaches 0.

The other classes are defined iteratively. Each class contains elements so that the ratios between their lengths and the lengths of elements in lower classes approaches infinity, the ratios between the lengths of elements in the class are uniformly bounded, and the ratios between their lengths and lengths of elements in higher classes approaches 0. By passing to a further subsequence we may assume that the ratios between elements in the same equivalence class (above the lowest one), and between short elements converge to positive constants.

Suppose first that all the bases that contain elements that belong to all the equivalence classes above the lowest class (the one that contains short elements), are covered exactly twice. i.e., that two such bases overlap only in elements that are in the lowest class. In that case we modify the procedure that was used in the proof of proposition 2.4 (the IET case).

We modify the Dehn twists that appear in the procedure that was used in the IET case, to gradually reduce the number of long elements, and while doing that add a bounded number of secondary short elements.

Let I be the interval that supports the pseudogroup $\hat{U}_t$ ($\hat{U}_t$ is in fact supported on finitely many subintervals that are contained in I). We consider all the elements that belong to classes that are above the lowest class to be long, and all the elements in the lowest class to be short or secondary short.

We further divide the short and secondary short elements in $\hat{U}_t$ into several classes. We say that two such elements $\hat{v}^t_{i_1}$, $\hat{v}^t_{i_2}$ are in the same class if there is a sequence of short and secondary short elements that leads from $\hat{v}^t_{i_1}$ to $\hat{v}^t_{i_2}$ and the length between the endpoints of consecutive elements in the sequence is bounded by $2c_4(c_p, f_t)$ times the maximal length of a short element. This divides the short and secondary short elements into at most f_t equivalence classes. Each such equivalence class is supported on some subinterval of I. Our strategy along the process that we present (that modifies the one that is used in the proof of proposition 2.4), is to gradually reduce the number of these equivalence classes (of short and secondary short elements) and the number of long bases (i.e. bases that contain long elements), while creating a bounded number of new secondary short and short elements, such that that the lengths of the new short elements are bounded below by a fixed fraction of the the lengths of the previous short elements where the fraction and the the numbers of new short and secondary short elements are bounded by a constant that depends only on the number of elements in the pseudogroup $\hat{U}_t$, with which we start this part of the procedure.

Let pv be the vertex at the positive end of I. On the given pseudogroup $\hat{U}_t$ we perform the following operations:

(1) Suppose that pv does not belong to a subinterval that supports one of the equivalence classes of short and secondary short elements. In particular, there are only two long bases that are adjacent to pv. Suppose that one of them is b_1 and the other is b_2. b_1 and b_2 can not be a pair of bases, since by part (2) of lemma 2.11 no non-degenerate subinterval is fixed by a non-trivial word.

Suppose that b_1 is longer than b_2, and that the other endpoint of b_1 is contained in a subinterval that supports an equivalence class of short and secondary short elements that contains the endpoint of just one additional long base, and this long base does not overlap with b_1 in a long element. Let b_3 be the base that supports the subinterval that supports the class that contains the (other, not pv) endpoint of b_1.

In that case we first transfer all the long elements, starting with b_2 until b_3 (not including b_3), and the short and secondary short elements that are supported on the interval that supports b_1, until the class that includes the endpoint of b_3 that we denote q (q is contained in the subinterval that supports b_1). We transfer these elements from the subinterval that supports b_1 to the subinterval that supports its paired base.

We cut b_1 at the point q, the endpoint of b_3, and throw away the part of b_1 between pv and q from b_1 and from its paired base. We further add a marking point on what was left from the base b_1, at the point that was the limit of the subinterval that supported the equivalence class of short and secondary short elements, that contained q. Note that this point is marked on the base that is paired with the new (what was left from) b_1. Note that by adding this marking, and the additional element that is associated with it, we guarantee that the semigroup that is generated by the elements that are associated with the new pseudogroup contains the semigroup that is associated with the previous semigroup. The group that is generated by the two semigroups remains the same.

We end this step by checking if there are new elements of length that are bounded by c_4 times the length of a short element. In case there are such elements we declare them to be secondary short. We also divide the short and secondary short elements into equivalence classes as we did before this step. The number of such equivalence classes did not increase after this step (it may decreased). The number of bases did not change after this step.

(2) Suppose that pv does not belong to a subinterval that supports one of the equivalence classes of short and secondary short elements (as in case (1)). Suppose the b_1 is longer than b_2, and that the other endpoint of b_1 is contained in a subinterval that supports an equivalence class of short and secondary short elements that contains the endpoint of a long base that overlaps with b_1 in a long element. Let b_3 be the long base that overlaps with b_1 along some long elements and such that its endpoint is supported by the same subinterval that supports the class of short and secondary short elements and supports the endpoint of b_1.

Like in part (1) we first transfer all the long bases, starting with b_2 until b_3 (not including b_3), and the short and secondary short elements that are supported on the interval that supports b_1, until the class that includes the endpoints of b_3 and b_1, to the subinterval that supports the base that is paired with b_1.

Let q_1 be the endpoint of b_1 and q_3 be the endpoint of b_3, be the endpoints that are supported by the same subinterval that supports an equivalence class of short and secondary short elements. Suppose that q_3 is supported on the interval that supports b_1. In that case we transfer b_3 using b_1. We further cut b_1 (and its paired base) into 3 new (paired) bases. One from pv to the beginning of the subinterval that supports the equivalence class of short and secondary short elements that contain q_1 and q_3, the second from this point to q_3 and the third from q_3 to q_1. We erase the first part of b_1 and its paired base, i.e., the part from pv to the beginning of the subinterval that supports the equivalence class that contains q_1 and q_3. The other two parts that are left from b_1 are set to be secondary short.

Suppose that q_1 is supported by the subinterval that supports b_3. In that case we cut the base b_3 at q_1 and declare the part between q_1 and q_3 to be secondary short. We do that same to the base is paired with b_3. We transfer the part of b_3 that contains a long element using b_1. We further cut b_1 into two bases, One from pv to the beginning of the subinterval that supports the equivalence class of short and secondary short elements that contains q_1 and q_3, and the other from that point to q_1. We erase the first part of b_1 and its paired base.

In both cases, we added two new pairs of secondary short bases, and get rid of one long pair. Hence, step of the type (2) can occur only boundedly many times (bounded by the number of long pairs in $\hat{U}_t$). We end this step by checking if there are new short or secondary short elements, and dividing the short and secondary short elements to equivalence classes as we did before this step (and at the end of step (1)). The number of such equivalence classes did not increase after this step (it may decreased). In both cases the semigroup that is associated with the new pseudogroup contains the semigroup that is associated with the previous semigroup. the groups that are generated by both semigroups are identical.

(3) Suppose that pv belongs to a subinterval that supports one of the equivalence classes of short and secondary short elements. Let b_1 and b_2 be the long bases that are adjacent to the subinterval that supports the equivalence class that contains pv.

Suppose that b_1 is longer than b_2, and as in part (1), the other endpoint of b_1 is contained in a subinterval that supports an equivalence class of short and secondary short elements that does not contain the endpoint of a long base that overlaps with b_1 along a long element. Let b_3 be the base that supports the subinterval that supports the class that contains the (other, not pv) endpoint of b_1.

Let p_1 be the endpoint of b_1 and p_2 be the endpoint of b_2 that are closer to pv. Suppose that p_2 is supported on the interval that supports b_1. In that case we transfer b_2 using b_1. We transfer all the long bases that overlap with b_1, from b_2 to b_3 (not including b_3, and all the short and secondary bases and elements that are in the equivalence classes that are supported on b_1 from pv until the endpoint of b_3, and do not include the equivalence class of pv and the equivalence class of the endpoint of b_3 that is supported on b_1, using b_1. We cut b_1 into 3 bases, one from p_1 to p_2, and the second from p_2 to the endpoint of the subinterval that supports the equivalence class of short and secondary short elements that contains p_1 and p_2. The third is from that point to the endpoint of b_1. We set the first two parts of b_1 to be secondary short.

Suppose that p_1 is supported by the subinterval that supports b_2. In that case we cut the base b_2 at p_1 and declare the part between p_1 and p_2 to be secondary short. We do the same to the base that is paired with b_2. We transfer the part of b_2 that contains a long element using b_1, and transfer the other long elements that are supported on b_1 (not including b_3), and the short and secondary short elements that are in equivalence classes that are supported on b_1 from pv until the endpoint of b_3, and do not contain p_1 or the endpoint of b_3, using b_1. Finally we cut b_1 into two bases, one from p_1 to the endpoint of the subinterval that supports the equivalence class of short and secondary short elements that contains p_1 and p_2. The second is from this point to the endpoint of b_1.

We continue as in part (1). Let q be the endpoint of b_3 that is supported by b_1. We cut b_1 at the point q, the endpoint of b_3, and throw away the part of b_1 between the endpoint of the subinterval that supports the equivalence class of short and secondary short elements that contains pv and q from b_1 and from its paired base. We further add a marking point on what was left from the base b_1, at the point that was the limit of the subinterval that supported the equivalence class of short and secondary short elements, that contained q.

As in the first two parts, the semigroup that is generated by the elements that are associated with the new pseudogroup contains the semigroup that is associated with the previous semigroup. The group that is generated by the two semigroups remains the same.

We end this step by checking if there are new short or secondary short elements, and dividing the short and secondary short elements to equivalence classes as we did before this step. The number of such equivalence classes that participate in the next steps of the process decreased by at least 1 after this step, hence, it may occur only boundedly many times (bounded by the number of such classes in the original pseudogroup $\hat{U}_t$).

(4) Suppose that pv belongs to a subinterval that supports one of the equivalence classes of short and secondary short elements. Let b_1 and b_2 be the long bases that are adjacent to the subinterval that supports the equivalence class that contains pv.

Suppose that b_1 is longer than b_2, and as in part (2), the other endpoint of b_1 is contained in a subinterval that supports an equivalence class of short and secondary short elements that contains the endpoint of a base that overlaps with b_1 along a long element. Let b_3 be the long base that overlaps with b_1 along some long elements and such that its endpoint is supported by the same subinterval that supports the class of short and secondary short elements and supports the endpoint of b_1. In this case we combine what we did in step (3) on one side of b_1, and in step (2) in the other side of b_1.

First we use the endpoint p_1 of b_1, and p_2 of b_2, to cut b_1 and b_2 as we did in part (3). Then we transfer all the long bases from b_2 to b_3 (excluding b_3), and all the short and secondary short bases that belong to equivalence classes, from the one that contains pv to the one that contains the endpoints of b_1 and b_3, excluding the initial class that includes pv and excluding the terminal one that contains the endpoints of b_1 and b_3.

We continue as in part (2). We use q_1, the endpoint of b_1, and q_3 the endpoint of b_3, to cut the bases b_1 and b_3 as we did in part (2). Then we transfer the long part of (what left from) b_3 using b_1, and erase the long part of (what is left from) b_1, precisely as we did in part (2).

In this case, we added at most 4 new pairs of short and secondary short bases, and got rid from at least one pair of long bases (b_1 and its paired base). Also, in the active part of the pseudogroup, the number of equivalence classes of short and secondary short elements is reduced by at least 1. Hence, part (4) can occur boundedly many times (bounded by the number of long pairs in $\tilde{U}_t$). As in the previous cases, the semigroup that is associated with the new pseudogroup contains the semigroup that is associated with the previous pseudogroup. the groups that are generated by both semigroups is identical.

As in the previous parts, we end this step by checking if there are new secondary short elements and finally dividing the short and secondary short elements to equivalence classes as we did before this step. The number of such equivalence classes in the active part of the pseudogroup decreased by at least 1 after this step.

Since in parts (2) and (4) the number of pairs of long bases is reduced, and in step (3) the number of equivalence relations of short and secondary short elements is reduced, these steps can occur only boundedly many times (where the bound can be taken to be the number of elements in the initial pseudogroup $\hat{U}_t$, which is uniformly bounded for all t). Therefore, only step of type (1) can occur finitely but unboundedly many times. In part (1), the endpoint of the interval I that supports the bases of the corresponding pseudogroup, pv, is the endpoint of the two long bases that are adjacent to it, and the equivalence classes of short and secondary short bases are supported on subintervals that do not contain the endpoint pv.

By construction, a step of type (1) produces no new bases, but it may add marks on a long base. In case a base that is adjacent to pv was marked, and the marked part of the base is erased, then the marking is inherited by the other base that is adjacent to pv. Note that the number of markings on each long base is bounded by the number of endpoints of long bases, hence, it is bounded by twice the number of long bases in the initial pseudogroup $\hat{U}_t$, which is uniformly bounded.

We denote the pseudogroup that is obtained when the second process terminates U_t^{fn}. It contains no long elements, and the total number of elements is bounded in terms of the number of elements in $\hat{U}_t$, so by passing to a further subsequence we may assume that it is fixed. The terminal pseudogroup $\hat{U}_t$ and its set of generators satisfy the conclusions of proposition 2.16, precisely in the same way it is was argued in the IET case in proposition 2.4.

So far we assumed that in the initial pseudogroup, $\hat{U}_t$, there is no element, that is not in the lowest equivalence class that consists of short and secondary short elements, and is contained in more than 2 bases. Suppose that there exists such element.

Suppose first that there exists a subsequence of indices t, for which there exists a long element in the highest equivalence class that is contained in more than 2 bases. With each of the pseudogroups we can naturally associate an action of the free group F on a real tree, and after appropriate rescaling and further passing to a subsequence, we can assume that these actions converge into an action of F on a real tree Y_∞. By the Gromov-Hausdorff convergence, The pseudogroups U_t converge into a pseudogroup U_∞, from which it is possible to reconstruct the action of F on U_∞.

In the pseudogroup U_∞, every element that is not in the highest class in the pseudogroups $\hat{U}_t$ degenerates to a point, and every element in the highest class converges into a non-degenerate element in U_∞. By our assumptions on the pseudogroups $\hat{U}_t$, there exists a non-degenerate element in U_∞ that is covered by at least 3 bases (of U_∞).

At this point we apply the second process of the Rips (or the Makanin) machine to the pseudogroup U_∞. Since each segment is covered at least twice, the second process in the Rips machine is applied (see section 7.3 in [Be-Fe]). By part (2) of lemma 2.11 no non-trivial word in the generators of the pseudogroups U_t (hence, also $\hat{U}_t$) fixes a non-degenerate subsegment in the original tree. The periodicity of the words that are associated with the various bases in $\hat{U}_t$ is assumed to be bounded. Hence, when the second process of the Rips machine is applied to the pseudogroup U_∞ there can not be a step in which a base is supported on precisely the same subinterval as its paired base.

Therefore, along the applications of the second process of the Rips machine to U_∞ only entire transformations are applied at each step. This means that every point in the interval that supports the pseudogroup that is constructed at each step is covered at least twice, and since there are subintervals that are covered more than twice by bases in U_∞, there are non-degenerate subintervals that covered more than twice in the pseudogroups that are constructed in each step of the process.

With the pseudogroup U_∞ there is an associated faithful action of the free group F on some real tree. Since F is free, the action contains no axial components. Hence, when the second process of the Rips machine is applied to the pseudogroup U_∞, no toral (axial) component can be discovered at any step of the process. Therefore, by proposition 7.6 in [Be-Fe] after finitely many steps each point which is not an endpoint of one of the bases in the constructed pseudogroup must be covered exactly twice. However, the in the process that is applied to U_∞ there are always non-degenerate subintervals that are covered more than twice, and we get a contradiction.

So far we can deduce that in the pseudogroups $\hat{U}_t$ (perhaps after passing to a further subsequence), every element that belongs to the highest equivalence class is covered exactly twice. Let v_{i_0} be an element that is covered more than twice, for which there are no elements in a higher class that are covered more than twice. In that case we modify the procedure that we used in case every element that is not in the lowest class is covered twice, to reduce the longer elements to the length of the elements in the equivalence class of the element that is covered more than twice.

We divide the equivalence classes of the elements in $\hat{U}_t$ into 3 categories. The elements in classes that are lower than the class of v_i are set to be *small*. The elements that are in classes that are higher then v_i are set to be *big*, and the elements in the class of v_i are set to be *intermediate*.

We further divide the small and intermediate elements into several classes. We say that two small elements $\hat{v}^t_{i_1}$, $\hat{v}^t_{i_2}$ are in the same class if there is a sequence of small elements that leads from $\hat{v}^t_{i_1}$ to $\hat{v}^t_{i_2}$ and the length between the endpoints of consecutive elements in the sequence is bounded by $2c_4(c_p, f_t)$ times the maximal length of a small element. We will refer to these classes as *small* classes of elements.

Similarly we divide the collection of small and intermediate elements into classes. We say that two such elements are in the same class if there exists a sequence of small and intermediate elements that leads from one such element to another, and the length between the endpoints of consecutive elements is bounded by $2c_4$ times the length of the maximal intermediate element. We will refer to such classes as *intermediate* classes. Clearly, the numbers of small and intermediate classes is bounded by f_t, and every small class is contained in an intermediate class.

Our strategy along the process is to gradually reduce the number of long elements and the number of intermediate classes, while keeping the intermediate elements that are covered more than twice. When the procedure terminates, the length of the terminal long elements is bounded by a constant times the maximal length of an intermediate element, hence, intermediate elements belong to the highest equivalence class, at least one of them is covered more than twice, and we will get a contradiction by the argument that was used in case an element in the highest equivalence class is covered more than twice.

Let pv be the vertex at the positive end of I. On the given pseudogroup $\hat{U}_t$ we perform the following operations:

(1) Suppose that pv does not belong to a subinterval that supports one of the intermediate equivalence classes. Let b_1 and b_2 be the two big bases that end at pv, suppose that b_1 is longer, and that b_1 ends in a point that belongs to an equivalence class of intermediate elements that contains no endpoint of a big base that overlaps with b_1 along a big element.

In that case we act precisely as in part (1) of the procedure in which every element which is not short is covered exactly twice. Note that if an intermediate element was covered more than twice (by intermediate and big bases), then that same element (perhaps after it was transfered) is covered more than twice (by intermediate and big bases) after the move.

We end this step by checking if there are new elements of length that are bounded by c_4 times the maximal length of an intermediate element. If the length of such new element is bounded by c_4 times the maximal length of a small element, we consider it to be small. Otherwise it will be considered as an intermediate element. Finally we divide the small and the intermediate elements into equivalence classes as we did before the initial step. Note that if the number of equivalence classes is the same as before the step, no new small or intermediate elements are created. Also, that we don't add any markings to long elements, as our goal is to get a contradiction and deduce that the whole process that we use in this case is in fact not needed.

(2) Suppose that pv does not belong to a subinterval that supports one of the equivalence classes of intermediate equivalence classes. Let b_1 and b_2 the two big bases that have pv as an endpoint, let b_1 be the longer, and suppose that b_1 ends in a point that belongs to an equivalence class of intermediate elements, and that class contains the endpoint of a big base that overlaps with b_1 along a big element, that we denote b_3.

In that case we act in a similar way to what we did in part (2) of the previous procedure. First, we transfer all the big bases, starting with b_2 until b_3 (not including b_3), and all the small and intermediate bases that are supported on the interval that supports b_1, until the intermediate class that includes the endpoints of b_3 and b_1, to the subinterval that supports the base that is paired with b_1.

Let q_1 be the endpoint of b_1 and q_3 be the endpoint of b_3, that are contained in a subinterval that supports the same intermediate class. Let p be the endpoint of that intermediate class that is closer to pv. We cut b_1 and b_3 at the point p, and cut their paired bases accordingly. We transfer the part of b_3 that starts at the endpoint that is not q_3 and ends at p using b_1. We further erase the part of b_1 from pv to p, and its paired base.

In this case we added two pairs of intermediate (or small) bases, but erased a big pair of bases. Hence, step of the type (2) can occur only boundedly many times. Note that if there was an intermediate element that was covered more then twice before this step, then there exists an intermediate element that is covered more than twice after applying step (2). As at the end of step (1) we check if there are new small or intermediate elements, and update the division of small and intermediate elements into equivalence classes.

(3) Suppose that pv belongs to a subinterval that supports one of the equivalence classes of small or intermediate equivalence classes of elements. Let b_1 and b_2 be the big bases that are adjacent to the subinterval that supports the small or intermediate equivalence class that contains pv.

Suppose that b_1 is longer than b_2, and as in part (1), the other endpoint of b_1 is contained in a subinterval that supports an intermediate class does not contain the endpoint of a big base that overlaps with b_1 along a big element. Let b_3 be the big base that supports the subinterval that supports the intermediate class that contains the (other, not pv) endpoint of b_1.

Let p_1 be the endpoint of b_1 and p_2 be the endpoint of b_2 that are closer to pv, and let p be the endpoint of the interval that supports the intermediate class that supports pv (and is not pv). We first cut b_1 and b_2 at the point p, and accordingly their paired bases. We transfer all the big bases from what is left from b_2 until b_3 (not including b_3, using b_1. We further transfer all the small and intermediate classes from the class the intermediate class that includes pv (but not including this class of pv) until the class that includes the endpoints of both b_1 and b_3 using b_1.

We continue as in part (1). Let q be the endpoint of b_3 that is supported by b_1. We cut what is left from b_1 at q. We throw away the part of b_1 between the points p and q (and the corresponding part from its paired base). Note that since there was an intermediate element that is covered by at least 3 bases, there is still such an intermediate element after applying step (3).

As in steps (1) and (2), we check if there are new small or intermediate elements after this step. We further update the equivalence classes of small and intermediate elements. The number of intermediate classes that participate in the next steps of the process decreased by at least 1 after applying step (3), so it may occur only boundedly many times.

(4) Suppose that pv belongs to a subinterval that supports one of the intermediate classes. Let b_1 and b_2 be the big bases that are adjacent to the subinterval that supports the intermediate class that contains pv.

Suppose that b_1 is longer than b_2, and as in part (2), the other endpoint of b_1 is contained in a subinterval that supports an intermediate class that contains the endpoint of a big base, that we denote b_3, that overlaps with b_1 along a big element.

In this case we combine what we did in part (3), along the subinterval that supports that class of pv, and in part (2), along the subinterval that supports the class that contains the endpoints of b_1 and b_3 (cf. part (4) in case there was no long element that is covered more than twice).

In this case, we added at most 4 new pairs of small or intermediate bases, and got rid from at least one pair of big bases (b_1 and its paired base). Also, in the active part of the pseudogroup, the number of intermediate classes is reduced by at least 1. Hence, part (4) can occur boundedly many times. Note that as in the previous steps, since there was an intermediate element that is covered more than twice, there such an intermediate element after applying step (4).

As in the previous parts, we end this step by checking if there are new small or intermediate elements. We also update the collections of small and intermediate equivalence classes of elements.

Parts (2)-(4) can occur boundedly many times, so as the number of big elements or the number of intermediate classes reduces in each of them. All the steps preserve the existence of intermediate elements that are covered more than twice. When the procedure terminates the lengths of all the elements are bounded by a constant times the length of an intermediate element. Hence, when we rescale the length so that the maximal length of an element is 1, and pass to a convergent sequence of pseudogroups, the convergent sequence converges into a sequence in there exists a non-degenerate subinterval that is covered more than twice. Therefore, the argument that leads to a contradiction, in case there is an element in the highest equivalence class that is covered more than twice (and the periodicity is bounded) leads to a contradiction in this case as well. This implies that in the bounded periodicity case, all the elements that are not in the lowest class in $\hat{U}_t$, i.e., the class that contains only short and secondary short elements, are all covered exactly twice, and the procedure that web used in this case constructs a subsequence of pseudogroups that satisfy the conclusions of proposition 2.16.

□

As in the IET case, proposition 2.16 enables the proof of the key claim in the proof of theorem 2.10 - the combinatorial bounded cancellation along the process that constructs the new sets of generators: $u_1^t, \dots, u_g^t$. Since proposition 2.16 is proved under the bounded periodicity assumption, we first prove the bounded cancellation assuming bounded periodicity.

Proposition 2.17. *Suppose that there exists an integer c_p, such that the periodicity of the elements $h_n(s_1), \dots, h_n(s_r)$ is bounded by c_p for all integers n.*

With the notation of proposition 2.16, there exists a constant $C > 0$, so that for a subsequence of the indices t, that for brevity we still denote t, the words z_j^t, $1 \leq j \leq r$, can be replaced by words: $\hat{z}_j^t$ with the following properties:

1. *As elements in ambient free group F: $\hat{z}_j^t(u_1^t, \dots, u_g^t) = z_j^t(u_1^t, \dots, u_g^t)$.*
2. *$\hat{z}_j^t$ is obtained from z_j^t by eliminating distinct pairs of subwords. Each pair of eliminated subwords corresponds to two subpaths of the path $p_{z_j^t}$ that lie over the same segment in the tree T_{w_j}, where the two subpaths have opposite orientations.*
3. *With the word $\hat{z}_j^t(u_1^t, \dots, u_g^t)$ we can naturally associate a path in the tree Y, that we denote, $p_{\hat{z}_j^t}$. The path $p_{\hat{z}_j^t}$ can be naturally divided into subsegments according to the appearances of the subwords u_i^t in the word $\hat{z}_j^t$.*

   Let $DB_{\hat{z}_j^t}$ be the number of such subsegments that are associated with subwords u_i^t in $p_{\hat{z}_j^t}$, that at least part of them is covered more than once by the path $p_{\hat{z}_j^t}$. Then for every $t > t_0$ and every j, $1 \leq j \leq r$, $DB_{\hat{z}_j^t} \leq C$.

Proof: In case of bounded periodicity, proposition 2.17 follows from proposition 2.16 by exactly the same argument that proposition 2.5 follows from proposition 2.4 in the IET case.

□

Under the bounded periodicity assumption, proposition 2.17 and part (3) of proposition 2.16, that proves that the tuples $u_1^t, \dots, u_g^t$ generate groups with similar presentations, imply the conclusions of theorem 2.10 by exactly the same argument that was used in the IET case (theorem 2.2).

As in the IET case, to omit the bounded periodicity assumption, we modify the statements and the arguments that were used in the proofs of propositions 2.16 and 2.17, in a similar but a slightly different way than in the IET case, to include long periodic subwords, or in the limit, to include non-degenerate segments with non-trivial stabilizers.

We start with a generalization of proposition 2.16, which is the analogue of proposition 2.7 in the Levitt case. Recall that the aim of proposition 2.16 was to replace the generators $v_1^t, \dots, v_{f_t}^t$ by a new (possibly larger set of) generators so that the ratios between their lengths is globally bounded.

Proposition 2.18 (cf. proposition 2.7). *There exists a subsequence of indices t, that for brevity we still denote t, for which the finite set of generators: $v_1^t, \dots, v_{f_t}^t$ can be replaced by elements $u_1^t, \dots, u_g^t$ that satisfy properties (1)-(4) in proposition 2.16. Properties (5) and (6) in proposition 2.16 are replaced by the following properties:*

(6) *there exist a real number $d_2 > 1$ and a subset of indices $1 \leq i_1 < i_2 < \dots < i_b \leq g$, such that for every index i, $1 \leq i \leq g$ for which $i \neq i_m$, $m = 1, \dots, b$:*

$$10g \cdot d_2 \cdot length(u_i^t) \leq length(u_{i_1}^t)$$

(7) *there exists an integer ℓ, $0 \leq \ell \leq b$, and a positive real number d_1 such that for every $\ell + 1 \leq m_1 < m_2 \leq b$:*

$$d_1 \cdot length(u_{i_{m_1}}^t) \leq length(u_{i_{m_2}}^t) \leq d_2 \cdot length(u_{i_{m_1}}^t)$$

For every $m_1 \leq \ell$ and $\ell + 1 \leq m_2 \leq b$:

$$d_1 \cdot length(u_{i_{m_2}}^t) \leq length(u_{i_{m_1}}^t)$$

(8) *For every t, and every index m, $1 \leq m \leq \ell$, there exist distinct indices $1 \leq j_1, \dots, j_{e_m} \leq g$ that do not belong to the set $i_1, \dots, i_b$, such that: $w_m = u_{j_1}^t \dots u_{j_{e_m}}^t$, and $u_m^t = \alpha w_m^{p_m}$ where α is a suffix of w_m.*

(9) *for each index t, in the words z_j^t, $1 \leq j \leq r$, in a bounded distance either before or after the occurrence of an element u_i^t, for $i \neq i_1, \dots, i_b$, appears one of the elements $u_{i_m}^t$, $1 \leq m \leq b$.*

Proof: To prove proposition 2.18 we start with the same (first) procedure that was used in the proof of proposition 2.16 (the bounded periodicity case). Hence, we divide the generators $v_1^t, \dots, v_{f_t}^t$ into finitely many sets according to their length, call only the elements in the shortest group *short* and all the other (longer) elements *long*.

At this point we apply the modification of the first process in the Rips machine (the Makanin algorithm) that was used in the bounded periodicity case. Note that the bounded periodicity assumption is not used nor mentioned along this modified first process. Once the modified first process terminates, every long element is covered at least twice, though there may still be short or secondary short elements that are covered only once.

To analyze the pseudogroup that is the output of the modified first process, we use modifications that combine the procedures that were used in the proofs of propositions 2.16 and 2.8. If there exists a subsequence of the indices t, for which there exists a constant $c_3 > 0$, so that the maximal length of a long element (after the first process) is bounded by c_3 times the maximal length of a short element, the conclusion of the theorem follows. Hence, in the sequel we will assume that there is no such subsequence.

As in the proof of proposition 2.16, we pass to a further subsequence, and divide the elements $\hat{v}_1^t, \dots, \hat{v}_{\hat{f}_t}^t$, the generators of the semigroup that is associated with the pseudogroup that was constructed after the first modified procedure, into finitely many equivalence classes according to their lengths. We may also assume that $\hat{f}_t$, the number of elements, is fixed.

Suppose first that all the bases that contain elements that belong to all the equivalence classes that are longer than the lowest equivalence class (the one that contains short elements), are covered exactly twice. i.e., that two such bases overlap only in elements that are in the lowest class. In that case the procedure that was used in the proof of proposition 2.16, that modifies the procedures that were used in the proofs of proposition 2.4 (in the bounded periodicity case) and in proposition 2.8 (in the general IET case) prove the conclusions of the proposition.

Suppose that there exists an element that belongs to an equivalence class that is longer than the lowest class that contains the short and secondary short elements, that is covered by more than 2 bases. Suppose first that there exists a long element in the highest equivalence class that is contained in more than 2 bases. As in the proof of proposition 2.16, after an appropriate rescaling and passing to a subsequence, the given pseudogroups, $\hat{U}_t$, converge into a pseudogroup U_∞, that is associated with a faithful action of the free group F on some real tree Y_∞.

In the pseudogroup U_∞, every element that is not in the highest class in the pseudogroups $\hat{U}_t$ degenerates to a point, and every element in the highest class converges into a non-degenerate element in U_∞. By our assumptions on the pseudogroups $\hat{U}_t$, every non-degenerate element that supports some of the bases of U_∞ supports at least two bases, and there exists a non-degenerate element in U_∞ that is covered by at least 3 bases (of U_∞).

Starting with U_∞ we apply the second process of the Rips (or the Makanin) machine, since by our assumption every non-degenerate segment in the interval that supports U_∞ is covered at least twice. The second process apply a sequence of entire transformations, until there is base that is identified with its dual.

Since the free group F is free, the action of F on the real tree Y_∞ contains no axial components. Hence, by proposition 7.6 in [Be-Fe], either after a finite time we get a pseudogroup in which a base is identified with its dual, or every non-degenerate segment is covered exactly twice. Since we started with U_∞ in which there was a non-degenerate segment that is covered more than twice, after finitely many applications of entire transformations we must get to a pseudogroup in which a base is identified with its dual.

Recall that by part (2) of lemma 2.11 no non-trivial word in the generators of the pseudogroups U_t (hence, also $\hat{U}_t$) fixes a non-degenerate subsegment in the original tree. Hence, when a base is identified with its dual, it follows that the part of the words that correspond to such a base has to be unboundedely periodic (the length of the period is not bounded below by a positive constant times the length of the base).

Let b be the base that is identified with its dual after finitely many entire transformations. If the subinterval that supports b, supports only b and its dual, we continue with the rest of the pseudogroup, and by our assumption there must be a non-degenerate subsegment that supports at least 3 bases that are not b nor its dual.

Let I_b be the subinterval that supports the base b and its dual. Suppose that there exists a non-degenerate subinterval of I_b that is transfered by a non-trivial word in the that involve all the bases except for b and its dual to a non-degenerate subinterval of I_b. Then there exists a non-trivial word, the commutator of this (non-trivial) word and the transfer from b to its dual, that acts trivially on a non-degenerate segment in the original tree, a contradiction to part (2) in lemma 2.11.

Note that the subinterval I_b is contained in the simplicial (discrete) part of the real tree Y_∞, and it has a non-trivial stabilizer. Hence, the part of every base that is supported on I_b has a non-trivial stabilizer, and is contained in the simplicial part of Y_∞. At this point we continue applying the Rips machine. We erase the bases b and its dual. and if there are subintervals of I_b that are covered only once (after the erasing of b and its dual), we apply the first process in the Rips machine.

Since I_b is contained in the simplicial part of Y_∞ the first process of the Rips machine terminates after finitely many steps (it erases subintervals that are all contained in the simplicial part). We start the process by erasing subintervals from basis that are (partly) supported on I_b, and we never get back to a subinterval that is supported on I_b, since as we argued before, in such a case we get a non-trivial word that stabilizes a non-degenerate segment, a contradiction to part (2) in lemma 2.11. By the same reason the subintervals that we erase from the various bases must have disjoint supports except (possibly) for their endpoints.

Since the subintervals that we erase must have distinct supports, and the erasing procedure terminates after finitely many steps, we are left with a new pseudogroup, that has a bigger Euler characteristic (smaller in absolute value, smaller complexity in Makanin's terminology), in which every point that is not an endpoint of a base is covered at least twice, and in which there exists a subsegment that supports a subinterval of a base that was erased, and that subsegment can be mapped (using a word in the generators of the old pseudogroup) to the subinterval I_b. In particular, this subinterval has non-trivial stabilizer, and it belongs to the simplicial part of Y_∞.

We continue by applying the second process of the Rips machine. After finitely many steps there must exist a new basis that is identified with its dual. Let I_c be the subinterval that supports that new basis that is identified with its dual. If a subinterval of I_c is identified with another subinterval of I_c using a non-trivial word that does not involve the new base and its dual, we get a non-trivial word that acts trivially on a non-degenerate segment, a contradiction to part (2) of lemma 2.11. If a non-degenerate subinterval of I_c can me mapped into a subinterval of I_c using elements that do not include the base b and its dual, we also get a non-trivial word that acts trivially on a non-degenerate segment.

We erase the base and its dual that I_c supports, and apply the first part the

Rips machine. The Euler characteristic of the remaining pseudogroup increases (Makanin's complexity decreases). As we argued after erasing I_b, the application of the first part of the Rips machine terminates after finitely many steps. If I_c supports parts of more bases, the supports of the subintervals that are erased are disjoint, and they must have trivial intersection with the subintervals that can be mapped into I_b.

We continue iteratively. After erasing I_c and the subintervals that are erased after the application of the first part of the Rips machine, there are still subintervals that covered at least twice, and be mapped into I_b, hence, belong to the simplicial part of Y_∞. Hence, after finitely many entire transformations (the second part of the Rips machine), there must exist a new base that is identified with its dual.

Each time such a base and its dual are identified, they are removed and the Euler characteristic increases (Makanin's complexity decreases). Hence, this process has to stop after finitely many steps. By the arguments that we already used, when it stops there must be exist a non-trivial word in the elements of the original pseudogroup $\hat{U}_t$, that acts trivially on a non-degenerate segment, a contradiction to part (2) in lemma 2.11. Therefore, all the long elements in the pseudogroup $\hat{U}_t$ must be covered exactly twice, except perhaps at their endpoints.

So far we can deduce that in the pseudogroups $\hat{U}_t$ (perhaps after passing to a further subsequence), every element that belongs to the highest equivalence class is covered exactly twice. Suppose that there is an element that belongs to an intermediate class, i.e., not to the highest class and not to the class that contains the short and secondary short elements, that is covered more than twice.

In that case we use the procedure that was applied in the proof of proposition 2.16 in that case. After applying this procedure we replace the generators of the pseudogroup $\hat{U}_t$ (possibly after passing to a subsequence) by generators in which the highest class contains the intermediate class that by assumption contains an element that is covered more than twice. By the argument that we presented above, the highest class can not contain such elements. Hence, in the pseudogroups $\hat{U}_t$ all the elements in all the equivalence classes that are longer than the lowest one, that contains short and secondary short elements, are covered exactly twice. Since we already treated this case, the conclusion of the proposition follows.

□

In a similar way to the IET case, Proposition 2.18 replaces proposition 2.16 in the general case (i.e., when there is no periodicity assumption). To obtain the same conclusions as in proposition 2.17, we further modify the tuples, $u_1^t, \dots, u_g^t$ in a similar way to what we did in proposition 2.8.

Proposition 2.19. *With the notation of proposition 2.16, for a subsequence of the indices t, that for brevity we still denote t, it is possible to further modify the tuple of elements $u_1^t, \dots, u_g^t$, by performing Dehn twists on some of the semi-periodic elements (the elements $u_1^t, \dots, u_\ell^t$ that satisfy part (8) in proposition 2.18), so that there exists a constant $C > 0$, for which for the modified tuples, that we still denote: $u_1^t, \dots, u_g^t$, for every index t the words z_j^t, $1 \leq j \leq r$, can be replaced by words: $\hat{z}_j^t$ that satisfy properties (1)-(3) in proposition 2.17.*

Proof: The proof is similar to the proof of proposition 2.8, though it needs to be modified since in the proof of proposition 2.8 we used the fact that a surface group is freely indecomposable.

Suppose that such a constant C does not exist (for any possible application of Dehn twists on the semiperiodic elements in the tuples: $u_1^t, \ldots, u_g^t$). Then for every positive integer m, there exists an index t_m, so that for every possible choice of Dehn twists to the semiperiodic elements in the tuple, $u_1^{t_m}, \ldots, u_g^{t_m}$, at least one of the words $\hat{z}_j^{t_m}$ that satisfy parts (1) and (2) (in proposition 2.5), part (3) is false for the constant $C = m$.

For each index t_m, we denote by $length_m$ the minimal length of a long element. For each semi-periodic element $u_1^{t_m}, \ldots, u_\ell^{t_m}$ we denote the length of its period by $lper_m^i$.

For each index m, and every i, $1 \leq i \leq \ell$, we look at the ratios: $\frac{lper_m^i}{length_m}$. We can pass to a subsequence of the indices m, for which (up to a change of order of indices): $0 < \epsilon < \frac{lper_m^i}{length_m}$ for some positive $\epsilon > 0$, and $i = 1, \ldots, \ell'$. And for every i, $\ell' < i \leq \ell$, the ratios $\frac{lper_m^i}{length_m}$ approaches 0. We perform Dehn twists along the semiperiodic elements $u_1^{t_m}, \ldots, u_{\ell'}^{t_m}$, so that all these semiperiodic elements have lengths bounded by a constant times the length of a long element. These elements will be treated as long elements and not as semiperiodic elements in the sequel.

First, suppose that $\ell' = \ell$, i.e., that there exists an $\epsilon > 0$, such that for every i, $1 \leq i \leq \ell$, $0 < \epsilon < \frac{lper_m^i}{length_m}$. In that case, after applying Dehn twists to the semiperiodic elements, all the elements $u_1^{t_m}, \ldots, u_g^{t_m}$ are either long or secondary short. By the argument that was used to prove proposition 2.5, either:

(i) the number of dual positions of the different elements is globally bounded (for the entire subsequence $\{t_m\}$).
(ii) there exists a subsequence (still denoted $\{t_m\}$), and a fixed positive word in the free group F that has roots of unbounded order.

Since there is a bound on the order of a root of a fixed element in a free group, part (ii) does not happen. Therefore, in case $\ell = \ell'$ the conclusion of the proposition follows as in the bounded periodicity case, and the same argument remains valid if $\ell' < \ell$ but the lengths of the semiperiodic elements $u_1^{t_m}, \ldots, u_\ell^{t_m}$ can be bounded by a constant times the length of a long element.

Suppose that there exists a subsequence of indices (still denoted $\{t_m\}$), for which along a paired subpaths p_1 and p_2, at least one of the appearances of a semiperiodic element $u_i^{t_m}$, $\ell' < i \leq \ell$ (along p_1 or p_2), overlaps with an unbounded number of elements (along p_2 or p_1 in correspondence). In that case, there exists a subsequence (still denoted $\{t_m\}$), and a fixed positive word in a positive number of either long elements or semiperiodic elements and possibly some secondary short elements, which is a periodic word, and the ratio between the length of the period and the length of the element that is represented by the positive word approaches 0.

By a theorem of P. Reynolds [Re] if a f.g. group G acts indecomposably on a real tree, and H is a f.g. subgroup of G that acts indecomposably on its minimal subtree, then H is finite index in G. Hence, if F acts on a real tree and the action is of Levitt type, the action of F extends to an indecomposable action of a group G, then F is of finite index in G, The action of G is of Levitt type as well, and in particular, G is free.

A given f.g. free group can be a finite index subgroup in finitely many free groups (that are all of strictly smaller rank). In particular, if the Levitt action of F extends to an indecomposable action of G, then there is a bound on the index of F in G

(the bound depends only on the rank of F).

Since F is of finite index in the group that include both F and the period, F contains a subgroup of bounded index in the subgroup that is generated by the period. Hence, if $\ell' < \ell$ and there is no bound on the number of elements that overlap with a semi-periodic element, there exists an element in the free group F with an unbounded root, a contradiction.

Therefore, there exists a global bound on the number of elements that overlap with a semiperiodic element that appears along a paired subpaths p_1 and p_2. In this last case, once again either part (i) or part (ii) hold, and since a free group contain no non-trivial elements with a root of unbounded order, part (i) holds. Given part (i), i.e., a global bound on the number of dual positions between two overlapping elements, the proposition follows by the same argument that was used to prove proposition 2.17 (in the bounded periodicity case).

□

Given propositions 2.18 and 2.19, that generalize propositions 2.15 and 2.16 and are the analogue of propositions 2.7 and 2.8 in the Levitt case, the rest of the proof of theorem 2.10 (the Levitt case) follows precisely as in the bounded periodicity case, and precisely as in the IET case.

□

§3. The (Canonical) JSJ decompositions of (some) Pairs

In theorem 1.1 we have shown that with any given f.g. semigroup, S, and its set of homomorphisms into the free semigroup FS_k, $Hom(S, FS_k)$, there is an associated (canonical) finite collection of pairs, $(S_1, L_1), \dots, (S_m, L_m)$, where each pair consists of a limit group, L_i, and a semigroup, S_i, that is embedded in the limit group L_i (as a subsemigroup), and generates L_i as a group. By the construction of the pairs, (S_i, L_i), each of them is obtained as a (maximal) limit from a sequence of homomorphisms from the set, $Hom(S_i, FS_k)$.

Once we associated the canonical set of maximal pairs, $(S_1, L_1), \dots, (S_m, L_m)$, with a f.g. semigroup S, to analyze the structure of the set of homomorphisms, $Hom(S, FS_k)$, it is sufficient (and equivalent) to analyze the set of homomorphisms of each of the limit groups, L_i, into the free group, F_k, that restrict to homomorphisms of the subsemigroups, S_i, into the free semigroup, FS_k. We denote this set of homomorphisms (of pairs), $Hom((S_i, L_i), (FS_k, F_k))$.

In the case of free groups, to analyze the set of homomorphisms, $Hom(L, F_k)$, where L is a free group, we used Grushko free decomposition to factor L into a free product, and then associated the canonical JSJ decomposition with each of the factors. With the JSJ decomposition we used its associated modular group to twist (or shorten) homomorphisms, that allowed us to associate finitely many (maximal) shortening quotients with the limit group. Repeating this procedure iteratively we obtained the Makanin-Razborov diagram, in which every path (called a resolution) terminates in a free group, and the set of homomorphisms from a free group into the coefficient group F_k, can be naturally presented as $(F_k)^s$, where s is the rank of the free group.

In the case of a semigroup, the geometric tools that are needed in order to analyze the set of homomorphisms, $Hom((S_i, L_i), (FS_k, F_k))$, are based on the analogous tools for groups, but they need to be further refined, as the modular automorphisms that can be used to modify (shorten) automorphisms are required to ensure that the image of the subsemigroup S_i remains a subsemigroup of the standard free semigroup FS_k. To ensure that we analyzed axial and IET actions of groups (or rather pairs) on oriented trees in the previous section.

As in analyzing homomorphisms into a free group, the basic object that we need to associate with a pair, (S, L), where L is a freely indecomposable limit group and S is a subsemigroup of L that generates L, is an analogue of a JSJ decomposition.

Unfortunately, we manage to construct a direct analogue of the JSJ decomposition for groups only under further restrictions on a pair (S, L). For general pairs (S, L) in which the limit group L is freely indecomposable, we replace the JSJ decomposition with finitely many sequences of decompositions, i.e., with finitely many *resolutions* or *towers*.

For pairs for which we construct a JSJ decomposition, the JSJ decomposition is (at least partly) canonical, but unlike the group analogue it is not unique, i.e., we associate a (canonical) finite collection of decompositions with a pair (S, L).

Let (S, L) be a pair consisting of a freely-indecomposable limit group, L, and a subsemigroup S of L that generates L as a group. We look at all the sequences of homomorphisms, $\{h_n : L \to F_k\}$, that restrict to semigroup homomorphisms of S into FS_k, and converge into the pair (S, L). By a theorem of F. Paulin [Pa], each such sequence subconverges into an action of the limit group L on a real tree Y, and this action is stable (and even super stable in the sense of Guirardel [Gu]) by lemma 1.3 in [Se1]. By works of M. Bestvina and M. Feighn [Be-Fe1], [Se3], and V. Guirardel [Gu], with a superstable action of the limit group L on the real tree Y it is possible to associate (canonically) a graph of groups decomposition.

Therefore, with the pair (S, L) we can associate a collection of graphs of groups decompositions of L, i.e., those graphs of groups that are associated with actions of L on real trees, where these actions are obtained as a limit from convergent sequences of homomorphisms. Note that since all of these graphs of groups are abelian decompositions, they can be all obtained from the abelian JSJ decomposition of the limit group L, by cutting QH vertex groups along some s.c.c. and then possibly collapse and fold some parts of the obtained graphs of groups.

Also, note that if a cyclic subgroup C of L stabilizes a non-degenerate segment in a real tree that is obtained as a limit of actions of L that correspond to homomorphisms of L into F_k, then the unique maximal cyclic subgroup of the limit group L that contains C stabilizes this segment as well. Similarly, if an abelian subgroup $A < L$ stabilizes a non-degenerate segment in such a real tree, then the unique direct summand that contains A as a subgroup of finite index, in the unique maximal abelian subgroup of L that contains A, stabilizes this non-degenerate segment as well.

On these graphs of groups we can naturally define a partial order. We say that given two graphs of groups, Λ_1 and Λ_2, $\Lambda_1 > \Lambda_2$, if Λ_1 is a proper refinement of Λ_2, or alternatively, Λ_2 is obtained from Λ_1 by (possibly) cutting some QH vertex groups along a finite collection of s.c.c. and then (possibly) performing some collapses and foldings.

Proposition 3.1. *Let (S, L) be a pair of a freely indecomposable limit group L, and its subsemigroup S that generates L. Then there exist maximal abelian decompositions of the pair (S, L). Every strictly increasing sequence of abelian decompositions that are associated with (S, L), $\Lambda_1 < \Lambda_2 < \Lambda_3 < \dots$, terminates after finitely* *many steps.*

Proof: All the abelian decompositions, Λ_i, are obtained from the abelian JSJ decomposition of the freely-indecomposable limit group L, by cutting QH vertex groups along a finite (possibly empty) collection of s.c.c. and then (possibly) collapse and fold the obtained abelian decomposition. Since there is a bound on the size of a collection of disjoint non-homotopic non null homotopic s.c.c. on the surfaces that are associated with the QH vertex groups in the JSJ decomposition of L, and the abelian edge groups of the JSJ decomposition are all finitely generated, the proposition follows.

□

Proposition 3.1 proves the existence of maximal elements in the set of abelian decompositions that are associated with the pair, (S, L), with their natural partial order. To construct an analogue of a JSJ decomposition for the pair, (S, L), we further prove that under further assumptions on the pair (S, L), there are only finitely many (equivalence classes) of such maximal elements.

Theorem 3.2. *Suppose that L is freely indecomposable, and that all the maximal abelian decompositions, $\{\Lambda_i\}$, that are associated with the pair (S, L), correspond to simplicial actions of (S, L) on real trees. Then there exist only finitely many (equivalence classes of) maximal abelian decompositions of the pair (S, L).*

Proof: Suppose that there are infinitely many (equivalence classes of) maximal abelian decompositions of a pair (S, L). Let $\{\Lambda_i\}_{i=1}^{\infty}$ be the collection of these (inequivalent) maximal decompositions. Under the assumptions of the theorem, they are all simplicial.

With the maximal pair (S, L) we associate a sequence of homomorphisms, $\{h_n : (S, L) \to (FS_k, F_k)\}$, that converges into (S, L). To define the sequence of homomorphisms, we fix a (symmetric) generating set of the limit group L, $\{g_1, \dots, g_m\}$, that contains a generating set, $s_1, \dots, s_r$, of the semigroup, S. With the given generating set of the pair, (S, L), we naturally associate its Cayley graph, that we denote X. For every positive integer n, we denote the ball of radius n in the Cayley graph X, B_n.

Each maximal decomposition, Λ_i, that is associated with the pair (S, L), is obtained from a sequence of homomorphisms, $\{f_i(j)\}_{j=1}^{\infty}$, from (S, L) into (FS_k, F_k). For each index i, the sequence of actions of (S, L) on the Cayley graph of (FS_k, F_k), that are associated with the sequence of homomorphisms, $\{f_i(j)\}_{j=1}^{\infty}$, converges in the Gromov-Hausdorff topology, after rescaling the metric so that the maximal length of the image of a generator (under $f_i(j)$) is 1, to an action of the pair (S, L) on a real tree, that we denote T_i.

For each index n, we choose a homomorphism $h_n : (S, L) \to (FS_k, F_k)$ that satisfies the following conditions:

1. h_n is one of the homomorphisms, $\{f_n(j)\}_{j=1}^{\infty}$.
2. let g be a word of length at most n in the fixed set of generators, $g_1, \dots, g_m$, of the limit group L. Then $h_n(g) = 1$ if and only if $g = 1$ in L.
3. Let Y_n be the Cayley graph of (FS_k, F_k) after rescaling the metric so that the maximal length of the image of a generator (under h_n) is 1, and let t_n be the base point in T_n. Then for every $\ell_1, \ell_2 \in B_n$:

$$d_{T_n}(\ell_1(t_n), \ell_2(t_n)) - \frac{1}{n} \leq d_{Y_n}(h_n(\ell_1), h_n(\ell_2)) \leq d_{T_n}(\ell_1(t_n), \ell_2(t_n)) + \frac{1}{n}.$$

With each homomorphism h_n, there is naturally an associated action of the pair (S,L) on the simplicial tree Y_n that is obtained from the Cayley graph of the coefficient free group F_k by rescaling the metric so that the maximal length of the image of a generator (under h_n) is 1. From the sequence of homomorphisms, $\{h_n\}$, it is possible to extract a subsequence, so that the sequence of actions of the pair (S,L) on the trees $\{Y_n\}$, converges into a faithful action of the pair (S,L) on a limit real tree Y. Note that this action is precisely the limit of the corresponding subsequence of actions of the limit group L on the limit real trees T_n.

The action of (S,L) on the limit tree Y is non-trivial, has abelian stabilizers of non-trivial segments, and is super-stable in the sense of [Gu]. Hence, with this action it is possible to associate a (graph of groups) decomposition Δ_1 with trivial and abelian edge stabilizers. Since L was assumed to be freely indecomposable, all the edge stabilizers in Δ_1 are non-trivial abelian. Furthermore, under the assumptions of the theorem, the abelian decompositions, $\{\Lambda_i\}$ and Δ_1, must be simplicial, and they are all dominated by the JSJ decomposition of the (freely indecomposable) limit group L.

We divide the edges in the abelian decomposition Δ_1, and the vertices that are adjacent to these edges into families. The stabilizer of every edge group in Δ_1 is an abelian subgroup of L, and each vertex group in Δ_1 is finitely presented. We fix a finite generating set for each of the vertex groups in Δ_1.

Definition 3.3. *Let E be an edge in Δ_1, and let A be its edge group. We say that an edge group A in Δ_1 is elliptic, if every element $a \in A$, is elliptic in almost all the abelian decompositions Λ_n (i.e., in all but at most finitely many decompositions). Otherwise we say that A is hyperbolic.*

Suppose that A is a hyperbolic edge group and let V be a vertex group that is adjacent to the edge E. We fix a generating set $v_1, \dots, v_\ell$ of V. For an element $f \in F_n$ we denote the length of the conjugacy class of f by $|f|$. We say that the edge group A is periodic in the vertex group V, if for every element $a \in A$, there exists a positive constant $c_a > 0$, such that for almost every index n (i.e., for all except finitely many values of n), every point in the Cayley graph T of the coefficient group F_k is moved by at least one of the elements, $h_n(v_1), \dots, h_n(v_\ell)$, a distance of at least $c_a \cdot n \cdot |h_n(a)|$.

We say that a hyperbolic edge group A is non-periodic in V, if there exists a non-trivial element $a \in A$, and a positive constant $c_a > 0$, so that for all but finitely many indices n, there exists a point in the Cayley graph T of the coefficient group F_k that is moved by each of the elements, $h_n(v_1), \dots, h_n(v_\ell)$, a distance that is bounded by $c_a \cdot |h_n(a)|$.

By passing to a subsequence of the maximal abelian decompositions, $\{\Lambda_n\}$, we may assume that every edge group in Δ_1 is either elliptic or hyperbolic. By passing to a further subsequence, we may assume that every hyperbolic edge group in Δ_1 is either periodic or non-periodic in the one or two vertices that are adjacent to it. We continue with such a subsequence of the abelian decompositions $\{\Lambda_n\}$.

At this point we gradually refine the abelian decomposition Δ_1. Let V be a vertex group in Δ_1, that is connected only to either elliptic or periodic edge groups. If V is not elliptic in a subsequence of the abelian decompositions,$\{\Lambda_n\}$, we pass to this subsequence, and analyze the actions of the f.g. subgroup V on the Cayley graph of the coefficient group F_k via the homomorphisms $\{h_n\}$. By passing to a further subsequence, these actions of V do converge into a non-trivial action of V on a real tree. Since all the edge groups that are connected to V in Δ_1 are either elliptic or periodic, the abelian decomposition that is associated with the limit action of V can be further extended to an abelian decomposition of the pair (S, L), that strictly refines the abelian decomposition Δ_1.

We repeat this refinement procedure for every non-elliptic vertex group in the obtained (refined) abelian decomposition, that is connected only to elliptic or periodic edge groups, and is a point stabilizer in the corresponding action on a real tree, i.e., that is not a vertex group that is associated with an IET or an axial component. By the accessibility for small splittings of f.p. groups [Be-Fe1] (or alternatively, by acylindrical accessibility ([Se],[De],[We])), this refinement procedure terminates after finitely (in fact boundedly) many steps, and we obtain an abelian decomposition that we denote Δ_2. In Δ_2 every vertex group V that is connected only to elliptic and periodic edge groups, is either associated with an IET or an axial component, or it is elliptic itself, in which case all the edge groups that are connected to it are elliptic as well.

If all the vertex groups in Δ_2 are elliptic, then for almost all the indices n, the abelian decomposition Λ_n is dominated by the abelian decomposition Δ_2 (i.e., Δ_2 is a (possibly trivial) refinement of Λ_n for almost every index n). But in this case Δ_2 dominates only finitely many non-equivalent abelian decompositions, a contradiction to the existence of the infinite sequence of non-equivalent maximal abelian decompositions $\{\Lambda_n\}$ (from which Δ_2 was obtained).

If Δ_2 contains a vertex that is associated with an axial or an IET component. Note that such a vertex group is connected to only elliptic and periodic edge groups. Then there exists a sequence of homomorphisms of pairs that converges into a non-simplicial faithful action of L on a real tree (and contradicts the assumption of theorem 3.2).

Proposition 3.4. *Suppose that Δ_2 contains a vertex group that is associated with either an axial or an IET component. Then there exists a sequence of homomorphisms of pairs: $\{\nu_n : (S, L) \to (FS_k, F_k)\}$ that converges into a faithful action of L on a real tree, and this action contains either an axial or an IET component.*

Proof: Suppose that Δ_2 contains a vertex group A that is associated with an axial component. In that case $L = V *_{A_0} A$. By theorem 2.1 the vertex group A can be written as $A = A_0 + < a_1, \ldots, a_\ell >$, where A_0 is the point stabilizer of the axial component, $\ell \geq 2$, there exists some index n_0 so that for every $n > n_0$, $h_n(a_i) \in FS_k$, and for each j, $1 \leq j \leq r$, s_j can be written as a word:

$$s_j \;=\; v_1^j w_1^j v_2^j w_2^j \ldots v_{b_j}^j w_{b_j}^j$$

where each of the elements w_i^j is a positive word in the basis elements $a_1, \ldots, a_\ell$, $v_i^j \in V$, and for every index $n > n_0$, $h_n(v_i^j) \in FS_k$.

In that case, for every $n > n_0$, we can modify the homomorphisms h_n, by preserving the images of elements in the vertex group V, and modifying the images of the elements $a_1, \ldots, a_\ell$. By the properties of the homomorphisms h_n, for every $n > n_0$ we can set $\nu_n(a_i)$ to be arbitrary elements in FS_k that commute with $h_n(A_0)$. Clearly, we can choose these images of $h_n(a_1), \ldots, h_n(a_\ell)$, so that in the limit they will be independent over the rationals, and so that in the limit the whole vertex group V will stabilize a point. Therefore, the limit of the constructed homomorphisms $\{\nu_n\}$ will contain a single axial component with an associated group A. In particular the limit action is faithful and contains an axial component.

Suppose that Δ_2 contains an IET component. Let Q be a (hyperbolic) surface group and let (S, Q) be a pair. Let $\{h_n : (S, Q) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converges to a free action of Q on a real tree. Let $s_1, \dots, s_r$ be a given set of generators of the subsemigroup S.

If Q, the subgroup that is associated with the IET component in Δ_2 is a (closed) surface group, then since L is freely indecomposable, $L = Q$, and L admits an IET action on a real tree, so proposition 3.4 follows. Hence, we may assume that Q is a punctured surface group. Let Q be a punctured surface group that is associated with an IET component in Δ_2. Let V_Q be the vertex that is stabilized by Q in Δ_2.

By proposition 2.4 it is possible to find a sequence of tuples of elements $\{u_1^t, \dots, u_g^t\}$ such that:

1. (1) the tuples belong to the same isomorphism class, and satisfy the properties that are listed in proposition 2.4.
2. (2) for a fixed t, for large enough index n, $h_n(u_i^t) \in FS_k$, $1 \leq i \leq g$.
3. (3) the elements u_i^t generate some natural extension of Q, that we denote $\hat{Q}$. The generators $s_1, \dots, s_r$ can be written as words in elements of $\hat{Q}$ that we denote $v_1, \dots, v_f$, and elements that lie outside the vertex that is stabilized by Q in Δ_2.
4. (4) for each t, and each index j, $1 \leq j \leq f$, there exists a word $\hat{z}_j^t$, such that $v_j = \hat{z}_j^t(u_1^t, \dots, u_j^t)$.
5. (5) For every pair of indices t_1, t_2, and every j, $1 \leq j \leq f$, the elements $h_n(\hat{z}_j^{t_1}(u_1^{t_2}, \dots, u_j^{t_2})) \in FS_k$ for large enough index n.

Given an index $t > 1$, we define an automorphism of the natural extension of Q that restricts to an automorphism of Q, by sending the tuple of generators $u_1^t, \dots, u_g^t$ to $u_1^1, \dots, u_g^1$. We denote this automorphism (of the natural extension of Q) ψ_t. ψ_t maps each of the elements v_j, $1 \leq j \leq f$ to: $\psi_t(v_j) = \hat{z}_j^t(u_1^1, \dots, u_j^1)$.

By part (5), for large enough n, $h_n \circ \psi_t(v_j) \in FS_k$, $1 \leq j \leq f$. Suppose that for every index t we choose an index n_t, such that n_t grows to ∞, and $h_{n_t} \circ \psi_t(v_j) \in FS_k$, $1 \leq j \leq f$. A subsequence of the sequence $h_{n_t} \circ \psi_t$ converges into an action of Q on some real tree. By construction and the properties of the elements $u_1^t, \dots, u_g^t$ (that are listed in proposition 2.4), this action is faithful, and the action has to be bi-Lipschitz equivalent to the limit IET action of Q on a real tree that is obtained from the restrictions of the homomorphisms $\{h_n\}$ to Q. In particular, the limit action has to be an IET action of Q on a real tree.

The restrictions of the automorphisms ψ_t to Q extend naturally to automorphisms of the limit group L (viewed as the fundamental group of the abelian decomposition Λ). We denote these automorphisms of L, φ_t. By construction, the sequence of homomorphisms $h_{n_t} \circ \varphi_t$ has a subsequence that converges into a faithful action of L on a real tree, and this action contains an IET component that is associated with the action of the subgroup Q. This concludes the proof of proposition 3.4.

□

After constructing the refined abelian decomposition Δ_2, we use (part of) its modular group to shorten the sequence of homomorphisms, $\{h_n\}$, that were used to construct the abelian decompositions Δ_1 and Δ_2, to obtain a new abelian decomposition of the pair (S, L). After repeating this shortening procedure iteratively finitely many times, we are able to replace the "machine" that is used in constructing the JSJ decomposition for groups [Ri-Se], and by the existence of such a machine, eventually deduce the finiteness of the maximal decompositions of the given pair (S, L).

In the abelian decomposition Δ_2 no non-elliptic vertex group is connected to only elliptic and periodic edge groups. We fix a finite set of generators for each of the vertex groups in Δ_2. We order the vertex groups in Δ_2 by the order of magnitude of the displacements of their fixed sets of generators. Let $v_1, \dots, v_\ell$ be the fixed set of generators of a vertex group V. For each index n, we associate with V the minimal displacement of a point in the Cayley graph T of the coefficient group F_k, under the action of the tuple of elements $h_n(v_1), \dots, h_n(v_\ell)$. We denote this minimal displacement $disp_n(V)$.

After passing to a subsequence, and up to certain equivalence relation on the displacement functions, the displacement functions define an order on the vertex groups in Δ_2. We say that two displacement functions, $disp_n(V_1), disp_n(V_2)$, are *comparable* if there exists positive constants c_1, c_2, so that for every index n, $c_1 disp_n(V_1) < disp_n(V_2) < c_2 disp_n(V_1)$. We say that $disp_n(V_1)$ dominates $disp_n(V_2)$ if $disp_n(V_2) = o(disp_n(V_1)$.

Since there are only finitely many vertex groups in Δ_2, there is a finite collection of vertex groups in Δ_2 with comparable displacement functions, that dominate the displacement functions of all the other vertex groups in Δ_2. Note that none of the vertex groups in the dominating set can be elliptic. By our construction of Δ_2, for each vertex group in this dominating subset of vertex groups, there exists a non-periodic edge group that is connected to it. Furthermore, a non-periodic edge group that is connected to a vertex group in the dominating set , must be connected only to vertex groups in the dominating subset.

We set $Mod(\Delta_2)$ to be the modular group of the pair (S, L) that is associated with the abelian decomposition Δ_2. We set $MXMod(\Delta_2)$ to be the subgroup of $Mod(\Delta_2)$, that is generated by Dehn twists only along non-periodic (hyperbolic) edge groups for which their corresponding edges connect between dominating vertex groups.

In the graph of groups Δ_2, the edges with non-periodic hyperbolic edge groups that connect between dominating vertex groups, are grouped in several connected components, that we denote: $\Gamma_1, \dots, \Gamma_u$. We fix finite sets of generators for the fundamental groups of each of the connected subgraphs, $\Gamma_1, \dots, \Gamma_u$.

For each index n, we replace the homomorphism, h_n, with a homomorphism, h_n^1, that has similar properties to those of h_n, and choose an automorphism $\varphi_n \in MXMod(\Delta_2)$, so that h_n^1, φ_n and the twisted homomorphism, $h_n^2 = h_n^1 \circ \varphi_n$, have the following properties:

(1) Let T_n be the Bass-Serre tree that is associated with the abelian decomposition Λ_n. φ_n is chosen so that:
   (i) for large enough j, $f_n(j) \circ \varphi_n : L \to F_k$, is a homomorphism of the pair (S, L) into the pair (FS_k, F_k) (i.e., $f_n(j) \circ \varphi_n$ sends the positive cone in L into the standard positive cone FS_k in F_k).
   (ii) for each connected component, Γ_i, $i = 1, \dots, u$, there exists a point $t_i(n) \in T_n$, so that the displacement of $t_i(n)$ under the action of the

fixed finite collection of generators of the fundamental group of Γ_i twisted by φ_n, is the shortest among all the points in T_n and all the automorphisms $\varphi \in MXMod(\Delta_2)$ that satisfy property (i).

(2) Having fixed φ_n, we choose h_n^1, to be one of the homomorphisms $f_n(j)$, for which both h_n^1 and $h_n^1 \circ \varphi_n$ satisfy the conditions that h_n was required to satisfy, with respect to the given action of L on T_n, $\lambda_n : L \times T_n \to T_n$, and with respect to the twisted action $\lambda_n \circ \varphi_n : L \times T_n \to T_n$, in correspondence.

The sequence of homomorphisms, $\{h_n^1\}$, converges into a faithful action of the limit group L on the same real tree as the sequence, $\{h_n\}$, i.e., to the action of L on the real tree Y. With the action of L on Y we have associated the abelian decomposition, Δ_2. We set $h_n^2 = h_n^1 \circ \varphi_n$. As the automorphisms φ_n were chosen to be from the modular group $MXMod(\Delta_2)$, the sequence h_n^2 converges into a faithful action of L on a real tree Y_1.

With the action of L on the real tree Y_1 we can naturally associate an abelian decomposition Δ_3. Since we assumed that all the maximal abelian decompositions that are associated with the pair (S, L) are simplicial, the abelian decomposition Δ_3 has to be simplicial as well. Starting with Δ_3, we can possibly successively refine it and obtain an abelian decomposition Δ_4, in the same way that we successively refined the abelian decomposition Δ_1 and obtained the abelian decomposition Δ_2.

By construction, all the edges in Δ_2 that are in the complement of the connected subgraphs, $\Gamma_1, \dots, \Gamma_u$, remain edges in Δ_4. Hence, the edge groups of these edges are elliptic in Δ_4. Unlike the abelian decomposition Δ_2, it may be that all the vertex and edge groups in Δ_4 are elliptic. Since we assumed that all the maximal abelian decompositions of the pair (S, L) are simplicial, proposition 3.4 implies that it can not be that Δ_4 contains an axial or a QH vertex group.

Since Δ_3 is obtained from Δ_2 by shortening edges that are stabilized by non-periodic edge groups and connect between dominating vertex groups, the edge groups in the connected subgraphs, $\Gamma_1, \dots, \Gamma_u$, are not contained in vertex groups in Δ_4. Therefore, there exists at least one edge group in Δ_2 that is not elliptic in Δ_4.

The abelian decomposition Δ_2 of the limit group L was obtained from an action of L on a real tree Y, that is itself obtained as a limit of a sequence of homomorphisms of L into the free group F_k. Hence, every edge group in Δ_2 that is associated with a stabilizer of a non-degenerate segment in the real tree Y, is either the centralizer of itself, or it is a direct summand in its centralizer.

We start by assuming that all the edge groups in Δ_2 that have cyclic centralizers can be conjugated into vertex groups in Δ_4. Let E_1 be an edge in Δ_2 with a non-periodic abelian edge group A_1, so that the edge E_1 is contained in one of the connected subgraphs of groups $\Gamma_1, \dots, \Gamma_u$ (so E_1 connects between dominating vertex groups, and A_1 is not contained in a vertex group in Δ_4). By our assumption the centralizer of A_1 in L is non-cyclic. First, we further assume that A_1 is a strict direct summand in its centralizer. Since the action of L on the real tree Y is assumed to be simplicial, and A_1 stabilizes pointwise a line in Y, Δ_2 contains a circle so that all the edge groups in this circle are stabilized by A_1, and the Bass-Serre generator that is associated with that circle, that we denote t_1, commutes with A_1, and the centralizer of A_1 is the direct sum of A_1 and $< t_1 >$.

Since the abelian decompositions, Δ_3 and Δ_4, were obtained by shortening the homomorphisms, $\{h_n^1\}$, using the subgroup of modular automorphisms, $MXMod(\Delta_2)$, that shortens only edges with non-periodic edge groups that connect between dominating vertex groups, the maximal abelian subgroup $A = A_1 + < t_1 >$, and in particular its subgroup A_1, do not fix a vertex in Δ_4. Hence, there is a circle in Δ_4 so that all the edges in this circle are stabilized by a direct summand $A_2 < A$, and $A_1 \cap A_2$ is a direct summand of A that is also a strict summand in both A_1 and A_2. Therefore, there exists an element $t_2 < A_1$, such that $A_1 = (A_1 \cap A_2) + < t_2 >$.

If there are no vertex groups along the circle that is stabilized by A_1 in Δ_2, or along the circle that is stabilized by A_2 in Δ_4, then the limit group L has to be a non-cyclic free abelian group. In that case L admits an axial action on a real tree that is obtained as a limit of a sequence of homomorphisms of pairs from (S, L) into (FS_k, F_k), and such an action contradicts our assumption that every action of (S, L) that is obtained as a limit of a sequence of homomorphisms from (S, L) into (FS_k, F_k) is simplicial. Hence, we may suppose that the circle that is stabilized by A_1 in Δ_2 and by A_2 in Δ_4 contain non-trivial vertex groups.

If $A_1 \cap A_2$ is the trivial subgroup of A, then the fundamental groups of the connected components that are obtained from Δ_2 by deleting the circle that is stabilized by A_1, inherit non-trivial free decompositions from Δ_4, and these free decompositions extend to a (non-trivial) free decomposition of the ambient limit group L, contradicting our assumption that L is freely indecomposable. Hence, we may assume that $A_1 \cap A_2$ is a non-trivial direct summand of A.

Let $V_1, \dots, V_\ell$ be the vertex groups in Δ_2, that are placed in the circle that is stabilized by A_1 in Δ_2. Each of these vertex groups inherits a graph of groups decomposition from the abelian decomposition (of the ambient group L) Δ_4. Each of these graphs of groups of the vertex groups, $V_1, \dots, V_\ell$, contains a circle, so that each edge in that circle is stabilized by the direct summand $A_1 \cap A_2$.

The limit group L is assumed to be freely indecomposable, and the centralizer of the edge group $A_1 \cap A_2$ is the non-cyclic abelian subgroup A. Hence, by the existence of an abelian JSJ decomposition for limit groups, an edge group C in Δ_2, which is not conjugate to A_1, and is a subgroup of a vertex group V_i (that is contained in the circle that is stabilized by A_1 in Δ_2), is elliptic with respect to a decomposition of L along edges that are stabilized by $A_1 \cap A_2$. Therefore, C, can be conjugated into a connected subgraph of the graph of groups that V_i inherits from Δ_4, where this subgraph does not contain any of the edges in the circle that is stabilized by $A_1 \cap A_2$ in the graph of groups that is inherited by V_i.

Each of the given set of generators $s_1, \dots, s_r$ of the subsemigroup S of the limit group L, can be written in a normal form with respect to the graph of groups Δ_2. These normal forms contain elements from the vertex groups $V_1, \dots, V_\ell$ of Δ_2. We can further write each of the elements of $V_1, \dots, V_\ell$, that appear in the normal forms of $s_1, \dots, s_r$, in a normal form with respect to the graph of groups that the vertex groups $V_1, \dots, V_\ell$ inherit from the abelian decomposition Δ_4. If we substitute each of these last normal forms in the normal forms of $s_1, \dots, s_r$ with respect to Δ_2, we represented each of the elements $s_1, \dots, s_r$ as (fixed) words in vertex groups and Bass-Serre generators of Δ_2 and of the vertex groups in the graphs of groups that are inherited by $V_1, \dots, V_\ell$ from Δ_4. In particular, these fixed words contain powers of the elements t_1 and t_2 that are contained in the abelian subgroup A, that are Bass-Serre generators in Δ_2 and Δ_4, in correspondence.

At this point we can finally modify the sequence of homomorphisms, $\{h_n^1\}$, in order to get a new sequence of homomorphisms of the pair (S, L) into the pair (FS_k, F_k) that converges into a faithful action of the pair (S, L) on a real tree, and this action contains an axial component, a contradiction to our assumption that every such limit action is simplicial.

With each of the elements, $s_1,\dots,s_r$, we have associated a (fixed) word, that was constructed from a normal form of the element s_i with respect to the abelian decomposition Δ_2, and normal forms of elements from the vertex groups $V_1,\dots,V_\ell$ with respect to the graphs of groups that these vertex groups inherit from Δ_4. These words contain powers of the elements t_1 and t_2. We set d to be the sum of the absolute values of powers of the element t_2 that appear in all the (fixed) words that we have associated with the elements $s_1,\dots,s_r$.

We choose $\ell-1$ positive irrational numbers, $\alpha_1,\dots,\alpha_{\ell-1}$, so that $1,\alpha_1,\dots,\alpha_{\ell-1}$ are independent over the rationals, and additional positive real numbers α_ℓ and β so that:

(1) $\alpha_1+\alpha_2+\ldots+\alpha_\ell=1$.
(2) for every i, $1\leq i\leq \ell$:

$$|\alpha_i-\frac{1}{\ell}| <, \frac{1}{10\ell}.$$

(3) $\frac{1}{4d\ell}<\beta<\frac{1}{3d\ell}$.
(4) β is not in the subspace that is spanned by $1,\alpha_1,\dots,\alpha_{\ell-1}$ over the rationals.

Given the positive irrational numbers, $\alpha_1,\dots,\alpha_\ell$ and β, we modify the sequence of homomorphisms $\{h_n^1\}$. First, for every index n, we precompose the homomorphism h_n^1 with a modular automorphism $\psi_n\in MXMod(\Delta_2)$. The automorphisms ψ_n only conjugate the vertex groups $V_2,\dots,V_\ell$, and the vertex groups that are connected to them in Δ_2 and are located outside the circle that is stabilized by A_1, by different powers of the element t_2. The conjugations that determine the automorphisms $\{\psi_n\}$, are chosen to guarantee that in the limit tree that is obtained from the sequence of homomorphisms $\{h_n^1\circ\psi_n\}$, by using the same rescaling constants as those that were used for the sequence $\{h_n^1\}$, the vertex groups $V_1,\dots,V_\ell$ fix points, and the distances in the limit tree (that we denote) $\hat{Y}$ satisfy, $d_{\hat{Y}}(Fix(V_i),Fix(V_{i+1}))=\alpha_i tr(t_1)$, for every i, $i=1,\dots,\ell-1$, and where $tr(t_1)$ is the displacement of the element t_1 along its axis (that is stabilized by A_1) in $\hat{Y}$. Since the homomorphisms $\{h_n^1\}$ are homomorphisms of the pair (S,L), for large enough index n, the homomorphisms $h_n^1\circ\psi_n$ are homomorphisms of the pair (S,L) as well, and in the limit, we get a faithful action of the pair (S,L) on the real tree $\hat{Y}$ (note that the action of (S,L) on the real tree $\hat{Y}$ is similar to the action of (S,L) on the real tree Y, where the difference is only in the lengths of the segments between the points that are stabilized by the vertex groups $V_1,\dots,V_\ell$).

At this point we further modify the homomorphisms $\{h_n^1\circ\psi_n\}$, by changing only the image of the element t_2. Each of the vertex groups, $V_1,\dots,V_\ell$, inherits an abelian decomposition from the graph of groups Δ_4. In each of these inherited graphs of groups there is a circle that is stabilized by the subgroup $A_1\cap A_2$, and with a Bass-Serre generator t_2. Hence, we can modify each of these graphs of groups of the vertex groups, $V_1,\dots,V_\ell$, and replace the circle that is stabilized by $A_1\cap A_2$, with a new vertex group that is stabilized by A_1, that is connected to the vertex V_ℓ with an edge that is stabilized by $A_1\cap A_2$.

Therefore, each of the vertex groups, $V_1,\dots,V_\ell$, can be written as an amalgamated product, $V_i=W_i *_{A_1\cap A_2} A_1$. Furthermore, all the edge groups in Δ_2 of edges that are connected to the vertex that is stabilized by V_i in Δ_2 can be conjugated into W_i. This implies that the graph of groups Δ_2 can be modified and collapsed to an amalgamated product: $L = W *_{A_1 \cap A_2} A$, where A is the centralizer of A_1 and A_2, and $A = (A_1 \cap A_2) + < t_2 > + < t_1 >$.

Now, we finally modify the homomorphisms $\{h_n^1 \circ \psi_n\}$ by changing the images of the element t_2. We define a sequence of homomorphisms $\{h_n^2 : (S, L) \to (FS_k, F_k)\}$ as follows. For each index n and every element $w \in W$ we set: $h_n^2(w) = h_n^1 \circ \psi_n(w)$. We further set $h_n^2(t_1) = h_n^1 \circ \psi_n(t_1)$, and $h_n^2(t_2)$ to be an element that commutes with $h_n^1(t_2)$, and so that in the limit tree $\tilde{Y}$, that is obtained from the sequence of homomorphisms $\{h_n^2\}$, by using the same rescaling constants as those that were used for the sequence $\{h_n^1\}$, $tr(t_2) = \beta tr(t_1)$. By our choice of the constants $\alpha_1, \dots, \alpha_{\ell-1}$ and β, the homomorphisms h_n^2 map the fixed generators of the semigroup S, $s_1, \dots, s_r$, to elements in the standard free semigroup FS_k, hence, h_n^2 are homomorphisms of the pair (S, L) into the pair (FS_k, F_k).

By construction, the subgroup W acts faithfully on the limit tree $\tilde{Y}$, since it acts faithfully on the real tree $\hat{Y}$, and the restrictions of the homomorphisms h_n^2 and h_n^1 to the subgroup W are identical. Since β was chosen to be irrational, the subgroup $< t_1, t_2 >$ acts indiscretely, with a dense orbit, on a line in $\tilde{Y}$. Since the real numbers, $1, \alpha_1, \dots, \alpha_{\ell-1}, \beta$, are independent over the rationals, L modulo the kernel of the action of it on $\tilde{Y}$ can be written as an amalgamated product: $W *_{A_1 \cap A_2} A$, hence, L acts faithfully on the real tree $\tilde{Y}$. Finally, the real tree $\tilde{Y}$ was constructed from a sequence of homomorphisms of the pair (S, L) into (FS_k, F_k), that we denoted $\{h_n^2\}$, and $\tilde{Y}$ contains an axial component (the component that contains the axis of t_1 and t_2), a contradiction to the assumption (of proposition 2.3) that every such limit action is discrete.

So far we assumed that all the edge groups in Δ_2 that have cyclic centralizers can be conjugated into vertex groups in Δ_4, and that there exists an edge in Δ_2 with a non-periodic abelian edge group A_1, that is contained in one of the connected subgraphs, $\Gamma_1, \dots, \Gamma_u$ (i.e., that this non-periodic edge group connects between dominating vertex groups), and so that A_1 is a (strict) direct summand in its centralizer, i.e., that the centralizer of A_1 is not contained in a vertex in Δ_2.

We have already pointed out that since we have started with an infinite sequence of inequivalent graphs of groups of L, not all the edge groups in Δ_2 are elliptic. Suppose that all the edge groups in Δ_2 that have cyclic centralizers, are not in the connected subgraphs of groups, $\Gamma_1, \dots, \Gamma_u$, so they can all be conjugated into vertex groups in Δ_4. Hence, the subgraphs $\Gamma_1, \dots, \Gamma_u$ contain only edges with edge groups that have non-cyclic centralizers. Suppose that there is no edge E_1 in Δ_2, with an edge group A_1, for which:

(1) E_1 is contained in one of the subgraphs $\Gamma_1, \dots, \Gamma_u$ (hence, A_1 has a non-cyclic centralizer).
(2) A_1 is a proper subgroup (a proper direct summand) in its centralizer.

In this case Δ_4 does contain edge groups with non-cyclic stabilizers, so that their centralizers can not be conjugated into vertex groups in Δ_4. At this stage we can not apply the argument that we used previously and we continue as follows. If all the edge groups in Δ_4 are elliptic, then the abelian decompositions $\{\Lambda_n\}$, from which we constructed the abelian decompositions Δ_i, $i = 1, \dots, 4$, belong to only finitely many equivalence classes, a contradiction to our assumptions. Hence, at least one edge group in Δ_4 is not elliptic. Therefore, we can start with Δ_4, and apply the construction of Δ_3 and Δ_4, to get new abelian decompositions of L, that we denote Δ_5 and Δ_6.

If there exists an edge E in Δ_4 with an edge group A that has a cyclic centralizer, so that A is non-periodic and E connects between two dominating vertex groups in Δ_4, A can not be conjugated into a vertex group in Δ_6. In that case we show in the sequel (proposition 3.5) that it's possible to find a sequence of homomorphisms of (S, L) into (FS_k, F_k) that converges into a faithful action of L on a real tree, and this action contains an axial or an IET component, a contradiction to the assumptions of theorem 3.2. Hence, we may assume that all the edges in Δ_4 that have edge groups with cyclic centralizers, are either periodic or they don't connect between dominating vertex groups.

If there exists an edge E in Δ_4 with an abelian edge group A that has a non-cyclic centralizer, the centralizer can not be conjugated into a vertex group in Δ_4, A is non-periodic and connects between two dominating vertex groups in Δ_4, we can apply the previous argument to the edge group A and its centralizer, and to the two abelian decompositions, Δ_4 and Δ_6, and obtain an action of L on a real tree with an axial component, a contradiction to the assumptions of theorem 3.2.

Therefore, we may assume that all the non-periodic edge groups in Δ_4 that connect between dominating vertex groups, have non-cyclic centralizers, and their centralizers can be conjugated into a vertex group in Δ_4. In that case, there exists at least one edge in Δ_4 and at least 2 edges in Δ_6, that have edge groups with non-cyclic stabilizers, and these stabilizers can not be conjugated into vertex groups in Δ_4 and Δ_6 in correspondence.

By starting with Δ_6 and iteratively constructing abelian decompositions, we either obtain:

(1) an abelian decomposition in which all the edge and vertex groups are elliptic - a contradiction to the assumption that there are infinitely many inequivalent maximal abelian decompositions of (S, L), $\{\Lambda_n\}$.
(2) and edge E with a stabilizer A that connects between two dominating vertex group in Δ_{2i}, such that A has non-cyclic centralizer, A is non-periodic, and the stabilizer of A is not contained in a vertex group in Δ_{2i}. In that case we apply the construction that we used in case Δ_2 has such an edge, and associate a sequence of pair homomorphisms with the pair (S, L) that converges into an action of L on a real tree and the action contains an axial component. A contradiction to theorem 3.2.
(3) an edge group with cyclic centralizer in Δ_{2i}, so that this centralizer can not be conjugated into a vertex group in $\Delta_{2(i+1)}$ - in that case we prove in the sequel (proposition 3.5) that there exists a sequence of homomorphisms of the pair (S, L) into (FS_k, F_k) that converges to a faithful action of (S, L) on a real tree, and this limit real tree contains an axial or an IET component, a contradiction to the assumption of theorem 3.2.
(4) if cases (1)- (3) do not occur, then the abelian decomposition Δ_{2i} contains at least $(i-1)$ edges with edge groups that have non-cyclic centralizers, and these centralizers can not be conjugated into vertex groups in Δ_{2i}.

By the accessibility for small splittings of a f.p. group [Be-Fe], or by acylindrical accessibility ([Se],[De],[We]), there is a global bound on the number of edges in the abelian decompositions Δ_{2i}, that depends only on L (in fact on the number of generators of L [We]). Hence, if cases (1) - (3) do not occur, there is a global bound on the number of steps that case (4) can occur.

Therefore, to complete the proof of theorem 3.2, we need the following basic proposition, that motivates our approach to the construction of the JSJ decompositions for pairs.

Proposition 3.5. *Suppose that for some index i, there exists an edge E in Δ_{2i} with an edge group A that has a cyclic centralizer, so that A is non-periodic and E connects between two dominating vertex groups in Δ_{2i}, and A can not be conjugated into a vertex group in $\Delta_{2(i+1)}$. Then there exists a sequence of homomorphisms of (S, L) into (FS_k, F_k) that converges into a faithful action of L on a real tree, and this action contains an IET component.*

Proof: By the properties of the JSJ decomposition of the limit group L, the subgroup A corresponds to a s.c.c. on some maximal QH subgroup MSQ in the JSJ of L. Hence, to prove proposition 3.5 it is possible to use geometric properties of (oriented) curves and arcs on surfaces (i.e., on the surface that is associated with MSQ), together with properties of the simplicial actions of L on the real trees that are associated with the abelian decompositions Δ_{2i} and $\Delta_{2(i+1)}$.

The abelian decomposition Δ_{2i} was obtained from a limit of a sequence of homomorphisms of pairs that we denote: $\{h_n : (S, L) \to (FS_k, F_k)\}$. The abelian decomposition $\Delta_{2(i+1)}$ is obtained from a sequence of shortened homomorphisms, $\{h_n^s : (S, L) \to (FS_k, F_k0\}$, i.e., from the homomorphisms $\{h_n\}$ that are pre-composed with automorphisms in the modular group that is associated with Δ_{2i} (generated by Dehn twists along the edges in Δ_{2i}), $h_n^s = h_n \circ \varphi_n$, where $\varphi_n \in MXMod(\Delta_{2i}$.

Since the sequnce of shortened homomorphisms, $\{h_n^s\}$, converges into an action of L on a real tree with an associated abelian decomposition, $\Delta_{2(i+1)}$, after possibly passing to a subsequence, each of the shortened homomorphisms, h_n^s, can be written as a composition: $h_n^s = h_n^b \circ \nu_n$, where:

(1) $\nu_n \in Mod(\Delta_{2(i+1)})$.
(2) h_n^b is a pair homomorphism: $h_n^b(S, L) \to (FS_k, F_k)$.
(3) for each index n:

$$\max_{1 \leq j \leq r} length(h_n^s(s_j)) \geq n \cdot \max_{1 \leq j \leq r} length(h_n^b(s_j)).$$

Let $A_1, \ldots, A_t$ be the edge groups in Δ_{2i} that are non-periodic, have cyclic stabilizers, and can not be conjugated into a vertex group in Δ_{2i+1}. Let $a_1, \ldots, a_t$ be the positive generators of $A_1, \ldots, A_t$, i.e., the generators of the cyclic groups $A_1, \ldots, A_t$ that have conjugates that are mapped to FS_k, by each of the homomorphisms $\{h_n\}$.

With the limit group L there is an associated JSJ decomposition. Since the subgroups $A_1, \ldots, A_\ell$ have cyclic centralizers, and they can not be conjugated into vertex groups in $\Delta_{2(i+1)}$, their positive generators $a_1, \ldots, a_t$ correspond to oriented s.c.c. $C_1, \ldots, C_t$ on surfaces (that are associated with MQH subgroups) in the JSJ decomposition of L.

Each of edge subgroups $A_1, \ldots, A_t$ can not be conjugated into a vertex group in $\Delta_{2(i+1)}$. Hence, with each of the edge group $A_1, \ldots, A_t$ we can associate a minimal, non-degenerate subgraph of $\Delta_{2(i+1)}$, that contains a conjugate of it. Let $B_1, \ldots, B_\ell$ be the edge groups in the union of these minimal non-degenerate subgraphs. By the properties of the JSJ decomposition of the limit group L, each of the subgroups

$B_1, \dots, B_\ell$ is cyclic, has a cyclic centralizer, and is associated with (the subgroup generated by) a s.c.c. on a surface in the JSJ decomposition of L (as a limit group).

Let $b_1, \dots, b_\ell$ be the positive generators of the cyclic groups $B_1, \dots, B_\ell$. These positive generators correspond to oriented s.c.c. $c_1, \dots, c_\ell$ on surfaces in the JSJ decomposition of the limit group L. Furthermore, since the generators $a_1, \dots, a_t$ and $b_1, \dots, b_\ell$ are positive, if two oriented curves C_i and c_j intersect, each of their intersection points is positively oriented.

With each of the given generators $s_1, \dots, s_r$ we can naturally associate a (possibly empty) finite collection of oriented arcs and curves on the surfaces in the JSJ decomposition of the limit group L. Since the graph of groups Δ_{2i} and $\Delta_{2(i+1)}$ were constructed from pair homomorphisms, the elements $s_1, \dots, s_r$ act positively on both Δ_{2i} and $\Delta_{2(i+1)}$. Hence, if an oriented arc and or curve from the finite collections that are associated with each of the elements $s_1, \dots, s_r$ intersect with any of the oriented s.c.c. $C_1, \dots, C_t$ or $c_1, \dots, c_\ell$, then every such intersection point is positively oriented.

With each homomorphism from a (punctured) surface group into a free group it is possible to associate a map from the (punctured) surface into a bouquet of circles. With a pair homomorphism the image of such a map is a bouquet of oriented circles. By homotoping such a map to be transversal at the midpoints of the (oriented) circles, we may assume that the preimages of the midpoints of the oriented circles are finite collections of oriented arcs and s.c.c. on the (punctured) surface.

The pair homomorphisms h_n^s are compositions $h_n^s = h_n^b \circ \nu_n$, where $\nu_n \in Mod(\Delta_{2(i+1)}$, and the homomorphisms h_n^b are much shorter than the homomorphisms h_n^s. With each of the homomorphisms h_n^b there is an associated map from each of the (punctured) surfaces in the JSJ decomposition of L into a bouquet of oriented circles. As we indicated, we may assume that the preimages of the midpoints of these oriented circles are finite collections of oriented s.c.c. and arcs. Up to conjugacy, each of the oriented s.c.c. c_j (that are associated with the positive generators $b_1, \dots, b_\ell$ of edge groups in $\Delta_{2(i+1)}$) are mapped by h_n^b into FS_k. Hence, we may homotope the map from the surface to the bouquet of oriented circles so that the preimages of the midpoints of the oriented circles are oriented s.c.c. and arcs that intersect the s.c.c. $c_1, \dots, c_\ell$ only in positive orientation.

We set the (modular) automorphism $\psi \in MXMod(\Delta_{2i}$ to be the composition of positive Dehn twists along the edges that are associated with the edge groups $A_1, \dots, A_t$ in Δ_{2i}. We set the (modular) automorphism $\varphi \in Mod(\Delta_{2(i+1)}$ to be the composition of positive Dehn twists along the edges that are associated with the edge groups $B_1, \dots, B_\ell$.

We look at homomorphisms of the form $h_n^b \circ \varphi^m$ for large m. Since the homomorphisms h_n^s map the elements $s_1, \dots, s_r$ and $C_1, \dots, C_t$ into FS_k, so do the homomorphisms $h_n^b \circ \varphi^m$. We look at the map from the surfaces that are associated with the JSJ decomposition of L into the bouquet of oriented circles that is associated with homomorphisms $h_n^b \circ \varphi^m$. The preimages of the midpoints of the oriented circles under this map, are obtained from the preimages of the midpoints of these circles under the maps that are associated with the homomorphisms h_n^b, by performing m powers of Dehn twists along the s.c.c. $c_1, \dots, c_\ell$.

Since the homomorphisms h_n^s) where obtained as shortenings, we may assume that $h_n^s \circ \psi^v(C_i) \in FS_k$ and $h_n^s \circ \psi^v(s_j) \in FS_k$, for $1 \leq i \leq t$, $1 \leq j \leq r$, and

$v = 0, 1, \dots, v_0$ for some fixed (previously chosen) positive integer m_0. Hence, for every every large m, $h_n^b \circ \varphi^m \circ \psi^v(C_i) \in FS_k$ and $h_n^s \circ \varphi^m \circ \psi^v(s_j) \in FS_k$, for $1 \leq i \leq t$, $1 \leq j \leq r$, and every positive integer v.

Because the intersection points between the s.c.c. c_i and C_j, $1 \leq i \leq t$, $1 \leq j \leq \ell$, are all positively oriented, and because the high power φ^m of positive Dehn twists along the curves $\{c_j\}$, surrounds each such s.c.c. with long positive (periodic) words, it follows that for every large positive pair m_1, m_2, and for every large m that is much bigger than both m_1 and m_2: $h_n^b \circ \varphi^m \circ \psi^{m_2} \circ \varphi^{m_1}(c_j) \in FS_k$, $1 \leq j \leq \ell$, and $h_n^b \circ \varphi^m \circ \psi^{m_2} \circ \varphi^{m_1}(C_i) \in FS_k$, $1 \leq i \leq t$.

Now, with each of the elements $s_1, \dots, s_r$ we can associate oriented arcs and curves on the surfaces that are associated with the JSJ decomposition of L. Whenever these curves intersect the curves $C_1, \dots, C_\ell$ and $c_1, \dots, c_t$, they intersect them positively. Since $h_n^b(s_j) \in FS_k$, and $h_n^s(s_j) \in FS_k$, for $1 \leq j \leq r$, for every large positive pair m_1, m_2, and for every large m that is much bigger than both m_1 and m_2: $h_n^b \circ \varphi^m \circ \psi^{m_2} \circ \varphi^{m_1}(s_j) \in FS_k$, $1 \leq j \leq r$.

Furthermore, the positive intersection numbers, and the high power of positive Dehn twists along the curves $\{c_j\}$, imply that for every positive integer p, and every large tuple of positive integers: $e_1, f_1, \dots, e_p, f_p$, and for every large m that is much bigger than the sum of these positive integers:

$$h_n^b \circ \varphi^m \circ \psi^{f_p} \circ \varphi^{e_p} \circ \psi^{f_1} \circ \varphi^{e_1}(C_i) \in FS_k \ , \ 1 \leq i \leq t$$

$$h_n^b \circ \varphi^m \circ \psi^{f_p} \circ \varphi^{e_p} \circ \psi^{f_1} \circ \varphi^{e_1}(c_j) \in FS_k \ , \ 1 \leq j \leq \ell$$

$$h_n^b \circ \varphi^m \circ \psi^{f_p} \circ \varphi^{e_p} \circ \psi^{f_1} \circ \varphi^{e_1}(s_j) \in FS_k \ , \ 1 \leq j \leq r.$$

Therefore, the sequence of powers that is used in constructing the JSJ decomposition for groups (theorem 4.5 in [Ri-Se]), when taken to consist of only positive powers (so that the twisted homomorphisms are indeed pair homomorphisms), can be used to construct a sequence of homomorphisms that converges into a faithful action of the limit group L on a real tree, and this limit action contains an IET component, as proposition 3.5 claims.

□

Proposition 3.5 completes the proof of theorem 3.2.

□

Theorem 3.2 proves that if every faithful action of a freely indecomposable limit group L on a real tree Y, that is obtained as a limit from a sequence of homomorphisms of the pair (S, L) into (FS_k, F_k), is discrete (or simplicial), then the pair (S, L) has only finitely many maximal abelian decompositions, that we view as its (finite collection of) canonical JSJ decompositions. A similar statement is valid in case every such action contains only simplicial and axial components.

Theorem 3.6. *Suppose that L is freely indecomposable, and that all the maximal abelian decompositions, $\{\Lambda_i\}$, that are associated with the pair (S, L), correspond to faithful actions of L on a real tree, where these actions contain only simplicial and axial components.*

Then there exist only finitely many (equivalence classes of) maximal abelian decompositions of the pair (S, L).

Proof: Suppose that there are infinitely many (equivalence classes of) maximal abelian decompositions of a pair (S, L). Let $\{\Lambda_i\}_{i=1}^{\infty}$ be the collection of these (inequivalent) maximal decompositions.

All the abelian decompositions, Λ_i, are dominated by the JSJ decomposition of the freely indecomposable limit group L. If there exists an infinite subsequence of the decompositions, $\{\Lambda_i\}$, that is dominated by an abelian decomposition, Θ_1, that is strictly dominated by the JSJ decomposition of L, we pass to that subsequence. We continue iteratively. If the infinite subsequence of the decompositions, $\{\Lambda_i\}$, that is dominated by Θ_1, has a further infinite subsequence that is dominated by an abelian decomposition, Θ_2, that is strictly dominated by Θ_1, we pass to that subsequence. Since every strictly decreasing sequence of abelian decompositions of the limit group L terminates after finitely many steps, the sequence of maximal abelian decompositions of the pair, (S, L), contains an infinite subsequence, that is dominated by an abelian decomposition, Θ, and no infinite subsequence of that infinite subsequence is dominated by an abelian decomposition, Θ', that is strictly dominated by Θ. We (still) denote this infinite subsequence of maximal abelian decompositions, $\{\Lambda_i\}$, and their dominating abelian decomposition of L, Θ.

Our goal is to show that the abelian decomposition, Θ, that strictly dominates the entire sequence of maximal abelian decompositions of the pair (S, L), $\{\Lambda_i\}$, is itself a maximal abelian decomposition of (S, L), a contradiction to the maximality of each of the abelian decompositions, Λ_i, hence, a contradiction to the existence of an infinite sequence of maximal abelian decompositions that are associated with the pair (S, L), and the theorem follows.

We start with the infinite sequence of inequivalent maximal abelian decompositions $\{\Lambda_i\}$, that are all dominated by the abelian decomposition Θ. With the sequence of abelian decompositions, $\{\Lambda_i\}$, we have associated a sequence of homomorphisms $\{h_n\}$ that satisfy properties (1)-(3) that are listed in the beginning of the proof of theorem 3.3. By possibly passing to a subsequence of the homomorphisms $\{h_n\}$ (still denoted $\{h_n\}$), we obtain a convergent sequence, that converges into a superstable action of the limit group L on a real tree Y, and with this action there is an associated abelian decomposition of L that we denoted Δ_1.

Starting with Δ_1 we (possibly) refine it to get an abelian decomposition Δ_2, precisely as we did in case all the actions on real trees are simplicial, i.e., precisely as we did in the course of proving theorem 3.2. Since all the abelian decompositions Λ_i are dominated by Θ, Δ_2 is dominated by Θ as well.

Proposition 3.7. *If Δ_2 is equivalent to the abelian decomposition Θ, then there exists a sequence of homomorphisms $u_n : (S, L) \to (FS_k, F_k)$ that converges into a superstable action of the limit group L on a real tree $\hat{Y}$, and the abelian decomposition that is dual to this action is Θ itself.*

Proof: Θ can not be simplicial since it dominates an infinite sequence of inequivalent maximal abelian decompositions of L. Hence, Θ contains vertex groups that are associated with either axial or IET components. Δ_2 that is assumed to be equivalent to Θ is obtained from a sequence of pair homomorphisms: $\{h_n : (S, L) \to (FS_k, F_k)\}$. If Θ does not contain vertex groups that are associated with IET components, we vary the homomorphisms $\{h_n\}$ by precomposing them with Dehn twists along edge groups (that connect between non-axial vertex groups), and vary the values of a prefered basis of generators of vertex groups that are associated with axial components using theorem 2.1, as we did in the proof of proposition 3.4 in case Δ_2 contains an axial component. Such modifications gives a new sequence of homomorphisms $\{\hat{h}_n\}$ that converges into a faithful action of the limit tree L on a real tree, where the abelian decomposition that is associated with the limit action is Δ_2 that is equivalent to Θ.

If Δ_2 contains vertex groups that are associated with IET components, then we modify the homomorphisms $\{h_n\}$ by performing Dehn twists along edges that connect between non-axial non-QH vertex groups, vary the values of prefered basis of axial vertex groups using theorem 2.1, and precomposing the obtained homomorphisms with automorphisms that are extensions of automorphisms of the QH vertex groups - precisely as we did in proving proposition 3.4 (in case Δ_2 contains a QH vertex group). Again, the obtained sequence of homomorphisms converges into a faithful action of L on a real tree, where the abelian decomposition that is associated with this action is Δ_2 that is assumed to be equivalent to Θ.

□

Since Θ dominates all the maximal abelian decompositions, $\{\Lambda_n\}$, proposition 3.7 implies that if Δ_2 is equivalent to Θ we obtained a contradiction, and theorem 3.2 follows. Hence, we may assume that Θ strictly dominates Δ_2.

We continue in a similar way to what we did in proving theorem 3.2. If all the edge groups in Δ_2 are elliptic, and the only non-elliptic vertex groups are axial, then for almost all the indices i, the abelian decomposition Λ_i is dominated by the abelian decomposition Δ_2 (i.e., Δ_2 is a (possibly trivial) refinement of Λ_i for almost every index i). Hence, there is a subsequence of the maximal abelian decompositions, $\{\Lambda_i\}$, that is dominated by the abelian decomposition Δ_2, that is strictly dominated by the abelian decomposition Θ, a contradiction to our choice of Θ. Therefore, Δ_2 must contain a hyperbolic edge group.

Suppose that Δ_2 contains an axial vertex group $A = A_1 + A_2$, so that Δ_2 collapses to the abelian decomposition $L = L_1 *_{A_1} A$. If in addition A_1 is elliptic, then it is elliptic in almost all the abelian decomposition $\{\Lambda_i\}$, so it must be elliptic in the dominating abelian decomposition Θ, and therefore, A is an axial vertex group in Θ as well. If A_1 is not elliptic, it can not be elliptic in Θ, and therefore, $rk(A_1) \geq 2$. In both cases we continue by essentially analyzing the subgraph that is obtained from Δ_2 by taking out the vertex group A (and continue with the limit group L_1).

Conversely, if A is an axial vertex group in Δ_2, $A = A_1 + A_2$, so that Δ_2 collapses to an abelian decomposition $L = L_1 *_{A_1} A$, and A is conjugate to an axial vertex group in Θ, that is connected to other vertex groups in Θ by subgroups that generate a conjugate to A_1, then A_1 is elliptic in all the abelian decompositions $\{\Lambda_i\}$.

From the graph of groups Δ_2 we take out the edges with elliptic edge groups, and the axial vertex groups. What left are several connected subgraphs of Δ_2, that we denote: $C_1, \ldots, C_u$. For each of these subgraphs we fix a finite generating set of its fundamental group.

As in the proof of theorem 3.2, we shorten the homomorphisms $\{h_n\}$ (using Dehn twists along dominating edge groups), and pass to a subsequence of them for which the shortenings converge into a (new) faithful action of the limit group L on some real tree. We further restrict each of the shortened homomorphisms to the fundamental groups of the connected components (that are subgraphs in Δ_2), C_i, $i = 1, \ldots, u$. After possibly passing to a subsequence, the restrictions of the shortened homomorphisms converge to a faithful action of that group on a real tree, that we denote Y_i. We denote the abelian decomposition that is associated with these actions, Δ_3^i, $i = 1, \dots, u$. Given each of the abelian decompositions Δ_3^i we further refine it in the same way in which we refined the abelian decomposition Δ_1 to obtain the abelian decomposition Δ_2. We denote the obtained refined abelian decompositions, Δ_4^i, $1 \leq i \leq u$.

The edges that were taken out from Δ_2 to obtain the connected components, C_i, $i = 1, \dots, u$, are either elliptic or contained in abelian subgroups of rank at least 2 in the fundamental groups of the connected components C_i. Hence, these edge groups can be assumed to be either elliptic in the abelian decomposition, Δ_4^i, $i = 1, \dots, u$, or the abelian decompositions Δ_4^i can be modified to guarantee that all the non-cyclic abelian subgroups are elliptic, so that the edge groups that were taken out from Δ_2 are elliptic. Therefore, from the abelian decomposition Δ_4^i, $1 \leq i \leq u$, possibly after a modification that guarantees that all the non-cyclic abelian subgroups are elliptic, it is possible to obtain an abelian decomposition of the ambient limit group L, by adding the elliptic edge groups, and the QH and (axial) abelian edge groups that were taken out from Δ_2. We denote the obtained abelian decomposition Δ_4.

Since the abelian decomposition Θ strictly dominates Δ_2, the construction of Δ_4 from (almost) shortest homomorphisms of the connected components C_i of Δ_2 implies that at least one of the following possibilities must hold:

(1) there exists a QH vertex group in Δ_4.
(2) there exists an edge group with a cyclic centralizer in Δ_4 that can not be conjugated into an edge group nor into a vertex group in Δ_2. Equivalently, there is an edge group with cyclic centralizer in Δ_2 that can not be conjugated into a vertex group or into an edge group in Δ_4.
(3) there exists a non-trivial abelian subgroup A_1 of an abelian subgroup A of rank at least 2 in L, so that A_1 was contained in a non-QH, non-axial vertex group in Δ_2 and A was not contained in such a vertex group in Δ_2, and A_1 can not be conjugated into a non-QH, non-axial vertex group in any of the abelian decompositions Δ_4^i.
(4) there exists a non-trivial abelian subgroup A of rank at least 2 in L, so that A was contained in a non-QH, non-axial vertex group in Δ_2, and A can not be conjugated into a vertex group in any of the abelian decompositions Δ_4^i.

If case (1) occurs we get a contradiction to the assumptions of theorem 3.6 according to proposition 3.4. If cases (3) or (4) occur we apply the argument that was used in the proof of theorem 3.2, and obtain a proper refinement of Δ_2. If case (2) occurs it is possible to construct a sequence of homomorphisms of the pair (S, L) that converge into a faithful action of L on a real tree, where this real tree contains an IET component, a contradiction to the assumptions of theorem 3.6.

As long as the obtained abelian decomposition is strictly dominated by Θ we can continue refining the obtained abelian decomposition iteratively. Therefore, after finitely many refinements we get the abelian decomposition Θ, and a sequence of homomorphisms of the pair (S, L) that converge into a faithful action of L on a real tree with an associated abelian decomposition Θ. Since Θ dominates the sequence of maximal abelian decompositions $\{\Lambda_i\}$, we obtained a contradiction to their maximality, and theorem 3.6 follows.

□

In case L is freely indecomposable, and all the faithful actions of L on a real tree that are obtained as limit of sequences of homomorphisms of the pair (S, L) contain only simplicial and axial components, we set each of the finitely many maximal abelian decompositions that are associated with such a pair (S,L), to be a *JSJ decomposition* of the pair (S,L). Hence, with such a pair (S,L) we have canonically assigned finitely many (abelian) JSJ decompositions, that are all obtained from the abelian JSJ decomposition of the limit group L, by possibly cutting some of the QH vertex groups along finitely many s.c.c. and further collapsing and folding.

Unfortunately, we were not able to generalize theorem 3.6 to all pairs (S,L) in which the limit group L is freely indecomposable. i.e., we were not able to prove that given a pair (S,L) in which L is freely indecomposable, there are only finitely many maximal abelian decompositions that are associated with it. Such a statement will enable one to associate canonically finitely many abelian JSJ decompositions with such a pair (that are all dominated by the abelian JSJ decomposition of the limit group L). The existence of these JSJ decompositions will simplify (and enrich) considerably the structure theory that we develop in the sequel, including the structure of the Makanin-Razborov diagram that we associate with a pair (S,L).

Question. *Let (S,L) be a pair in which L is freely indecomposable. Are there only finitely maximal abelian decompositions that are associated with the pair (S,L)? (see proposition 3.1 for the definition and existence of maximal abelian decompositions).*

§4. Limit sets

One of the fundamental objects that is associated with a Kleinian group is its limit set. For such a (non-elementary) group the limit set is a non-empty closed subset of S^2, that often has a fractal structure. In this section we define a natural limit set that is associated with a pair (S,L), where L is a freely indecomposable limit group, and one of its associated (canonical) JSJ decompositions.

Given a (Gromov) hyperbolic group or a limit group one can study all the small stable faithful actions of such a group on a real tree. For a freely indecomposable torsion-free hyperbolic group, the properties of its canonical JSJ decompositions, together with a theorem of Skora [Sk], and Thurston's compactification of the Teichmuller space, imply that the set of small stable faithful actions of such a group on a real tree up to dilatation, can be naturally viewed as a topological space, that is homeomorphic to a finite union of (possibly trivial) products of finitely many spheres and a (possibly trivial) Euclidean factor. The properties of the abelian decomposition of a limit group, implies the same conclusion for the structure of the set of small stable faithful actions of a freely indecomposable limit group on a real tree up to dilatation.

Definition 4.1. *Let (S,L) be a pair in which L is a freely indecomposable limit group. Look at all the sequences of pair homomorphisms $\{h_n:(S,L)\to(FS_k,F_k)\}$ that converge into a faithful action of L on a real tree. Note that the action must be small and (super) stable.*

The set of all these limit actions up do dilatation, is a subset of all the faithful small stable actions of L on real trees up to dilatation, and can naturally be equipped with the induced topology. We call this topological space the limit set of the pair (S,L).

It is natural to ask if such a limit set is homeomorphic to a finite simplicial complex. If so it may be described more explicitly, perhaps even computed. In the sequel, for the purposes of encoding the structure of all pair homomorphisms of a given pair, we don't need a detailed understanding of this limit set, apart from some of its global properties.

If it is homeomorphic to a simplicial complex, when is freely indecomposable, one can ask the same question for general limit groups L, omitting the indecomposability requirement.

§5. Enlarging the Positive Cone

The JSJ decomposition a freely indecomposable limit group plays an essential role in the geometric construction of the Makanin-Razborov diagram that encodes the set of solutions to a system of equations over a free group ([Se1]). In section 3 we managed to associate canonically finitely many JSJ decompositions with pairs (S, L) only is some special cases. Therefore, to construct a Makanin-Razborov diagram that encodes the solutions to a system of equations over a free semigroup, we need to modify the construction that is used over groups.

The main object that we use in the case of semigroups is a *resolution.* A resolution has finitely many steps, it starts with the given pair (S, L), and continues with quotients of it that are not necessarily proper quotients. Later on we show that there exist finitely many such resolutions that encode all the pair homomorphisms of a given pair (S, L).

To construct resolutions we start with the *extended cone* that we associate with an abelian decomposition, that was already used in the shortenings in section 2. We start with a strengthening of theorem 2.1 which is not needed in the sequel, but is of independent interest.

Theorem 5.1. *Let A be a f.g. free abelian group, let $rk(A) = \ell$, and let S be a f.g. subsemigroup of A that generates A as a group. Let $s_1, \dots, s_r$ be a fixed generating set of S.*

Then there exist finitely many positive collections of (free) bases of A, $a^i_1, \dots, a^i_\ell$, $1 \leq i \leq c$, such that for any sequence of homomorphisms $h_n : (S, A) \to (FS_k, F_k)$, that converges into a free action of A on a real tree, there exists an index i, $1 \leq i \leq c$, and an index n_0, for which for every index $n > n_0$:

(1) *(1) $h_n(a^i_j) \in FS_k$ for $1 \leq j \leq \ell$.*
(2) *(2) for each of the (fixed set of) generators $s_1, \dots, s_r$ of the semigroup S, there are fixed words, that depend only on the index i, $1 \leq i \leq c$, so that for every m, $1 \leq m \leq r$:*

$$s_m = w_{i,m}(a^i_1, \dots, a^i_\ell)$$

where the words $w_{i,m}$ are (fixed) positive words in the elements $a^i_1, \dots, a^i_\ell$.

Proof: The theorem is immediate if $\ell = 1$, hence, we may assume that $\ell > 1$. Given a sequence of homomorphisms $\{h_n : (S, A) \to (FS_k, F_k)\}$, that converges into a free action of A on a real tree, theorem 2.1 implies that there exists a free basis of A, $a_1, \dots, a_\ell$, and an index n_0, so that for every $n > n_0$, the homomorphisms, $\{h_n\}$, that satisfy properties (1) and (2) in the statement of the theorem.

To prove that there exists a finite collection of positive words that suffices for all the convergent sequences we use a compactness argument. We have already shown that given a sequence of homomorphisms $\{h_n : (S, A) \to (FS_k, F_k)\}$ that converges into a free action of A on a real tree Y, there exists a collection of positive words and a basis of A for which the conclusions of the theorem hold.

Suppose that finitely many such collections of free bases together with positive words do not suffice. There exist only countably many possible collections of positive words, so we order the infinite set of the collections of free bases and positive words that are associated with convergent sequences of homomorphisms that satisfy the assumptions of the theorem.

For each positive integer t, let $\{h_n^t : (S, A) \to (FS_k, F_k)\}$ be a sequence of homomorphisms that satisfies the assumption of the theorem (i.e., it converges to a free action of A on a real tree). Suppose further that the sequence $\{h_n^t\}$ does not satisfy the conclusions of the theorem with respect to first t collections of free bases and positive words.

For each positive integer t, the sequence $\{h_n^t\}$ converges into a free action of the (free) abelian group A on an oriented line Y_t, $\lambda_t : A \times Y_t \to Y_t$. From the sequence of actions λ_n^t it is a possible to extract a convergent subsequence, that converge into a non-trivial, not necessarily faithful, action of the abelian group A on a directed real line, $\alpha_0 : A \times L_0 \to L_0$.

If the action α_0 is not free, let A_1 be the kernel of the action α_0. A_1 is a direct summand in A. If A_1 is non-trivial, then we can pass to a further subsequence, for which the sequence of actions λ_t restricted to the direct summand A_1 converges into a non-trivial, not necessarily faithful action of A_0 on a directed line: $\alpha_1 : A_0 \times L_1 \to L_1$.

If the action α_1 is not free, we continue iteratively by restricting the actions λ_t to the kernel of the previous action (which is a direct summand in the previous summand), and pass to a convergent subsequence. Since A is a f.g. free abelian group this iterative procedure terminates after finitely many steps.

Let $\{\lambda_{t_d}\}$ be the final convergent subsequence. By the argument that was used to prove theorem 2.1, the conclusions of theorem 2.1 hold for the subsequence λ_{t_d}. Hence, there exists a fixed free basis of A that act positively on Y_{t_d} for large enough d, and a fixed set of words $\{w_m\}$, $1 \leq m \leq r$, that satisfy part (ii) in the statement of the theorem.

This free basis of A, together with the words $\{w_m\}$, appear in the ordered list of bases and words. Hence, for large enough index d, the sequences of homomorphisms $\{h_n^{t_d}\}$ are assumed not to satisfy the conclusion of theorem 2.1 with respect to the free basis and the collection of words that we associated with the convergent sequence $\{\lambda_{t_d}\}$. But since this free basis and the collection of words $\{w_m\}$ were associated using the argument of theorem 2.1 with the convergent sequence $\{\lambda_{t_d}\}$, for large enough d, and large enough n, the sequence of homomorphisms $\{h_n^{t_d}\}$ does satisfy the conclusion of theorem 2.1 with respect to this free basis and collection of words, a contradiction.

Therefore, finitely many free bases together with a finite collection of words suffice for all the convergent sequences that satisfy the assumption of the theorem, and the conclusion of the theorem follows.

□

For a pair (S, Q) in which Q is a (closed) surface group, S is a f.g. subsemigroup that generates Q as a group, and (S, Q) is obtained as a limit of a sequence of pair homomorphisms into (FS_k, F_k), we state a weaker statement that associates a *standard cone* with such a pair.

Lemma 5.2. *Let (S,Q) be a pair of a closed (hyperbolic) surface group Q and a subsemigroup S that generates Q as a group. Let $s_1,\ldots,s_r$ be a fixed generating set of S.*

Let $\{h_n : (S,Q) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converge into a free action of Q on a real tree. Then there exists a set of standard generators, $q_1,\ldots,q_\ell$ of Q, such that:

1. *the presentation of Q with respect to $q_1,\ldots,q_\ell$ is one of the finitely many possible (positively oriented) interval exchange type generating sets of Q (i.e., they generate Q and the presentation is the one obtained from a permutation of finitely many positively oriented subintervals (that intersect only in their endpoints) of an ambient positively oriented interval).*
2. *there exists an index n_0 such that for every $n > n_0$: $h_n(q_b) \in FS_k$ for $b = 1,\ldots,\ell$.*

Proof: Since the sequence of homomorphisms $\{h_n\}$ converges into a minimal IET action on a real tree, by the Rips machine, or the Makanin procedure, there exists a standard set of generators that satisfies the conclusion of the lemma.

□

Theorem 5.1 and lemma 5.2 generalize to pairs (S,L) in which the ambient limit group L is freely indecomposable. The generalization is crucial in constructing the Makanin-Razborov diagram of a pair.

Theorem 5.3. *Let (S,L) be a pair of a freely indecomposable limit group L and a subsemigroup S that generates L as a group. Let $s_1,\ldots,s_r$ be a fixed generating set of S. Let $\{h_n : (S,L) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converge into a faithful action of L on a real tree with an associated abelian decomposition Λ. Suppose that A is the stabilizer of an axial component in the real tree (and a vertex group in Λ), and Q is a QH vertex group in Λ that is associated with an IET component.*

Let $A_0 < A$ be the direct summand of A that contains the subgroup that is generated by the edge groups that are connected to the vertex that is stabilized by A in Λ as a subgroup of finite index. Suppose that: $rk(A) - rk(A_0) = \ell$. Let $\hat{g}$ be the number of vertices in Λ, that are not stabilized by the abelian group A, for which their vertex groups have conjugates that stabilize points in the axis of A (these are all the vertices that are adjacent to the vertex that is stabilized by A in Λ). Let $g = max(\hat{g}-1, 0)$. Let $EA = A + < e_1,\ldots,e_g >$ be a free abelian group of rank $rk(A) + g$.

With the QH subgroup Q we associate a natural extension. The associated natural extension of Q is generated by Q and finitely many elements that map a point with non-trivial stabilizer in the subtree that is stabilized by Q, to another point with a non-trivial stabilizer in that subtree that is not in the same orbit under the action of Q. By adding these elements all the points with non-trivial stabilizers in the subtree that is stabilized by Q are in the same orbit under the natural extension of Q.

Each of the given generators of the semigroup S, $s_1,\ldots,s_r$, can be written as a word (in a normal form) in elements that lie in the natural extensions of the surface group Q (that are associated with the subpaths of the paths that are associated with $s_1,\ldots,s_r$ that are contained in subtrees that are stabilized by conjugates of Q), and *elements that lie outside these extensions. Let $t_1, \dots, t_d$ be the elements that lie in the natural extensions of Q that appear as subwords in the normal form of the elements $s_1, \dots, s_r$.*

Then there exist:

(i) *$\ell + g$ elements that are part from a (free) basis of EA, $a_1, \dots, a_{\ell+g}$, such that $EA =< a_1, \dots, a_{\ell+g} > +A_0$.*

(ii) *standard generators, $q_1, \dots, q_\ell$, of a natural extension of Q, that satisfy properties that are analogous to properties (1) and (2) in the statement of lemma 5.2.*

iii) *each of the generators s_j can be written as a word in terms of elements $\{t^j_d\}$ in the natural extension of Q (i.e., t^j_d itself is a word in $q_1, \dots, q_\ell$, elements $\{u^j_e\}$ that are positive words in $a_1, \dots, a_{ell+g}$, and elements in the other vertex and edge groups in Λ.*

Such that there exists an index n_0 for which for every $n > n_0$ the following properties hold:

(1) *$h_n(a_b) \in FS_k$ for $1 \leq b \leq \ell + g$.*

(2) *$h_n(q_b) \in FS_k$, $1 \leq b \leq \ell+g$, $h_n(a_i) \in FS_k$, $1 \leq i \leq \ell+g$, and $h_n(t^j_d) \in FS_k$ for every possible pair of indices (j, d).*

Proof: Follows by the same arguments that were used to prove theorem 5.1 and lemma 5.2.

□

§6. A Makanin-Razborov diagram - the freely indecomposable case

In the previous section we associated a standard set of generators with an abelian decomposition. In this section we use shortenings that were constructed in section 2, together with the machine for the construction of the JSJ decomposition for groups, to associate finitely many resolutions with a given pair (S, L) in which the limit group L is freely indecomposable. These resolutions enable one to encode the set of all the pair homomorphisms of such a pair using the set of pair homomorphisms of finitely many proper quotient pairs.

For presentation purposes we will start by proving the main theorem in case the limit group L is freely indecomposable and contains no non-cyclic abelian subgroups, and then combine it with arguments that were used in the construction of the JSJ decompositions of pairs (in special cases) in section 3 to omit the assumption on abelian subgroups. In the sequel we generalize the construction of resolutions to all pairs, omitting the freely indecomposable assumption. These resolutions are the building block of the Makanin-Razborov diagram that encodes all the pair homomorphisms of a given pair, or alternatively, all the solutions to a system of equations over a free semigroup.

Theorem 6.1. *Let (S, L) be a pair, where L is a freely indecomposable limit group, and let $s_1, \dots, s_r$ be a fixed generating set of the semigroup S. Suppose that the limit group L contains no non-cyclic abelian subgroup. Let $\{h_n : (S, L) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on a real tree Y.*

Then there exists a resolution:

$$(S_1, L_1) \to (S_2, L_2) \to \ldots \to (S_m, L_m) \to (S_f, L_f)$$

that satisfies the following properties:

(1) $(S_1, L_1) = (S, L)$, *and* $\eta_i : (S_i, L_i) \to (S_{i+1}, L_{i+1})$ *is an isomorphism for* $i = 1, \ldots, m-1$ *and* $\eta_m : (S_m, L_m) \to (S_f, L_f)$ *is a proper quotient map.*

(2) *with each of the pairs* (S_i, L_i), $1 \leq i \leq m$, *there is an associated abelian decomposition that we denote* Λ_i.

(3) *there exists a subsequence of the homomorphisms* $\{h_n\}$ *that factors through the resolution. i.e., each homomorphism* h_{n_r} *from the subsequence, can be written in the form:*

$$h_{n_r} \;=\; \hat{h}_r \circ \varphi_r^m \circ \ldots \circ \varphi_r^1$$

where $\hat{h}_r : (S_f, L_f) \to (FS_k, F_k)$, *each of the automorphisms* $\varphi_r^i \in Mod(\Lambda_i)$, *and each of the homomorphisms:*

$$h_{n_r}^i \;=\; \hat{h}_r \circ \varphi_r^m \circ \ldots \circ \varphi_r^i$$

is a pair homomorphism $h_{n_r}^i : (S_i, L_i) \to (FS_k, F_k)$.

Proof: If the abelian decomposition Λ that is associated with the action of L on the limit tree Y is equivalent to the abelian JSJ decomposition of the limit group L (as a group), then the theorem follows from the possibility to shorten using $Mod(\Lambda)$ that was proved in section 2, and claim 5.3 in [Se1]. i.e., in that case every shortening quotient is a proper quotient, so a subsequence of shortenings of the homomorphisms $\{h_n\}$ converge into a proper quotient of the pair (S, L), and the conclusion of the theorem follows with a resolution of length 1: $\eta_1 : (S, L) \to (S_f, L_f)$.

Hence, we may assume that the abelian decomposition Λ, that is associated with the action of L on the limit tree Y, is strictly dominated by the abelian JSJ decomposition of L. In particular, Λ does not correspond to a closed surface, so it contains edges.

Assume that the sequence $\{h_n\}$ converges into a faithful action of the limit group L on some real tree Y with an associated abelian decomposition Λ, L is freely indecomposable, and contains no non-cyclic abelian subgroups.

We start by refining the abelian decomposition Λ precisely as we did in proving theorem 3.2. Using definition 3.3 we divide the edges that are adjacent to a given non-QH vertex group in Λ to periodic and non-periodic. If there is a non-QH vertex group in Λ that is adjacent only to periodic edge groups, we pass to a subsequence of the homomorphisms $\{h_n\}$ for which their restrictions to this vertex group converges to a non-trivial faithful action on a real tree. The abelian decomposition that is associated with this action (in which all the edge groups that are connected to this vertex group in Λ are elliptic), enable us to further refine Λ. By Bestvina-Feighn accessibility [Be-Fe1], or alternatively by acylindrical accessibility [Se3], this refinement process terminates after finitely many steps, and we obtain an abelian decomposition Λ_1. Note that in Λ_1 every non-QH vertex group is adjacent to at least one non-periodic edge group.

In section 3 we divided the edges and vertices in Λ_1 into several equivalence classes, according to the growth of the translation lengths of fixed sets of generators of the corresponding edge groups. We fix a finite set of generators for each of the vertex groups and edge groups in Λ_1. Recall that we say that two displacement functions of two vertex groups or edge groups, $disp_n(V_1), disp_n(V_2)$, are *comparable* if there exists positive constants c_1, c_2, so that for every index n, $c_1 \cdot disp_n(V_1) < disp_n(V_2) < c_2 \cdot disp_n(V_1)$. We say that $disp_n(V_1)$ *dominates* $disp_n(V_2)$ if $disp_n(V_2) = o(disp_n(V_1))$.

After passing to a subsequence of the homomorphisms $\{h_n\}$ this defines an order on the equivalence classes of the edge groups and vertex groups in Λ_1. In particular, there exists a collection of edge groups and vertex groups with comparable displacement functions that dominate all the other (displacement functions of) vertex groups and edge groups in Λ_1.

Suppose first that Λ_1 does not contain QH vertex groups. In that case we set $Mod(\Lambda_1)$ to be the modular group of the pair (S, L) that is associated with the abelian decomposition Λ_1, and $MXMod(\Lambda_1)$ to be the subgroup of $Mod(\Lambda_1)$, that is generated by Dehn twists only along dominating edge groups.

We start by using the full modular group $Mod(\Lambda_1)$. For each index n, we set the pair homomorphism: $h_n^1 : (S, L) \to (FS_k, F_k)$ and $h_n^1 = h_n \circ \varphi_n$ where $\varphi_n \in Mod(\Lambda_1)$, to be the shortest pair homomorphism that is obtained from h_n by precomposing it with a modular automorphism from $Mod(\Lambda_1)$. If there exists a subsequence of the homomorphisms $\{h_n^1\}$ that converge into a proper quotient of the pair (S, L), we set the limit of this subsequence to be (S_f, L_f), and the conclusion of the theorem follows with a resolution of length 1.

Therefore, we may assume that every convergent subsequence of the homomorphisms $\{h_n^1\}$ converges into a faithful action of the limit group L on some real tree. In that case we use only Dehn twists along dominant edge groups. For each index n, we set $h_n^1 = h_n \circ \varphi_n$, where $\varphi_n \in MXMod(\Lambda_1)$, and h_n^1 is one of the shortest homomorphisms that is obtained by precomposing h_n with a modular automorphism from $MXMod(\Lambda_1)$. We pass to a subsequence of the homomorphisms $\{h_n^1\}$ that converges into (necessarily faithful) action of L on some real tree with an associated abelian decomposition Δ_2. We further refine Δ_2 by analyzing actions of non-QH vertex groups in Δ_2 that are connected only to periodic edge groups and obtain a (possibly) refinement of Δ_2 that we denote Λ_2.

For presentation purposes we assume that Λ_2 contains no QH vertex groups. Note that by construction every dominant edge group in Λ_1 is not elliptic in Λ_2. With Λ_2 we associate its modular group $Mod(\Lambda_2)$, and the modular group that is associated only with dominant edge groups that we denote $MXMod(\Lambda_2)$.

We continue iteratively. First we shorten using the full modular group, $Mod(\Lambda_i)$, and check if there a subsequence of shortened homomorphisms that converge into a proper quotient of the pair (S, L). If there is such a subsequence we obtained a finite resolution. If not we shorten only along edges with dominant edge groups, i.e., using automorphisms from $MXMod(\Lambda_i)$, pass to a convergent subsequence and further refine the obtained abelian decomposition. For presentation purposes we assume that all the obtained abelian decompositions, $\{\Lambda_i\}$ do not contain QH vertex groups.

Definition 6.2. *Let $\Lambda_1, \ldots, \Lambda_i, \ldots$ be the infinite sequence of abelian decompositions of the limit groups L that are constructed along the iterative procedure. Note* *that we assume that all these abelian decompositions do not contain QH vertex groups, and that the limit group L in the pair (S, L) is freely indecomposable and contains no non-cyclic abelian subgroups. By construction, a dominant edge group in Λ_i is not elliptic in Λ_{i+1}.*

All the abelian decompositions Λ_i are dominated by the abelian JSJ decomposition of L. For each index i, we set Θ_i to be the minimal abelian decomposition (w.r.t. to the natural partial order that we defined on abelian decompositions of L) that dominates all the abelian decompositions, $\Lambda_i, \Lambda_{i+1}, \ldots$.

The sequence of abelian decompositions $\{\Theta_i\}$ is non-increasing, and every strictly decreasing sequence of abelian decompositions of L has to terminate, hence, there exists some index i_0, such that for every $i > i_0$, $\Theta_{i_0} = \Theta_i$. We call Θ_{i_0} the stable dominant abelain decomposition of the sequence $\Lambda_1, \Lambda_2, \ldots$.

The abelian decompositions Λ_i were assumed to have no QH vertex groups. Since every dominant edge group in Λ_i is hyperbolic in Λ_{i+1}, each of the dominant abelian decompositions, Θ_i, dominates a pair of hyperbolic-hyperbolic splittings of the limit group L. Hence, it must contain a QH vertex group. In particular, the stable dominant abelian decomposition, Θ_{i_0}, must contain a QH vertex group.

The next proposition is crucial in our approach to proving theorem 6.1. It enables the substitution of the entire suffix of the sequence of abelian decompositions $\{\Lambda_i\}$, $\Lambda_{i_0}, \Lambda_{i_0+1}, \ldots$, with a single abelian decomposition - the stable dominant decomposition, Θ_{i_0}.

Proposition 6.3. *There exists a subsequence of shortened pair homomorphisms $\{h^i_{n_i}\}$, $i \geq i_0$, and a sequence of automorphisms: $\psi^i, \nu^i \in Mod(\Theta_{i_0})$, $i \geq i_0$, with the following properties:*

(1) *$h^i_{n_i}$ is obtained from the pair homomorphism h_{n_i}, by shortening h_{n_i} using a sequence of elements from the dominant modular groups: $MXMod(\Lambda_1), \ldots, MXMod(\Lambda_i)$. i.e.,*

$$h_{n_i} \;=\; h^i_{n_i} \circ \varphi^i_{n_i} \circ \ldots \circ \varphi^1_{n_i}$$

where $\varphi^j_{n_i} \in MXMod(\Lambda_j)$, $1 \leq j \leq i$ and for each j, $1 \leq j \leq i-1$, the homomorphism:

$$h^j_{n_i} \;=\; h^i_{n_i} \circ \varphi^i_{n_i} \circ \ldots \circ \varphi^{j+1}_{n_i}$$

is a pair homomorphism $h^j_{n_i} : (S, L) \to (FS_k, F_k)$.

(2) *For each index i we define a pair homomorphism:*

$$f_i \;=\; h^i_{n_i} \circ \nu^{i+1} \circ \psi^i \circ \ldots \circ \psi^{i_0}.$$

The sequence of pair homomorphisms, $\{f_i\}$, converges into a faithful action of the limit group L on a real tree with an associated abelian decomposition Θ_{i_0}. In particular, the limit action contains an IET component.

Proof: To prove the proposition we basically imitate the construction of the JSJ decomposition as it appears in [Se1] and [Ri-Se]. We fix a generating set $< s_1, \ldots, s_r >$ of the semigroup S, that also generate the limit group L (as a group). We start with the subsequence of pair homomorphisms that are obtained using a (finite) iterative sequence of shortenings from a subsequence of the given sequence of homomorphisms $\{h_n\}$, using the dominant modular groups $MXMod$ that are associated with the abelian decompositions $\Lambda_1, \ldots, \Lambda_{i_0-1}$. By our assumptions the sequence $\{h_{n_t}^{i_0-1}\}$ converges into a faithful action of L on a real tree, with an associated abelian decomposition that can be further refined (by restricting the homomorphisms to non-QH vertex groups that are connected only to periodic edge groups) to the abelian decomposition Λ_{i_0}.

We set the automorphisms $\nu_{n_t}^{i_0} \in Mod(\Lambda_{i_0})$ to be automorphisms that guarantee that the compositions $h_{n_t}^{i_0-1} \circ \nu_{n_t}^{i_0}$ converge into a faithful action of L on a real tree Y_{i_0}, in which:

(1) the abelian decomposition that is dual to the action of L on the new real tree Y_{i_0} is Λ_{i_0}. In particular, all the edges in Λ_{i_0} correspond to non-degenerate segments in Y_{i_0}.
(2) for some pre-chosen (and arbitrary) positive constant ϵ_{i_0}, and after rescalings, the lengths of the segments $[y_0, s_j(y_0)]$, $1 \leq j \leq r$, differ by at most ϵ_{i_0} from the length of that segment in the original real tree, which is the limit of the sequence of homomorphisms $h_{n_t}^{i_0-1}$. Furthermore, the length of each segment in the quotient graph of groups that is associated with the action of L on Y_{i_0} differ by at most ϵ_{i_0} from the length of the corresponding segment in the graph of groups that is associated with the action of L on the real tree that is obtained as a limit of the sequence $h_{n_t}^{i_0-1}$.
(3) the segments $[y_0, s_j(y_0)]$ are degenerate or positively oriented, and for each t, $h_{n_t}^{i_0-1} \circ \nu_{n_t}^{i_0} \in FS_k$.

By the properties of the abelian decomposition Λ_{i_0}, such automorphisms $\nu_{n_t}^{i_0}$ can be constructed.

We continue iteratively. In analyzing the next level, the one in which the sequence $\{h_{n_t}^{i_0}\}$ converges into a limit action on a real tree that is associated with Λ_{i_0+1}, we set the automorphisms $\psi^{i_0} \in Mod(\Lambda_{i_0})$ and $\nu_{n_t}^{i_0+1} \in Mod(\Lambda_{i_0+1})$, to be automorphisms that guarantee that the compositions $h_{n_t}^{i_0} \circ \nu_{n_t}^{i_0+1}$ converge into a faithful action of L on a real tree Y_{i_0+1}, in which:

(1) the abelian decomposition that is dual to the action of L on the new real tree Y_{i_0+1} is Λ_{i_0+1}. In particular, all the edges in Λ_{i_0+1} correspond to non-degenerate segments in Y_{i_0+1}.
(2) we fix finite generating sets for all the vertex groups in Λ_{i_0} that are connected to dominant edge groups, and for all the dominant edge groups in Λ_{i_0}.

    For some pre-chosen (and arbitrary) positive constant ϵ_{i_0+1}, after rescalings, the lengths and the translation lengths of all the elements in these fixed finite generating sets when acting on Y_{i_0+1}, differ by at most ϵ_{i_0+1} from the corresponding lengths and translation lengths in the limit tree that is obtained from the sequence of homomorphisms $\{h_{n_t}^{i_0}\}$.
(3) after rescaling, the lengths of the segments $[y_0, \psi^{i_0}(s_j)(y_0)]$, $1 \leq j \leq r$, in Y_{i_0+1}, differ by at most ϵ_{i_0+1} from the lengths of the corresponding segments, $[y_0, s_j(y_0)]$ in the limit tree Y_{i_0}.
(4) the segments $[y_0, s_j(y_0)]$ are degenerate or positively oriented in Y_{i_0+1}, and for each t, $h_{n_t}^{i_0} \circ \nu_{n_t}^{i_0+1}(s_j) \in FS_k$ and $h_{n_t}^{i_0} \circ \nu_{n_t}^{i_0+1} \circ \psi^{i_0}(s_j) \in FS_k$.

In analyzing the next level, the one in which the sequence $\{h_{n_t}^{i_0+1}\}$ converges into a limit action on a real tree that is associated with Λ_{i_0+2}, we consider all the elements in the ball of radius 2, B_2, in the Cayley graph of the limit group L w.r.t. the generating set $s_1, \dots, s_r$.

We set the automorphisms $\psi^{i_0+1} \in MXMod(\Lambda_{i_0+1})$ and $\nu_{n_t}^{i_0+2} \in Mod(\Lambda_{i_0+2})$ to be automorphisms that guarantee that the compositions $h_{n_t}^{i_0+1} \circ \nu_{n_t}^{i_0+2}$ converge into a faithful action of L on a real tree Y_{i_0+2}, in which:

(1) the abelian decomposition that is dual to the action of L on the new real tree Y_{i_0+2} is Λ_{i_0+2}. In particular, all the edges in Λ_{i_0+2} correspond to non-degenerate segments in Y_{i_0+2}.

(2) we fix finite generating sets for all the vertex groups in Λ_{i_0+1} that are connected to dominant edge groups, and for all the dominant edge groups in Λ_{i_0+1}.

For some pre-chosen (and arbitrary) positive constant ϵ_{i_0+2}, after rescalings, the lengths and the translation lengths of all the elements in these fixed finite generating sets when acting on Y_{i_0+2}, differ by at most ϵ_{i_0+2} from the corresponding lengths and translation lengths in the limit tree that is obtained from the sequence of homomorphisms $\{h_{n_t}^{i_0+1}\}$.

(3) after rescaling, for all the elements $u \in B_2$, the lengths of the segments $[y_0, \psi^{i_0+1} \circ \psi^{i_0}(u)(y_0)]$ in Y_{i_0+2}, differ by at most ϵ_{i_0+2} from the lengths of the corresponding segments, $[y_0, \psi^{i_0}(u)(y_0)]$ in the limit tree Y_{i_0+1}.

(4) the segments $[y_0, s_j(y_0)]$ are degenerate or positively oriented in Y_{i_0+2}, and for each t, $h_{n_t}^{i_0+1} \circ \nu_{n_t}^{i_0+2}(s_j) \in FS_k$ and $h_{n_t}^{i_0+1} \circ \nu_{n_t}^{i_0+2} \circ \psi^{i_0+1} \circ \psi^{i_0}(s_j) \in FS_k$.

We continue iteratively, constructing at each level i the automorphisms $\{\nu_{n_t}^{i+1}\}$, and ψ^i. Note that in the iterative construction the automorphisms ψ^i are fixed at step i, and the sequence $\{\nu_{n_t}^{i+1}\}$ is constructed at step i. All these automorphisms are from the modular group $Mod(\Theta_{i_0})$.

We set the homomorphisms f_i, the sequence of homomorphisms that appear in part (2) of the statement of the proposition, to be:

$$f_i \;=\; h_{n_i}^i \circ \nu^{i+1} \circ \psi^i \circ \dots \circ \psi^{i_0}$$

for a suitable strictly increasing sequence of indices: $\{n_i\}$, such that the sequence $\{f_i\}$ converges into an action of the limit group L on a real tree Y. By construction, every non-trivial element of L is mapped to a non-trivial element of F_k for large enough i, since the homomorphisms: $h_{n_t}^i \circ \nu_{n_t}^{i+1}$ converge into a faithful action of L on a real tree. Hence, the action of L on the limit tree Y is faithfull. Furthermore, by the construction of the automorphisms ψ^i and $\nu_{n_t}^{i+1}$, every element $g \in L$ that is not elliptic in the abelian decomposition Θ_{i_0} acts hyperbolically on the real tree Y. Clearly, every element that is elliptic in Θ_{i_0} fixes a point in Y, since it is elliptic in all the abelian decompositions Λ_i, $i \geq i_0$, and the autmorphisms ψ^i map it to a conjugate. Therefore, the abelian decomposition that is associated with the (faithful) action of L on the limit tree Y is the stable dominant abelian decomposition Θ_{i_0}.

□

Proposition 6.3 enables one to replace a suffix of the sequence of abelian decompositions: $\Lambda_1, \dots$ that was constructed by iteratively shortening the given sequence of homomorphisms $\{h_n\}$ (and further pass to subsequences), with a single abelian decomposition that minimally dominates the suffix.

Proposition 6.3 assumes that all the constructed abelian decompositions $\{\Lambda_i\}$ do not contain QH vertex groups. To prove theorem 6.1 in case the limit group L contains no non-cyclic abelian vertex groups, we need to generalize the construction of the abelian decompositions $\{\Lambda_i\}$, and the conclusion of proposition 6.3, in case the abelian decompositions Λ_i do contain QH vertex groups.

Suppose that L is freely indecomposable with no non-cyclic abelian subgroups. Recall that Λ is the abelian decomposition that L inherits from its action on the real tree Y, that it is obtained as a limit from the convergent sequence of homomorphisms $\{h_n\}$. We further refine Λ. If a non-QH vertex group is connected only to periodic edge groups (and in particular it is not connected to a QH vertex group), we restrict the homomorphisms $\{h_n\}$ to such a vertex group and obtain a non-trivial splitting of it in which all the previous (periodic) edge groups are elliptic. Hence, the obtained abelian decomposition of the non-QH vertex group can be used to refine the abelian decomposition Λ. Repeating this refinement procedure iteratively, we get an abelian decomposition that we denote Λ_1.

We fix finite generating sets of all the edge groups and all the non-QH vertex groups in Λ_1. We divide the edge and non-QH vertex groups in Λ_1 into finitely many equivalence classes of their growth rates as we did in case Λ_1 contained no QH vertex groups. Two non-QH vertex groups or edge groups (or an edge and a non-QH vertex group) are said to be in the same equivalence class if the maximal length of the image of their finite set of generators have comparable lengths, i.e., the length of a maximal length image of one is bounded by a (global) constant times the maximal length of an image of other, and vice versa. After passing to a subsequence of the homomorphisms $\{h_n\}$ (that we still denote $\{h_n\}$), the non-QH and edge groups in Λ_1 are divided into finitely many equivalence classes. We say that one class dominates another, if the maximal length of an image of a generator of a group from the first class dominates the maximal length of an image of the second, but not vice versa. By definition, the classes are linearly ordered (possibly after passing to a further subsequence). There exists a class that dominates all the other classes, that we call the *dominant* class, that includes (possibly) dominant edge groups and (possibly) dominant non-QH vertex groups.

As in the case in which Λ_1 contains no QH vertex groups, we denote by $Mod(\Lambda_1)$ the modular group that is associated with Λ_1. We set $MXMod(\Lambda_1)$ to be the *dominant* subgroup that is generated by Dehn twists along dominant edge groups, and modular groups of those QH vertex groups that the lengths of the images of their (fixed) set of generators grows faster than a constant times the length of the images of the generators of a dominant edge or vertex group. We call these QH vertex groups, *dominant* QH vertex groups.

As we did in the simplicial case, we start by using the full modular group $Mod(\Lambda_1)$. For each index n, we set the pair homomorphism: $h_n^1 : (S, L) \to (FS_k, F_k)$, $h_n^1 = h_n \circ \varphi_n$, where $\varphi_n \in Mod(\Lambda_1)$, to be a shortest pair homomorphism that is obtained from h_n by precomposing it with a modular automorphism from $Mod(\Lambda_1)$. If there exists a subsequence of the homomorphisms $\{h_n^1\}$ that converges into a proper quotient of the pair (S, L), we set the limit of this subsequence to be (S_f, L_f), and the conclusion of the theorem follows with a resolution of length 1.

Therefore, we may assume that every convergent subsequence of the homomorphisms $\{h_n^1\}$ converges into a faithful action of the limit group L on some real tree. In that case we use only elements from the dominant modular group, $MXMod(\Lambda_1)$.

First, we shorten the action of each of the dominant QH vertex groups using the procedure that is used in the proof of propositions 2.7 and 2.8. For each of the dominant QH vertex groups, the procedures that are used in the proofs of these propositions give us an infinite collections of positive generators, $u_1^m, \ldots, u_g^m$, with similar presentations of the corresponding QH vertex groups, which means that the sequence of sets of generators belong to the same isomorphism class. Furthermore, with each of these sets of generators there are associated words, w_j^m, $j = 1, \ldots, r$, of lengths that increase with m, such that a given (fixed) set of positive elements can be presented as: $y_j = w_j^m(u_1^m, \ldots, u_g^m)$. The words w_j^m are words in the generators u_i^m and their inverses. However, they can be presented as positive words in the generators u_j^m, and unique appearances of words t_ℓ, that are fixed words in the elements u_j^m and their inverses (i.e., the words do not depend on m), and these elements $t_\ell^m(u_1^m, \ldots, u_g^m)$ are positive for every m.

Given the output of the procedures that were used in propositions 2.7 and 2.8, with each of the dominant QH vertex groups we associate a system of generators $u_1^1, \ldots, u_g^1$ (the integer g depends on the QH vertex group). Since an IET action of a QH vertex group is indecomposable in the sense of $[Gu]$, finitely many (fixed) translates of each of the positive paths that are associated with the positive paths, $u_{i_1}^1$, cover the positive path that is associated with $u_{i_2}^1$, and the positive paths that are associated the words $t_\ell(u_1^1, \ldots, u_g^1)$. The covering of the elements $u^1 i_1$ by finitely many translates of elements $u_{i_2}^1$ guarantee that the ratios between their lengths along the iterative (shortening) procedure that we use remain globally bounded.

With a QH vertex group Q, and the abelian decomposition Λ_1, we can naturally associate an abelian decomposition Γ_Q, that is obtained by collapsing all the edge groups that are not connected to Q in Λ_1, Γ_Q contains one QH vertex group, Q, and all the other vertex groups are connected only to the vertex stabilized by Q. If a generator s_j from the fixed set of generators of the semigroup S, $s_1, \ldots, s_r$, is not elliptic in Γ_Q, or it is contained in a conjugate of Q, then finitely many translates of the path that is associated with s_j in the limit tree Y covers the paths that are associated with the elements: t_ℓ and $u_1^1, \ldots, u_g^1$ that generate Q.

These coverings of the paths that are associated with the elements $u_1^1, \ldots, u_g^1$ by translates of the path that is associated with s_j, will guarantee that the lengths of the paths that are associated with s_j, along the entire shortening process that we present, are bounded below by some (global) positive constant times the lengths of the paths that are associated with the elements, $u_1^1, \ldots, u_g^1$ along the process.

For all the homomorphisms $\{h_n\}$, except perhaps finitely many of them, the images of the elements $u_1^1, \ldots, u_g^1$ that are associated with the various dominant QH vertex groups, and the elements $t_\ell(u_1^1, \ldots, u_g^1)$, and the generators of the edge groups in Λ_1, are all in free semigroup FS_k.

At this point we shorten the homomorphisms $\{h_n\}$ using the dominant modular group $MXMod(\Lambda_1)$. For each homomorphism h_n, we pick the shortest homomorphism after precomposing with an element from $MXMod(\Lambda_1)$ that keeps the positivity of the given set of the images of the generators $s_1, \ldots, s_r$, and the positivity of the elements $u_1^1, \ldots, u_g^1$ and t_ℓ, and keeps their lengths to be at least the maximal length of the image of a generator of a dominant edge or vertex group (the generators are chosen from the fixed finite sets of generators of each of the vertex and edge groups). We further require that after the shortening, the image of each of the elements $u_1^1, \ldots, u_g^1$ and t_ℓ, will be covered by the finitely many translates of them and of the paths that are associated with the relevant generators $s_1, \ldots, s_r$, that cover them in the limit action that is associated with Λ_1. We further require that if a path that is associated with one of the generators, $s_1, \ldots, s_r$, passes through an edge with a dominant edge group in Λ_1, then after shortening the path that is associated with such a generator contains at least a subpath that is associated with the dominant edge group. This guarantees that the length of such a generator remains bigger than the length of the dominant edge group along the entire procedure. We (still) denote the obtained (shortened) homomorphisms $\{h_n^1\}$.

By the shortening arguments that are proved in propositions 2.7 and 2.8, the lengths of the images, under the shortened homomorphisms $\{h_n^1\}$, of the elements $u_1^1, \ldots, u_g^1$ and t_ℓ, that are associated with the various dominant QH vertex groups, and the lengths of the elements, $s_1, \ldots, s_r$, are bounded by some constant c_1 (that is independent of n) times the maximal length of the images of the fixed generators of the dominant vertex and edge groups.

We pass to a subsequence of the homomorphisms $\{h_n^1\}$ that converges into an action of L on some real tree with an associated abelian decomposition Δ_2. If the action of L is not faithful, the conclusions of theorem 6.1 follow, hence, we may assume that the action of L is faithful. We further refine Δ_2, by restricting (a convergent subsequence of) the homomorphisms to non-QH vertex groups that are connected only to periodic edge groups, as we did with Λ, and construct an abelian decomposition that we denote $\hat{\Delta}_2$.

Suppose that there exists a QH vertex group Q in Λ_1, so that all its boundary elements are elliptic in $\hat{\Delta}_2$. i.e., every boundary element is contained in either an edge group or in a non-QH vertex group in $\hat{\Delta}_2$. In that case, by the properties of the JSJ decomposition of the freely indecomposable (limit) group L, there exists an abelian decomposition that possibly refines $\hat{\Delta}_2$ and contains Q as a QH vertex group. We further refine $\hat{\Delta}_2$, so that it contains all the QH vertex groups in Λ_1 for which all their boundary components are elliptic in $\hat{Delta}_2$. We denote the obtained decomposition (refinement), Λ_2.

By construction, a dominant edge group, a dominant boundary component of a QH vertex group, a dominant non-QH vertex group, and a QH vertex group with a dominant boundary component in Λ_1 can not be elliptic in Λ_2 (i.e., they can not be contained in a non-QH vertex group or an edge group in Λ_2). Furthermore, a QH vertex group that has a dominant boundary element in Λ_1, is not elliptic nor a QH vertex group in Λ_2.

With Λ_2 we associate its modular group $Mod(\Lambda_2)$. We further associate with Λ_2 its dominant edge groups and non-QH vertex groups, and the modular group that is associated only with dominant edge groups, and with dominant QH vertex groups, that we denote $MXMod(\Lambda_2)$.

We continue in a similar way to what we did with the sequence of homomorphisms $\{h_n\}$ and with Λ_1. We first shorten using automorphisms from the ambient modular group $Mod(\Lambda_2)$. If there exists a subsequence of shortened homomorphisms that converges into a proper quotient of L, the conclusion of theorem 6.1 follows. If there is no such subsequence, we restrict the shortenings to automorphisms from the dominant modular group $MXMod(\Lambda_2)$, and modify what we did in shortening the homomorphisms $\{h_n\}$ (and the abelian decomposition Λ_1).

First, we associate sets of positive generators with each of the new QH vertex groups in Λ_2, i.e., those QH vertex groups in Λ_2 that are not QH vertex groups in Λ_1. We further choose finitely many translates of the paths that are associated with each generator that cover the paths that are associated with the other generators. These translates guarantee that the ratios between the lengths of the paths that are associated with these generators along the iterative (shortening) procedure that we use remain globally bounded.

With a QH vertex group Q, and the abelian decomposition Λ_2, we can naturally associate an abelian decomposition Γ_Q^2, as we associated with QH vertex groups in Λ_1. Γ_Q^2 is obtained by collapsing Λ_2, and it contains one QH vertex group, Q, and all the other vertex groups are connected only to the vertex stabilized by Q.

If any of the fixed set of generators $u_1^1,\dots,u_g^1$ of a QH vertex groups in Λ_1, or a generator of a dominant edge group in Λ_1, or an element in the ball of radius 2 in the Cayley graph of L w.r.t. the generating set: $s_1,\dots,s_r$, is not elliptic in Γ_Q, or it is contained in a conjugate of Q, then finitely many translates of the path that is associated with s_j in the limit tree Y covers the paths that are associated with the elements: t_ℓ and $u_1^1,\dots,u_g^1$ that generate Q.

Note that unlike the (fixed) positive generators of the QH vertex groups and dominant edge groups in λ_1, and unlike the generating set, $s_1,\dots,s_r$, path that are associated with elements in the ball of radius 2 in the Cayley graph of L may contain positively and negatively oriented subpaths. In case such an element is not elliptic with respect to Γ_Q, finitely many translates of a fixed positively or negatively oriented subpath of the path that is associated with such element suffice to cover the (positively oriented) paths that are associated with the fixed set of generators of the QH vertex group Q.

These coverings of the paths that are associated with the fixed set of generators of the QH vertex groups in Λ_2 by translates of the paths that are associated with generators of QH vertex groups in Λ_1, generators of dominant edge groups in Λ_1, and elements in the ball of radius 2 in the Cayley graph of L, will guarantee that the lengths of the paths that are associated with these elements along the entire shortening process that we present, are bounded below by some (global) positive constant times the lengths of the paths that are associated with the fixed set of generators of the QH vertex group.

At we did in the first step, we shorten the homomorphisms $\{h_n^1\}$ using the dominant modular group $MXMod(\Lambda_1)$. For each homomorphism h_n^1, we pick the shortest homomorphism after precomposing with an element from $MXMod(\Lambda_1)$ that keeps the positivity of the given set of the images of the generators $s_1,\dots,s_r$, and the positivity of the fixed sets of generators of the QH vertex groups in both Λ_1 and Λ_2, and the generators of the dominant edge groups in Λ_1. We further keep the positivity and negativity of the positively and negatively oriented subpaths in the paths that are associated with elements in the ball of radius 2 in the Cayley graph of L w.r.t. $s_1,\dots,s_r$. We require the shortened homomorphisms to keep the lengths of the fixed set of generators of the QH vertex groups in Λ_2 to be at least the maximal length of the image of a generator of a dominant edge or vertex group in Λ_2 (the generators are chosen from the fixed finite sets of generators of each of the vertex and edge groups).

We further require that after the shortening, if a path in the limit tree Y_2 that is associated with Λ_2 and with one of the following:

(1) a generator of a QH vertex group in Λ_1 or a generator of the dominant edge

groups in Λ_1.

(2) a positively or negatively oriented subpath in the paths that are associated with elements in the ball of radius 2 in the Cayley graph of L w.r.t. $s_1, \ldots, s_r$.

passes through an edge with a dominant edge group in Λ_2, then after shortening the path that is associated with such a generator contains at least a subpath that is associated with the dominant edge group. This guarantees that the length of the corresponding element remains bigger than the length of the dominant edge group along the entire procedure. We denote the obtained (shortened) homomorphisms $\{h_n^2\}$.

Note that the shortening procedures that we presented in section 2 preserve the positivity of positively oriented paths. However, the way they are constructed can be used to keep the positivity and the negativity of finitely many subpaths in a given path (that may not be oriented, but can be divided into finitely many positively and negatively oriented subpaths). Furthermore, the shortening procedure is constructed to keep the non-cancellability between the positive and negative subpaths of a given (embedded) path.

By the shortening arguments that are proved in propositions 2.7 and 2.8, the lengths of the images, under the shortened homomorphisms $\{h_n^2\}$, of the fixed set of generators of the dominant QH vertex groups are bounded by some constant c_1 (that is independent of n) times the maximal length of the images of the fixed generators of the dominant vertex and edge groups.

We pass to a subsequence of the homomorphisms $\{h_n^2\}$ that converges into an action of L on some real tree with an associated abelian decomposition Δ_3. If the action of L is not faithful, the conclusions of theorem 6.1 follow. Hence, we may assume that the action of L is faithful. We refine Δ_3 to an abelian decomposition Λ_3, precisely as we refined Δ_2 to obtain Λ_2.

With Λ_3 we associate its modular group $Mod(\Lambda_3)$ and dominant modular group, $MXMod(\Lambda_3)$. We first shorten using $Mod(\Lambda_3)$, and if every convergent shortened subsequence converges into a faithful action of L, we further use the dominant modular group $MXMod(\Lambda_3)$.

In shortening using $MXMod(\Lambda_3)$, we keep the positivity of all the fixed (positive) sets of generators of the QH vertex groups in Λ_1 and Λ_2, and the generators of the dominant edge groups in Λ_1 and Λ_2. We also keep the positivity and negativity of all the finitely many positive and negative subpaths of the paths that are associated with the elements in the ball of radius 3 in the Cayley graph of L w.r.t. the generating set $s_1, \ldots, s_r$.

As we did in shortening using Λ_2, for each of the above elements (generators of QH vertex groups and dominant edge groups in Λ_1 and Λ_2, and elements in the ball of radius 3 in the Cayley graph of L) that are not elliptic in an abelian decomposition Γ_Q, that is associated with a QH vertex group Q in Λ_3, and is obtained by collapsing Λ_3, we use elements from L to demonstrate that each of the fixed set of generators of Q is covered by finitely many translates of the path that are associated with these elements. These translates will demonstrate that the lengths of the paths that are associated with these elements will be at least a (fixed) positive constant times the length of the path that are associated with the fixed generators of Q along the rest of the procedure.

We denote the obtained (shortened) homomorphisms h_n^3, and continue iteratively. If in all steps the obtained actions are faithful, we get an infinite sequence of abelian decompositions, $\Lambda_1, \Lambda_2, \dots$. Given the infinite sequence of abelian decompositions, we define the stable dominant abelian decomposition, Θ_{i_0}, in the same way as we did in the simplicial case (definition 6.2). From the convergent sequences $\{h^i_n\}_{n=1}^{\infty}$, we choose a subsequence $\{f_i\}$ that has a subsequence that converges into a faithful action of the limit groups L, with an associated abelian decomposition Θ_{i_0}.

Proposition 6.4. *Suppose that the limit group L is freely indecomposable, and contains no non-cyclic abelian subgroups. It is possible to choose a sequence of pair homomorphisms, $f_i : (S, L) \to (FS_k, F_k)$, such that:*

1. *for each i, f_i is a homomorphism from the sequence $\{h^i_n\}$.*
2. *the sequence f_i has a subsequence that converges into a faithful action of L on a real tree Y.*
3. *With the action of L on the limit tree Y there is an associated abelian decomposition Δ. By possibly refining Δ, using restrictions of the homomorphisms $\{f_i\}$ to some of its non-QH vertex groups, it is possible to obtain the stable abelian decomposition, Θ_{i_0}.*

Proof: For each index i, we choose f_i to be a homomorphism h^i_n (from the sequence $\{h^i_n\}$), that maps the fixed positive generators of all the QH vertex groups and the generators of all the edge groups in $\Lambda_1, \dots, \Lambda_i$ to FS_k. We further require that the ratios between the lengths of the images of these generators, and the ratios between the lengths of the images of all the elements in the ball of radius i in the Cayley graph of L w.r.t. the generating set $s_1, \dots, s_r$ will be approximately the ratios between the lengths of the paths that are associated with these elements in the corresponding limit tree Y_{i+1} (the trees that are obtained as the limits of the sequences $\{h^i_n\}$. We further require that f_i maps the elements in a ball of radius i in the Cayley graph of L monomorphically into F_k.

The sequence $\{f_i\}$ has a subsequence that converges into a faithful action of L on a real tree Y. Let Δ be the abelian decomposition that is associated with the action of L on Y. Since every elliptic element in the stable abelian decomposition Θ_{i_0} must fix a point in Y, Δ is dominated by Θ_{i_0}.

Lemma 6.5. *Let Q be a QH vertex group in Θ_{i_0} that does not appear in any of the abelian decompositions Λ_i, for $i \geq i_0$.*

If there is an non-peripheral element in Q that fixes a point in Y, then the entire QH vertex group Q fixes a point in Y.

Proof: Suppose that a non-peripheral element $q \in Q$ fixes a point in Y. q is contained in some ball B_m in the Cayley graph of L w.r.t. $s_1, \dots, s_r$. Θ_{i_0} is the stable dominant abelian decomposition of the sequence of abelian decompositions: $\Lambda_1, \dots$. Hence, there must exist an abelian decomposition Λ_i, for some $i > max(i_0, m)$, for which either:

(i) q is a non-peripheral element in some QH vertex group in Λ_i.
(ii) q is hyperbolic in the abelian decomposition Λ_i.

According to the procedure that was used to construct the abelian decompositions, $\Lambda_1, \dots$, if either (i) or (ii) hold for Λ_i, then the traces and the lengths of q in its actions on the tress that are associated with the abelian decompositions $\Lambda_{i+1}, \dots$

(i.e., the limit trees $Y_{i+1}, \dots$), are bounded below by either a (global) positive constant times the lengths of the fixed set of generators of the QH vertex group that contains q in Λ_i (in case (i)), or by either a a (global) positive constant times the lengths of a fixed set of generators in a QH vertex group in Λ_i, or the length of a generator of an edge group in Λ_i that becomes a dominant edge group in some $\Lambda_{i'}$ for some $i' \geq i$ in case (ii).

By the properties of the stable abelian decomposition Θ_{i_0}, and the structure of the procedure for the construction of the abelian decompositions $\Lambda_1, \dots$, for some index $j_0 > i$, the lengths of the fixed set of generators of a QH vertex group in Λ_i, or an edge group in Λ_i, that are contained in a QH vertex group Q in Θ_{i_0}, that does not appear as a QH vertex group in any of the abelian decompositions Λ_j, $j \geq i_0$, multiplied by some positive constants, have to be bigger than the lengths of the fixed sets of generators of all the QH vertex groups and the lengths of generators of all the edge groups in the abelian decompositions Λ_j, $j \geq j_0$, for edge groups and QH vertex groups that are contained in the QH vertex group Q in Θ_{i_0}.

Hence, if the element q is elliptic in the limit action on the real tree Y, all the QH vertex groups and all the edge groups that appear in Λ_j for $j \geq j_0$, and are contained in Q, must fix points in Y, which means that the entire QH vertex group Q in Θ_{i_0} that contains the non-peripheral element q must fix a point in Y.

□

Edge groups and the boundary elements of QH vertex groups in Θ_{i_0}, that appear in some abelian decomposition Λ_j, for some $j \geq i_0$, have to be fix points in the limit tree Y. From the graph of groups Θ_{i_0} we take out all the edge groups and all the QH vertex groups that appear in some abelian decomposition Λ_j for some $j \geq i_0$. We denote the obtained (possibly disconnected) graph of groups $\hat{\Theta}$. $\hat{\Theta}$ contains QH vertex groups and non-QH vertex groups that are connected to them. We restrict the homomorphisms $\{f_i\}$ to the fundamental groups of each of the connected components in $\hat{\Theta}$. By lemma 6.5 if a component of $\hat{\Theta}$ contains a QH vertex group, then the restrictions of the sequence $\{f_i\}$ to the fundamental group of that component subconverges into an action of a real tree, such that the abelian decomposition that is associated with that action, contains one or more QH vertex groups from $\hat{\Theta}$, that are also QH vertex groups in Θ_{i_0}.

We continue the refinement process by erasing these QH vertex groups from $\hat{\Theta}$, and restrict the homomorphisms $\{f_i\}$ to the fundamental components of connected components in the remaining graph of groups. By lemma 6.5, after finitely many such revisions of the graph of groups $\hat{\Theta}$ (i.e. erasing the QH vertex groups that were visible), we uncover all the QH vertex groups in Θ_{i_0} that do not appear as QH vertex groups in an abelian decomposition Λ_j for $j \geq i_0$.

Now, starting with the abelian decomposition Δ, that was read from the original faithful action of the limit group L on the limit tree Y, possibly refining Δ using the QH vertex groups and the edge groups from Θ_{i_0} that appear in some abelian decomposition Λ_j for some $j \geq i_0$, and further refining the obtained decomposition using the uncovered QH vertex groups, we finally obtain the stable dominant abelian decomposition Θ_{i_0}. This concludes the proof of proposition 6.4.

□

Let $j_0 \geq i_0$ be an index for which all the QH vertex groups in Θ_{i_0} that appear in some Λ_j for some $j \geq i_0$, already appear in Λ_j for some $j_0 \leq j \leq i_0$. Proposition 6.4 enables us to replace the suffix of the sequence of abelian decompositions, $\Lambda_{i_0}, \dots$, with a finite resolution $\Lambda_{i_0}, \dots, \Lambda_{j_0}, \Theta_{i_0}$, and hence the entire sequence $\Lambda_1, \dots$ with the finite resolution: $\Lambda_1, \dots, \Lambda_{j_0}, \Theta_{i_0}$. Furthermore, we may continue with the sequence of pair homomorphisms $\{f_i\}$, that are obtained from a subsequence of the pair homomorphisms that we started with $\{h_n\}$, by precomposing them with automorphisms from the modular groups that are associated with the abelian decompositions that appear along the finite resolution.

We continue to the next step starting with the sequence of homomorphisms $\{f_i\}$, and the abelian decomposition Θ_{i_0}. In the abelian decomposition that is obtained from a subsequence of homomorphisms that were obtained from the homomorphisms $\{f_i\}$ by precomposing them with automorphisms from the (dominant) modular group of Θ_{i_0}, either a dominant edge group or a dominant non-QH vertex group is not elliptic.

At this point we repeat the whole construction of a sequence of abelian decompositions. If the sequence terminates after a finite number of steps the conclusion of theorem 6.1 follows. Suppose it ends up with an infinite sequence of abelian decompositions.

Note that for any abelian decomposition Λ_i along this sequence, after finitely many steps there exists an abelian decomposition $\Lambda_{i'}$ for some $i' \geq i$, with a dominant edge group. This means that the dominant edge group in $\Lambda_{i'}$ is not elliptic in $\Lambda_{i'+1}$. Furthermore, as long as the boundary elements of a QH vertex group Q in Λ_i remain elliptic in the next abelian decompositions, the QH vertex group Q remains a QH vertex group in the next abelian decompositions.

Using proposition 6.4, we can replace the suffix of the sequence with a finite resolution that stably dominates the entire sequence (see definition 6.2).

We continue iteratively. At each step we start with a sequence of homomorphisms that are obtained from a subsequence of the homomorphisms $\{h_n\}$ by precomposing them with automorphisms from the modular groups of the abelian decompositions that appear along the previously constructed finite resolution. We continue for a single step, so that at least one dominant edge group or a dominant vertex group in the previous abelian decomposition is not elliptic. Then we either associate with the shortened sequence of homomorphisms a finite resolution, that completes the proof of theorem 6.1, or we associate with it an infinite sequence of abelian decompositions. By proposition 6.4 a suffix of this last infinite sequence can be replaced by a finite resolution that ends with a stable dominant abelian decomposition of it.

If this iterative procedure terminates after finitely many steps, the conclusion of theorem 6.1 follows. Otherwise we obtained an infinite sequence of abelian decompositions. Note that the infinite sequence of abelian decompositions contains a subsequence of abelian decompositions that contain QH vertex groups, and dominant edge groups in the abelian decompositions from the subsequence are not elliptic in the abelian decomposition that appears afterwards in the sequence.

Now, we apply proposition 6.4 to the sequence of abelian decompositions that we constructed. By proposition 6.4 a suffix of the sequence can be replaced with a finite resolution that terminates with the abelian decomposition that stably dominates the original suffix of the sequence. Since the sequence of abelian decompositions contains a subsequence with QH vertex groups, the stable dominant abelian decomposition must contain a QH vertex group as well. Since every abelian decomposition in this subsequence contains a (dominant) edge that is not elliptic in the next abelian decomposition, the stable dominant abelian decomposition contains either 2 QH vertex groups, or a single QH vertex group with an associated surface group S that is either:

(i) an orientable surface with $\chi(S) \leq -2$.
(ii) a non-orientable surface with $genus(S) + bnd(S) \geq 3$, where $bnd(S)$ is the number of boundary components of the surface S.

We repeat the whole construction starting with the abelian decomposition that we obtained and the subsequence of homomorphisms that is associated with it according to the construction that is used in the proof of proposition 6.4. Either the construction terminates in finitely many steps, or we get a sequence of abelian decompositions that has a subsequence that satisfies the properties that the previous stably dominant abelian decomposition satisfied, and in each abelian decomposition from the subsequence there exists a dominant edge group that is not elliptic in the next abelian decomposition.

Once again we apply proposition 6.4 to the constructed sequence of abelian decompositions. The stable dominant abelian decomposition of the sequence has to contain QH vertex groups with topological complexity that is bounded below by a larger lower bound. Hence, it either contains at least 3 QH vertex groups, or 2 QH vertex groups so that at least one of them satisfies properties (1) or (2), or a single QH vertex group with an associated surface group S that is either orientable with $\chi(S) \leq -3$ or non-orientable with $genus(S) + bnd(S) \geq 4$.

We continue iteratively. Since after each iteration (of the entire construction) we obtain an abelian decomposition that contain QH vertex groups with topological complexity that is bounded below by larger and larger bounds, and since the obtained abelian decompositions are all dominated by the JSJ decomposition of the freely-indecomposable limit group L, the procedure has to terminate after finitely many iterations, and the obtained resolution satisfies the conclusions of theorem 6.1.

□

So far we assumed that L is freely indecomposable and contains no non-cyclic abelian groups. To get a conclusion similar to the one that appears in theorem 6.1 in the presence of non-cyclic abelian subgroups, we need to slightly modify it. Instead of using only the modular group or the dominant modular group, we need to further allow generalized Dehn twists in the presence of a non-cyclic abelian group, i.e., we further allow Dehn twists in roots of the the values of a generator of an edge group (in case the edge group belongs to a non-cyclic maximal abelian subgroup), and not just modular automorphisms.

Definition 6.6. *Let (S, L) be a pair in which L is freely indecomposable limit group, and let Λ be an abelian decomposition that is associated with this pair (L is the fundamental group of Λ). Let $h : (S, L) \to (FS_k, F_k)$ and $f : (S, L) \to (FS_k, F_k)$ be two pair homomorphisms. We say that f is obtained from h using generalized Dehn twists if:*

(i) *there exists a pair homomorphism $\hat{h} : (S, L) \to (FS_k, F_k)$ that is obtained from h by precomposing it with a modular automorphism of Λ: $\hat{h} = h \circ \varphi$, $\varphi \in Mod(\Lambda)$.*
(ii) *suppose that $A < L$ is a non-cyclic maximal abelian subgroup, and A is not elliptic in Λ. Let A_0 be the maximal abelian subgroup in A that is elliptic in Λ, and let $A = A_0 + < a_1, \dots, a_\ell >$. Then f is obtained from $\hat{h}$ by modifying the values of the (non-elliptic) generators $a_1, \dots, a_\ell$, that is*

replacing $\hat{h}(a_i)$ by elements in F_k that are in the maximal cyclic subgroup that contains $\hat{h}(A)$, in (possibly) all the (finitely many conjugacy classes of) non-cyclic maximal abelian subgroups $A < L$ that are not elliptic in Λ.

We say that a pair homomorphism f is obtained from a homomorphism h using dominant generalized Dehn twists if:

(iii) *there exists a pair homomorphism $\hat{h} : (S, L) \to (FS_k, F_k)$ that is obtained from h by precomposing it with a dominant modular automorphism of Λ: $\hat{h} = h \circ \varphi$, $\varphi \in MXMod(\Lambda)$.*

(iv) *f is obtained from $\hat{h}$ by modifying the values of the (non-elliptic) generators $a_1, \ldots, a_\ell$, that is replacing $\hat{h}(a_i)$ by elements in F_k that are in the maximal cyclic subgroup that contains $\hat{h}(A)$, in (possibly) all the (finitely many conjugacy classes of) non-cyclic maximal abelian subgroups $A < L$ that are not elliptic in Λ, and in which the maximal elliptic subgroup $A_0 < A$ is dominant.*

Note that in general f is not obtained from h by a precomposition with a modular automorphism, and that apart from the degenerate case in which $f(A)$ is trivial, the relation of being obtained using (dominant) generalized Dehn twists is symmetric. Also, note that generalized Dehn twists are used in studying systems of equations with parameters over a free group (see sections 9-10 in [Se1]).

The addition of generalized Dehn twists enable us to use our treatment of non-cyclic abelian subgroups in sections 2 (theorem 2.1) and 3 and generalize the statement of theorem 6.1 to include all pairs (S, L) with L a general freely indecomposable limit group.

Theorem 6.7. *Let (S, L) be a pair, where L is a freely indecomposable limit group, and let $s_1, \ldots, s_r$ be a fixed generating set of the semigroup S. Let $\{h_n : (S, L) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on a real tree Y.*

Then there exists a resolution:

$$(S_1, L_1) \to (S_2, L_2) \to \ldots \to (S_m, L_m) \to (S_f, L_f)$$

that satisfies the following properties:

(1) *$(S_1, L_1) = (S, L)$, and $\eta_i : (S_i, L_i) \to (S_{i+1}, L_{i+1})$ is an isomorphism for $i = 1, \ldots, m-1$ and $\eta_m : (S_m, L_m) \to (S_f, L_f)$ is a proper quotient map.*

(2) *with each of the pairs (S_i, L_i), $1 \leq i \leq m$, there is an associated abelian decomposition that we denote Λ_i.*

(3) *there exists a subsequence of the homomorphisms $\{h_n\}$ that factors through the resolution. i.e., each homomorphism h_{n_r} from the subsequence, is obtained from a homomorphism of the terminal pair (S_f, L_f) using a composition of a modification that uses generalized Dehn twists that are associated with Λ_m, and modular automorphisms that are associated with $\Lambda_1, \ldots, \Lambda_{m-1}$.*

Proof: Suppose that L is freely indecomposable and does contain a non-cyclic abelian group. If all the non-cyclic abelian groups in L remain elliptic in all the abelian decompositions that are constructed along the iterative procedure that was used in the proof of theorem 6.1 in case there is no non-cyclic abelian groups, i.e., in the iterative applications of the construction that is used in the proof of proposition 6.4, the same construction proves the conclusion of theorem 6.1.

Suppose that at some step along an application of the iterative procedure that is used in the proof of proposition 6.4, a non-cyclic (maximal) abelian subgroup $A < L$ is non-elliptic in an abelian decomposition Λ_i, that is associated with a corresponding (faithful) action of L on a corresponding limit tree Y_i.

In that case A is either the set stabilizer of an axial component or it is the set stabilizer of a line in the simplicial part of Y_i. Let $A_0 < A$ be the point stabilizer of the axial component with set stabilizer A or of the axis of A.

A_0 is the stabilizer of an edge in Λ_i. If A_0 is not dominant, we don't include Dehn twists along elements of A in the dominant modular group of Λ_i, and proceed with the procedure that is used in the proof of theorem 6.1 as long as A_0 is not dominant. The abelian decompositions that we consider in all the steps of the procedure in which A_0 is not dominant, are abelian decompositions relative to A_0.

If A_0 remains not dominant along the entire procedure, the procedure that is used in proving proposition 6.4 and its conclusions remain valid. Suppose that at some step j, A_0 is dominant. $A = A_0 + < a_1, \dots, a_\ell >$. By theorem 2.1 in the axial case, and in case A acts simplicially, when A_0 is dominant we can modify a subsequence of the homomorphisms $\{h_n\}$ using generalized Dehn twists along A (that do not change A_0), so that the images of $a_1, \dots, a_\ell$ under the modified homomorphisms are identified with fixed elements in A_0. Hence the modified sequence of homomorphisms converges into a limit group in which the image of A is A_0. Therefore, the modified sequence of homomorphisms converges into a proper quotient of the limit group L, and the conclusion of theorem 6.6 follows.

□

Theorem 6.7 proves that given a pair (S, L) in which L is freely indecomposable, and a sequence of pair homomorphisms $\{h_n : (S, L) \to (FS_k, F_k)\}$, that converges into a faithful action of L on some real tree, it is possible to extract a subsequence that factors through a finite resolution of the pair (S, L) that terminates in a proper quotient of the pair (S, L). Using a compactness argument it is not difficult to apply the conclusion of theorem 6.7 and get a Makanin-Razborov diagram for a pair (S, L), when L is a restricted limit group with no freely decomposable restricted limit quotients.

Let $FS_k =< a_1, \dots, a_k >$ be a free semigroup that generates the free group F_k, and let (S, L) be a restricted pair, i.e., a pair that contains the subpair (FS_k, F_k). Suppose that the pair (S, L) does not admit a quotient restricted map $\eta : (S, L) \to (\hat{S}, \hat{L})$ with the following properties:

(1) η maps the subpair $(FS_k, F_k) < (S, L)$ monomorphically onto the corresponding subpair $(FS_k, F_k) < (\hat{S}, \hat{L})$.
(2) $\hat{L}$ admits a non-trivial free decomposition in which $\eta(F_k)$ is contained in a factor.

The restricted pair (S, L) is in particular freely indecomposable with respect to the subpair (FS_k, F_k), and so is every restricted quotient of (S, L). Given a sequence of homomorphisms $\{h_n : (S, L) \to (FS_k, F_k)\}$, it is possible to extract a subsequence that converges into a faithful action of some restricted quotient (S_0, L_0) of (S, L). By theorem 6.7 it is possible to further extract a subsequence that factors through a resolution that terminates in a proper quotient of (S_0, L_0).

Applying theorem 6.7 iteratively, finitely many times, it is possible to extract a further subsequence that factors through a resolution:

$$(S_0, L_0) \to (S_2, L_2) \to \ldots \to (S_m, L_m) \to (FS_k, F_k)$$

that terminates in the standard pair (FS_k, F_k), and in which some of the epimorphisms: $\eta_i : (S_i, L_i) \to (S_{i+1}, L_{i+1})$ are isomorphisms and some are proper quotient maps. with each of the pairs (S_i, L_i), $1 \leq i \leq m$, there is an associated abelian decomposition that we denote Λ_i, with which we naturally associate a modular group.

Although f.g. subsemigroups of limit groups need not be f.p. in general, a pair (S, L) in which L is a limit group and S is a f.g. subsemigroup of L is naturally a finitely presented object. Hence, a resolution of the form:

$$(S_0, L_0) \to (S_2, L_2) \to \ldots \to (S_m, L_m) \to (FS_k, F_k)$$

together with the abelian decompositions that are associated with the various restricted pairs (S_i, L_i) can be encoded using a finite amount of data. In particular, there are only countably many such resolutions (that are associated with convergent restricted pair homomorphisms of a given pair (S, L)), and we can order these resolutions using their finite encoding.

Theorem 6.8. *Let (S, L) be a restricted pair, that has no restricted quotient $(\hat{S}, \hat{L})$ in which $\hat{L}$ admits a free decomposition relative to (the embedding of) F_k. Then there are finitely many resolutions of the form that is constructed in theorem 6.7:*

$$(S_0, L_0) \to (S_2, L_2) \to \ldots \to (S_m, L_m) \to (FS_k, F_k)$$

where (S_0, L_0) is restricted quotient pair of (S, L) , such that:

1. *every restricted pair homomorphism, $h : (S, L) \to (FS_k, F_k)$, factors through at least one of these finitely many resolutions.*
2. *for each of the resolutions in the collection, there exists a sequence of homomorphisms: $\{h_n : (S, L) \to (FS_k, F_k)\}$, that converges into a faithful action of the initial pair (S_0, L_0) on a real tree with an associated abelian decomposition (after the refinement that is used in the proof of proposition 6.4) Λ_0.*

   Furthermore, the sequence of homomorphisms $\{h_n\}$ can be modified using modular automorphisms and generalized Dehn twists that are associated with the abelian decompositions: $\Lambda_0, \ldots, \Lambda_m$, to get sequences of pair homomorphisms $\{h_n^1\}, \ldots, \{h_n^m\}$. Each of these modified sequences of homomorphisms $\{h_n^i\}$ converges into a faithful action of the pair (S_i, L_i) on a real tree.

Proof: Let $\{h \,|\, h : (S, L) \to (FS_k, F_k)\}$ be the collection of all the restricted pair homomorphisms of the pair (S, L). We look at all the possible subsequences of such homomorphisms $\{h_n\}$, that converge into quotients of (S, L), with which we can associate a resolution:

$$(S_0, L_0) \to (S_2, L_2) \to \ldots \to (S_m, L_m) \to (FS_k, F_k)$$

by iteratively applying theorem 6.7. Note that each of these constructed resolutions satisfy part (2) of the theorem.

Each such resolution can be encoded using finite amount of data, in particular, there are countably many such resolutions, and we can order them using the finite encoding. We argue that the collection of all the pair homomorphisms of (S, L) factor through a finite collection of these resolutions.

Suppose that finitely many do not suffice. Then there exists a sequence of restricted pair homomorphisms: $\{h_n\}$, such that for every index n, h_n does not factor through the first n resolutions from the ordered countable set of resolutions that were constructed. By iteratively applying theorem 6.7, from the sequence $\{h_n\}$ we can extract a subsequence $\{h_{n_r}\}$, that factors through a resolution of form we previously constructed:

$$(S_0, L_0) \to (S_2, L_2) \to \ldots \to (S_m, L_m) \to (FS_k, F_k)$$

But this resolution appears in the countable set of resolutions we associated with the pair (S, L), hence, it appears in the ordered list. Therefore, for some index r_0, and for every $r > r_0$, the subsequence of homomorphisms: $\{h_{n_r}\}$ factors through a resolution that appears in the ordered list of resolutions, a contradiction to the choice of the homomorphisms $\{h_n\}$.

□

Theorem 6.8 constructs a diagram that encodes all the homomorphisms from a pair (S, L) into the standard pair (FS_k, F_k). However, the construction of the resolutions in the diagram, that mainly uses the iterative procedure that was used in proving theorem 6.1 and proposition 6.4, does not guarantee that there exist sequences of homomorphisms that factor through them for which the corresponding shortened homomorphisms converge into the pairs that appear along the resolutions. i.e., the construction of the resolutions does not guarantee the existence of test sequences or generic points for the resolutions in the diagram. To guarantee the existence of such test sequences, we need to slightly modify the sequences of homomorphisms that are used in the construction of the resolutions, i.e., those that are used in the proof of proposition 6.4.

Theorem 6.9. *Let (S, L) be a restricted pair, that has no restricted quotient $(\hat{S}, \hat{L})$ in which $\hat{L}$ admits a free decomposition relative to (the embedding of) F_k. Then there are finitely many resolutions of the form that is constructed in theorem 6.7:*

$$(S_0, L_0) \to (S_2, L_2) \to \ldots \to (S_m, L_m) \to (FS_k, F_k)$$

where (S_0, L_0) is restricted quotient pair of (S, L) , such that parts (1) and (2) in theorem 6.8 hold for these resolutions, and in addition:

(3) *for each of the resolutions in the collection, there exists a sequence of homomorphisms: $\{h_n : (S, L) \to (FS_k, F_k)\}$, that can be modified using modular automorphisms and generalized Dehn twists that are associated with the abelian decompositions: $\Lambda_0, \ldots, \Lambda_m$, to get sequences of pair homomorphisms $\{h_n^1\}, \ldots, \{h_n^m\}$. Each of these modified sequences of homomorphisms $\{h_n^i\}$ converges into a faithful action of the pair (S_i, L_i) on a real tree with an associated abelian decomposition (after an appropriate refinement) Λ_i.*

Proof:

□

We call a finite collection of resolution that satisfies properties (1) and (2) in theorem 6.8 and part (3) in theorem 6.9 a *Makanin-Razborov diagram* of the restricted pair (S, L). Note that such a diagram is not canonical in general. We view sequences of pair homomorphisms that satisfy part (3) with respect to one of the resolutions in the diagram as generic points in the variety that is associated with the pair (S, L). We later use such sequences of homomorphisms as a replacement to *test sequences* that were used in [Se2] to construct formal solutions and obtain generalized Merzlyakov theorems for AE sentences and formulas that are defined over a given variety over a free semigroup.

§7. A Makanin-Razborov diagram

In section 6 we analyzed the collection of homomorphisms from a restricted pair (S, L) into the standard pair (FS_k, F_k), in case the pair (S, L) has no restricted quotients $(\hat{S}, \hat{L})$, in which the restricted limit group $\hat{L}$ is freely decomposable (with respect to the coefficient subgroup F_k), i.e., in which L is freely decomposable as a restricted limit group.

In that case we managed to associate a Makanin-Razborov diagram with such a restricted pair, that encodes all its pair homomorphisms, such that every resolution in the diagram has a collection of generic points of homomorphisms that factor through it (see theorem 6.9). Each of the resolutions in such a diagram terminates with the standard pair (FS_k, F_k).

In this section we generalize the construction of the Makanin-Razborov diagram to include all possible pairs (S, L). To do that we need to generalize theorems 6.1 and 6.6 to include pairs with freely decomposable limit groups. We first present such generalizations in case there are no Levitt components in the actions on the real trees that we consider, and then omit this assumption, and allow Levitt components. In both cases we use the machinery that was presented in section 6, to construct a JSJ-like decompositions, that in the general case considers and encodes Levitt components.

Definition 7.1. *Let (S, L) be a pair. We say that (S, L) is Levitt-free if every sequence of pair homomorphisms: $\{h_n : (S, L) \to (FS_k, F_k)\}$ that converges into a faithful action of L on some real tree, contains no Levitt (thin) components (for a detailed description of Levitt components see [Be-Fe1] who call them thin).*

Theorem 7.2. *Let (S, L) be a Levitt-free pair, and suppose that the limit group L contains no non-cyclic abelian subgroup. Let $\{h_n : (S, L) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on a real tree Y.*

Then there exists a resolution:

$$(S_1, L_1) \to \ldots \to (S_f, L_f)$$

that satisfies the following properties:

1. *$(S_1, L_1) = (S, L)$, and $\eta_i : (S_i, L_i) \to (S_{i+1}, L_{i+1})$ is an isomorphism for $i = 1, \ldots, f-2$ and $\eta_{f-1} : (S_{f-1}, L_{f-1}) \to (S_f, L_f)$ is a quotient map.*

(2) *with each of the pairs* (S_i, L_i), $1 \le i \le f$, *there is an associated abelian decomposition that we denote* Λ_i. *The abelian decompositions* $\Lambda_1, \dots, \Lambda_{f-1}$ *contain edges with trivial and cyclic edge stabilizers, QH vertex groups that are associated with IET components, and rigid vertex groups.*

(3) *either* η_{f-1} *is a proper quotient map, or the abelian decomposition* Λ_f *contains separating edges with trivial edge groups. Each separating edge is oriented.*

(4) *there exists a subsequence of the homomorphisms* $\{h_n\}$ *that factors through the resolution. i.e., each homomorphism* h_{n_r} *from the subsequence, can be written in the form:*

$$h_{n_r} \;=\; \hat{h}_r \circ \varphi_r^{f-1} \circ \dots \circ \varphi_r^1$$

where $\hat{h}_r : (S_f, L_f) \to (FS_k, F_k)$, *each of the automorphisms* $\varphi_r^i \in Mod(\Lambda_i)$, *where* $Mod(\Lambda_i)$ *is generated by the modular groups that are associated with the QH vertex groups, and by Dehn twists along edge groups with cyclic stabilizers in* Λ_i.

Each of the homomorphisms:

$$h_{n_r}^i \;=\; \hat{h}_r \circ \varphi_r^m \circ \dots \circ \varphi_r^i$$

is a pair homomorphism $h_{n_r}^i : (S_i, L_i) \to (FS_k, F_k)$.

(5) *if* (S_f, L_f) *is not a proper quotient of* (S, L), *then the pair homomorphisms* $\hat{h}_r$ *are compatible with* Λ_f. *Let* $R_1, \dots, R_v$ *be the connected components of* Λ_f *after deleting its (oriented) separating edges. The homomorphisms* $\hat{h}_r$ *are composed from homomorphisms of the fundamental groups of the connected components* $R_1, \dots, R_v$, *together with assignments of values from* FS_k *to the oriented separating edges. The homomorphisms of the fundamental groups of the connected components* $R_1, \dots, R_v$ *converge into a faithful action of these groups on real trees with associated abelian decompositions:* $R_1, \dots, R_v$.

Proof: We modify the procedure that was used in proving theorem 6.1. Let Λ be the abelian decomposition that is associated with the action of L on the limit tree Y (that is obtained from the convergent sequence $\{h_n\}$). If Λ contains a segment in its simplicial part, and that segment has a trivial stabilizer, the conclusions of the theorem follow.

By our assumptions, (S, L) is Levitt free, so as long as (S, L) is not replaced by a proper quotient, none of the faithful actions of L on the limit trees that are constructed along the procedure contain Levitt components. Suppose that the action of L on the corresponding real tree is not geometric (in the sense of [Be-Fe]). In that case we associate with the non-geometric action (with no Levitt components), an approximating resolution (in the sense of [Be-Fe]) according to the one that we constructed in theorem 2.9. The graph of groups that is associated with that approximating resolution contains edges with trivial stabilizers, and a subsequence of the given sequence of homomorphisms satisfies parts (4) and (5) of the theorem according to theorem 2.9.

Therefore, we may assume that the action of L on the limit tree Y is geometric. Hence, Y contains only IET and discrete components, and every segment in the discrete part of Y can be divided into finitely many non-degenerate segments with (non-trivial) cyclic stabilizers. Therefore, if L is a free product of non-cyclic, freely indecomposable limit groups, and Λ is the JSJ decomposition of L, L is replaced by a proper quotient after shortening along the modular group of L, and the conclusion of the theorem follows.

We start by refining the abelian decomposition Λ in a similar way to what we did proving theorems 3.2 and 6.1. If a non-QH vertex group is connected only to periodic edge groups (and in particular it is not connected to a QH vertex group), we restrict the homomorphisms $\{h_n\}$ to such a vertex group and obtain a non-trivial splitting of it in which all the previous (periodic) edge groups are elliptic. Hence, the obtained abelian decomposition of the non-QH vertex group can be used to refine the abelian decomposition Λ. Repeating this refinement procedure iteratively, we get an abelian decomposition that we denote Λ_1.

We fix finite generating sets of all the edge groups and all the non-QH vertex groups in Λ_1. We divide the edge and non-QH vertex groups in Λ_1 into finitely many equivalence classes of their growth rates as we did the proof of theorem 6.1. Two non-QH vertex groups or edge groups (or an edge and a non-QH vertex group) are said to be in the same equivalence class if the maximal length of the images of their finite set of generators have comparable lengths, i.e., the length of a maximal length image of one is bounded by a (global) positive constant times the maximal length of an image of other, and vice versa. After possibly passing to a subsequence, there exists a class that dominates all the other classes, that we call the *dominant* class, that includes (possibly) dominant edge groups and (possibly) dominant non-QH vertex groups.

As in the freely indecomposable case, we denote by $Mod(\Lambda_1)$ the modular group that is associated with Λ_1. We set $MXMod(\Lambda_1)$ to be the *dominant* modular group that is generated by Dehn twists along dominant edge groups and modular groups of dominant QH vertex groups, i.e., those QH vertex groups that the lengths of the images of their fixed sets of generators grow faster than the lengths of the images of the fixed generators of dominant edge and vertex groups.

We start by using the full modular group $Mod(\Lambda_1)$. For each index n, we set the pair homomorphism: $h_n^1 : (S, L) \to (FS_k, F_k)$, $h_n^1 = h_n \circ \varphi_n$, where $\varphi_n \in Mod(\Lambda_1)$, to be a shortest pair homomorphism that is obtained from h_n by precomposing it with a modular automorphism from $Mod(\Lambda_1)$. If there exists a subsequence of the homomorphisms $\{h_n^1\}$ that converges into a proper quotient of the pair (S, L), or that converges into a non-geometric action of L on a real tree, or into an action that contains a segment in its simplicial part, and this segment can not be divided into finitely many subsegments with trivial stabilizers, we set the limit of this subsequence to be (S_f, L_f), and the conclusion of the theorem follows with a resolution of length 1.

Therefore, we may assume that every convergent subsequence of the homomorphisms $\{h_n^1\}$ converges into a faithful action of the limit group L on some real tree. In that case we use only elements from the dominant modular group, $MXMod(\Lambda_1)$.

Before we continue to the next abelian decomposition, we need to check if the original sequence of homomorphisms, $\{h_n\}$, does not contain a subsequence of *separable* homomorphisms. We are going to look for subsequences of separable homomorphisms in every step of the iterative procedure for the construction of the sequence of abelian decompositions $\Lambda_1, \ldots$. The existence of such subsequence will lead to a termination of the procedure, with an abelian decomposition that satisfies the conclusions of theorem 7.2.

Definition 7.3. *Let (S, L) be a pair, and let: $\{u_n : (S, L) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on some real tree Y. We say that the sequence $\{u_n\}$ is separable if there exists a (reduced) graph of groups Δ with fundamental group L with the following properties and additional data:*

1. *the graph of groups Δ is non-trivial and all its edges have trivial stabilizers. We further assign orientation with each edge in Δ.*
2. *with each non-trivial vertex group in Δ we associate a base point. In addition there is a base point for the fundamental group L, that is placed in the interior of one of the edges or it is one of the basepoints that are associated with the vertex groups.*
3. *with each edge in Δ we associate a label. If the basepoint of L is in the interior of an edge, then with that edge there are two labels that are associated with the two parts of the edge.*
4. *the homomorphisms $\{u_n\}$ are composed from homomorphisms of the vertex groups into F_k, and assignments of values in F_k to the labels that are associated with the edges in Δ. Each element in L can be considered as a path in Δ that starts and ends in its basepoint. Hence, from the values that are assigned to the labels, and the homomorphisms of the vertex groups (with their basepoints), it is possible to read (uniquely) a homomorphism of L into F_k.*
5. *we extend the set of generators of the standard pair (FS_k, F_k), by adding a new free generator to the standard semigroup FS_k for each label in Δ. We denote the extended standard pair (FS_m, F_m).*

   Given each of the homomorphisms u_n, it is possible to replace the values that are assigned with each of the labels that are associated with the edges in Δ, to values that contain a single (positively oriented) appearance of the generator that is associated with each label, and no appearances of generators that are associated with the other labels, without changing the homomorphisms of the vertex groups in Δ to obtain a homomorphism $\hat{u}_n$. The homomorphism $\hat{u}_n$ should coincide with the original homomorphism u_n if we map the (new) generators that are associated with the labels to the identity. Furthermore, and for each index n, the homomorphism $\hat{u}_n$ is a pair homomorphism: $\hat{u}_n : (S, L) \to (FS_m, F_m)$.

If the sequence of homomorphisms, $\{h_n\}$, contains a separable subsequence, the conclusion of theorem 7.2 follows.

Proposition 7.4. *Let (S, L) be a Levitt-free pair, and suppose that the limit group L contains no non-cyclic abelian subgroup. Let $\{h_n : (S, L) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on a real tree Y.*

Let $\{u_t : (S, L) \to (FS_k, F_k)\}$ be a separable subsequence of the sequence $\{h_n\}$. Then part (5) of theorem 7.2 holds for the subsequence $\{u_t\}$.

Proof: With this separable subsequence there is an associated abelian decomposition Δ that satisfies all the properties that are listed in definition 7.3. Given Δ, and the separable sequence, $\{u_t\}$, for every index t, we can associate with the homomorphism u_t, an extended pair homomorphism: $\hat{u}_t : (S, L) \to (FS_m, F_m)$ that satisfies the properties that are listed in part (5) of definition 7.3.

The sequence $\{\hat{u}_t\}$ has a convergent subsequence (still denote $\{\hat{u}_t\}$). This convergent subsequence converges into a faithful action of L on a real tree, $\hat{Y}$. The abelian decomposition, Θ, that is associated with the action of L on $\hat{Y}$ contains separating edges, that appear because of the extra free generators that appear (once) in the values that are assigned to the labels that are associated with the edges in Δ. In particular, with each edge in Δ there is a corresponding separating edge in Θ. Hence, the conclusion of theorem 7.2 holds for the abelian decomposition, Θ.

□

For presentation purposes we first assume that Λ_1, and all the next abelian decompositions that are obtained along the iterative procedure do not contain any QH vertex groups. In that case we continue iteratively, precisely as we did in the freely indecomposable case.

First we shorten the homomorphisms $\{h_n\}$ using the dominant modular group, $MXMod(\Lambda_1)$. We denote the shortened sequence of homomorphisms, $\{h_n^1\}$, and after passing to a subsequence, assume that the obtained sequence converges into a faithful action on a real tree Y_2. If the action of the limit group L on the real tree Y_2 is not faithful, or if there is a segment in the simplicial part of Y_2 that can not be divided into subsegments with non-trivial stabilizers, or if the action of L on Y_2 is not geometric, the conclusions of theorem 7.2 follow.

Hence, we may assume that the action of L on Y_2 is geometric and faithful. With the action of L on Y_2 there is an associated abelian decomposition Δ_2. By our standard procedure Δ_2 can be (possibly) refined to an abelian decomposition Λ_2. By our assumptions, Λ_2 contains no QH vertex groups.

We first check if the sequence of abelian decomposition $\{h_n^1\}$ contains a separable subsequence. If it does theorem 7.2 follows. Otherwise, we associate with Λ_2 its modular group, $Mod(\Lambda_2)$, and its dominant modular group, $MXMod(\Lambda_2)$. We first shorten using the ambient modular group, and if the action is faithful, we shorten using the dominant modular group.

We continue iteratively, and either terminate after finite time, or obtain the sequence of abelian decompositions, $\Lambda_1, \ldots,$ and the corresponding sequences of convergent shortened homomorphisms, that have no separable subsequences. As in the freely indecomposable case, with the sequence of abelian decompositions, $\Lambda_1, \ldots$ we associate its stable dominant (abelian) decomposition.

Definition 7.5. *Let (S, L) be a pair, and let $\Lambda_1, \ldots$ be a sequence of abelian decompositions of L. With the sequence we associate its stable dominant abelian decompositions, that generalizes the one presented in definition 6.2, in the case of freely decomposable groups.*

Note that given two abelian decompositions Δ_1 and Δ_2 of a limit group L, their common refinement is the multi-graded abelian JSJ decomposition of L with respect to the collection of (finitely many conjugacy classes of) subgroups that are elliptic in both splittings, Δ_1 and Δ_2. The multi-graded abelian JSJ decomposition is an abelian decomposition of a freely decomposable group, and hence includes free products, and starts with the multi-graded Grushko decomposition. For the definition of the multi-graded JSJ decomposition see section 12 in [Se1].

Associating an abelian decomposition (a common refinement) with every pair of abelian decompositions of L, enables one to associate such a common refinement with every finite sequence of abelian decompositions of L. Given a sequence of abelian decompositions, $\Lambda_1, \ldots$, the (natural) complexities of the common refinements of the prefixes, $\Lambda_1, \ldots, \Lambda_\ell$, does not decrease with ℓ, and all these decompositions are bounded by the abelian JSJ decomposition of the limit group L (the abelian JSJ of a freely decomposable group that starts with the Grushko decomposition, and then associates the abelian JSJ decomposition with each of the non-cyclic freely indecomposable factors). Hence, the common refinements of the prefixes of the sequence $\Lambda_1, \ldots$ stabilize, and it is possible to associate a common refinement with the entire sequence.

Therefore, it is possible to associate an abelian decomposition (common refinement) with every suffix of the sequence, $\Lambda_\ell, \ldots$. The complexities of these abelian decompositions do not increase with ℓ, and every sequence of strictly decreasing abelian decompositions of L terminates after a finite (in fact, bounded) time. We define the stable dominant abelian decomposition of the sequence $\Lambda_1, \ldots$, to be the minimal abelian decomposition that is associated with a suffix $\Lambda_\ell, \ldots$, where the minimum is with respect to all the suffixes of the sequence.

Let Θ_{i_0} be the stable dominant abelian decomposition of the sequence $\Lambda_1, \ldots$. Suppose that Θ_{i_0} does not collapse to any free factors, i.e., no edges with trivial stabilizers (only QH and rigid vertex groups, and edges with non-trivial cyclic stabilizers).

We assumed that the sequence of abelian decompositions $\Lambda_1, \ldots$ contains no QH vertex groups. Hence, in case Θ_{i_0} contains no free products, the procedure that was used to prove theorem 6.1 in case the abelian decompositions $\Lambda_1, \ldots$ contain no QH vertex groups (cf. proposition 6.3), proves the existence of a sequence of homomorphisms that converges into an action of L on a real tree, with an associated abelain decomposition, that after a possible refinement, is identical to Θ_{i_0}. Therefore, in this case the infinite sequence $\Lambda_1, \ldots$ can be replaced by a finite sequence that ends with Θ_{i_0}, precisely as we did in the freely indecomposable case.

Suppose that Θ_{i_0} collapses to free products. In that case, by the structure of the iterative procedure for the construction of the abelian decompositions $\Lambda_1, \ldots$ (proposition 6.3), there exists a sequence of shortened homomorphisms that converges into a geometric and faithful action of L on a real tree with an associated abelian decomposition that can be further refined to be Θ_{i_0}. Since (S, L) is Levitt-free the action on L on the limit tree contains only simplicial part and IET components. Using an argument that we explain in more detail in the presence of QH vertex groups in the sequel, the limit action of L must be geometric and the limit tree contains no segment in its simplicial part, that can not be divided into segments with non-trivial stabilizers. Furthermore, there must exist an index $i_1 \geq i_0$, such that all the edge groups in the abelian decompositions Λ_i, $i > i_1$, either appear as edges in Θ_{i_0}, or they correspond to s.c.c. in a QH vertex groups in Θ_{i_0}. In particular, the modular groups that are associated with all the abelian decompositions, $\Lambda_{i_1}, \ldots$, are contained in the modular group that is associated with Θ_{i_0}. Therefore, the sequence of abelian decompositions, $\Lambda_1, \ldots$ can be replaced by a finite sequence that terminates with Θ_{i_0}.

At this point we still assume that the pair (S, L) is Levitt-free, but allow the abelian decompositions in the sequel to be arbitrary, i.e., to contain QH vertex groups. We have already assumed that starting with the sequence $\{h_n\}$ contains no separable subsequence, and that shortening it using the ambient modular group $Mod(\Lambda_1)$, we get a sequence for which every subsequence is not separable (definition 7.3), and every convergent subsequence converges into a faithful action of L.

We use only the dominant modular group, $MXMod(\Lambda_1)$. We modify what we did in the freely indecomposable case. First, we shorten the action of each of the dominant QH vertex groups using the procedure that is used in the proof of propositions 2.7 and 2.8. For each of the dominant QH vertex groups, the procedures that are used in the proofs of these propositions give us an infinite collections of positive generators, $u_1^m, \ldots, u_g^m$, with similar presentations of the corresponding QH vertex groups, which means that the sequence of sets of generators belong to the same isomorphism class. Furthermore, with each of these sets of generators there associated words, w_j^m, $j = 1, \ldots, r$, of lengths that increase with m, such that a given (fixed) set of positive elements can be presented as: $y_j = w_j^m(u_1^m, \ldots, u_g^m)$. The words w_j^m are words in the generators u_i^m and their inverses. However, they can be presented as positive words in the generators u_j^m, and unique appearances of words t_ℓ, that are fixed words in the elements u_j^m and their inverses (i.e., the words do not depend on m), and these elements $t_\ell^m(u_1^m, \ldots, u_g^m)$ are positive for every m.

Given the output of the procedures that were used in propositions 2.7 and 2.8, with each of the dominant QH vertex groups we associate a system of generators $u_1^1, \ldots, u_g^1$ (the integer g depends on the QH vertex group). Since an IET action of a QH vertex group is indecomposable in the sense of $[Gu]$, finitely many (fixed) translates of each of the positive paths that are associated with the positive paths, $u_{i_1}^1$, cover the positive path that is associated with $u_{i_2}^1$, and the positive paths that are associated the words $t_\ell(u_1^1, \ldots, u_g^1)$. As in the procedure in the freely indecomposable case, the covering of the elements $u^1 i_1$ by finitely many translates of elements $u_{i_2}^1$ guarantee that the ratios between their lengths along the iterative (shortening) procedure that we use remain globally bounded.

Let Q be a QH vertex group in Λ_1. With each of the generators s_j, $1 \leq j \leq r$, we associate the (positive) path in the limit tree Y from the base point to the image of the base point under s_j (the path may be degenerate). Since the action of L on the limit tree Y is geometric, such a path contains finitely many subpaths that are contained in the orbit of an IET component that is associated with the QH vertex group Q. Since the action of Q on its associated IET component is indecomposable, given each non-degenerate subpath of the path that corresponds to a generator s_j and is contained in the IET component that is associated with Q, finitely many translates of this non-degenerate subpath cover the paths that are associated with the elements: t_ℓ and $u_1^1, \ldots, u_g^1$ that generate Q.

These coverings of the paths that are associated with the elements $u_1^1, \ldots, u_g^1$ by translates of the subpaths that is associated with s_j, will guarantee that the lengths of the subpaths that are associated with s_j, along the entire shortening process that we present, are bounded below by some (global) positive constant times the lengths of the paths that are associated with the elements, $u_1^1, \ldots, u_g^1$ along the process.

At this point we shorten the homomorphisms $\{h_n\}$ using the dominant modular group $MXMod(\Lambda_1)$. For each homomorphism h_n, we pick the shortest homomorphism after precomposing with an element from $MXMod(\Lambda_1)$ that keeps the positivity of the given set of the images of the generators $s_1, \ldots, s_r$, and the positivity of the elements $u_1^1, \dots, u_g^1$ and t_ℓ, and keeps their lengths to be at least the maximal length of the image of a generator of a dominant edge or vertex group (the generators are chosen from the fixed finite sets of generators of each of the vertex and edge groups). We further require that after the shortening, the image of each of the elements $u_1^1, \dots, u_g^1$ and t_ℓ, will be covered by the finitely many translates of them and of the paths that are associated with the relevant generators $s_1, \dots, s_r$, that cover them in the limit action that is associated with Λ_1. We further require that if a path that is associated with one of the generators, $s_1, \dots, s_r$, passes through an edge with a dominant edge group in Λ_1, then after shortening the path that is associated with such a generator contains at least a subpath that is associated with the dominant edge group. This guarantees that the length of such a generator remains bigger than the length of the dominant edge group along the entire procedure. We (still) denote the obtained (shortened) homomorphisms $\{h_n^1\}$.

By the shortening arguments that are proved in propositions 2.7 and 2.8, the lengths of the images, under the shortened homomorphisms $\{h_n^1\}$, of the elements $u_1^1, \dots, u_g^1$ and t_ℓ, that are associated with the various dominant QH vertex groups, and the lengths of the elements, $s_1, \dots, s_r$, are bounded by some constant c_1 (that is independent of n) times the maximal length of the images of the fixed generators of the dominant vertex and edge groups.

As in the freely indecomposable case, we pass to a subsequence of the homomorphisms $\{h_n^1\}$ that converges into an action of L on some real tree with an associated abelian decomposition Δ_2. If the action of L is not faithful, the conclusions of theorem 7.2 follow, hence, we may assume that the action of L is faithful. If the action of L is not geometric or contains a non-degenerate subsegment in its simplicial part, that can not be divided into finitely many subsegments with non-trivial stabilizers, the conclusion of theorem 7.2 follow. If there exists a subsequence of the homomorphisms $\{h_n^1\}$ that are separable (see definition 7.3), theorem 7.2 follow. Hence, we may assume that there $\{h_n^1\}$ does not contain a separable subsequence, that the action of L is geometric, and that every segment in the simplicial part can be divided into finitely many segments with non-trivial stabilizers.

We further refine Δ_2 as we did in the freely indecomposable case., We restrict (a convergent subsequence of) the homomorphisms $\{h_n^1\}$ to non-QH vertex groups that are connected only to periodic edge groups, as we did with Λ, and construct an abelian decomposition that we denote $\hat{\Delta}_2$.

Let Q be a dominant QH vertex group in Λ_1, so that all its boundary elements are elliptic in $\hat{\Delta}_2$. i.e., every boundary element is contained in either an edge group or in a non-QH vertex group in $\hat{\Delta}_2$. With the original sequence of homomorphisms, $\{h_n\}$, and the shortened sequence, $\{h_n^1\}$, we associate a new (intermediate) sequence of homomorphisms. Each homomorphism, h_n^1, is obtained from h_n, by precomposition with a (shortened) automorphism from the dominant modular group, $\varphi_n \in MXMod(\Lambda_1)$. φ_n is a composition of elements from the modular groups of dominant QH vertex group in Λ, and Dehn twists along edges with dominant edge groups. We set $\tau_n \in MXMod(\Lambda_1)$ to be a composition of the same elements from the modular groups of dominant QH vertex groups in Λ_1, that are not the dominant QH vertex group Q, and the same Dehn twists along edges with dominant edge group as the shortened homomorphism φ_n. Let $\mu_n = h_n \circ \tau_n$.

The sequence of homomorphisms μ_n, converges into a faithful action of L on a real tree, with an associated abelian decomposition Γ_Q, that has a single QH vertex group, a conjugate of Q, and possibly several edges with non-trivial edge groups, that are edge in both Λ_1 and $\hat{\Delta}_2$.

Suppose that the abelian decompositions, $\hat{\Delta}_2$ and Γ_Q, have a common refinement, in which a conjugate of Q appears as a QH vertex group, and all the edges and QH vertex groups in $\hat{\Delta}_2$ that do not correspond to s.c.c. or proper QH subgroups of Q also appear in the common refinement (the elliptic elements in the common refinement are precisely those elements that are elliptic in both Γ_Q and $\hat{\Delta}_2$). This, in particular, implies that all the QH vertex groups that can not be conjugated into Q, and all the edge groups in $\hat{\Delta}_2$ that can not be conjugated into non-peripheral elements in Q are elliptic in Γ_Q. In case there exists such a common refinement, we replace $\hat{\Delta}_2$ with this common refinement. We repeat this possible refinement for all the QH vertex groups in Λ_1 that satisfy these conditions (the refinement procedure does not depend on the order of the QH vertex groups in Λ_1 that satisfy the refinement conditions). We denote the obtained abelian decomposition, Λ_2.

At this point, we modify the shortened homomorphisms, $\{h^1_n\}$, that converge into a faithful limit action of L on a limit tree from which the abelian decomposition Λ_2 is obtained, by precomposing them with a fixed automorphism, ψ_1, from the dominant modular group $MXMod(\Lambda_1)$. This precomposition is needed to guarantee the validity of certain inequalities between the lengths, or the ratios of lengths, of a finite set of elements. This finite set include:

(1) the fixed set of generators of dominant QH vertex groups and generators of dominant edge groups in Λ_1.
(2) the (fixed) generators, $s_1, \dots, s_r$.
(3) the fixed set of generators of some of the QH vertex groups and some of the edge groups in Λ_2.

Let $Q^2_1, \dots, Q^2_f$, and $E^2_1, \dots, E^2_v$, be those QH vertex groups and edge groups in Λ_2, for which at least one of the fixed set of generators of the dominant QH vertex groups, and dominant edge groups in Λ_1, is not elliptic with respect to them. i.e., if we collapse Λ_2 to contain a single QH vertex group Q^2_i, $\Gamma_{Q^2_i}$, or a single edge group E^2_i, $\Gamma_{E^2_i}$, then there exists a generator of a dominant QH vertex group or a dominant edge group in Λ_1 that is mapped to a non-elliptic element in that collapsed abelian decomposition.

Note that every non-dominant QH vertex group or edge group in Λ_1 is a QH vertex group or an edge group in Λ_2. Hence, in choosing a fixed automorphism that we use in order to modify the shortened homomorphisms, $\{h^1_n\}$, we are not concerned with these non-dominant vertex and edge groups.

Each generator s_j, $j = 1, \dots, r$, can be written in a normal form with respect to Λ_1. Let $b_1, \dots, b_t \in L$ be the collection of elements in the (fixed) normal forms of $s_1, \dots, s_r$, that are contained in non-QH vertex groups, or in an edge group that is adjacent only to QH vertex groups in Λ_1. Each of the elements $b_1, \dots, b_t$ can be represented in a normal form with respect to the abelian decomposition Λ_2. In particular, with each of the elements $b_1, \dots, b_t$, it is possible to associate a (possibly empty) collection of paths in the IET components that are associated with $Q^2_1, \dots, Q^2_f$, and segments in the simplicial part of Y_2 (the tree that is associated with Λ_2), and are associated with $E^2_1, \dots, E^2_v$.

Before shortening the fixed set of generators of the dominant QH vertex groups in Λ_1, and the generators, $s_1, \ldots, s_r$, we used the shortening procedure that was applied in proving propositions 2.7 and 2.8. These propositions give us a sequence of automorphisms of these QH vertex groups, that preserve positivity and the demonstration of the indecomposibility by a cover of finitely many (fixed) translates, that we can now use to make the set of generators of the QH vertex groups longer, as well the paths in the corresponding IET components in the associated limit tree Y_1, that are associated with the set of generators, $s_1, \ldots, s_r$.

For each dominant QH vertex group Q in Λ_1 we denote such an automorphism φ_Q. For each dominant edge E in Λ_1 we denote the corresponding (positive) Dehn twist by φ_E. We set the (fixed) automorphism $\psi_1 \in MXMod(\Lambda_1)$, that we precompose with the sequence of shortened homomorphisms $\{h_n^1\}$, to satisfy the following properties:

(1) Let $Q_1, \ldots, Q_\ell$ be the dominant QH vertex groups in Λ_1, and $E_1, \ldots, E_s$ be the dominant edge groups in Λ_1. Then for some positive integers α_1, α_ℓ and $\beta_1, \ldots, \beta_s$:

$$\psi_1 = \varphi_{Q_1}^{\alpha_1} \circ \ldots \circ \varphi_{Q_\ell}^{\alpha_\ell} \circ \varphi_{E_1}^{\beta_1} \circ \ldots \circ \varphi_{E_s}^{\beta_s}.$$

(2) with each element $g \in L$ it is possible to associate (possibly empty) finite collection of paths in the IET components that are associated with (conjugates of) $Q_1^2, \ldots, Q_f^2$ (QH vertex groups in Λ_2), and segments in the simplicial part that are associated with $E_1^2, \ldots, E_v^2$.

Let $u_1, \ldots, u_g$ be the fixed set of generators of a dominant QH vertex group Q in Λ_1. If there are some non-degenerate segments that are associated with u_j in the IET component that is associated with a (conjugate of a) QH vertex group Q_i^2, then the total lengths of the segments that are associated with $\varphi_Q^{\alpha_Q}(u_j)$ in the IET components that are associated with conjugates of Q_i^2, are at least twice the total length of the segments that are associated with all the elements $b_1, \ldots, b_t$ in the IET component that are associated with conjugates of Q_i^2. Furthermore, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these paths. This demonstration guarantees that the lower bounds on the ratios remain valid along the entire process.

If the path that is associated with u_j in the limit tree that is associated with Λ_2, contains a subsegment that is associated with (a conjugate of) an edge group E_i^2, then the number of such subsegments (that are associated with conjugates of E_i^2) in the path that is associated with $\varphi_Q^{\alpha_Q}(u_j)$ is at least twice the total appearances of such subsegments (associated with conjugates of E_i^2) in the paths that are associated with: $b_1, \ldots, b_t$.

(3) Let s_j be one of the fixed set of generators $s_1, \ldots, s_r$. Suppose that there are some non-degenerate subsegments that are contained in the path that is associated with s_j in the IET components that are associated with conjugates of a dominant QH vertex group Q in Λ_1.

Suppose that in the path that is associated a generator from the fixed (finite) set of generators of Q in the real tree that is associated with Λ_2, there exist some subpath in an IET component that is associated with a QH vertex group Q_i^2 in Λ_2.

Then the total lengths of the subpaths that are associated with $\varphi_Q^{\alpha_Q}(s_j)$ in the IET components that are associated with conjugates of Q_i^2, are at least twice the total length of the subpaths that are associated with all the elements $b_1, \ldots, b_t$ in the IET components that are associated with conjugates of Q_i^2. Furthermore, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these subpaths. This demonstration guarantees that these lower bounds on the ratios remain valid along the entire process.

Suppose that in the path that is associated with a generator from the fixed (finite) set of generators of Q in the real tree that is associated with Λ_2, there is a non-degenerate subsegment in the simplicial part of the real tree that is associated with a conjugate of an edge group E_i^2 in Λ_2.

Then the number of such subsegments (that are associated with conjugates of E_i^2) in the path that is associated with $\varphi_Q^{\alpha_Q}(s_j)$ is at least twice the total appearances of such subsegments (associated with conjugates of E_i^2) in the paths that are associated with: $b_1, \ldots, b_t$.

(4) Suppose that the path that is associated with s_j in the tree that is associated with Λ_1, contains non-degenerate subsegments that are associated with conjugates of a dominant edge group E in Λ_1. Suppose that in the path that is associated a generator of E in the real tree that is associated with Λ_2, there exist some subpath in an IET component that is associated with a QH vertex group Q_i^2 in Λ_2.

Then with the path that is associated with the element $\varphi_E^{\beta_E}(s_j)$ in the limit tree that is associated with Λ_2, the total lengths of the subpaths that are in the IET components that are associated with conjugates of Q_i^2, are at least twice the total length of the subpaths that are associated with all the elements $b_1, \ldots, b_t$ in the IET components that are associated with conjugates of Q_i^2. Furthermore, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these subpaths.

Suppose that in the path that is associated a generator of E in the real tree that is associated with Λ_2, there is a non-degenerate subsegment in the simplicial part of the real tree that is associated with a conjugate of an edge group E_i^2 in Λ_2.

Then the number of such subsegments (that are associated with conjugates of E_i^2) in the path that is associated with $\varphi_E^{\beta_E}(s_j)$ is at least twice the total appearances of such subsegments (associated with conjugates of E_i^2) in the paths that are associated with: $b_1, \ldots, b_t$.

This concludes the construction of the homomorphisms that are associated with the second level, that are set to be: $\{h_n^1 \circ \psi_1\}$. Note that each homomorphism h_n^1 can be presented as: $h_n^1 = h_n \circ \nu_n^1$, where ν_n^1 and ψ_1 are automorphisms from the dominant modular group $MXMod(\Lambda_1)$.

We assign weight 1 with every QH vertex group and every edge group in Λ_2 that are also a QH vertex group or an edge group in Λ_1. We also assign weight 1 with every QH vertex group and every edge group in Λ_2, for which there a generator of a dominant QH vertex group or a dominant edge group in Λ_1, such that the path that is associated with that generator in the tree that is associated with Λ_2, has a subpath in an IET component that is associated with a conjugate of the QH vertex group in Λ_2, or a subpath in the simplicial part of that tree that is associated with the edge group in Λ_2. We assign weight 2 with every other QH vertex group or edge group in Λ_2.

With the abelian decomposition Λ_2 we associate its modular group, $Mod(\Lambda_2)$, and its dominant modular group, $MXMod(\Lambda_2)$. As we did in the freely indecomposable case, and in the first case, we first shorten the homomorphisms, $\{h_n^1 \circ \psi_1\}$, using the ambient modular group, $Mod(\Lambda_2)$. We denote the obtained sequence of homomorphisms, $\{h_n^2\}$. If there exists a subsequence of the homomorphisms, $\{h_n^2\}$, that is either separable, or that converges into a non-faithful action of L on some real tree, the conclusions of theorem 7.2 follow for this subsequence, and the procedure terminates.

Hence, in the sequel we assume that $\{h_n^2\}$ has no separable subsequence or a subsequence that converges into a non-faithful action of L on a real tree. In this case we use only the dominant modular group, $MXMod(\Lambda_2)$, and modify what we did in the first step.

First, the shortening of the homomorphisms, $\{h_n^1 \circ \psi_1\}$, using the dominant modular group, $MXMod(\Lambda_2)$, requires to preserve the positivity of the fixed set of generators of the semigroup S, $s_1, \dots, s_r$, and the fixed sets of generators of the QH and edge groups in the abelian decompositions, Λ_1 and Λ_2. In particular, it is required to preserve the positivity of the paths that are associated with these elements and are contained in IET components that are associated with conjugates of dominant QH vertex groups in Λ_2. In addition the shortening is required to preserve the finitely many equivariance restrictions that we imposed on these paths, i.e., the overlaps of these paths with translates of themselves, that sample the indecomposibility of the IET components, and guarantee that certain inequalities between lengths of these paths will hold in the sequel, and in particular after the shortenings.

As we did in the freely indecomposable case, in addition to these elements, in shortening using $MXMod(\Lambda_2)$, we also consider the elements in the ball of radius 2 in the Cayley graph of L w.r.t. the generating set $s_1, \dots, s_r$. Note that elements of length 2 need not be positive nor negative, but they may rather contain a positive and a negative subsegments. Still, the shortening procedure that is presented in section 2 for abelian, QH, and Levitt components, works precisely in the same way, when the path that is associated with an element contains finitely many oriented (positively or negatively) subsegments, and not only in case the entire path is oriented.

As in the freely indecomposable and in the first step, we first shorten the action of each of the dominant QH vertex groups in Λ_2 using the procedure that is used in the proof of propositions 2.7 and 2.8. This, in particular, associates a fixed set of generators with each dominant QH vertex group, and a further finite collection of words, so that their positivity implies the positivity of a finite collection of given paths under an infinite sequence of modular automorphisms (see the description in the freely indecomposable case in the proof of proposition 6.4).

As in the first step, using the indecomposibility of an IET components, finitely many (fixed) translates of each of the positive paths that are associated with each of the fixed generators of a QH vertex group cover the paths that are associated with the other generators. This coverings guarantee that the ratios between their lengths along the iterative (shortening) procedure that we use remain globally bounded. We do the same to paths that are associated with elements in the ball of radius 2 of L and pass through an IET component, and to paths that are associated with generators of QH vertex groups in Λ_1 and pass through IET components.

At this point we shorten the homomorphisms $\{h_n^1 \circ \psi_1\}$ using the dominant modular group $MXMod(\Lambda_2)$, precisely as we did in the first step, but keeping the positivity and the equivariance of a larger set of elements that is specified in the beginning of this step. We (still) denote the obtained (shortened) homomorphisms $\{h_n^2\}$. We pass to a subsequence of the homomorphisms $\{h_n^2\}$ that converges into an action of L on some real tree with an associated abelian decomposition Δ_3. If the action of L is not faithful, or if the sequence $\{h_n^2\}$ contains a separable subsequence, the conclusions of theorem 7.2 follow and the procedure terminates. Hence, we may assume that the action of L is faithful, that $\{h_n^2\}$ contains no separable subsequence, and that the action of L on the limit tree is geometric, and the tree contains no segment in its simplicial part that can not be divided into finitely many subsegments with non-trivial stabilizers. We further refine Δ_3 to $\hat{\Delta}_3$ and finally to an abelian decomposition Λ_3, precisely as we did in the first step.

As in the first step, we modify the shortened homomorphisms, $\{h_n^2\}$, that converge into a faithful limit action of L on a limit tree from which the abelian decomposition Λ_3 is obtained, by precomposing them with a fixed automorphism, ψ_2, from the dominant modular group $MXMod(\Lambda_1)$. This precomposition is needed to guarantee the validity of certain inequalities between the lengths, or the ratios of lengths, of a finite set of elements. This finite set include:

(1) the fixed set of generators of dominant QH vertex groups and generators of dominant edge groups in Λ_1 and Λ_2.
(2) the elements in a ball of radius 2 in the Cayley graph of L w.r.t. the generators: $s_1, \dots, s_r$.
(3) the fixed set of generators of some of the QH vertex groups and some of the edge groups in Λ_3.

Let $Q_1^3, \dots, Q_{f_2}^3$, and $E_1^3, \dots, E_{v_2}^3$, be those QH vertex groups and edge groups in Λ_3, for which at least one of the fixed set of generators of the dominant QH vertex groups of weight 2, and dominant edge groups of weight 2 in Λ_2, is not elliptic with respect to them.

Each element in the ball of radius 2 in the Cayley graph of L (w.r.t. $s_1, \dots, s_r$), and each of the fixed generators of a QH vertex group or an edge group in Λ_1, can be written in a normal form with respect to Λ_2. Let $b_1^2, \dots, b_{t_2}^2 \in L$ be the collection of elements in the (fixed) normal forms of these elements, that are contained in non-QH vertex groups, or in an edge group that is adjacent only to QH vertex groups in Λ_2. Each of the elements $b_1^2, \dots, b_{t_2}^2$ can be represented in a normal form with respect to the abelian decomposition Λ_3. In particular, with each of the elements $b_1^2, \dots, b_{t_2}^2$, it is possible to associate a (possibly empty) collection of paths in the IET components that are associated with $Q_1^3, \dots, Q_{f_2}^3$, and segments in the simplicial part of Y_3 (the tree that is associated with Λ_3), and are associated with $E_1^3, \dots, E_{v_2}^3$.

As in the first step, before shortening the fixed set of generators of the dominant QH vertex groups in Λ_2, we used the shortening procedure that was applied in proving propositions 2.7 and 2.8. These propositions give us a sequence of automorphisms of these QH vertex groups, that preserve positivity of positive elements and the orientation (positive or negative) of subpaths of paths that are associated with elements from the finite set that include elements from the ball of radius 2 in the Cayley graph of L, and (fixed) generators of QH and abelian vertex groups in Λ_1 and Λ_2.

For each dominant QH vertex group Q^2 in Λ_2 we denote such an automorphism φ_{Q^2}. For each dominant edge group E^2 in Λ_2 we denote the corresponding (positive) Dehn twist by φ_{E^2}. Before constructing a fixed automorphism from the dominant modular group $MXMod(\Lambda_2)$ that is going to be used to twist (precompose) the sequence of homomorphisms, $\{h^2_n\}$, We construct a fixed automorphism that is associated only with the dominant QH vertex groups and dominant edge groups in Λ_2 that are of weight 2. We denote this automorphism ψ^2_2, and in a similar way to what we did in the first step we require it to satisfy the following properties:

(1) Let $Q^2_1,\dots,Q^2_{\ell_2}$ be the dominant QH vertex groups of weight 2 in Λ_2, and $E^2_1,\dots,E^2_{s_2}$ be the dominant edge groups of weight 2 in Λ_2. Then for some positive integers $\alpha^2_1,\alpha^2_{\ell_2}$ and $\beta^2_1,\dots,\beta^2_{s_2}$:

$$\psi^2_2 = \varphi_{Q^2_1}^{\alpha^2_1} \circ \ldots \circ \varphi_{Q^2_{\ell_2}}^{\alpha^2_{\ell_2}} \circ \varphi_{E^2_1}^{\beta^2_1} \circ \ldots \circ \varphi_{E^2_{s_2}}^{\beta^2_{s_2}}.$$

(2) with each element in L it is possible to associate (possibly empty) finite collection of paths in the IET components that are associated with (conjugates of) $Q^3_1,\dots,Q^3_{f_2}$ (QH vertex groups in Λ_3), and segments in the simplicial part that are associated with $E^3_1,\dots,E^3_{v_2}$.

As in (part (2) in) the first step, if there are some non-degenerate segments, that are associated with a fixed generator, u_j, of a dominant QH vertex group, Q^2, of weight 2 in Λ_2, in the IET component that is associated with a (conjugate of a) QH vertex group Q^3_i, then the total lengths of the segments that are associated with $\varphi_{Q^2}^{\alpha^2_{Q^2}}(u_j)$ in the IET components that are associated with conjugates of Q^3_i, are at least four times the total length of the segments that are associated with all the elements $b^2_1,\dots,b^2_{t_2}$ in the IET components that are associated with conjugates of Q^3_i. Furthermore, as in the first step, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these paths. This demonstration guarantees that these lower bounds on the ratios remain valid along the entire process.

If the path that is associated with u_j in the limit tree that is associated with Λ_3, contains a subsegment that is associated with (a conjugate of) an edge group E^3_i, then the number of such subsegments (that are associated with conjugates of E^3_i) in the path that is associated with $\varphi_{Q^2}^{\alpha^2_{Q^2}}(u_j)$ is at least four times the total appearances of such subsegments (associated with conjugates of E^3_i) in the paths that are associated with: $b^2_1,\dots,b^2_{t_2}$.

(3) Let g be one of the elements in the ball of radius 2 in the Cayley graph of L (w.r.t. $s_1,\dots,s_r$), or one of the fixed set of generators of QH vertex groups in Λ_1 or a generator of an edge group in Λ_1. Note that g need not be an oriented element, but the path that is associated with g may contain a positive and a negative subpaths.

Suppose that there are some non-degenerate subsegments that are contained in the path that is associated with g in the IET components that are associated with conjugates of a dominant QH vertex group Q^2 of weight 2 in Λ_2.

Suppose that in the path that is associated with a generator from the fixed (finite) set of generators of Q^2 in the real tree that is associated with Λ_3, there exist some subpath in an IET component that is associated with a QH vertex group Q_i^3 in Λ_3.

With the path that is associated with the element $\varphi_{Q^2}^{\alpha^2_{Q^2}}(g)$ in the limit tree that is associated with Λ_2, one can associate finitely many subpaths that are contained in (finitely many) IET components that are associated with conjugates of Q^2. Note that the number of such subpaths is the same as the number of such subpaths in the path that is associated with g.

Then the total lengths of the subpaths that are associated with the image of each of these subpaths in the IET components that are associated with conjugates of Q_i^3, are at least four times the total length of the subpaths that are associated with all the elements $b_1^2,\dots,b_{t_2}^2$ in the IET components that are associated with conjugates of Q_i^3. Furthermore, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these subpaths. This demonstration guarantees that these lower bounds on the ratios remain valid along the entire process.

Suppose that in the path that is associated with a generator from the fixed (finite) set of generators of Q^2 in the real tree that is associated with Λ_3, there is a non-degenerate subsegment in the simplicial part of the real tree that is associated with a conjugate of an edge group E_i^3 in Λ_3.

Then the number of such subsegments (that are associated with conjugates of E_i^3) in the path that is associated with $\varphi_{Q^2}^{\alpha^2_{Q^2}}(g)$ is at least four times the total appearances of such subsegments (associated with conjugates of E_i^3) in the paths that are associated with: $b_1^2,\dots,b_{t_2}^2$.

(4) Suppose that the path that is associated with g in the tree that is associated with Λ_2, contains non-degenerate subsegments that are associated with conjugates of a dominant edge group E^2 of weight 2 in Λ_2. Suppose that in the path that is associated with a generator of E^2 in the real tree that is associated with Λ_3, there exist some subpath in an IET component that is associated with a QH vertex group Q_i^3 in Λ_3.

Then with the path that is associated with the element $\varphi_{E^2}^{\beta^2_{E^2}}(g)$ in the limit tree that is associated with Λ_2, the image of the subpaths that are stabilized by conjugates of the dominant edge group E^2, contain subpaths that are in the IET components that are associated with conjugates of Q_i^3, and the total lengths of these subpaths are at four times the total length of the subpaths that are associated with all the elements $b_1^2,\dots,b_{t_2}^2$ in the IET components that are associated with conjugates of Q_i^3. Furthermore, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these subpaths.

Suppose that in the path that is associated a generator of E^2 in the real tree that is associated with Λ_3, there is a non-degenerate subsegment in the simplicial part of the real tree that is associated with a conjugate of an edge group E_i^3 in Λ_3.

Then the number of such subsegments (that are associated with conjugates of E_i^3) in the path that is associated with $\varphi_{E^2}^{\beta_{E^2}^2}(g)$ is at least four times the total appearances of such subsegments (associated with conjugates of E_i^3) in the paths that are associated with: $b_1^2, \ldots, b_{t_2}^2$.

This concludes the treatment of dominant QH vertex groups and dominant edge groups of weight 2 in Λ_2, and the construction of a fixed automorphism, $\psi_2^2 \in MXMod(\lambda_2)$, that is associated with them. After ψ_2^2 is constructed we can treat in a similar way the dominant QH vertex groups and dominant edge groups of weight 1 in Λ_2, and finally construct the fixed automorphism ψ_2, that is used in precomposing the sequence of shortened homomorphisms, $\{h_n^2\}$.

Let $Q_1^3, \ldots, Q_{f_1}^3$, and $E_1^3, \ldots, E_{v_1}^3$, be those QH vertex groups and edge groups in Λ_3, for which at least one of the fixed set of generators of the dominant QH vertex groups of weight 1, and dominant edge groups of weight 1 in Λ_2, is not elliptic with respect to them.

Let $Col\Lambda_2$ be the abelian decomposition that is obtained by collapsing all the QH vertex groups of weight 2 in Λ_2 and the edges that are connected to them, and all the edge groups of weight 2 in Λ_2. $Col\Lambda_2$ contains only QH vertex groups and edge groups of weight 1.

Each element in the ball of radius 2 in the Cayley graph of L, and each of the fixed generators of a QH vertex group or an edge group in Λ_1, can be written in a normal form with respect to $Col\Lambda_2$. Let $b_1^1, \ldots, b_{t_1}^1 \in L$ be the collection of elements in the (fixed) normal forms of these elements, that are contained in non-QH vertex groups, or in an edge group that is adjacent only to QH vertex groups in $Col\Lambda_2$. Each of the elements $b_1^1, \ldots, b_{t_1}^1$ can be represented in a normal form with respect to the abelian decomposition Λ_3. In particular, with each of the elements $b_1^1, \ldots, b_{t_1}^1$, it is possible to associate a (possibly empty) collection of paths in the IET components that are associated with $Q_1^3, \ldots, Q_{f_1}^3$, and segments in the simplicial part of Y_3 (the tree that is associated with Λ_3), and are associated with $E_1^3, \ldots, E_{v_1}^3$.

At this point we construct the automorphism ψ_2 in a similar way to the construction of the automorphism ψ_2^2, and the automorphism ψ_1 in the first step. Recall that given a dominant QH vertex group, Q, in Λ_2, we used the procedure that was applied in proving propositions 2.7 and 2.8 to associate with Q an automorphism, φ_Q, that preserve the orientation of the oriented subpaths in the paths that are associated with the elements from the finite set that include elements from the ball of radius 2 in the Cayley graph of L, and (fixed) generators of QH and abelian vertex groups in Λ_1 and Λ_2.

The automorphism ψ_2 is required to satisfy similar properties as ψ_2^2 and ψ_1:

(1) Let $Q_1^1, \ldots, Q_{\ell_1}^1$ be the dominant QH vertex groups of weight 1 in Λ_2, and $E_1^1, \ldots, E_{s_1}^1$ be the dominant edge groups of weight 1 in Λ_2. Then for some positive integers $\alpha_1^1, \alpha_{\ell_1}^1$ and $\beta_1^1, \ldots, \beta_{s_1}^1$:

$$\psi_2^1 = \varphi_{Q_1^1}^{\alpha_1^1} \circ \ldots \circ \varphi_{Q_{\ell_1}^1}^{\alpha_{\ell_1}^1} \circ \varphi_{E_1^1}^{\beta_1^1} \circ \ldots \circ \varphi_{E_{s_1}^1}^{\beta_{s_1}^1}$$

and $\psi_2 = \psi_2^1 \circ \psi_2^2$.

(2) with each element in L it is possible to associate (possibly empty) finite collection of paths in the IET components that are associated with (conjugates

of) $Q^3_1,\dots,Q^3_{f_1}$ (QH vertex groups in Λ_3), and segments in the simplicial part that are associated with $E^3_1,\dots,E^3_{v_1}$.

As in (part (2) in) the first step, if there are some non-degenerate segments, that are associated with a fixed generator, u_j, of a dominant QH vertex group, Q^1, of weight 1 in Λ_2, in the IET component that is associated with a (conjugate of a) QH vertex group Q^3_i, then the total lengths of the segments that are associated with $\varphi_{Q^1}^{\alpha^1_{Q^1}}(u_j)$ in the IET components that are associated with conjugates of Q^3_i, are at least four times the total length of the segments that are associated with all the elements $\psi^2_2(b^1_1),\dots,\psi^2_2(b^1_{t_1})$ in the IET components that are associated with conjugates of Q^3_i. Furthermore, as in the first step, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these paths. This demonstration guarantees that these lower bounds on the ratios remain valid along the entire process.

If the path that is associated with u_j in the limit tree that is associated with Λ_3, contains a subsegment that is associated with (a conjugate of) an edge group E^3_i, then the number of such subsegments (that are associated with conjugates of E^3_i) in the path that is associated with $\varphi_{Q^1}^{\alpha^1_{Q^1}}(u_j)$ is at least four times the total appearances of such subsegments (associated with conjugates of E^3_i) in the paths that are associated with: $\psi^2_2(b^1_1),\dots,\psi^2_2(b^1_{t_1})$.

(3) Let g be one of the elements in the ball of radius 2 in the Cayley graph of L (w.r.t. $s_1,\dots,s_r$), or one of the fixed set of generators of QH vertex groups in Λ_1 or a generator of an edge group in Λ_1.

Suppose that there are some non-degenerate subsegments that are contained in the path that is associated with g in the IET components that are associated with conjugates of a dominant QH vertex group Q^1 of weight 1 in Λ_2.

Suppose that in the path that is associated with a generator from the fixed (finite) set of generators of Q^1 in the real tree that is associated with Λ_3, there exist some subpath in an IET component that is associated with a QH vertex group Q^3_i in Λ_3.

With the path that is associated with the element $\varphi_{Q^1}^{\alpha^1_{Q^1}}(g)$ in the limit tree that is associated with Λ_2, one can associate finitely many subpaths that are contained in (finitely many) IET components that are associated with conjugates of Q^1.

Then the total lengths of the subpaths that are associated with the image of each of these subpaths in the IET components that are associated with conjugates of Q^3_i, are at least four times the total length of the subpaths that are associated with all the elements $\psi^2_2 9b^1_1),\dots,\psi^2_2(b^1_{t_1})$ in the IET components that are associated with conjugates of Q^3_i. Furthermore, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these subpaths. This demonstration guarantees that these lower bounds on the ratios remain valid along the entire process.

Suppose that in the path that is associated with a generator from the fixed (finite) set of generators of Q^1 in the real tree that is associated with Λ_3, there is a non-degenerate subsegment in the simplicial part of the real tree that is associated with a conjugate of an edge group E_i^3 in Λ_3.

Then the number of such subsegments (that are associated with conjugates of E_i^3) in the path that is associated with $\varphi_{Q^1}^{\alpha^1_{Q^1}}(g)$ is at least four times the total appearances of such subsegments (associated with conjugates of E_i^3) in the paths that are associated with: $\psi_2^2(b_1^2), \ldots, \psi_2^2(b_{t_2}^2)$.

(4) Suppose that the path that is associated with g in the tree that is associated with Λ_2, contains non-degenerate subsegments that are associated with conjugates of a dominant edge group E^1 of weight 1 in Λ_2. Suppose that in the path that is associated with a generator of E^1 in the real tree that is associated with Λ_3, there exist some subpath in an IET component that is associated with a QH vertex group Q_i^3 in Λ_3.

Then with the path that is associated with the element $\varphi_{E^1}^{\beta^1_{E^1}}(g)$ in the limit tree that is associated with Λ_2, the image of the subpaths that are stabilized by conjugates of the dominant edge group E^1, contain subpaths that are in the IET components that are associated with conjugates of Q_i^3, and the total lengths of these subpaths are at four times the total length of the subpaths that are associated with all the elements $\psi_2^2(b_1^1), \ldots, \psi_2^2(b_{t_1}^1)$ in the IET components that are associated with conjugates of Q_i^3. Furthermore, this lower bound on the ratios between the total lengths of paths can be demonstrated by finitely many translations of subpaths in these subpaths.

Suppose that in the path that is associated a generator of E^1 in the real tree that is associated with Λ_3, there is a non-degenerate subsegment in the simplicial part of the real tree that is associated with a conjugate of an edge group E_i^3 in Λ_3.

Then the number of such subsegments (that are associated with conjugates of E_i^3) in the path that is associated with $\varphi_{E^1}^{\beta^1_{E^1}}(g)$ is at least four times the total appearances of such subsegments (associated with conjugates of E_i^3) in the paths that are associated with: $\psi_2^2(b_1^1), \ldots, \psi_2^2(b_{t_1}^1)$.

This concludes the construction of the homomorphisms that are associated with the third level, that are set to be: $\{h_n^2 \circ \psi_2\}$. Note that each homomorphism h_n^2 can be presented as: $h_n^2 = h_n^1 \circ \nu_n^2$, where ν_n^2 and ψ_2 are automorphisms from the dominant modular group $MXMod(\Lambda_2)$.

We assign weight 1 or 2 with every QH vertex group and every edge group in Λ_3 that are also a QH vertex group or an edge group of a similar weight in Λ_2. We assign weight 1 with every QH vertex group and every edge group in Λ_3, for which there is a generator of a dominant QH vertex group or a dominant edge group of weight 1 in Λ_2, such that the path that is associated with that generator in the tree that is associated with Λ_3, has a subpath in an IET component that is associated with a conjugate of the QH vertex group in Λ_3, or a subpath in the simplicial part of that tree that is associated with the edge group in Λ_3.

We assign weight 2 with every QH vertex group and every edge group in Λ_3, which is not already of weight 1, and for which there is a generator of a dominant QH vertex group or a dominant edge group of weight 2 in Λ_2, such that the path that is associated with that generator in the tree that is associated with Λ_3, has a subpath in an IET component that is associated with a conjugate of the QH vertex group in Λ_3, or a subpath in the simplicial part of that tree that is associated with the edge group in Λ_3. We assign weight 3 with every other QH vertex group or edge group in Λ_3, i.e., with the other QH vertex groups or edge groups that were not assigned weight 1 or 2.

We continue iteratively. The abelian decompositions at step i, is constructed from shortened homomorphisms $\{h_n^{i-1}\}$, that are further twisted (or precomposed) with a fixed automorphism ψ_{i-1} that is contained in the dominant modular group $MXMod(\Lambda_{i-1})$. The QH vertex groups and edge groups in Λ_i have weights in the set $1, \ldots, i$.

With the abelian decomposition Λ_i we associate its modular group, $Mod(\Lambda_2)$, and its dominant modular group, $MXMod(\Lambda_2)$. As we did in first steps, we first shorten the homomorphisms, $\{h_n^{i-1} \circ \psi_{i-1}\}$, using the ambient modular group, $Mod(\Lambda_i)$. We denote the obtained sequence of homomorphisms, $\{h_n^i\}$. If there exists a subsequence of the homomorphisms, $\{h_n^i\}$, that is either separable, or that converges into a non-faithful action of L on some real tree, the conclusions of theorem 7.2 follow for this subsequence, and the procedure terminates.

If $\{h_n^i\}$ has no separable subsequence or a subsequence that converges into a non-faithful action of L on a real tree, we use only the dominant modular group, $MXMod(\Lambda_i)$. The shortening of the sequence of homomorphisms, $\{h_n^{i-1} \circ \psi_{i-1}\}$, is required to preserve the positivity of the fixed set of generators, $s_1, \ldots, s_r$, and all the fixed sets of generators of QH vertex groups and abelian edge groups in all the abelian decompositions, $\Lambda_1, \ldots, \Lambda_i$. It further requires to preserve the orientation of the oriented (positive or negative) subpaths in the paths that are associated with all the elements in a ball of radius i in the Cayley graph of L w.r.t. $s_1, \ldots, s_r$. The shortening procedure is also suppose to preserve the finite equivariance that was used to (finitely) demonstrate the indecomposability of paths that are associated with the fixed generators of QH vertex groups in $\Lambda_1, \ldots, \Lambda_i$, and the finite equivariance that was used to demonstrate certain inequalities between subpaths in the trees that are associated with $\Lambda_1, \ldots, \Lambda_i$.

Before shortening the (fixed) set of generators of each of the dominant QH vertex groups in Λ_i, we use the procedure that was applied in the proofs of propositions 2.7 and 2.8, to associate an automorphism φ_Q with each such dominant QH vertex group in Λ_i. We further use the indecomposability of IET components, and associate finitely many translates of the fixed generators of each of the dominant QH vertex groups, that demonstrate the indecomposability, and provide lower and upper bounds on the ratios between lengths of the fixed set of generators that will be kept along the next steps of the procedure.

After shortening the homomorphisms, $\{h_n^{i-1} \circ \psi_{i-1}\}$, using the dominant modular group $MXMod(\Lambda_i)$, we pass to a convergent subsequence that we denote $\{h_n^i\}$ (note that at each step the shortening is required to preserve the orientation of a larger number of subpaths that are associated a larger number of elements). We pass to a subsequence that we still denote, $\{h_n^i\}$ that converges into an action of L on some real tree with an associated abelian decomposition Δ_{i+1}. If the action of L is not faithful, or if the sequence $\{h_n^i\}$ contains a separable subsequence, the conclusions of theorem 7.2 follow and the procedure terminates. Hence, we may assume that the action of L is faithful, and we further refine Δ_{i+1}, precisely as we did in the first two steps, and obtain the abelian decomposition Λ_{i+1}.

As in the first two steps, we further construct a fixed automorphism, ψ_i, that is used in modifying (by precomposition) the homomorphisms, $\{h_n^i\}$. This precomposition is needed to guarantee the validity of certain inequalities between the lengths, or the ratios of lengths, of a finite set of elements. This finite set include:

(1) the fixed set of generators of dominant QH vertex groups and generators of dominant edge groups in $\Lambda_1, \dots, \Lambda_i$.
(2) the elements in a ball of radius i in the Cayley graph of L w.r.t. the generators: $s_1, \dots, s_r$.
(3) the fixed set of generators of some of the QH vertex groups and some of the edge groups in Λ_{i+1}.

We start with dominant QH vertex groups and dominant edge groups in Λ_i that are of highest weight (in Λ_i). We look at all the QH vertex groups and all the edge groups in Λ_{i+1} for which at least one of the fixed set of generators of the dominant QH vertex groups of highest weight in Λ_i, or of dominant edge groups of highest weight in Λ_i, is not elliptic with respect to them.

Each element in the ball of radius i in the Cayley graph of L (w.r.t. $s_1, \dots, s_r$), and each of the fixed generators of a QH vertex group or an edge group in $\Lambda_1, \dots, \Lambda_i$, can be written in a normal form with respect to Λ_i. Let w_1, $1 \leq w_1 \leq i$, be the highest weight in Λ_i, and let $b_1^{w_1}, \dots, b_{t_{w_1}}^{w_1} \in L$ be the collection of elements in the (fixed) normal forms of these elements, that are contained in non-QH vertex groups, or in an edge group that is adjacent only to QH vertex groups in Λ_i. We require that a fixed automorphism, $\psi_m^{w_1} \in MXMod(\Lambda_i)$ will satisfy properties (1)-(4) that are listed in the first and second steps, w.r.t. the QH vertex groups and the edge groups in Λ_{i+1} for which at least one of the fixed set of generators of the dominant QH vertex groups of highest weight in Λ_i, or of dominant edge groups of highest weight in Λ_i, is not elliptic with respect to them, and w.r.t. the elements, $b_1^{w_1}, \dots, b_{t_{w_1}}^{w_1} \in L$.

Once the automorphism $\psi_i^{w_1}$ is fixed, we treat dominant QH vertex groups and dominant edge groups in Λ_i that are of the next highest weight, w_2, $1 \leq w_2 < w_1$.

Let $Col^{w_1}\Lambda_i$ be the abelian decomposition that is obtained by collapsing all the QH vertex groups of weight w_1 in Λ_i and the edges that are connected to them, and all the edge groups of weight w_1 in Λ_i. $Col^{w_1}\Lambda_i$ contains only QH vertex groups and edge groups of lower weight.

Each element in the ball of radius i in the Cayley graph of L, and each of the fixed generators of a QH vertex group or an edge group in $\Lambda_1, \dots, \Lambda_i$, can be written in a normal form with respect to $Col^{w_1}\Lambda_i$. Let $b_1^{w_2}, \dots, b_{t_{w_2}}^{w_2} \in L$ be the collection of elements in the (fixed) normal forms of these elements, that are contained in non-QH vertex groups, or in an edge group that is adjacent only to QH vertex groups in $Col^{w_1}\Lambda_i$. We require that a fixed automorphism, $\psi_m^{w_2} \in MXMod(\Lambda_i)$ will satisfy properties (1)-(4) that are listed in the first and second steps, w.r.t. the QH vertex groups and the edge groups in Λ_{i+1} for which at least one of the fixed set of generators of the dominant QH vertex groups of weight w_2 in Λ_i, or of dominant edge groups of weight w_2 in Λ_i, is not elliptic with respect to them, and w.r.t. the elements, $\psi_i^{w_1}(b_1^{w_1}), \dots, \psi_i^{w_1}(b_{t_{w_1}}^{w_1}) \in L$.

We continue iteratively to lower weight dominant QH vertex groups and dominant edge groups in Λ_i, until a a fixed automorphism $\psi_i \in MXMod(\Lambda_i)$ is constructed. This automorphism enable us to replace the sequence of homomorphisms, $\{h_n^i\}$ by precomposing them with ψ_i: $\{h_n^i \circ \psi_n^i\}$. Hence, the new sequence of homomorphisms is obtained from the sequence in the previous level as $\{h_n^{i-1} \circ \nu_n^i \circ \psi_i$, where ν_n^i and ψ_i are automorphisms from the dominant modular group $MXMod(\Lambda_i)$.

We assign the weight in Λ_i to every QH vertex group and every edge group in Λ_{i+1} that are also a QH vertex group or an edge group in Λ_i. We assign weight w with every QH vertex group and every edge group in Λ_{i+1}, for which there is a generator of a dominant QH vertex group or a dominant edge group of weight w in Λ_i, such that the path that is associated with that generator in the tree that is associated with Λ_{i+1}, has a subpath in an IET component that is associated with a conjugate of the QH vertex group in Λ_{i+1}, or a subpath in the simplicial part of that tree that is associated with the edge group in Λ_{i+1}, and there is no such generator of a dominant QH vertex group or a dominant edge group in Λ_i that has weight smaller than w. We assign weight $i+1$ with every other QH vertex group or edge group in Λ_{i+1}, i.e., with those QH vertex groups and edge groups for which no weight (bounded by i) was assigned to them.

We continue iteratively. If in all steps the obtained actions are faithful and geometric, and the sequences of homomorphisms contain no separable subsequence, we get an infinite sequence of abelian decompositions, $\Lambda_1, \Lambda_2, \ldots$. Given the infinite sequence of abelian decompositions, we associate with it its stable dominant abelian decomposition (see definition 7.5), Θ_{i_0}.

Given the sequence of abelian decompositions, $\Lambda_1, \ldots$, and its stable abelian decomposition, Θ_{i_0}, our goal is to show that there exists an index $i_1 \geq i_0$, such that all the abelian decompositions, $\Lambda_{i_1}, \ldots$, are dominated by Θ_{i_0}. i.e., all the modular groups, $Mod(\Lambda_i)$, $i \geq i_1$, are contained in the modular group $Mod(\Theta_{i_0})$, where $Mod(\Theta_{i_0}$ is generated by the modular groups of the QH vertex groups in Θ_{i_0}, and the Dehn twists along edges (with non-trivial stabilizers) in Θ_{i_0}. This will enable us to replace the infinite sequence of abelian decompositions, $\Lambda_1, \ldots$, with the finite sequence, $\Lambda_1, \ldots, \Lambda_{i_1-1}, \Theta_{i_0}$.

To show the existence of such an index i_1, we first need to modify the sequences of homomorphisms, $\{h_n^i\}$, that will enable us to appropriately collapse some of the QH vertex groups and some of the edge groups in the abelian decompositions, $\Lambda_{i_0}, \ldots$.

By the accessibility of f.p. groups [Be-Fe], or by acylindrical accessibility ([Se],[De],[We]), the number of edge (and vertex groups) in each of the abelian decompositions Λ_i is universally bounded. With each QH vertex group and each edge group in the abelian decompositions, Λ_i, we have associated a weight. A QH vertex group or an edge group in Λ_i that are not dominant pass to the next level, as the same QH vertex group or an edge group with the same weight. Hence, all the abelian decompositions, Λ_i, must have either a QH vertex group or an edge group with weight 1.

Definition 7.6. *Let w be a positive integer. We say that w is a stable weight of the sequence of abelian decompositions: $\Lambda_1, \ldots$, if the sequence contains a subsequence that contains either a QH vertex group or an edge group of weight w. This is equivalent to the existence of a QH vertex group or an edge group of weight w in each of the abelian decompositions: $\Lambda_w, \ldots$. 1 is always a stable weight, and the accessibility of f.p. groups implies that there are only finitely many stable weights.*

Given Λ_i, we define the unstable modular group, $USMOd(\Lambda_i)$, to be the modular group that is generated by modular groups of QH vertex groups of unstable weight in Λ_i, and Dehn twists along edge groups with unstable weight in Λ_i.

Given the stable weights we modify the sequences of homomorphisms, $\{h_n^i\}$.

For each index i, we precompose the homomorphisms $\{h_n^i\}$ with automorphisms $\tau_n^i \in USMod(\Lambda_i)$, such that $h_n^i \circ \tau_n^i$ preserve all the positivity and the (finite) equivariance properties that h_n^i was supposed to preserve, and is the shortest among all the homomorphisms of the form: $h_n^i \circ \tau$ that preserve the positivity and the finite equivariance requirements, and in which: $\tau \in USMod(\Lambda_i)$. We denote the homomorphisms $h_n^i \circ \tau_n$, sh_n^i.

If for some index i, the sequence $\{sh_n^i\}$ contains a separable subsequence, or if it contains a subsequence that converges into a proper quotient of the limit group L, the conclusions of theorem 7.2 follow. Hence, in the sequel we may assume that $\{sh_n^i\}$ contain no such subsequences.

Lemma 7.7. *The stable dominant abelian decomposition Θ_{i_0} (possibly) contains several QH vertex groups and (possibly) several edge groups with non-trivial stabilizers.*

(i) *for every QH vertex group Q in Θ_{i_0} there exists at least one stable weight w, so that for all $i \geq max(w, i_0)$, Λ_i contains a QH vertex group or an edge group with weight w that are contained in Q.*

(ii) *for any edge E in Θ_{i_0} that has a non-trivial stabilizer, there exists a stable weight w, so that (a conjugate of) E appears as an edge group in Λ_i for all $i \geq max(w, i_0)$.*

Proof: Let Q be a QH vertex group in Θ_{i_0}. Q inherits an abelian decomposition from each of the abelian decompositions, Λ_i, $i \geq i_0$, an abelian decomposition in which the boundary elements in Q are elliptic (elliptic elements in the stable dominant abelian decomposition Θ_{i_0} are elliptic in all the abelian decompositions, Λ_i, $i \geq i_0$).

Non-peripheral elements in Q that do not have roots in Q do not have roots in the ambient limit group L. Hence, every QH vertex group that appears in one of the abelian decompositions, Λ_i, can either be conjugated into a subsurface of Q, or every conjugate of that QH vertex group intersects Q trivially or only in conjugates of peripheral elements of Q. Similarly every edge group in Λ_i, $i \geq i_0$, is either conjugate into a non-peripheral s.c.c. in Q, or it can not be conjugated into Q, or it can be conjugated only to peripheral subgroups in Q.

Since all the non-peripheral elements in Q are not elliptic in the stable dominant abelian decomposition, Θ_{i_0}, there must exists some index $j \geq i_0$, so that Λ_j contains a QH vertex group that can be conjugated into a subsurface of Q, or Λ_j contains an edge group that can be conjugated into a non-peripheral s.c.c. in Q. Such a QH vertex group or an edge group, has weight w that is bounded by j. Furthermore, such a QH vertex group or an edge group are not elliptic in an abelian decomposition Λ_i, $i > j$, only with respect to QH vertex groups or edge groups that can be conjugated into subsurfaces or non-peripheral s.c.c. in Q.

Therefore, by the structure of the procedure that constructs the abelian decompositions, $\{\Lambda_i\}$, if such a QH vertex group or an edge group of weight w is not elliptic in Λ_i, $i > j$, the next abelian decomposition, Λ_{i+1}, must contain QH vertex groups QH vertex groups or edge groups that can be conjugated into subsurfaces or non-peripheral s.c.c. in Q, and have weight that is bounded above by w. This implies part (i).

To prove (ii), note that an edge group E in Θ_{i_0} must be elliptic in all the abelian decompositions Λ_i, $i \geq i_0$. Since there is an edge e with stabilizer E in Θ_{i_0}, and E is elliptic in Λ_i for all $i \geq i_0$, there must exist an index $i_1 \geq i_0$, for which Λ_{i_1} contains the (splitting that corresponds to the) edge e. The edge group E is elliptic in all the abelian decompositions Λ_i, $i \geq i_1$. Hence, it is not a dominant edge group in any of these splittings, so the edge e is inherited by all the abelian decompositions, λ_i, for $i \geq i_1$.

□

Lemma 7.8. *Let w be a stable weight. Suppose that for every large enough i, Λ_i contains a QH vertex group or an edge group of weight w that are not conjugate to a QH vertex group or an edge group in Θ_{i_0}, and w is the minimal such stable weight. Then for every large i, there must exist a dominant QH vertex group of weight w in Λ_i, or an edge group of weight w in Λ_i, such that the length of a generator of a dominant edge group or the length of a generator of a non-QH dominant vertex group in Λ_i is infinitesimal in comparison to the the length of the edge of weight w in the real tree from which Λ_i was obtained.*

Furthermore, if for every large i there exist a QH vertex group or an edge group of weight w in Λ_i that are properly contained in a QH vertex group Q in Θ_{i_0}, and w is the minimal such stable weight, then for every large i, there exists a dominant QH vertex group of weight w in λ_i, or an edge of weight w in Λ_i, for which the length of a generator of a dominant edge group or a dominant non-QH vertex group is infinitesimal with respect to its length.

Proof: If Λ_i, $i \geq w$, contains a QH vertex group or and edge group of stable weight w or an edge group of stable weight w, and these are not conjugate to an edge group or a QH vertex group in Θ_{i_0}, then for some $i_1 \geq w$, Λ_{i_1} contains s dominant such edge group or QH vertex group.

Since w is assumed to be the minimal stable weight with these properties, there exists some index $i_2 \geq i_1$, such that for every $i \geq i_2$, if Λ_i contains a dominant QH vertex group or a dominant edge group of weight w, that are not conjugate to a QH vertex group or an edge group in Θ_{i_0}, then at least one of the QH vertex groups in Λ_{i+1} is dominant, or the length of all the fixed generators of the dominant edge groups and the non-QH vertex groups in Λ_{i+1} is infinitesimal in comparison with the length of an edge with weight w that is not conjugate to an edge in Θ_{i_0}.

Given a QH vertex group in Θ_{i_0}, we apply the same argument to the QH vertex groups and edge groups that can be conjugated into that QH vertex group in Θ_{i_0}, and same consequence holds for the minimal weight stable weight w w.r.t. a fixed QH vertex group in Θ_{i_0}.

□

Lemmas 7.7 and 7.8 imply that the sequence of homomorphisms, $\{sh_n^i\}$, subconverges into an action on a real tree from which an abelian decomposition, $s\Lambda_i$, can be obtained (using the refinement procedure that was used in construction Λ_i), where $s\Lambda_i$ is obtained from Λ_i by collapsing the following:

(1) QH vertex groups and edge groups with unstable weight.
(2) non-dominant QH vertex groups of stable weight that are not conjugate to QH vertex groups in Θ_{i_0}.
(3) edge groups of stable weight that are not conjugate to edges in Θ_{i_0}, and for which the length of the corresponding edge in the real tree into which $\{h_n^i\}$ converges, is bounded by a constant times the length of a generator of a dominant edge group or a dominant non-QH vertex group.

For each index i, it is possible to choose a homomorphism, f_i, from the sequnce $\{h_n^i\}$, with the following properties:

(1) the set of abelian decompositions, Δ, with fundamental group L, and in which all the edges have trivial stabilizers, is clearly countable. Hence, we can define an (arbitrary but fixed) order on this set, and we denote the corresponding sequence of abelian decompositions $\{\Delta_m\}$. Note that every abelian decomposition with fundamental group L and trivial edge groups appears in this sequence.

For each index $i \geq i_0$, we choose the homomorphism f_i, to be a homomorphism from the sequence $\{sh_n^i\}$ that is not separable with respect to the abelian decompositions: $\Delta_1, \ldots, \Delta_i$ (see definition 7.3).

(2) The sequence of homomorphisms $\{sh_n^i\}$ subconverges into a faithful action of L on the limit tree sY_{i+1}. We require f_i to approximate the action on the limit tree sY_{i+1}, of all the elements in a ball of radius i in the Cayley graph of L (w.r.t. the given generating set $s_1, \ldots, s_r$), of all the fixed sets of generators of the QH vertex groups and edge groups in $\Lambda_1, \ldots, \Lambda_i$, and of all the (finitely many) elements that were chosen to demonstrate the mixing property of the dominant QH vertex groups and dominant edge groups in $\Lambda_1, \ldots, \Lambda_i$.

By construction, the sequence of homomorphisms, $\{f_i\}$, subconverges into a faithful action of the limit group L on a real tree Y_∞. Since the sequence $\{f_i\}$ contains no separable subsequence, the action of L on Y_∞ must be geometric, and contains no non-degenerate segments in its simplicial part that can not be divided into finitely many segments with non-trivial (cyclic) stabilizers. Since (S, L) is assumed to be Levitt-free, the action of L on Y_{infty} contains (possibly) only IET components and (possibly) a simplicial part.

Let Γ_∞ be the abelian decomposition that is associated with the action of L on Y_∞. Γ_{infty} has to be compatible with the stable dominant abelian decomposition Θ_{i_0}, i.e., every elliptic element in Θ_{i_0} must be elliptic in Γ_∞. Our goal is to show that Γ_∞ can be further refined, by restricting the homomorphisms, $\{f_i\}$, to non-QH vertex groups in Γ_∞, and passing to a further subsequence, to the stable dominant abelian decomposition, Θ_{i_0}.

We start with the following claim that is similar to lemma 6.5 in the freely indecomposable case.

Lemma 7.9. *Let Q be a QH vertex group in Θ_{i_0} that does not appear in any of the abelian decompositions Λ_i, for $i \geq i_0$.*

If there is a non-peripheral element in Q that fixes a point in Y_∞, then the entire QH vertex group Q fixes a point in Y_∞.

Proof: Suppose that a non-peripheral element $q \in Q$ fixes a point in Y. q is contained in some ball B_m in the Cayley graph of L w.r.t. $s_1, \ldots, s_r$. Θ_{i_0} is the stable dominant abelian decomposition of the sequence of abelian decompositions: $\Lambda_1, \ldots$. Hence, there must exist an abelian decomposition Λ_{i_1}, for some $i_1 > max(i_0, m)$, for which either:

(i) q is a non-peripheral element in some QH vertex group in Λ_{i_1}.

(ii) q is hyperbolic in the abelian decomposition Λ_{i_1}.

This implies that q is either non-peripheral element in QH vertex groups, or a hyperbolic element in all the abelian decompositions, Λ_i, for $i \geq i_1$. These abelian decompositions, Λ_i, $i > i_1$, may contain both stable QH vertex groups and stable edge groups. By the construction of the procedure that produces the abelian decompositions, $\{\Lambda_i\}$, for such a non-peripheral element q in a QH vertex group, or such an element q that is hyperbolic w.r.t. Λ_{i_1}, for which $q \in B_m$, $m \leq i_1$, there must exist an index $i_2 \geq i_1$, such that:

(i) q is hyperbolic w.r.t. all the abelian decompositions, Λ_i, for $i \geq i_2$.

(ii) q is hyperbolic w.r.t. all the abelian decompositions that are obtained by collapsing all the edges and all the QH vertex groups in Λ_i, $i \geq i_2$, except for a single QH vertex group or a single edge group that can be conjugated into Q. In particular, q is hyperbolic in the abelian decomposition $s\Lambda_i$ for $i \geq i_2$.

According to the procedure, if either (i) or (ii) hold for Λ_{i_2}, for some $i_2 \geq i_1$, then it remains true for all $i \geq i_2$. Now, as in the proof of lemma 6.5, the traces and the lengths of q in its actions on the trees that are associated with the abelian decompositions $s\Lambda_{i_2+1}, \ldots$, are bounded below by either a (global) positive constant times the lengths of the fixed set of generators of s QH vertex groups in $\Lambda_{i'}$ and by lengths of the generators of edge groups in $\Lambda_{i'}$, $i_2 \leq i' < i$ when acting on the tree that is associated with $s\Lambda_i$.

This implies that if q is elliptic in the action of L on the limit tree Y_∞, then all the QH vertex groups and all the edge groups that can be conjugated into Q in all the abelian decompositions, Λ_i, $i \geq i_2$, must be elliptic as well. Hence, by the structure of the stable dominant abelian decomposition, Θ_{i_0}, Q must be elliptic as well.

□

Lemma 7.9 guarantees that in the action of L on Y_∞, a QH vertex group in the stable dominant abelian decomposition, Θ_{i_0}, is either elliptic or is associated with an IET component. The next proposition proves that the abelian decomposition Γ_∞ that is associated with the action of L on the limit tree Y_∞, is dominated by Θ_{i_0}. i.e., every QH vertex group in Γ_∞ is an edge group in Θ_{i_0}, and every edge (with non-trivial stabilizer) in Γ_∞ is conjugate to an edge in Θ_{i_0}, which by part (ii) of lemma 7.7 implies that such an edge appears in all the abelian decompositions Λ_i and $s\Lambda_i$ for large enough i.

Lemma 7.10. *Let Q be a QH vertex group, and let E be an edge group in Γ_∞, the abelian decomposition that is associated with the action of L on Y_∞. Then Q is conjugate to a QH vertex group in Θ_{i_0}, and E is conjugate to an edge group in Θ_{i_0}.*

Proof: Lemma 7.7 proves that for every large index i, every QH vertex group Q in Θ_{i_0}, and every edge group E in Θ_{i_0}, there exists a QH vertex group with stable weight in Λ_i that is conjugate to a subsurface in Q or an edge group with stable weight in Λ_i that is conjugate to either a s.c.c. in Q or to the edge group E.

By lemma 7.9 a QH vertex group in Θ_{i_0} is either elliptic in Γ_∞, or it is conjugate to a QH vertex group in Γ_∞. An edge e in Θ_∞ may appear as an edge in Γ_∞, and all the edge groups in Θ_{i_0} are elliptic in Γ_∞.

Suppose that for every large index i all the the QH vertex groups in Λ_i, and all the edge groups in Λ_i, can be conjugated to subsurfaces of QH vertex groups in Θ_{i_0}, or to s.c.c. in QH vertex group in Θ_{i_0}, or to edge groups (with non-trivial stabilizers) in Θ_{i_0}.

By the procedure for the construction of the abelian decompositions, $\{\Lambda_i\}$, if all the QH vertex groups in Θ_{i_0} fix points in Y_∞, then Γ_∞ contains only some of the edge groups with non-trivial stabilizers in Θ_{i_0}. These appear as edge groups with stable weight in all the abelian decompositions, Λ_i, for large index i. Hence, we may assume that at least one QH vertex group in Θ_{i_0} appears as a QH vertex group in Γ_∞.

Let $Q_1, \dots, Q_v$ be the QH vertex groups in Θ_{i_0} that appear as QH vertex groups in Γ_∞. Since the action of L on Y_∞ is geometric, and for large i, Λ_i contains only QH vertex groups and edge groups that can be conjugated into QH vertex groups and edge groups in Θ_{i_0}, for every $u \in L$, the path from the base point in Y_∞ to the image of the base point under the action of u, is composed from (possibly) finitely many subpaths that are contained in the IET components that are associated with $Q_1, \dots, Q_v$, and (possibly) finitely many edges with non-trivial stabilizers, that are in the simplicial part of Y_∞. This implies that Γ_∞ contains only conjugates the QH vertex groups, $Q_1, \dots, Q_v$, that are also QH vertex groups in Θ_{i_0}, and finitely many edges, that are all edges in Θ_{i_0}, and appear as edges in Λ_i for large enough i.

Suppose that not all the edge groups and the QH vertex groups in Λ_i, for large i, can be conjugated into edge groups and QH vertex groups in Θ_{i_0}. In that case, for large i, there exist QH vertex groups with stable weight, or edges with non-trivial edge groups with stable weight in Λ_i, that can not be conjugated into QH vertex groups nor into edge groups in Θ_{i_0}.

Since the pair (S, L) is Levitt free, the action of L on Y_∞ contains no Levitt components. Since the limit action is constructed from a sequence of (gradually) non-separable homomorphisms, the limit action of L on Y_∞ must be geometric. Therefore, Y_∞ contains only IET components and a simplicial part, with which there are associated QH vertex groups and edges with non-trivial stabilizers in Γ_∞.

The procedure for the construction of the abelian decompositions, $\{\Lambda_i\}$, forces finite equivariance on generators of dominant QH vertex groups and edge groups, that guarantees that an edge with non-trivial stabilizer exists in Γ_∞ if and only if conjugates of that edge exist in all the abelian decompositions, Λ_i, for large i. In particular, such an edge must be conjugate to an edge in the stable dominant abelian decomposition, Θ_{i_0}.

The finite equivariance that is forced on generators of dominant QH vertex groups and dominant edge groups, also implies that for large enough i, all the QH vertex groups in Λ_i, and all the edge groups in Λ_i, are either elliptic in Γ_∞, or they can be conjugated into QH vertex groups or s.c.c. in QH vertex groups in Γ_∞. This clearly implies that the boundaries of all the QH vertex groups in Γ_∞ are elliptic in all the abelian decompositions, Λ_i, for i large enough. Hence, all the QH vertex groups in Γ_∞ are in fact conjugate to QH vertex groups in Θ_{i_0}.

□

By lemma 7.10 all the QH vertex groups and all the edge groups in Γ_∞, are (conjugates of) QH vertex groups and edge groups in the stable dominant abelian decomposition Θ_{i_0}. Suppose that Θ_{i_0} contains QH vertex groups or edge groups that don't have conjugates in Γ_∞.

In that case we restrict the sequence of homomorphisms, $\{f_i\}$, to the (elliptic) vertex groups in Y_∞. Since Θ_{i_0} contains QH vertex groups that are not conjugate to QH vertex groups in Γ_∞, for all large i, the abelian decompositions Λ_i contain QH vertex groups or edge groups with stable weight that are elliptic in Γ_∞. Hence, from the restrictions of the homomorphisms $\{f_i\}$ to the point stabilizers in Y_∞ it is possible to associate a non-trivial abelian decomposition with at least one of the point stabilizers.

Therefore, we pass to a convergent subsequence of the sequnce $\{f_i\}$, and associate with the non-QH vertex groups in Γ_∞ abelian decompositions, that at least one of them is non-trivial. Since the QH vertex groups and the edge groups in Γ_∞ are conjugates of QH vertex groups and edge groups in the stable dominant abelian decomposition, Θ_{i_0}, all the edge groups in Γ_∞ are elliptic in the abelian decompositions of the various elliptic vertex groups. Hence, the abelian decompositions of the various vertex groups further refine the abelian decomposition Γ_∞.

Let Γ^1_∞ be the obtained refinement of Γ_∞. By the same argument that was used in proving lemma 7.10, Γ^1_∞ contains (new) QH vertex groups and edge groups that are not conjugate to QH vertex groups and edge groups in Γ_∞, but they are all conjugates of QH vertex groups and edge groups in Θ_{i_0}.

If Γ^1_∞ does not contain conjugates of all the QH vertex groups and all the edge groups in Θ_{i_0}, we repeat the refinement process, by restricting the sequence of homomorphisms, $\{f_i\}$, to the elliptic vertex groups in Γ^1_∞. After finitely many iterations we obtain an abelian decomposition Γ_f, such that the QH vertex groups and the edge groups in Γ_f and in Θ_{i_0} are conjugate. Furthermore, the (elliptic) non-QH vertex groups in Γ_f are conjugates of the non-QH vertex groups in Θ_{i_0}.

By the properties of the procedures for the constructions of Γ_f and the sequence of abelian decompositions, $\{\Lambda_i\}$, every element that is hyperbolic in Θ_{i_0} is hyperbolic in Γ_f. Hence, the collections of hyperbolic and elliptic elements in Γ_f and Θ_{i_0} are identical. To be able to replace a suffix of the sequence of the abelian decompositions, $\{\Lambda_i\}$, with the abelian decomposition Γ_f, we still need to prove the following proposition.

Proposition 7.11. *There exists an index $i_1 \geq i_0$, such that for every $i \geq i_1$, the modular group $Mod(\Lambda_i)$ is contained in the modular group $Mod(\Gamma_f)$.*

Proof: Note that the modular groups, $Mod(\Lambda_i)$ and $Mod(\Gamma_f)$, are generated by Dehn twists along edge groups and modular groups of QH vertex groups in the two abelian decompositions. We have already argued that for large i the edge groups in Λ_i can be conjugated into either edge groups in Γ_f, or into s.c.c. in QH vertex groups in Γ_f, and the QH vertex groups in Λ_i can be conjugated into QH vertex groups in Γ_f. Therefore, to prove the proposition we just need to analyze the branching points in IET components in the trees that are associated with the decompositions $\{\Lambda_i\}$ and Γ_f for large i.

Suppose first that there are no edges with trivial stabilizers that are connected to the QH vertex groups in Γ_f. This means that the IET components in the limit trees from which Γ_f was obtained, contain no branching points that are also branching points in other components, except for the orbits of points that are stabilized by peripheral elements in the QH vertex groups.

Recall that the abelian decomposition Γ_f was obtained using a (finite) successive refinement of an abelian decomposition Γ_∞. Let Y_∞ be the tree from which the abelian decomposition Γ_∞ was obtained. The action of L on the real tree Y_∞ is geometric, hence, to analyze the branching points in Y_∞ it is enough to look at the segments, $[y_\infty, s_j(y_\infty)]$, where y_∞ is the base point in Y_∞, and s_j, $1 \leq j \leq r$, are the fixed set of generators of the semigroups $S < L$. Since the actions of L on each of the trees Y_i, from which the abelian decompositions Λ_i were obtained, are all geometric, the same conclusion holds for these actions.

Since the action of L on Y_∞ is geometric, the path $[y_\infty, s_j(y_\infty)]$, is divided into (possibly) finitely many segments that are contained in IET components, and (possibly) finitely many segments with non-trivial stabilizers in the simplicial part of Γ_∞, where the last segments are associated with edges with non-trivial edge groups in Γ_∞. Once again, the actions of L on the real trees Y_i are geometric, hence, the same conclusion holds for the segments, $[y_i, s_j(y_i)]$, where y_i is the basepoint of the tree Y_i.

There exists an index $i_2 \geq i_0$, so that for every $i \geq i_2$ every QH vertex group in Λ_i is a subgroup of a conjugate of a QH vertex group in Γ_f, and every edge group in Λ_i is either conjugate to an edge group in Γ_f or it can be conjugated into a s.c.c. in a QH vertex group in Γ_f.

Suppose that there is a subsequence of indices i, such that QH vertex groups in Λ_i contain branching points that are not stabilized by one of their peripheral elements. We can pass to a subsequence of the indices, for which such branching points occur along the segments, $[y_i, s_j(y_i)]$, for some fixed j, $1 \leq j \leq r$. Suppose that we can pass to a further subsequence, so that there exist such branching points that are not stabilized by peripheral elements in IET components that are associated with conjugates of QH vertex groups in Λ_i, and these QH vertex groups can be conjugated into QH vertex groups in Γ_∞ (according to our assumptions there exist such branching points in IET components that are associated with QH vertex groups that can be conjugated into QH vertex groups in Γ_f. We assume that there is a subsequence in which the QH vertex groups in Λ_i that their associated IET components contain these branching points can be conjugated into QH vertex groups in Γ_∞).

The path $[y_\infty, s_j(y_\infty)]$ is divided into subpaths in IET components in Y_∞, and segments with non-trivial stabilizers in the simplicial part of Y_∞. The paths $[y_i, s_j(y_i)]$ can be divided into subpaths in IET components in Y_i and segments with non-trivial stabilizers in the simplicial part of Y_i. Furthermore, the sequence of subpaths in $[y_i, s_j(y_i)]$, can be divided into finitely many consecutive subsequences, such that the QH vertex groups and the edge groups in every subsequence can be jointly conjugated into the same QH vertex group or edge group in Γ_f.

By our assumption $[y_i, s_j(y_i)]$ contains a branching point in an IET component, that is not stabilized by a peripheral element in the QH vertex group that is associated with that IET component, and that QH vertex group can be conjugated into a QH vertex group in Γ_∞. By passing to a further subsequence we may assume that such a branching point exists in a subsequence of subpaths in $[y_i, s_j(y_i)]$ that is mapped into the same subpath in $[y_\infty, s_j(y_\infty)]$ that is contained in an IET component in Y_∞.

Let Q be the QH vertex group in Γ_∞ that is associated with that IET component. Let $u \in L$ be a peripheral element in Q that stabilizes the endpoint of the subpath in $[y_\infty, s_j(y_\infty)]$ that is contained in an IET component that is associated with Q. The point that is stabilized by u in that IET component is a branching point in the division of $[y_\infty, s_j(y_\infty)]$. By our assumption, for every index i from the subsequence, the subpath in $[y_i, s_j(y_i)]$ that is mapped into the subpath in the IET that is associated with Q in Y_∞, ends in a branching point that is not stabilized by u.

The element u is contained in a ball of radius i_3 in the Cayley graph of L w.r.t. the generators $s_1, \dots, s_r$. Hence, for all $i > max(i_3, i_2)$ the procedure that constructs the sequences of homomorphisms, $\{h_n^i\}$, takes into consideration the actions of the images of all the elements in the ball of radius i_3 in the Cayley graph of L. Since there exists an index $i > max(i_3, i_2)$ for which u does not stabilize the branching point at the end of the subpath in $[y_i, s_j(y_i)] \subset Y_i$, that is mapped into the subpath of $[y_\infty, s_j(y_\infty)] \subset Y_\infty$, the procedure that constructs the homomorphisms, $\{h_n^i\}$, guarantees that the endpoint of the subpath in Y_∞ into which the corresponding subpath in $[y_i, s_j(y_i)]$ is mapped can not be stabilized by the element u. This clearly contradicts the assumption that u does stabilize the endpoint of that subpath in Y_∞.

This argument proves that for large i, the subpaths of $[y_i, s_j(y_i)]$ that are mapped into IET components in Y_∞, start and end by either points that are stabilized by peripheral elements in conjugates of QH vertex groups in Λ_i, that can be conjugated into the associated QH vertex group in Γ_∞, or in non-cyclic vertex groups that are connected to edge groups in Λ_i, where these edge groups can be conjugated into s.c.c. in the associated QH vertex group in Γ_∞.

The abelian decomposition Γ_f was obtained from Γ_∞ by a finite refinement procedure. By following the steps of this refinement procedure, and repeating the argument that was used for subpaths that are mapped into subpaths in IET components in Y_∞, it follows that for large i the IET components in Λ_i contain no branching points that are not stabilized by peripheral elements. Hence proposition 7.11 follows in case the QH vertex groups in Γ_f are not connected to edges with trivial stabilizers.

Now, Suppose that QH vertex groups in Γ_f are connected to edges with trivial stabilizers. First, suppose that there exists a QH vertex group in Γ_∞ that is connected to edges with trivial stabilizers. As we already argued, the paths, $[y_\infty, s_j(y_\infty)]$, can be divided into finitely many subpaths that are either contained in IET components in Y_∞, or they are segments with non-trivial stabilizers in the simplicial part of Y_∞. The starting and ending points of these subpaths are either:

(1) associated with the beginning or the end of an edge in Γ_∞.
(2) stabilized by a peripheral element in a QH vertex group in Γ_∞.
(3) contained in an IET component but not stabilized by a peripheral element of that IET component.

The argument that we applied in case there were no branching points of type (3), implies that for large i, the branching points in Y_i that start or end subpaths of $[y_i, s_j(y_i)]$, $1 \leq j \leq r$, that are mapped into subpaths in Γ_∞ that start or end in branching points of type (1) or (2) in Γ_∞, have to be of type (1) or (2) in Y_i.

The germs of the (finitely many) branching points of type (3) in the segments, $[y_\infty, s_j(y_\infty)]$, belong to finitely many orbits (under the action of L on Y_∞). By the same argument that we used in order to analyze the branching points of types (1) and (2), if two starting or ending points of two sequences of consecutive subpaths in $[y_i, s_j(y_i)]$, $1 \leq j \leq r$, are mapped into starting or ending points of subpaths in QH components in $[y_\infty, s_j(y_\infty)]$, so that the germs of these branching points in Y_∞ are in the same orbit under the action of L, then the germs of the pair of branching (starting or ending) points in $[y_i, s_j(y_i)]$ are in the same orbit in Y_i under the action of L.

Using the finite refinement procedure that led from Γ_∞ into Γ_f, for large i, the same hold for starting and ending points subpaths of $[y_i, s_j(y_i)]$, $1 \leq j \leq r$, that are mapped into subpaths in IET components or segments with non-trivial stabilizers in one of the finitely many trees that were used to refine Γ_∞ and obtain Γ_f. In particular, the preimages of an orbit of branching points in Γ_∞ in Y_i, is in the same orbit under the action of L. Using the refinement process the same holds for preimages of orbits of branching points in Γ_f. This equivariance of the preimages of branching points in Γ_f, guarantees that for large i, the modular groups $Mod(\Lambda_i)$ are contained in the modular group $Mod(\Gamma_f)$.

□

In the freely indecomposable case, in the proof of theorem 6.1 proposition 6.4 enables us to replace an infinite sequence of abelian decompositions by a finite sequence that terminates in the stable dominant abelian decomposition, where the modular group of the stable dominant abelian decomposition was guaranteed to contain the modular groups of all the abelain decompositions of the (infinite) suffix of abelian decompositions that was removed from the original sequence.

Proposition 7.11 enables us to proceed in the same way in the freely decomposable, Levitt-free case. i.e., given the infinite sequence of abelian decompositions, $\{\Lambda_i\}$, we remove a suffix of it that is replaced by the abelian decomposition Γ_f. Hence, the sequence, $\{\Lambda_i\}$, is replaced by the finite sequence: $\Lambda_1, \dots, \Lambda_{i_1-1}, \Gamma_f$.

Note that unlike the freely indecomposable case, we replace the removed suffix by the abelian decomposition Γ_f, that is obtained from a limit of the sequence of homomorphisms, $\{f_i\}$, and not by the stable dominant abelian decomposition, Θ_{i_0}. In the freely decomposable case, these two abelian decompositions are equivalent, but in the freely decomposable, Levitt-free case, the modular groups that are associated with the two abelian decompositions may be different.

We continue to the next steps of the construction with the sequence of homomorphisms $\{f_i\}$ and the abelian decomposition that was obtained from their limit Γ_f. Recall, that the sequence $\{f^i\}$ was obtained from the sequences of pair homomorphisms that we denoted $\{h_n^i\}$. The sequences $\{h_n^i\}$ were constructed by shortening using elements from the dominant modular groups of the abelian decompositions, $\{\Lambda_i\}$. By proposition 7.11 the modular groups of the abelian decompositions, Λ_i, for $i \geq i_1$ are all contained in the modular group of the abelian decomposition Γ_f. Hence, the sequence of homomorphisms, $\{f_i\}$, is obtained from the sequence, $\{h_n^{i_1-1}\}$, by precompositions with automorphisms from the modular group of Γ_f.

We continue to the next step starting with the sequence of homomorphisms $\{f^i\}$, and the associated abelian decomposition Γ_f. Note that this abelian decomposition contains at least one QH vertex group. At this point we repeat the whole construction of a sequence of abelian decompositions. If the sequence terminates after a finite number of steps, the conclusion of theorem 7.2 follows. If it ends up with an infinite sequence of abelian decompositions, we replace it by a finite sequence of abelian decompositions using the procedure that we described and proposition 7.11.

We continue iteratively as we did in the proof of theorem 6.1. If this iterative procedure terminates after finitely many steps, the conclusion of theorem 7.2 follows. Otherwise we obtained an infinite sequence of abelian decompositions that do all contain QH vertex groups. In each of these abelian decompositions, either:

(1) there exists a dominant edge group that is not elliptic in an abelian decomposition that appears afterwards in the sequence.
(2) there exists a QH vertex group Q in the abelian decomposition, such that the abelian decomposition collapses to an abelian decomposition Γ_Q that con-

tains one QH vertex group, Q, and possible several non-QH vertex groups that are connected only to the vertex stabilized by Q by edges with trivial and cyclic edge groups. Γ_Q and an abelian decomposition that appears afterwards in the sequence do not have a common refinement (see the construction of the original sequence of abelian decompositions $\Lambda_1, \dots$).

Now, we apply the procedure that analyzed the original sequence of abelian decompositions, $\{\Lambda_i\}$, to the sequence of abelian decompositions that we constructed. By proposition 7.11 a suffix of the sequence can be replaced with an abelian decompositions that has the same elliptic subgroups as the stable dominant abelian decomposition of the sequence, and this abelian decomposition contains either

(i) more than one QH vertex group, or a QH vertex group with higher (topological) complexity (see the proof of theorem 6.1).

(ii) only a single QH vertex group, possibly of the same topological complexity (e.g. a once punctured torus), but this QH vertex has to be connected to the other vertex groups in the abelian decomposition with at least one edge with trivial stabilizer.

As in the proof of theorem 6.1, we repeat the whole construction starting with the (higher complexity) abelian decomposition that we obtained and the subsequence of homomorphisms that is associated with it. Either the construction terminates in finitely many steps, or a suffix of an infinite sequence of abelian decompositions can be replaced with an abelian decomposition of higher complexity. i.e., the total topological complexity of the QH vertex groups that appear in the abelian decompositions is bounded below by a higher lower bound, or the minimum number of edges with trivial stabilizers that are connected to the QH vertex groups is bigger. Repeating this construction itartively, by the accessibility of f.p. groups, or by acylindrical accessibility, we are left with a finite resolution that satisfies the conclusion of theorem 7.2 (see the proof of theorem 6.1).

□

So far we assumed that the pairs that we consider (S, L) are Levitt free, and L contains no non-cyclic abelian subgroups. To deal with pairs that are not Levitt free, we need the abelian decompositions that we construct to have special vertex groups that are free factors of the ambient limit group L, and we call *Levitt* vertex groups. If a limit group L admits a superstable action on a real tree, the action is geometric, and every non-degenerate segment in the simplicial part of the tree can be divided into finitely many segments with non-trivial abelian stabilizers, then using the Rips' machine, it is possible to associate with the action a graph of groups (see [Gu]). The vertex groups in this graph of groups are either QH or axial (abelian) or Levitt (thin) or point stabilizers. The stabilizers of an edge in this graph of groups are either trivial or abelian. In particular, the edges that are connected to Levitt vertex groups must have trivial stabilizers.

To generalize theorem 7.2 to pairs that are not necessarily Levitt free, we need to allow the abelian decompositions that are associated with pairs along a resolution to include *Levitt* vertex groups. This means that to the generators of the modular groups that are associated with such abelian decompositions we need to add automorphisms of the free factors (that are free groups) that are Levitt vertex groups. Furthermore, given a sequence of pair homomorphisms, to extract a subsequence that factor through a resolution, similar to the one that appears in theorem 7.2, we will need to generalize the JSJ machine that was used in the proof of theorem 7.2, to construct larger and larger Levitt vertex groups. i.e., the JSJ type decompositions that we consider need to include a new type of vertex groups (that was so far not needed over groups), the *Levitt* vertex groups and their modular groups.

Theorem 7.12. *Let (S, L) be a pair, and suppose that the limit group L contains no non-cyclic abelian subgroup. Let $\{h_n : (S, L) \to (FS_k, F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on a real tree Y.*

Then there exists a resolution:

$$(S_1, L_1) \to \ldots \to (S_f, L_f)$$

that satisfies the following properties:

(1) *$(S_1, L_1) = (S, L)$, and $\eta_i : (S_i, L_i) \to (S_{i+1}, L_{i+1})$ is an isomorphism for $i = 1, \ldots, f-2$ and $\eta_{f-1} : (S_{f-1}, L_{f-1}) \to (S_f, L_f)$ is a quotient map.*

(2) *with each of the pairs (S_i, L_i), $1 \leq i \leq f$, there is an associated abelian decomposition that we denote Λ_i. The abelian decompositions $\Lambda_1, \ldots, \Lambda_{f-1}$ contain edges with trivial or cyclic edge stabilizers, and QH, Levitt and rigid vertex groups.*

(3) *either η_{f-1} is a proper quotient map, or the abelian decomposition Λ_f contains separating edges with trivial edge groups. Each separating edge is oriented.*

(4) *there exists a subsequence of the homomorphisms $\{h_n\}$ that factors through the resolution. i.e., each homomorphism h_{n_r} from the subsequence, can be written in the form:*

$$h_{n_r} \ = \ \hat{h}_r \circ \varphi_r^{f-1} \circ \ldots \circ \varphi_r^1$$

where $\hat{h}_r : (S_f, L_f) \to (FS_k, F_k)$, each of the automorphisms $\varphi_r^i \in Mod(\Lambda_i)$, where $Mod(\Lambda_i)$ is generated by the modular groups that are associated with the QH vertex groups, modular groups that are associated with Levitt vertex groups (these are automorphisms of the Levitt free factors), and by Dehn twists along cyclic edge groups in Λ_i.

Each of the homomorphisms:

$$h_{n_r}^i \ = \ \hat{h}_r \circ \varphi_r^m \circ \ldots \circ \varphi_r^i$$

is a pair homomorphism $h_{n_r}^i : (S_i, L_i) \to (FS_k, F_k)$.

(5) *if (S_f, L_f) is not a proper quotient of (S, L), then the pair homomorphisms $\hat{h}_r$ are compatible with Λ_f. Let $R_1, \ldots, R_v$ be the connected components of Λ_f after deleting its (oriented) separating edges. The homomorphisms $\hat{h}_r$ are composed from homomorphisms of the fundamental groups of the connected components $R_1, \ldots, R_v$, together with assignments of values from FS_k to the oriented separating edges. The homomorphisms of the fundamental groups of the connected components $R_1, \ldots, R_v$ converge into a faithful action of these groups on real trees with associated abelian decompositions: $R_1, \ldots, R_v$.*

Proof: We modify the same procedure that was used in proving theorem 7.2 to include Levitt components. Recall that we started the proof of theorem 7.2 by iteratively constructing abelian decompositions, $\Lambda_1, \ldots$. As we pointed in the construction of these abelian decompositions in the Levitt free case, we may assume that all the faithful actions of L on real trees that we consider are geometric, and in the simplicial parts of these actions every non-degenerate segment can be divided into finitely many subsegments with non-trivial stabilizers.

Let Λ be the abelian decomposition that is associated with the faithful action of L on the real tree Y, that is obtained from a convergent subsequence of the given sequence of homomorphisms, $\{h_n\}$. Λ may contain rigid, QH and Levitt vertex groups, and the stabilizers of edge groups may be trivial or cyclic. A Levitt vertex group is adjacent only to edges with trivial stabilizers.

We start by refining the abelian decomposition Λ in a similar way to what we did in the proof of theorem 7.2. First we replace each non-QH, non-Levitt vertex group that is connected by periodic edge groups only to non-QH non-Levitt vertex groups, by restricting the homomorphisms $\{h_n\}$ to such a vertex group, and replace the vertex group with the obtained (non-trivial) abelian decomposition. Repeating this refinement procedure iteratively, we get an abelian decomposition that we denote Λ_1.

We fix finite generating sets of all the edge groups and all the non-QH, non-Levitt vertex groups in Λ_1. We divide the edge and non-QH, non Levitt vertex groups in Λ_1 into finitely many equivalence classes of their growth rates as we did the proof of theorems 6.1 and 7.2. After possibly passing to a subsequence, there exists a class that dominates all the other classes, that we call the *dominant* class, that includes (possibly) dominant edge groups and (possibly) dominant non-QH, non-Levitt vertex groups.

As in the freely indecomposable case, we denote by $Mod(\Lambda_1)$ the modular group that is associated with Λ_1. We set $MXMod(\Lambda_1)$ to be the *dominant* subgroup that is generated by Dehn twists along dominant edge groups, and modular groups of dominant QH and Levitt vertex groups, i.e., modular groups of those QH and Levitt vertex groups that the lengths of the images of their fixed sets of generators grow faster than the lengths of the images of the fixed generators of dominant edge and non-QH, non-Levitt vertex groups. Note that since a Levitt vertex group is a free factor of the ambient group L, the modular group of a Levitt vertex group is defined to be the natural extension of the automorphism group of that free factor to the ambient group L (acting by appropriate conjugations on all the other vertex and edge groups).

We start by using the full modular group $Mod(\Lambda_1)$. For each index n, we set the pair homomorphism: $h_n^1 : (S, L) \to (FS_k, F_k)$, $h_n^1 = h_n \circ \varphi_n$, where $\varphi_n \in Mod(\Lambda_1)$, to be a shortest pair homomorphism that is obtained from h_n by precomposing it with a modular automorphism from $Mod(\Lambda_1)$. If there exists a subsequence of the homomorphisms $\{h_n^1\}$ that converges into a proper quotient of the pair (S, L), or that converges into a non-geometric action of L on a real tree, or into an action that contains a segment in its simplicial part, and this segment has a trivial stabilizer, we set the limit of this subsequence to be (S_f, L_f), and the conclusion of theorem 7.12 follows with a resolution of length 1.

Therefore, we may assume that every convergent subsequence of the homomorphisms $\{h_n^1\}$ converges into a faithful geometric action of the limit group L on some real tree. In that case we use only elements from the dominant modular group, $MXMod(\Lambda_1)$.

We modify what we did in the proofs of theorems 6.1 and 7.2. First, we shorten the action of each of the dominant QH and Levitt vertex groups using the procedure that is used in the proofs of propositions 2.7, 2.8 and 2.18, 2.19. For each of the dominant QH and Levitt vertex groups, the procedures that are used in the proofs of these propositions give us an infinite collections of positive generators, $u_1^m, \ldots, u_g^m$, with similar presentations of the corresponding QH vertex groups, which means that the sequence of sets of generators belong to the same isomorphism class. Furthermore, with each of these sets of generators there associated words, w_j^m, $j = 1, \ldots, r$, of lengths that increase with m, such that a given (fixed) set of positive elements can be presented as: $y_j = w_j^m(u_1^m, \ldots, u_g^m)$. The words w_j^m are words in the generators u_i^m and their inverses. However, they can be presented as positive words in the generators u_j^m, and unique appearances of words t_ℓ, that are fixed words in the elements u_j^m and their inverses (i.e., the words do not depend on m), and these elements $t_\ell^m(u_1^m, \ldots, u_g^m)$ are positive for every m.

With each of the dominant QH and Levitt vertex groups we associate a system of generators $u_1^1, \ldots, u_g^1$). Since an IET action of a QH vertex group, and and a Levitt action are indecomposable in the sense of [Gu], finitely many (fixed) translates of each of the positive paths that are associated with the positive paths, $u_{i_1}^1$, cover the positive path that is associated with $u_{i_2}^1$, and cover the positive paths that are associated with the words $t_\ell(u_1^1, \ldots, u_g^1)$.

As in the proofs of theorems 6.1 and 7.2, we shorten the homomorphisms $\{h_n\}$ using the dominant modular group $MXMod(\Lambda_1)$. For each homomorphism h_n, we pick the shortest homomorphism after precomposing with an element from $MXMod(\Lambda_1)$ that keeps the positivity of the given set of the images of the generators $s_1, \ldots, s_r$, and the elements $u_1^1, \ldots, u_g^1$ and t_ℓ (for each dominant QH and Levitt vertex groups), and keeps their lengths to be at least the maximal length of a constant multiple c_1 of the maximal length of the image of a generator of a dominant edge or non-QH, non-Levitt vertex group (the generators are chosen from the fixed finite sets of generators of each of these vertex and edge groups). We further require that after the shortening, the image of each of the elements $u_1^1, \ldots, u_g^1$ and t_ℓ, will be covered by the finitely many translates that cover them in the limit action that is associated with Λ_1. We (still) denote the obtained (shortened) homomorphisms $\{h_n^1\}$.

By the shortening arguments that are proved in propositions 2.7, 2.8 and 2.18, 2.19, the lengths of the images, under the shortened homomorphisms $\{h_n^1\}$, of the elements $u_1^1, \ldots, u_g^1$ and t_ℓ, that are associated with the various dominant QH and Levitt vertex groups, are bounded by some constant c_2 (that is independent of n) times the maximal length of the images of the fixed generators of the dominant vertex and edge groups.

We pass to a subsequence of the homomorphisms $\{h_n^1\}$ that converges into an action of L on some real tree with an associated abelian decomposition Δ_2. If the action of L is not faithful, or non-geometric, or contains a non-degenerate segment in its simplicial part that can not be divided into finitely many segments with non-trivial stabilizers, the conclusions of theorem 7.12 follow. Hence, we may assume that the action of L is faithful, and the action is geometric. We further check if the sequence $\{h_n^1\}$ contains a separable subsequence (definition 7.3). If it does, we pass to this subsequence, and the conclusion of theorem 7.12 follows (cf. proposition 7.4).

We further refine Δ_2, precisely as we refined the first abelian decomposition Λ.

We denote the obtained (possibly) refined abelian decomposition $\hat{\Delta}_2$.

We further refine $\hat{\Delta}_2$ in a similar way to what we did in the Levitt free case. Let Q be a dominant QH vertex group in Λ_1, so that all its boundary elements are elliptic in $\hat{\Delta}_2$. Each shortened homomorphism, h_n^1, is obtained from h_n, by pre-composition with a (shortened) automorphism from the dominant modular group, $\varphi_n \in MXMod(\Lambda_1)$. φ_n is a composition of elements from automorphisms of free factors that are dominant Levitt components, modular groups of dominant QH vertex group in Λ, and Dehn twists along edges with dominant edge groups. We set $\tau_n \in MXMod(\Lambda_1)$ to be a composition of the same elements from automorphisms of Levitt components, and from the modular groups of dominant QH vertex groups in Λ_1, that are not the dominant QH vertex group Q, and the same Dehn twists along edges with dominant edge group as the shortened homomorphism φ_n. Let $\mu_n = h_n \circ \tau_n$.

The sequence of homomorphisms μ_n, converges into a faithful action of L on a real tree, with an associated abelian decomposition Γ_Q, that has a single QH vertex group, a conjugate of Q, and possibly several edges with non-trivial edge groups, that are edges in both Λ_1 and $\hat{\Delta}_2$.

Suppose that the abelian decompositions, $\hat{\Delta}_2$ and Γ_Q, have a common refinement, in which a conjugate of Q appears as a QH vertex group, and all the Levitt components and all the the edges and QH vertex groups in $\hat{\Delta}_2$ that do not correspond to s.c.c. or proper QH subgroups of Q also appear in the common refinement (the elliptic elements in the common refinement are precisely those elements that are elliptic in both Γ_Q and $\hat{\Delta}_2$). This, in particular, implies that all the QH vertex groups that can not be conjugated into Q, all the Levitt components, and all the edge groups in $\hat{\Delta}_2$ that can not be conjugated into non-peripheral elements in Q, are elliptic in Γ_Q. In case there exists such a common refinement, we replace $\hat{\Delta}_2$ with this common refinement. We repeat this possible refinement for all the QH vertex groups in Λ_1 that satisfy these conditions (the refinement procedure does not depend on the order of the QH vertex groups in Λ_1 that satisfy the refinement conditions). We denote the obtained abelian decomposition, $\tilde{\Delta}_2$.

We further refine $\tilde{\Delta}_2$ to possibly include Levitt vertex groups that appear in Λ_1. Let B be a dominant Levitt vertex group in Λ_1. We look for a refinement of $\tilde{\Delta}_2$ that will include B as a vertex group, and we do it in a similar way to what we did with dominant QH vertex group in Λ_1. For each index n, let $\varphi_n \in MXMod(\Lambda_1)$ be the automorphism that was used to shorten h_n, i.e., $h_n^1 = h_n \circ \varphi_n$. φ_n is a composition of elements from automorphisms of free factors that are dominant Levitt components, modular groups of dominant QH vertex group in Λ, and Dehn twists along edges with dominant edge groups. We set $\gamma_n \in MXMod(\Lambda_1)$ to be a composition of the same elements from automorphisms of dominant Levitt components, except for the dominant Levitt component B, and from the modular groups of dominant QH vertex groups in Λ_1, and the same Dehn twists along edges with dominant edge group as the shortened homomorphism φ_n. Let $\nu_n = h_n \circ \gamma_n$.

The sequence of homomorphisms ν_n, converges into a faithful action of L on a real tree, with an associated abelian decomposition Γ_B, that has a single Levitt component, a conjugate of B, no QH vertex groups, and possibly several edges with non-trivial edge groups, that are edges in both Λ_1 and $\hat{\Delta}_2$ (hence, also in $\tilde{\Delta}_2$).

Suppose that the abelian decompositions, $\tilde{\Delta}_2$ and Γ_B, have a common refinement, in which a conjugate of B appears as a Levitt component, and all the Levitt components and all the the edge groups and the QH vertex groups in $\tilde{\Delta}_2$ that can not be conjugated into B also appear in the common refinement (the elliptic elements in the common refinement are precisely those elements that are elliptic in both Γ_B and $\tilde{\Delta}_2$). This, in particular, implies that all the Levitt components, all the QH vertex groups and all the edge groups that can not be conjugated into B, are elliptic in Γ_B. In case there exists such a common refinement, we replace $\tilde{\Delta}_2$ with this common refinement. We repeat this possible refinement for all the Levitt vertex groups in Λ_1 that satisfy these conditions (the refinement procedure does not depend on the order of the Levitt vertex groups in Λ_1 that satisfy the refinement conditions). We denote the obtained abelian decomposition, Λ_2.

We continue as in the Levitt-free case. First we precompose the sequence of homomorphisms, $\{h^1_n\}$, with a fixed automorphism $\psi_1 \in MXMod(\Lambda_1)$, in the same way it was done in the Levitt free case. Then we associate weights with the edge groups, the QH vertex groups, and the Levitt vertex groups in Λ_2.

We proceed iteratively. First we shorten using the full modular group, $Mod(\Lambda_i)$. If the obtained sequence of homomorphisms has a separable subsequence, or a subsequence that convergence into a proper quotient of the limit group L, the conclusion of theorem 7.12 follows. Otherwise we use only the dominant modular group, $MXMod(\Lambda_i)$.

We shorten the homomorphisms, $\{h^{i-1}_n\}$, using the dominant modular group $MXMod(\Lambda_i)$. We denote the obtained shortened sequence, $\{h^i_n\}$. If the sequence $\{h^i_n\}$ contains a separable subsequence, or a sequence that converges into a proper quotient of L, the conclusion of theorem 7.12 follows. Otherwise we pass to a convergent subsequence, refine the obtained associated decomposition precsiely as we did in the first shortening step, and precompose the sequence, $\{h^i_n\}$, with an automorphism $psi_i \in MXMod(\Lambda_i)$. The automorphism ψ_i is constructed in the same way it was constructed in the Levitt-free case. We still denote the compositions, $\{h^i_n \circ \psi_i\}$, $\{h^i_n\}$. Finally, precisely as we did in the Levitt-free case, we assign weights with the edge groups with non-trivial stabilizers in Λ_i, and with QH and Levitt vertex groups in Λ_i, precsiely as we did it in the Levitt-free case.

If in all steps the obtained actions are faithful, the actions are all geometric, and the sequences of homomorphisms contain no separable subsequences, we get an infinite sequence of abelian decompositions, $\Lambda_1, \Lambda_2, \ldots$. As in the Levitt free case, our goal is to replace a suffix of this sequence with a single abelian decomposition that is obtained as a limit from a sequence of pair homomorphisms.

Given the sequence of abelian decompositions, $\Lambda_1, \ldots$, we associate with it its stably dominant decomposition Θ_{i_0}. Note that since the construction of the stably dominant decomposition (definition 7.5) does not consider and does not encode the precise factorization of the free factor which is a free group, one can not expect to replace a suffix of the sequence of abelian decompositions: $\Lambda_1, \ldots$, with Θ_{i_0} itself, but rather with a modification of it, that associates a further (free) decomposition with the free factor which is a free group (and in particular, specifies exactly the Levitt components up to conjugacy).

First, we associate with the sequence of abelian decompositions, $\Lambda_1, \ldots$, its finite set of stable weights (see definition 7.6). As in the Levitt free case, given the stable weights we modify the sequences of homomorphisms, $\{h^i_n\}$. Recall that the unstable modular group of an abelian decomposition, Λ_i, is generated by the modular groups of QH and Levitt vertex groups in Λ_i that have unstable weights, and Dehn twists along edges with non-trivial stabilizers in Λ_i that have unstable weights (see definition 7.6). For each index i, we precompose the homomorphisms $\{h_n^i\}$ with automorphisms $\tau_n^i \in USMod(\Lambda_i)$, such that $h_n^i \circ \tau_n^i$ preserve all the positivity and the (finite) equivariance properties that h_n^i was supposed to preserve, and is the shortest among all the homomorphisms of the form: $h_n^i \circ \tau$ that preserve the positivity and the finite equivariance requirements, and in which: $\tau \in USMod(\Lambda_i)$. We denote the homomorphisms $h_n^i \circ \tau_n$, sh_n^i.

If for some index i, the sequence $\{sh_n^i\}$ contains a separable subsequence, or if it contains a subsequence that converges into a proper quotient of the limit group L, the conclusions of theorem 7.2 follow. Hence, in the sequel we may assume that $\{sh_n^i\}$ contain no such subsequences.

Lemmas 7.7 and 7.8 (and their proof) remain valid in the presence of Levitt vertex groups. Hence, for all large i, and for every QH vertex group Q in the stable dominant abelian decomposition, Θ_{i_0}, and for every edge group E with non-trivial stabilizer in Θ_{i_0}, there exist QH vertex groups and edge groups with non-trivial stabilizers with stable weights in Λ_i, such that these QH vertex groups and edge groups with stable weights can be conjugated into Q or E.

As in the Levitt free case the sequence, $\{sh_n^i\}$, subconverges into an action on a real tree from which an abelian decomposition, $s\Lambda_i$, can be obtained (using the refinement procedure that was used in construction Λ_i), where $s\Lambda_i$ is obtained from Λ_i by collapsing the following:

(1) Levitt and QH vertex groups and edge groups with unstable weight.
(2) non-dominant Levitt and QH vertex groups of stable weight that are not conjugate to Levitt and QH vertex groups in Θ_{i_0}.
(3) edge groups of stable weight that are not conjugate to edges in Θ_{i_0}, and for which the length of the corresponding edge in the real tree into which $\{h_n^i\}$ converges, is bounded by a constant times the length of a generator of a dominant edge group or a dominant non-QH vertex group.

At this point we are ready to choose a sequence of homomorphisms, $\{f_i\}$, that will subconverge to an abelian decomposition that is going to replace a suffix of the sequence of decompositions, $\Lambda_1, \ldots$. For each index i, it is possible to choose a homomorphism, f_i, from the sequence $\{h_n^i\}$, that satisfies similar properties as in the Levitt free case.

For each index $i \geq i_0$, we choose the homomorphism f_i, to be a homomorphism from the sequence $\{sh_n^i\}$ that is not separable with respect to the abelian decompositions: $\Delta_1, \ldots, \Delta_i$, where the sequence, $\{\Delta_i\}$, enumerates all the possible graphs of groups with fundamental group L and trivial edge groups (see definition 7.3).

The sequence of homomorphisms $\{sh_n^i\}$ subconverges into a faithful action of L on the limit tree sY_{i+1}. We require f_i to approximate the action on the limit tree sY_{i+1}, of all the elements in a ball of radius i in the Cayley graph of L (w.r.t. the given generating set $s_1, \ldots, s_r$), of all the fixed sets of generators of the QH and Levitt vertex groups, and of edge groups with non-trivial stabilizers in $\Lambda_1, \ldots, \Lambda_i$, and of all the (finitely many) elements that were chosen to demonstrate the mixing property of the dominant QH and Levitt vertex groups and dominant edge groups in $\Lambda_1, \ldots, \Lambda_i$.

The sequence of homomorphisms, $\{f_i\}$, subconverges into a faithful action of the limit group L on a real tree Y_∞. Since the sequence $\{f_i\}$ contains no separable subsequence, the action of L on Y_∞ must be geometric, and contains no non-degenerate segments in its simplicial part that can not be divided into finitely many segments with non-trivial (cyclic) stabilizers.

Let Γ_∞ be the abelian decomposition that is associated with the action of L on Y_∞. Γ_∞ may have Levitt and QH vertex groups, non-QH non-Levitt vertex groups, and edges with trivial and with non-trivial stabilizers. Γ_{infty} has to be compatible with the stable dominant abelian decomposition Θ_{i_0}, i.e., every elliptic element in Θ_{i_0} must be elliptic in Γ_∞. Our goal is to show that Γ_∞ can be further refined, by restricting the homomorphisms, $\{f_i\}$, to non-QH vertex groups in Γ_∞, and passing to a further subsequence, to an abelian decomposition that have the same elliptic elements as the stable dominant abelian decomposition, Θ_{i_0}. This refinement of Γ_∞ is going to be used to replace a suffix of the sequence of abelian decompositions, $\Lambda_1, \ldots$.

Lemma 7.9 (and its proof) remain valid in the presence of Levitt components. Hence, a QH vertex group in the stable dominant abelian decomposition, Θ_{i_0}, is either elliptic in Γ_∞ or it is conjugate to a QH vertex group in Γ_∞. Lemma 7.10 is also valid in the presence of Levitt components, and its proof requires a slight modification.

Lemma 7.13. *Let Q be a QH vertex group, and let E be an edge group in Γ_∞, the abelian decomposition that is associated with the action of L on Y_∞. Then Q is conjugate to a QH vertex group in Θ_{i_0}, and E is conjugate to an edge group in Θ_{i_0}.*

Proof: By lemma 7.9 a QH vertex group in Θ_{i_0} is either elliptic in Γ_∞, or it is conjugate to a QH vertex group in Γ_∞. An edge e in Θ_∞ may appear as an edge in Γ_∞, and all the edge groups in Θ_{i_0} are elliptic in Γ_∞.

Let $Q_1, \ldots, Q_v$ be the QH vertex groups in Θ_{i_0} that are conjugate to QH vertex groups in Γ_∞. Suppose that for large i, Λ_i contains only QH vertex groups and edge groups that can be conjugated into QH vertex groups and edge groups in Θ_{i_0}. In that case for large i, Λ_i does not contain Levitt components, and Γ_∞ contains only conjugates of the QH vertex groups, $Q_1, \ldots, Q_v$, that are also QH vertex groups in Θ_{i_0}, and finitely many edges, that are all edges in Θ_{i_0}, and appear as edges in Λ_i for large enough i.

Suppose that not all the edge groups and the QH and Levitt vertex groups in Λ_i, for large i, can be conjugated into edge groups and QH vertex groups in Θ_{i_0} (Levitt vertex groups in Λ_i for large i can never be conjugated into QH vertex groups in Θ_{i_0}. In that case, for large i, there exist QH or Levitt vertex groups with stable weight, or edges with non-trivial edge groups with stable weight in Λ_i, that can not be conjugated into QH vertex groups nor into edge groups in Θ_{i_0}.

Since the limit action is constructed from a sequence of (gradually) non-separable homomorphisms, the limit action of L on Y_∞ must be geometric. Therefore, Y_∞ (possibly) contains only IET and Levitt components and a simplicial part, with which there are associated Levitt and QH vertex groups and edges with non-trivial stabilizers in Γ_∞.

The procedure for the construction of the abelian decompositions, $\{\Lambda_i\}$, forces finite equivariance on generators of dominant Levitt and QH vertex groups and edge groups, that guarantees that an edge with non-trivial stabilizer exists in Γ_∞ if and only if conjugates of that edge exist in all the abelian decompositions, Λ_i, for large i. In particular, such an edge must be conjugate to an edge in the stable dominant abelian decomposition, Θ_{i_0}.

The finite equivariance that is forced on generators of dominant QH and Levitt vertex groups and dominant edge groups, also implies that for large enough i, all the QH and Levitt vertex groups in Λ_i, and all the edge groups in Λ_i, are either elliptic in Γ_∞, or they can be conjugated into QH or Levitt vertex groups or s.c.c. in Γ_∞. This clearly implies that the boundaries of all the QH vertex groups in Γ_∞ are elliptic in all the abelian decompositions, Λ_i, for i large enough. Hence, all the QH vertex groups in Γ_∞ are in fact conjugate to QH vertex groups in Θ_{i_0}.

□

By lemma 7.13 all the QH vertex groups and all the edge groups in Γ_∞, are (conjugates of) QH vertex groups and edge groups in the stable dominant abelian decomposition Θ_{i_0}. Γ_∞ may also contain some Levitt vertex groups. As we have already indicated in the proof of lemma 7.13, for large i, every Levitt vertex group in Λ_i is either elliptic in Γ_∞ or it can be conjugated into a Levitt vertex group in Γ_∞. Every QH vertex group in Λ_i is either elliptic in Γ_∞ or it can be conjugated into either a Levitt vertex group or into a QH vertex group in Γ_∞. Every dominant edge group in Λ_i is either elliptic in Γ_∞, or it can be conjugated into a s.c.c. in a QH vertex group in Γ_∞ or into a Levitt component in Γ_∞. In the last case the stabilizer of the Levitt vertex group inherits a non-trivial abelian decomposition from Λ_i, an inherited abelian decomposition in which the edge with the dominant edge group in Λ_i appears as an edge.

Suppose that not all the elliptic elements in Γ_∞ are elliptic in Θ_{i_0} (i.e., suppose that non-QH non-Levitt vertex groups in Γ_∞ are not conjugates of non-QH vertex group in Θ_{i_0}).

In that case we restrict the sequence of homomorphisms, $\{f_i\}$, to the (elliptic) vertex groups in Y_∞. Since these elliptic subgroups are not elliptic in Θ_{i_0}, for large enough i, the abelian decompositions Λ_i contain QH or Levitt vertex groups or edge groups with stable weight that are elliptic in Γ_∞. Hence, from the restrictions of the homomorphisms $\{f_i\}$ to the point stabilizers in Y_∞ it is possible to associate a non-trivial abelian decomposition with at least one of the point stabilizers.

Therefore, we pass to a convergent subsequence of the sequnce $\{f_i\}$, and associate with the elliptic subgroups in Γ_∞ abelian decompositions, that at least one of them is non-trivial. Since the QH vertex groups and the edge groups in Γ_∞ are conjugates of QH vertex groups and edge groups in the stable dominant abelian decomposition, Θ_{i_0}, all the edge groups in Γ_∞ are elliptic in the abelian decompositions of the various elliptic vertex groups. Hence, the abelian decompositions of the various vertex groups further refine the abelian decomposition Γ_∞.

Let Γ^1_∞ be the obtained refinement of Γ_∞. Γ^1_∞ contains (possibly) new QH or Levitt vertex groups and edge groups that are not conjugate to QH and Levitt vertex groups and edge groups in Γ_∞. By the proof of lemma 7.13, all the new QH vertex groups in Γ^1_∞y are conjugates of QH vertex groups in Θ_{i_0}, and all the new edge groups in Γ^1_∞ are conjugates of edge groups in Θ_{i_0}.

If the elliptic vertex groups in Γ^1_∞ are not elliptic in Θ_{i-0} we repeat the refinement process, by restricting the sequence of homomorphisms, $\{f_i\}$, to the elliptic vertex groups in Γ^1_∞. After finitely many iterations we obtain an abelian decomposition Γ_f. The QH vertex groups and the edge groups in Γ_f are conjugates of the QH vertex groups and the edge groups in Θ_{i_0}. Γ_f may contain Levitt components. The elliptic (non-QH, non-Levitt) vertex groups in Γ_f are conjugates of the elliptic vertex groups in Θ_{i_0}.

As in the Levitt free case, to be able to replace a suffix of the sequence of the abelian decompositions, $\{\Lambda_i\}$, with the abelian decomposition Γ_f, we still need to prove an analogue of proposition 7.11 in the presence of Levitt components.

Proposition 7.14. *There exists an index $i_1 \geq i_0$, such that for every $i \geq i_1$, the modular group $Mod(\Lambda_i)$ is contained in the modular group $Mod(\Gamma_f)$.*

Proof: The modular groups, $Mod(\Lambda_i)$ and $Mod(\Gamma_f)$, are generated by automorphisms of Levitt factors, Dehn twists along edge groups and modular groups of QH vertex groups in the two abelian decompositions. We have already argued that for large i the edge groups in Λ_i can be conjugated into either edge groups in Γ_f, or into s.c.c. in QH vertex groups or into Levitt vertex groups in Γ_f, and in case it can be conjugated into a Levitt factor, the Levitt factor inherits a non-trivial decomposition along that edge group that it inherits from Λ_i. QH vertex groups in Λ_i can be conjugated into QH vertex groups or into Levitt factors in Γ_f, and Levitt factors in Λ_i can be conjugated into Levitt factors in Γ_f. Therefore, as in the Levitt free case (proposition 7.11), to prove the proposition we just need to analyze the branching points in IET and Levitt components in the trees that are associated with the decompositions $\{\Lambda_i\}$ and Γ_f.

The abelian decomposition Γ_f was obtained using a (finite) successive refinement of an abelian decomposition Γ_∞. Let Y_∞ be the tree from which the abelian decomposition Γ_∞ was obtained. The action of L on the real tree Y_∞ is geometric, hence, to analyze the branching points in Y_∞ it is enough to look at the segments, $[y_\infty, s_j(y_\infty)]$, where y_∞ is the base point in Y_∞, and s_j, $1 \leq j \leq r$, are the fixed set of generators of the semigroups $S < L$. Since the actions of L on each of the trees Y_i, from which the abelian decompositions Λ_i were obtained, are all geometric, the same conclusion holds for these actions.

Since the action of L on Y_∞ is geometric, the path $[y_\infty, s_j(y_\infty)]$, is divided into (possibly) finitely many segments that are contained in Levitt components, IET components, and (possibly) finitely many segments with non-trivial stabilizers in the simplicial part of Γ_∞, where the last segments are associated with edges with non-trivial edge groups in Γ_∞. The actions of L on the real trees Y_i are geometric, hence, the same conclusion holds for the segments, $[y_i, s_j(y_i)]$, where y_i is the basepoint of the tree Y_i.

There exists an index $i_2 \geq i_0$, so that for every $i \geq i_2$ every Levitt vertex vertex group in Λ_i is a subgroup of a conjugate of a Levitt vertex group in Γ_f, every QH vertex group in Λ_i is a subgroup of a conjugate of a Levitt or a QH vertex group in Γ_f, and every edge group in Λ_i is either conjugate to an edge group in Γ_f or it can be conjugated into a s.c.c. in a QH vertex group in Γ_f, or it can be conjugated into a Levitt component in Γ_f.

The path $[y_\infty, s_j(y_\infty)]$ is divided into subpaths in Levitt and IET components in Y_∞, and segments with non-trivial stabilizers in the simplicial part of Y_∞. The paths $[y_i, s_j(y_i)]$ can be divided into subpaths in Levitt and IET components in Y_i and segments with non-trivial stabilizers in the simplicial part of Y_i. Furthermore, the sequence of subpaths in $[y_i, s_j(y_i)]$, can be divided into finitely many consecutive subsequences, such that the Levitt and QH vertex groups and the edge groups in every subsequence can be jointly conjugated into the same Levitt or QH vertex group or edge group in Γ_f.

We first look at those sequences of consecutive subpaths in $[y_i, s_j(y_i)]$ that are not mapped to points in Y_∞. The starting and ending points of these consecutive subpaths are either:

(1) associated with the beginning or the end of an edge in Γ_∞.
(2) stabilized by a peripheral element in a QH vertex group in Γ_∞.
(3) contained in an IET component but not stabilized by a peripheral element of that IET component.
(4) contained in a Levitt component in Y_∞.

The argument that we applied in the Levitt free case, implies that for large i, the branching points in Y_i that start or end subpaths of $[y_i, s_j(y_i)]$, $1 \leq j \leq r$, that are mapped into subpaths in Γ_∞ that start or end in branching points of type (1) or (2) in Γ_∞, have to be of type (1) or (2) in Y_i.

The germs of the (finitely many) branching points of types (3) and (4) in the segments, $[y_\infty, s_j(y_\infty)]$, belong to finitely many orbits (under the action of L on Y_∞). By the same argument that we used in order to analyze the branching points of types (1) and (2) in the Levitt free case, if two starting or ending points of two sequences of consecutive subpaths in $[y_i, s_j(y_i)]$, $1 \leq j \leq r$, are mapped into starting or ending points of subpaths in QH or Levitt components in $[y_\infty, s_j(y_\infty)]$, so that the germs of these branching points in Y_∞ are in the same orbit under the action of L, then the germs of the pair of branching (starting or ending) points in $[y_i, s_j(y_i)]$ are in the same orbit in Y_i under the action of L.

Using the finite refinement procedure that led from Γ_∞ into Γ_f, for large i, the same hold for starting and ending points subpaths of $[y_i, s_j(y_i)]$, $1 \leq j \leq r$, that are mapped into subpaths in IET or Levitt components or segments with non-trivial stabilizers in one of the finitely many trees that were used to refine Γ_∞ and obtain Γ_f. In particular, the preimages of an orbit of branching points in Γ_∞ in Y_i, is in the same orbit under the action of L. Using the refinement process the same holds for preimages of orbits of branching points in Γ_f. This equivariance of the preimages of branching points in Γ_f, guarantees that for large i, the modular groups $Mod(\Lambda_i)$ are contained in the modular group $Mod(\Gamma_f)$.

□

As in the Levitt free case, proposition 7.14 enables us to remove a suffix of the sequence of abelian decompositions, $\Lambda_1, \dots$, and replace it by the finite sequence: $\Lambda_1, \dots, \Lambda_{i_1-1}, \Gamma_f$. We continue to the next steps with the abelian decomposition, Γ_f, and with the sequence of pair homomorphisms, $\{f^i\}$, that converges to a limit action from which Γ_f was obtained.

The terminal abelian decomposition of that finite resolution contains at least one QH vertex group or one Levitt component. We repeat the whole construction of a sequence of abelian decompositions starting with the sequence of pair homomorphisms $\{f^i\}$. If the sequence terminates after a finite number of steps, the conclusion of theorem 7.12 follows. If it ends up with an infinite sequence of abelian decompositions, we use the same construction as the one that was used in proving proposition 7.14.

We continue iteratively as we did in the proof of theorems 6.1 and 7.2. If this iterative procedure terminates after finitely many steps, the conclusion of theorem 7.12 follows. Otherwise we obtained an infinite sequence of abelian decompositions that do all contain QH or Levitt vertex groups. In each of these abelian decompositions, either:

(1) there exists a dominant edge group that is not elliptic in an abelian decomposition that appears afterwards in the sequence.

(2) there exists a QH vertex group Q in the abelian decomposition, such that the abelian decomposition collapses to an abelian decomposition Γ_Q that contains one QH vertex group, Q, and possibly several non-QH, non Levitt vertex groups that are connected only to the vertex stabilized by Q by edges with trivial and cyclic edge groups. Γ_Q and an abelian decomposition that appears afterwards in the sequence do not have a common refinement (see the construction of the original sequence of abelian decompositions $\Lambda_1, \dots$).
(3) there exists a Levitt vertex group B in the abelian decomposition, such that the abelian decomposition collapses to an abelian decomposition $\tilde{\Gamma}_B$ that contains one Levitt vertex group, B, and possibly several non-QH, non Levitt vertex groups that are connected only to the vertex stabilized by B by edges with trivial edge groups. Γ_B and an abelian decomposition that appears afterwards in the sequence do not have a common refinement.

Now, we apply proposition 7.14 to the sequence of abelian decompositions that we constructed. By proposition 7.14 a suffix of the sequence can be replaced with the abelian decomposition, such that all the modular groups that are associated with the abelian decompositions from the suffix are contained in the modular group of the abelian decomposition that the proposition produces and dominates the suffix. Hence, the abelian decomposition that is obtained using proposition 7.14 contains either:

(i) a Levitt vertex group.
(ii) a Levitt vertex vertex group and a QH vertex group, or two Levitt vertex groups, or two QH vertex groups.
(iii) more than one QH vertex group, or a QH vertex group with higher (topological) complexity (see the proof of theorem 6.1).
(iv) only a single QH vertex group, possibly of the same topological complexity as QH vertex groups that appear in the abelian decompositions in the suffix (e.g. a once punctured torus), but this QH vertex has to be connected to the other vertex groups in the abelian decomposition with at least one edge with trivial stabilizer.
(v) suppose that a subsequence of the abelian decompositions contain Levitt vertex groups. Then either the abelian decomposition that dominates a suffix of the entire sequence contains two Levitt vertex groups, or a Levitt vertex group of rank at least 3, or a single Levitt vertex group that is connected to other vertex groups in the abelian decomposition with at least two edge groups with trivial stabilizers.

As in the proof of theorems 6.1 and 7.2, we repeat the whole construction starting with the (higher complexity) abelian decomposition that we obtained and the subsequence of homomorphisms that is associated with it. Either the construction terminates in finitely many steps, or a suffix of an infinite sequence of abelian decompositions can be replaced with an abelian decomposition of higher complexity. i.e., either the ranks of the Levitt vertex groups (ordered lexichographically from highest to lowest ranks), or the topological complexities of the QH vertex groups (ordered lexichographically from high to low), that appear in the abelian decompositions, is bounded below by a higher lower bound, or the minimum number of edges with trivial stabilizers that are connected to the Levitt and QH vertex groups is bigger. Repeating this construction itartively, by the accessibility of f.p. groups, or by acylindrical accessibility, we are left with a finite resolution that satisfies the conclusion of theorem 7.12 (see the proof of theorem 6.1).

□

For presentation purposes, in theorems 7.2 and 7.12 we assumed that the limit group L contains no non-cyclic abelian groups. To include abelian vertex groups we use the same modifications as we used in the freely indecomposable case (see definition 6.6 and theorem 6.7).

Theorem 7.15. *Let (S,L) be a pair, and let $s_1,\dots,s_r$ be a fixed generating set of the semigroup S. Let $\{h_n : (S,L) \to (FS_k,F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on a real tree Y.*

Then there exists a resolution:

$$(S_1,L_1) \to (S_2,L_2) \to \dots \to (S_m,L_m) \to (S_f,L_f)$$

that satisfies the following properties:

1. *$(S_1,L_1)=(S,L)$, and $\eta_i : (S_i,L_i) \to (S_{i+1},L_{i+1})$ is an isomorphism for $i=1,\dots,m-1$ and $\eta_m : (S_m,L_m) \to (S_f,L_f)$ is a proper quotient map.*
2. *with each of the pairs (S_i,L_i), $1 \leq i \leq f$, there is an associated abelian decomposition that we denote Λ_i. The abelian decompositions $\Lambda_1,\dots,\Lambda_{f-1}$ contain edges with trivial or abelian edge stabilizers, and QH, Levitt, abelian (axial), and rigid vertex groups.*
3. *either η_{f-1} is a proper quotient map, or the abelian decomposition Λ_f contains separating edges with trivial edge groups. Each separating edge is oriented.*
4. *there exists a subsequence of the homomorphisms $\{h_n\}$ that factors through the resolution. i.e., each homomorphism h_{n_r} from the subsequence, is obtained from a homomorphism of the terminal pair (S_f,L_f) using a composition of a modification that uses generalized Dehn twists that are associated with Λ_m, and modular automorphisms that are associated with $\Lambda_1,\dots,\Lambda_{m-1}$.*
5. *if (S_f,L_f) is not a proper quotient of (S,L), then the (shortened) pair homomorphisms $\{h_n^f\}$ of (S_f,L_f) (that are obtained by shortening a subsequence of the homomorphisms $\{h_n\}$) are compatible with Λ_f. Let $R_1,\dots,R_v$ be the connected components of Λ_f after deleting its (oriented) separating edges. The homomorphisms $\hat{h}_n^f$ are composed from homomorphisms of the fundamental groups of the connected components $R_1,\dots,R_v$, together with assignments of values from FS_k to the oriented separating edges. The homomorphisms of the fundamental groups of the connected components $R_1,\dots,R_v$ converge into a faithful action of these groups on real trees with associated abelian decompositions: $R_1,\dots,R_v$.*

Proof: The theorem follows from theorem 7.7 using the same argument that was used to prove theorem 6.7 from theorem 6.1.

□

Theorem 7.15 generalizes theorem 6.7 to general pairs. Given a sequence of pair homomorphisms of a pair (S,L), and a sequence of pair homomorphisms $\{h_n : (S,L) \to (FS_k,F_k)\}$, that converges into a faithful action of L on some real tree, it proves that it is possible to extract a subsequence that factors through a finite resolution of the pair (S,L) that terminates in a proper quotient of the pair (S,L), or with a sequence of separable homomorphisms of (S,L). This allows one to continue the construction of a resolution iteratively and get a resolution that terminates with a graph of groups in which all the edges are separating edges and all the vertex groups are trivial.

Proposition 7.16. *Let (S,L) be a pair, and let $s_1,\dots,s_r$ be a fixed generating set of the semigroup S. Let $\{h_n : (S,L) \to (FS_k,F_k)\}$ be a sequence of pair homomorphisms that converges into a faithful action of L on a real tree Y.*

Then there exists a resolution:

$$(S_1,L_1) \to (S_2,L_2) \to \dots \to (S_f,L_f)$$

that satisfies the following properties:

1. *(S_{i+1},L_{i+1}) is a quotient of (S_i,L_i), but not necessarily a proper quotient. L_f is a free group.*
2. *with each of the pairs (S_i,L_i), $1 \leq i \leq f$, there is an associated abelian decomposition that we denote Λ_i. The abelian decompositions $\Lambda_1,\dots,\Lambda_{f-1}$ contain edges with trivial or abelian edge stabilizers, and QH, Levitt, abelian (axial), and rigid vertex groups. Some edges in these abelian decompositions may be separating edges, and separating edges in Λ_i are canonically mapped into separating edges in Λ_{i+1}.*
3. *Λ_f that is associated with the terminal pair, (S_f,L_f), is a graph (of groups) with trivial vertex groups, and all its edges are separating edges. In particular, they are oriented and with trivial stabilizers. The fundamental group of this graph is L_f, and the generators of the semigroup S, $s_1,\dots,s_r$, represent positive closed paths in the graph.*
4. *there exists a subsequence of the homomorphisms $\{h_n\}$ that factors through the resolution. i.e., each homomorphism h_{n_r} from the subsequence, is obtained from a homomorphism of the terminal pair (S_f,L_f) using a composition of generalized Dehn twists and modular automorphisms that are associated with $\Lambda_1,\dots,\Lambda_{f-1}$.*

Proof: We start with the sequence of homomorphisms $\{h_n\}$. By theorem 7.15 it is possible to extract a subsequence that factors to a finite resolution Res_1. The finite resolution Res_1 terminates in either a proper quotient of the original pair (S,L), or it ends with a pair that is isomorphic to (S,L), but the abelian decomposition that is associated with the terminal pair has separating edges (see definition 7.3).

Let $\{h^m_{n_k}\}$ be the sequence of homomorphisms that is obtained from a subsequence of the original sequence of homomorphisms $\{h_n\}$ after factoring through the resolution Res_1 (this subsequence, $\{h^m_{n_k}\}$ was used to construct the abelian decomposition that is associated with the terminal level of Res_1). If the terminal pair that is associated with the resolution of Res_1 is a proper quotient of (S,L), we continue by applying theorem 7.11 to the subsequence $\{h^m_{n_k}\}$.

Suppose that the terminal pair of Res_1 is isomorphic to (S,L). In that sequence $\{h^m_{n_k}\}$ is separable with respect to an abelian decomposition Δ_1, in which all the edges have trivial edge groups, and they are all separating edges. In that case we replace the homomorphisms, $\{h^m_{n_k}\}$, with a sequence of homomorphisms into a bigger coefficient group as described in part (5) of definition 7.3. Recall that n that part of definition 7.3, given each of the homomorphisms $h^m_{n_k}$, we replace the values that are assigned with each of the labels that are associated with the edges in Δ_1, to values that contain a single (positively oriented) appearance of the generator that is associated with each label, and no appearances of generators that are associated with the other labels, without changing the homomorphisms of the vertex groups in Δ_1 to obtain a homomorphism $\hat{h}^m_{n_k}$. Each such homomorphism $\hat{h}^m_{n_k}$ is a pair homomorphism: $\hat{h}^m_{n_k} : (S_m, L_m) \to (\hat{F}S, \hat{F})$, where (S_m, L_m) is the terminal limit group of Res_1, and $(\hat{F}_s, \hat{F})$ is a standard extension of the standard coefficient pair, (FS_k, F_k), that is obtained by adding a finite set of new generators.

Now, we apply theorem 7.15 to the modified homomorphisms $\{h^m_{n_k}\}$, where we analyze and modify (shorten) only the actions of the vertex groups in Δ. The conclusion of theorem 7.11 gives another resolution Res_2. Either the terminal pair of Res_2 is a proper quotient of the original pair (S, L), or the resolution terminates with an isomorphic pair with an associated separating abelian decomposition Δ_2, and the number of separating edges in Δ_2 is strictly bigger than the number of separating edges in Δ_1.

We proceed iteratively, and modify the homomorphisms that are the output of the construction of a resolution, whenever a new separating edges are associated with the terminal limit group of a resolution. By the d.c.c. property of limit groups, the construction terminates after finitely many steps. By construction, the terminal group has to be a free group, that is associated with a graph (of groups), in which all edge and vertex groups are trivial and all the edges are separating edges.

□

Proposition 7.16 enables us to apply a compactness argument, and associate a Makanin-Razborov diagram with a pair (S, L).

Theorem 7.17. *Let (S, G) be a pair of a group G and a f.g. subsemigroup S that generates G as a group. Then there exist finitely many resolutions of the form that is constructed in proposition 7.11:*

$$(S_0, L_0) \to (S_2, L_2) \to \ldots \to (S_f, L_f)$$

where (S_0, L_0) is a quotient pair of (S, G), and each L_i is a limit group, such that:

1. *every pair homomorphism, $h : (S, L) \to (FS_k, F_k)$, factors through at least one of these finitely many resolutions.*
2. *for each of the resolutions in the collection, there exists a sequence of homomorphisms: $\{h_n : (S, L) \to (FS_k, F_k)\}$, that converges into a faithful action of the initial pair (S_0, L_0) on a real tree with an associated abelian decomposition (after the refinement that is used in the proof of proposition 6.4) Λ_0.*

   Furthermore, the sequence of homomorphisms $\{h_n\}$ can be modified using modular automorphisms and generalized Dehn twists that are associated with the abelian decompositions: $\Lambda_0, \ldots, \Lambda_m$, and by adding separators to separating edges, to get sequences of pair homomorphisms $\{h^1_n\}, \ldots, \{h^f_n\}$. Each of these modified sequences of homomorphisms $\{h^i_n\}$ converges into a faithful action of the pair (S_i, L_i) on a real tree with an associated abelian decomposition (after an appropriate refinement) Λ_i.

Proof: Proposition 7.16 shows that given any sequence of pair homomorphisms of (S, G) into (FS_k, F_k), it is possible to extract a subsequence that factors through a (finite) resolution that satisfies part (2) of the theorem. Hence, the theorem follows by the same compactness argument that was used in the proof of theorem 6.8.

□

Theorem 7.17 associates a Makanin-Razborov diagram with a pair, and theorem 6.8 associates such a diagram with a freely indecomposable restricted pair. We end this chapter by associating a Makanin-Razborov diagram with a general restricted pair.

Theorem 7.18. *Let (S,G) be a restricted pair of a group G and a f.g. subsemigroup S that generates G as a group. i.e., the standard pair (FS_k,F_k) is a subpair of (S,G). Then there exist finitely many resolutions of the form:*

$$(S_0,L_0) \to (S_2,L_2) \to \ldots \to (S_f,L_f)$$

where (S_0,L_0) is a quotient restricted pair of (S,G), and each L_i is a restricted limit group, such that:

(1) *(S_{i+1},L_{i+1}) is a restricted quotient pair of (S_i,L_i), but not necessarily a proper quotient. L_f is a restricted free product of a (possibly trivial) free group and the coefficient group F_k and the coefficient group F_k.*
(2) *with each of the pairs (S_i,L_i), $1 \leq i \leq f$, there is an associated abelian decomposition that we denote Λ_i that satisfies part (2) in proposition 7.8. The coefficient group F_k is contained in a (distinguished) vertex group in each of the abelian decomposition Λ_i.*
(3) *Λ_f that is associated with the terminal pair, (S_f,L_f), is a graph of groups with a single non-trivial vertex group, which is the coefficient group F_k. All its edges are separating edges. In particular, they are oriented and with trivial stabilizers. The fundamental group of this graph is $L_f = F_f * F$.*
(4) *every restricted pair homomorphism, $h:(S,G) \to (FS_k,F_k)$, factors through at least one of these finitely many resolutions.*
(5) *part (2) in the statement of theorem 7.9 holds for the restricted diagram, where sequences of homomorphisms and their shortenings and modifications are replaced by sequences of restricted homomorphisms.*

Proof: Identical to the proof of theorem 7.17.

□

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