# DIOPHANTINE GEOMETRY OVER GROUPS VIII: STABILITY

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This paper is the eighth in a sequence on the structure of sets of solutions to systems of equations in free and hyperbolic groups, projections of such sets (Diophantine sets), and the structure of definable sets over free and hyperbolic groups. In the eighth paper we use a modification of the sieve procedure, presented in [Se6] as part of the quantifier elimination procedure, to prove that free and torsion-free (Gromov) hyperbolic groups are stable.

In the first 6 papers in the sequence on Diophantine geometry over groups we studied sets of solutions to systems of equations in a free group, and developed basic techniques and objects that are required for the analysis of sentences and elementary sets defined over a free group. The techniques we developed, enabled us to present an iterative procedure that analyzes $EAE$ sets defined over a free group (i.e., sets defined using 3 quantifiers), and shows that every such set is in the Boolean algebra generated by $AE$ sets ([Se6],41), hence, we obtained a quantifier elimination over a free group.

In 1983 B. Poizat [Po1] proved that free groups are not super-stable (W. Hodges pointed out to us that this was also known to Gibone around 1976). In this paper we use our analysis of definable sets, and the geometric structure they admit as a consequence from our quantifier elimination procedure, together with the tools and the techniques that are presented in the previous papers in the sequence, to prove that free groups are stable (Theorem 5.1 - for a definition of a stable theory see [Pi] or the beginning of section 5). Since in [Se8] it was shown that the structure of definable sets and the tools that were developed for the analysis of them generalize to non-elementary, torsion-free hyperbolic groups, the argument that we use for proving the stability of a free group generalizes to an arbitrary non-elementary, torsion-free hyperbolic group (Theorem 5.2).

The stability of free and hyperbolic groups gives a linkage between negative curvature in Riemannian and coarse geometry and in geometric group theory and stability theory. With stability it is possible to continue the study of the first order theories of free and hyperbolic groups using well-developed objects and notions from model theory. Furthermore, following Shelah, logicians often view stability as the border line between "controlled" and "wild" structures. From certain points of view, and in certain aspects, this border line is reflected in group theory (see [Po2],[Po3]). Negatively curved groups are stable. For non-positively curved groups we don't really know, but we suspect that there should be unstable non-positively

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[2]Partially supported by an Israel academy of sciences fellowship.

curved groups. For other classes of groups the question of stability is still wide open.

To prove the stability of free and hyperbolic groups, we start by analyzing a special class of definable sets that we call *minimal rank.* These sets are easier to analyze than general definable sets, and in section 1 we prove that minimal rank definable sets are in the Boolean algebra generated by equational sets (recall that equational sets and theories were defined by G. Srour. For a definition see the beginning of section 1 and [Pi-Sr]). Note that as we show in section 6 there are definable sets over a free group that do not belong to the Boolean algebra generated by equational sets.

In section 2 we slightly modify the sieve procedure that was presented in [Se6] (and used for quantifier elimination) to prove that Diophantine sets are equational. This is a key in obtaining stability for general definable sets in the sequel. In section 3 we present a basic object that we use repeatedly in proving stability - *Duo limit groups* (definition 3.1), and their *Duo families*. We further prove a boundedness property of duo limit groups and their families (Theorem 3.2), that is not required in the sequel, but still motivates our approach to stability. In fact, we managed to prove the boundedness property in the case of rigid limit groups and left the more general case of solid limit groups as an open question.

In section 4 we use duo limit groups and their families, together with the sieve procedure and the equationality of Diophantine sets, to prove the stability of some classes of definable sets, that are in a sense the building blocks of general definable sets. These include the set of specializations of the defining parameters of a rigid and solid limit groups, for which the rigid (solid) limit group has precisely $s$ rigid (strictly solid families of) specializations for some fixed integer $s$ (see section 10 in [Se1] and section 1 in [Se3] for these notions).

In section 5 we use the geometric structure of a general definable set that was proved using the sieve procedure in [Se6], together with the stability of the classes of definable sets that are considered in section 4, to prove the stability of a general definable set over a free group, hence, to obtain the stability of a free group (Theorem 5.1). Using the results of [Se8] we further generalize our results to a non-elementary, torsion-free (Gromov) hyperbolic group (Theorem 5.2).

Finally, in section 6, we apply once again the geometric structure of a definable set obtained in [Se6], to give examples of a definable set over a free group and over a non-elementary, torsion-free hyperbolic group, that do not belong to the Boolean algebra generated by equational sets, hence, to prove that the theories of free and non-elementary torsion-free (Gromov) hyperbolic groups are not equational.

Quite a few people have assisted us along the course of this work. In particular we would like to thank G. Cherlin, W. Hodges, O. Belegradek, A. Pillay, B. Zilber, and especially E. Hrushovski for their help and suggestions.

## §1. The Minimal (Graded) Rank Case

As we will show in the sequel, free and hyperbolic groups are stable but not equational. Before treating the stability of these groups, we study a subcollection of definable sets, that we call *minimal rank*, and prove that these sets are in the Boolean algebra generated by equational sets. Minimal (graded) rank sets were treated separately in our procedure for quantifier elimination ([Se5]-[Se6]), and it was indicated there that our procedure for quantifier elimination for minimal (graded) rank predicates is far easier than it is for general predicates.

In order to prove equationality for minimal rank definable sets, we introduce a collection of (minimal rank) *equational sets* for which:

(i) the Boolean algebra generated by the collection of equational sets contains the collection of minimal rank definable sets.

(ii) if $\varphi(p, q)$ is an equational set then there exists a constant $N_\varphi$, so that every sequence of specializations $\{q_i\}_{i=1}^m$, for which the sequence of intersections: $\{\wedge_{i=1}^j \varphi(p, q_i)\}_{j=1}^m$ is a strictly decreasing sequence, satisfies: $m \leq N_\varphi$ (O. Belegradek has pointed out to us that this is the definition of equationality that one needs to use in case the underlying model is not necessarily saturated).

To define the subcollection of equational sets, and prove the descending chain condition they satisfy, we study the Boolean algebra of minimal rank definable sets gradually.

(1) Diophantine sets - we show that minimal rank Diophantine sets are equational.

(2) Rigid limit groups are defined in section 10 of [Se1], and their rigid specializations are analyzed in sections 1-2 of [Se3]. With a given minimal rank rigid limit group $Rgd(x, p, q, a)$ (where $< p, q >$ is the parameters group), we associate a natural existential predicate $\varphi(p, q)$, that specifies those values of the defining parameters $p, q$ for which $Rgd(x, p, q, a)$ admits at least $m$ rigid specializations, for some fixed integer $m$. We show the existence of a collection of equational sets, so that the Boolean algebra generated by this collection contains all the predicates $\varphi(p, q)$ associated with all minimal rank rigid limit groups and an arbitrary integer $m$.

(3) Solid limit groups are defined in section 10 of [Se1], and their strictly solid families of specializations are analyzed in sections 1-2 of [Se3] (see definition 1.5 in [Se3]). With a given minimal rank solid limit group $Sld(x, p, q, a)$ we associate a natural $EA$ predicate $\varphi(p, q)$, that specifies those values of the defining parameters $p, q$ for which $Sld(x, p, q, a)$ admits at least $m$ strictly solid families of specializations, for some fixed integer $m$. As for rigid limit groups, we show the existence of a collection of equational sets, so that the Boolean algebra generated by this collection contains all the predicates $\varphi(p, q)$ associated with minimal rank solid limit groups and an arbitrary integer $m$.

(4) Given a graded resolution (that terminates in either a rigid or a solid limit group), and a finite collection of (graded) closures of that graded resolution, we define a natural predicate, $\alpha(p, q)$ (which is in the Boolean algebra of $AE$ predicates), that specifies those values of the defining parameters for which the given set of closures forms a covering closure of the given graded resolution (see definition 1.16 in [Se2] for a covering closure). We show the existence of a collection of equational sets, so that the Boolean algebra generated by this collection contains all the predicates $\alpha(p, q)$ associated with all the graded resolutions for which their terminal rigid or solid limit group is of minimal (graded) rank.

(5) Finally, we show the existence of a subcollection of equational sets that generate the Boolean algebra of minimal rank definable sets.

**Theorem 1.1.** *Let $F_k =< a_1, \dots, a_k >$ be a non-abelian free group, and let*

$$D(p,q) \ = \ \{ \, (p,q) \ \ \exists x \ \Sigma(x,p,q,a)=1 \, \}$$

*be a Diophantine set defined over $F_k$. Suppose that each of the (finite) maximal limit groups $Rlim_i(x,p,q,a)$, associated with the system of equations $\Sigma$, does not admit a (restricted) epimorphism onto a free group, $\eta: Rlim_i(x,p,q,a) \to F_k * F$, where $F$ is a non-trivial free group, and the subgroup $< q >$ is mapped into the coefficient group $F_k$.*
*Then $D(p,q)$ is equational.*

*Proof:* Let $\varphi(p,q)$ be the existential predicate associated with the minimal rank Diophantine set $D(p,q)$. We need to show that $\varphi(p,q)$ is equational, i.e., that there exists an integer $N_{\varphi}$, so that every sequence of specializations $\{q_i\}_{i=1}^m$, for which the sequence of intersections: $\{ \wedge_{i=1}^j \varphi(p,q_i) \}_{j=1}^m$ is a strictly decreasing sequence, satisfies: $m \leq N_{\varphi}$.

Let $D(p,q)$ be defined by the existential predicate:

$$D(p,q) \ = \ \{ \, (p,q) \mid \exists x \ \Sigma(x,p,q,a)=1 \, \}$$

and let: $L_1(x,p,q,a), \dots, L_t(x,p,q,a)$ be the canonical collection of maximal limit groups associated with the system of equations $\Sigma(x,p,q,a)=1$. By our assumptions each of the limit groups $L_i(x,p,q,a)$ is of minimal rank, when viewed as a graded limit group with respect to the parameter subgroup $< q >$. To prove the existence of a bound $N_{\varphi}$, we associate with the predicate $\varphi$ a finite diagram.

We start the construction of the diagram with each of the maximal limit groups $L_1(x,p,q,a), \dots, L_t(x,p,q,a)$ in parallel. With a limit group $L_i(x,p,q,a)$, viewed as a graded limit group with respect to the parameter subgroup $< q >$, we associate its strict graded Makanin-Razborov diagram (for the construction of the strict Makanin-Razborov diagram, see proposition 1.10 in [Se2]). With each resolution in the strict Makanin-Razborov diagram, we further associate its singular locus (see section 11 in [Se1]), and the graded resolutions associated with each of the strata in the singular locus. We conclude the first step of the construction of the diagram, by associating the (graded) completion with each of the graded resolutions in our finite collection, that we denote, $Comp(x,p,z,q,a)$, and with each graded completion we associate its complexity, according to definition 1.16 in [Se5].

We continue the construction of the second step of the diagram with each of the completions $Comp(x,p,z,q,a)$ in parallel. With each such completion we associate the collection of specializations $(x_1,x_2,p,z,q_1,q_2,a)$ for which:

1. $(x_1,p,z,q_1,a)$ factors through the completion, $Comp(x,p,z,q,a)$.
2. $(x_2,p,q_2,a)$ factors through at least one of the maximal limit groups, $L_i(x,p,q,a)$, associated with the system of equations $\Sigma(x,p,q,a)$.

By the standard arguments presented in section 5 of [Se1],with this system of specializations we can associate a canonical finite collection of maximal limit groups, $M_j(x_1,x_2,p,z,q_1,q_2)$, which we view as graded limit groups with respect to the parameter subgroup $< q_1, q_2 >$.

By our assumptions each of the completions, $Comp(x,p,z,q,a)$, is of minimal rank, and so are the limit groups, $L_i(x,p,q,a)$. Hence, we may apply the iterative procedure for the analysis of (graded) minimal rank resolutions presented in section 1 of [Se5] (which is based on the procedure presented in section 1 of [Se4]), and associate with each of the graded limit groups $M_j(x_1, x_2, p, z, q_1, q_2, a)$, a finite collection of minimal rank (graded) resolutions (with respect to the parameter subgroup $< q_1, q_2 >$). Furthermore, the complexity of each of these minimal rank resolutions is bounded above by the complexity of the resolution associated with the corresponding completion, $Comp(x, p, z, q, a)$, and in case of equality the obtained resolution is a graded closure of the corresponding completion, $Comp(x, p, z, q, a)$ (see definition 1.16 in [Se5] for the complexity of a (graded) minimal rank resolution, and theorem 1.17 in [Se5] for the reduction in the complexity of the obtained resolutions). Therefore, some of the obtained resolutions are (graded) closures of the completion, $Comp(x, p, z, a)$, and the other resolutions have strictly smaller complexity than the complexity of $Comp(x, p, z, a)$.

We continue to the third step only with those resolutions that have strictly smaller complexity than the completion we have started the second step with, and with those closures of the original completion that are not identical to it (i.e., only proper closures). We perform the same operations we have conducted in the second step. For each such resolution, we take its completion, and look at all the specializations that factor through that completion, and for which there exists a specialization $(x_3, q_3)$, so that the combined specialization, $(x_3, p, q_3, a)$, factors through at least one of the limit groups, $L_i(x, p, q, a)$, associated with the system of equations $\Sigma$. With the collection of these specializations, we canonically associate a finite collection of maximal limit groups, and we analyze the (minimal rank) resolutions associated with these limit groups using the iterative procedure presented in section 1 of [Se5]. The complexity of the obtained resolutions is bounded by the complexity of the resolution we have started the third step with, and we continue to the fourth step, only with those resolutions that have strictly smaller complexity than the resolution we have started the third step with, or with proper closures of the completions we have started the third step with.

We continue the construction iteratively. Since the number of steps we replace a completion of a resolution by a proper closure of it is bounded (for each completion that appears along the diagram), and since the complexities of the resolutions constructed along successive steps of this iterative procedure strictly decrease (and all the resolutions that appear along the procedure are of minimal rank), the construction of the diagram terminates after finitely many steps (cf. theorem 1.18 in [Se5]).

By theorems 2.5, 2.9 and 2.13 of [Se3] the number of rigid or strictly solid families of specializations associated with a given specialization of the defining parameters of a rigid or solid limit group is bounded by a bound that depends only on the rigid or solid limit group. This global bound on the number of rigid and strictly solid families, together with the finiteness of the constructed diagram, imply the equationality of Diophantine predicates of minimal rank. In fact, the equationality bound $N_\varphi$ for a minimal rank Diophantine predicate $\varphi$, can be easily computed from the finite diagram and the bounds on the numbers of rigid and strictly solid families associated with each strictly solid limit group along the diagram.

□

**Theorem 1.2.** *Let $F_k =< a_1, \dots, a_k >$ be a non-abelian free group, and let $Rgd(x, p, q, a)$ be a rigid limit group, with respect to the parameter subgroup $< p, q >$.*

*Let $s$ be a positive integer, and let $NR_s$ be the set of specializations of the defining parameters $<p,q>$ for which the rigid limit group, $Rgd(x,p,q)$, has at least $s$ rigid specializations.*

*There exists a collection of equational sets, so that the Boolean algebra generated by this collection contains the sets $NR_s$ for every possible integer $s$, and every minimal rank rigid limit group $Rgd(x,p,q,a)$, i.e., a rigid limit group $Rgd(x,p,q,a)$ that does not admit a (restricted) epimorphism onto a free group $\eta : Rgd(x,p,q,a) \to F_k * F$, where $F$ is a non-trivial free group, and the subgroup $<q>$ is mapped into the coefficient group $F_k$.*

*Proof:* We construct a collection of equational sets that generate a Boolean algebra that contains the sets of the form $NR_s$ iteratively. With a set of the form $NR_s$, we associate a minimal rank Diophantine set $D_1$, and show that $NR_s \cup D_1$ is equational. Clearly:

$$NR_s \,=\, ((NR_s \cup D_1) \setminus D_1) \,\cup\, (D_1 \cap NR_s)$$

Since by theorem 1.1 the minimal rank Diophantine set $D_1$ is equational, to prove the theorem we further need to study the set $D_1 \cap NR_s$. We study this set in the same way we treated the set $NR_s$, and we argue that the "complexity" of the set $D_1 \cap NR_s$ is strictly smaller than the complexity of the original set $NR_s$. This decrease in complexity forces the iterative procedure to terminate, hence, prove the theorem for the sets $NR_s$.

We start with the construction of the set $D_1$ associated with the set $NR_s$. To define $D_1$, we look at the collection of all the tuples of specializations, $(x_1,\dots,x_s,p,q,a)$, for which for every index $i$, $1 \leq i \leq s$, $(x_i,p,q,a)$ is a rigid specialization of the given rigid limit group $Rgd(x,p,q,a)$, and for every couple $i,j$, $1 \leq i < j \leq s$, $x_i \neq x_j$. By our standard arguments, with this collection of specializations we can associate canonically a finite collection of maximal limit groups, $M_j(x_1,\dots,x_s,p,q,a)$.

We continue with each of the limit groups $M_j(x_1,\dots,x_s,p,q,a)$ in parallel. With $M_j$ viewed as a graded limit group with respect to the parameter subgroup $<q>$, we associate its strict graded Makanin-Razborov diagram. With each resolution in the strict Makanin-Razborov diagram, we further associate its singular locus, and the graded resolutions associated with each of the strata in the singular locus. With each of the obtained graded resolutions we further associate its (graded) completion, and with each graded completion we associate its complexity, according to definition 1.16 in [Se5].

We continue with each of the completions in parallel. Given such a completion, we look at all the specializations that factor through it, and for which either one of the specializations that are suppose to be rigid is flexible, or those for which two rigid specializations that are suppose to be distinct coincide. With the collection of all such specializations we can associate a canonical finite collection of (graded) limit groups. Each such graded limit group is minimal rank by our assumptions, so we associate with it a finite collection of resolutions according to section 1 in [Se5]. Some of the obtained graded resolutions have the same structure as the original resolution (the maximal complexity ones), and the rest have strictly smaller complexity than the completion they are associated with. With each resolution that have strictly smaller complexity, we associate its completion, and we set the Diophantine set $D_1$ to be the disjunction of all the Diophantine sets associated with completions of those resolutions that are not of maximal complexity.

**Proposition 1.3.** *The set $NR_s \cup D_1$ is equational.*

*Proof:* To prove the proposition, we associate with the set $NR_s \cup D_1$ a finite diagram, that is constructed iteratively, in a similar way to the construction of the diagram associated with a Diophantine set that was used in the proof of theorem 1.1.

We start the construction of the diagram with the collection of all the tuples of specializations, $(x_1, \dots, x_s, p, q, a)$, for which for every index $i$, $1 \leq i \leq s$, $(x_i, p, q, a)$ is a rigid specialization of the given rigid limit group $Rgd(x, p, q, a)$, and the $x_i$'s are distinct, and the collection of all tuples $(u, p, q, a)$ that factor through the defining equations of the Diophantine set $D_1$. By our standard arguments, with this collection of specializations we can associate canonically a finite collection of maximal limit groups, $M_j(x_1, \dots, x_s, p, q, a)$ and $V_r(u, p, q, a)$.
We continue with each of the limit groups $M_j(x_1, \dots, x_s, p, q, a)$ and $V_r(u, p, q, a)$, in parallel. With a maximal limit group, $M_j$ or $V_r$, viewed as a graded limit group with respect to the parameter subgroup $< q >$, we associate its strict graded Makanin-Razborov diagram. With each resolution in the strict Makanin-Razborov diagram, we further associate its singular locus, and the graded resolutions associated with each of the strata in the singular locus. With each of the obtained graded resolutions we further associate its (graded) completion, and with each graded completion we associate its complexity, according to definition 1.16 in [Se5].

We continue the construction of the second step of the diagram with each of the completions, $Comp(x, p, z, q, a)$ and $Comp(u, p, q, a)$, in parallel. With each such completion $Comp(x, p, z, q, a)$ we associate the collection of specializations $(y_1, \dots, y_s, x_1, \dots, x_s, p, z, q_1, q_2, a)$ and $(u, x_1, \dots, x_s, p, q_1, q_2, a)$ for which:

(1) $(x_1, \dots, x_s, p, z, q_1, a)$ factors through the completion, $Comp(x_1, \dots, x_s, p, z, q, a)$, the $x_i$'s are distinct, and each of the specializations $(x_i, p, q_1, a)$ is a rigid specialization of $Rgd(x, p, q, a)$.

(2) the $y_i$'s are distinct, and each of the specializations $(y_i, p, q_2, a)$ is a rigid specialization of $Rgd(y, p, q, a)$.

(3) the specialization $(u, p, q_2, a)$ factors through one of the completions $V_r(u, p, q, a)$.

and similarly, we look at the collection of specializations: $(y_1, \dots, y_s, u, p, q_1, q_2, a)$ and $(u_1, u_2, p, q_1, q_2, a)$.
With this system of specializations we can associate a canonical finite collection of maximal limit groups, which we view as graded limit groups with respect to the parameter subgroup $< q_1, q_2 >$.

By our assumptions each of the completions, $Comp(x, p, z, q, a)$ and $Comp(u, p, q, a)$, is of minimal rank. Hence, we may apply the iterative procedure for the analysis of (graded) minimal rank resolutions, presented in section 1 of [Se5], and associate with each of the graded limit groups, a finite collection of minimal rank (graded) resolutions (with respect to the parameter subgroup $< q_1, q_2 >$). Furthermore, by theorem 1.17 in [Se5], the complexity of each of these minimal rank resolutions is bounded above by the complexity of the resolution associated with the corresponding completion (with which we have started the second step), $Comp(x, p, z, q, a)$ or $Comp(u, p, z, q, a)$, and in case of equality in complexity the obtained resolution is a graded closure of the corresponding completion. Therefore, some of the obtained resolutions are (graded) closures of the completions, $Comp(x, p, z, a)$ and

$Comp(u,p,q,a)$, and the other resolutions have strictly smaller complexity than the complexity of corresponding completion, $Comp(x,p,z,a)$ or $Comp(u,p,q,a)$.

We continue to the third step only with those resolutions that have strictly smaller complexity than the completion we have started the second step with, or with resolutions that are proper closures of one of these completions. We perform the same operations we have conducted in the second step for each of these resolutions, i.e., we take its completion, and look at all the specializations that factor through that completion, satisfy the corresponding rigidity conditions and for which there exists either a specialization $(t_1,\dots,t_s,q_3)$, so that the combined specializations, $(t_i,p,q_3,a)$, are distinct rigid specializations of $Rgd(x,p,q,a)$, or a specialization $(u,p,q_3,a)$ that factor through the limit groups associated with the Diophantine set $D_1$. With the collection of these specializations, we canonically associate a finite collection of maximal limit groups, and we analyze the (minimal rank) resolutions associated with these limit groups using the iterative procedure presented in section 1 of [Se5]. The complexity of the obtained resolutions is bounded by the complexity of the resolution we have started the third step with, and we continue to the fourth step, only with those resolutions that have strictly smaller complexity than the resolution we have started the third step with, or with proper closures of that completion.

We continue the construction iteratively. Since the complexities of the resolutions constructed along successive steps of this iterative procedure strictly decrease (and they are all of minimal rank), and we can replace a completion by a proper closure only finitely many times, the construction of the diagram terminates after finitely many steps (cf. theorem 1.18 in [Se5]).

Given an ungraded resolution that is covered by a graded resolution that appears along the finite diagram we constructed, that terminates in collecting $s$ distinct rigid specializations $x_1,\dots,x_s$, note that either:

(1) the ungraded resolution that was used in collecting the terminal distinct rigid specializations, $x_1,\dots,x_s$, is covered by a Makanin-Razborov graded resolution associated with one of the limit groups $M_j(x_1,\dots,x_s,p,q,a)$ (i.e., a graded resolution constructed in the initial step of the diagram), so that for all the specializations that factor through this ungraded resolution, either one of the $x_i$'s is flexible, or at least two of the $x_i$'s coincide. In this case for all the specializations that factor through the given ungraded resolution (that collects the terminal rigid specializations), either one of the $x_i$'s is flexible, or at least two of the $x_i$'s coincide. Hence, we can ignore the collection of such ungraded resolutions, and regard them as terminal points in applying the diagram we constructed for analyzing the equationality of the set $NR_s \cup D_1$.

(2) the ungraded resolution is not covered by a Makanin-Razborov graded resolution associated with one of the limit groups $M_j$, in which at least one of the $x_i$'s is flexible, or at least two of the $x_i$'s coincide. In this case all the specializations $(x_1,\dots,x_s,p,q,a)$ that factor through the given ungraded resolution, and for which either one of the $x_i$'s is flexible or at least two of the $x_i$'s coincide, factor through the defining limit group associated with the Diophantine set $D_1$.

By the above observation, either all the specializations that factor through an ungraded resolution that is covered by a graded resolution constructed along the

diagram factor through an ungraded resolution that is covered by a graded Makanin-Razborov resolution constructed in the initial step of the diagram, and all the specializations that factor through this Makanin-Razborov resolution do not satisfy the $NR_s$ predicate, or all the specializations that factor through the given ungraded resolution satisfy the $NR_s \cup D_1$ predicate.
Therefore, the finiteness of the constructed diagram, together with the global bound on the number of rigid and strictly solid families of specializations of rigid and solid limit groups ([Se3],theorems 2.5, 2.9 and 2.13), imply the equationality of the set $NR_s \cup D_1$, in the same way it was obtained for Diophantine sets in the proof of theorem 1.1.

□

Proposition 1.3 proves that the set $NR_s \cup D_1$ is equational. To prove theorem 1.2 we continue iteratively. With the set $NR_s \cap D_1$ we associate a Diophantine set $D_2$, precisely as we associated the set $D_1$ with $NR_s$. Note that $D_2 \subset D_1$, and $D_2$ is the disjunction of finitely many minimal rank Diophantine sets, each has strictly smaller complexity than the complexity of the set $D_1$ (where the complexities of the Diophantine sets $D_1$ and $D_2$ are set to be the maximal complexities of the minimal rank resolutions defining them). By the same argument used to prove proposition 1.3, the set $(NR_s \cap D_1) \cup D_2$ is equational. We continue to the third step with the set $NR_s \cap D_1 \cap D_2 = NR_s \cap D_2$ and treat it exactly in the same way. By the descending chain condition for complexities of minimal rank resolutions (cf. theorem 1.18 in [Se5]), this iterative process terminates after finitely many steps, that finally implies that the original set $NR_s$ is in the Boolean algebra of a collection of equational sets, and theorem 1.2 follows.

□

Essentially the same argument that was used to prove theorem 1.2 for the sets $NR_s$ associated with minimal rank rigid graded limit groups, can be used to prove a similar statement for sets of parameters for which a minimal rank solid limit group admits at least $s$ strictly solid families of specializations.

**Theorem 1.4.** *Let $F_k =< a_1, \dots, a_k >$ be a non-abelian free group, and let $Sld(x,p,a)$ be a solid limit group, with respect to the parameter subgroup $< p,q >$. Let $s$ be a positive integer, and let $NS_s$ be the set of specializations of the defining parameters $< p,q >$ for which the solid limit group, $Sld(x,p,q)$, has at least $s$ strictly solid families of specializations.*

*There exists a collection of equational sets, so that the Boolean algebra generated by this collection contains the sets $NS_s$, for every possible integer $s$, and every minimal rank solid limit group $Sld(x,p,q,a)$, i.e., a solid limit group $Sld(x,p,q,a)$ that does not admit a (restricted) epimorphism onto a free group $\eta : Sld(x,p,q,a) \to F_k * F$, where $F$ is a non-trivial free group, and the subgroup $< q >$ is mapped into the coefficient group $F_k$.*

*Proof:* The proof is similar to the proof of theorem 1.2. With the set $NS_s$ we associate a Diophantine set $D_1$, for which $NS_s \cup D_1$ is equational, and $D_1$ is "smaller" than $NS_s$. To construct the Diophantine set $D_1$, we look at the entire collection of specializations $(x_1, \dots, x_s, p, q, a)$ for which the $x_i$'s belong to distinct strictly solid families. With this collection we associate a canonical finite collection of maximal limit groups, that we view as graded with respect to the parameter subgroup $< q >$. With these graded limit groups we associate the (graded) resolutions that appear in their strict graded Makanin-Razborov diagrams, and the resolutions associated with the various strata in the singular loci of the diagrams.

Given a (graded) completion, $Comp(x_1, \dots, x_s, p, z, q, a)$, of one of these graded resolutions, we look at all the specializations that factor through that completion, and for which either (at least) one of the specializations that is supposed to be strictly solid is not strictly solid, or two such specializations belong to the same strictly solid family. Note that if a given specialization is not strictly solid, or if two specializations belong to the same strictly solid family, then the (ambient) specialization of the given completion has to satisfy at least one of finitely many (prescribed) Diophantine conditions that are associated with the given solid limit group, $Sld(x, p, q, a)$ (see definition 1.5 in [Se3]).
With this collection of specializations together with specializations of extra variables that are being added to demonstrate the Diophantine conditions they satisfy, we canonically associate a finite collection of graded limit groups, and the finite collection of graded (minimal rank) resolutions associated with these graded limit groups according to the iterative procedure for the analysis of minimal rank quotient resolutions presented in section 1 of [Se5]. By this iterative procedure, the complexity of each of the obtained resolutions is bounded above by the complexity of the completion we started with, and in case of equality in complexities, an obtained resolution has to be a graded closure of the completion we started with (see theorem 1.17 in [Se5]).

Furthermore, with the completion, $Comp(x_1, \dots, x_s, p, z, q, a)$, we associate the collection of all the sequences:

$$\{(f(n), x_1(n), \dots, x_s(n), p(n), z(n), q(n), a)\}_{n=1}^{\infty}$$

so that the restricted sequences, $\{(x_1(n), \dots, x_s(n), p(n), z(n), q(n), a)\}_{n=1}^{\infty}$, are test sequences with respect to the given completion (see definition 1.20 in [Se2]), and the specializations $f(n)$ were added to demonstrate that the corresponding specialization of the given completion satisfies one of the finitely many Diophantine conditions that demonstrate that either two of the specializations belong to the same strictly solid family or at least one of the specializations is not strictly solid (see definition 1.5 in [Se3] for these Diophantine conditions).

By the techniques that were used to analyze graded formal limit groups, presented in section 3 of [Se2], with this collection of sequences it is possible to canonically associate a finite collection of (graded) closures of the completion, $Comp(x_1, \dots, x_s, p, z, q, a)$, through which they factor. We will denote these graded closures, $Cl_i(x_1, \dots, x_s, p, f, z, q, a)$ (note that these closures are graded with respect to the parameter subgroup $< q >$).

We set the Diophantine set $D_1$ to be the disjunction of all the graded resolutions obtained through this process that have strictly smaller complexity than the completion they are associated with, and those closures, $Cl_i$, that are associated with the finite set of completions and are proper closures of the completion they are associated with. By the same argument that was used in proving theorem 1.3, the set $NS_s \cup D_1$ is equational.

As in analyzing rigid limit groups, we continue by analyzing the set $NS_s \cap D_1$. With this set we associate a Diophantine set $D_2$ in a similar way to the construction of the Diophantine set $D_1$. By the same argument that was used to prove theorem 1.3, $(NS_s \cap D_1) \cup D_2$ is equational, and $D_2$ is a union of finitely many Diophantine sets, corresponding to completions of resolutions that have strictly smaller

complexities than the complexities of the corresponding completions and closures that define the set $D_1$, together with some proper closures of the completions and closures that define $D_1$.
We continue iteratively, precisely as we did in proving theorem 1.2 in the rigid case. Since by the universality of the set of proper closures of resolutions that appear along this iterative process, only finitely many proper closures are associated with a given completion along it, for some step $n_0$, all the completions and closures that define the Diophantine set, $D_{n_0}$, have strictly smaller complexity than the maximal complexity of the completions and closures that define the Diophantine set $D_1$. Hence, the d.c.c. for complexities of minimal rank resolutions ([Se5],1.18) guarantees that the iterative process terminates after finitely many steps, which implies that the sets $NS_s$ are in the Boolean algebra of equational sets.

□

Theorem 1.1 proves that in the minimal (graded) rank case Diophantine sets are equational. Theorems 1.2 and 1.4 prove that (in minimal (graded) rank) sets for which a rigid or solid limit group have at least $s$ rigid (strictly solid families of) specializations, are in the Boolean algebra of equational sets. Before we analyze general definable sets that are of minimal (graded) rank, we need to analyze the (definable) set of specializations of the defining parameters for which a given (finite) collection of covers of a graded resolution forms a covering closure (see definition 1.16 in [Se2] for a covering closure).

**Theorem 1.5.** *Let $F_k =< a_1, \dots , a_k >$ be a non-abelian free group, let $G(x, p, q, a)$ be a graded limit group (with respect to the parameter subgroup $< p, q >$), and let $GRes(x, p, q, a)$ be a well-structured graded resolution of $G(x, p, q, a)$ that terminates in the rigid (solid) limit group, $Rgd(x, p, q, a)$ ($Sld(x, p, q, a)$).*
*Suppose that the terminating rigid (solid) limit group, $Rgd(x, p, q, a)$ ($Sld(x, p, q, a)$), is of minimal (graded) rank. i.e., it does not admit a (restricted) epimorphism onto a free group $\eta : Sld(x, p, q, a) \to F_k * F$, where $F$ is a non-trivial free group, and the subgroup $< q >$ is mapped into the coefficient group $F_k$.*
*Let $GCl_1(z, x, p, q, a), \dots , GCl_t(z, x, p, q, a)$ be a given set of graded closures of $GRes(x, p, q, a)$. Then the set of specializations of the parameters $< p, q >$ for which the given set of closures forms a covering closure of the graded resolution $GRes(x, p, q, a)$, $Cov(p, q)$, is in the Boolean algebra of equational sets.*

*Proof:* The proof is similar to the proofs of theorems 1.2 and 1.4. With the set $Cov(p, q)$ (i.e., the set of specializations of the defining parameters, $< p, q >$, for which the (ungraded) resolutions associated with the given (finite) set of closures form a covering closure of the (ungraded) resolutions associated with the graded resolution, $GRes(x, p, q, a)$) we associate a Diophantine set $D_1$, for which $Cov(p, q) \cup D_1$ is equational, and $D_1$ has smaller complexity than $Cov(p, q)$. To analyze the set $Cov(p, q)$ and construct the Diophantine set $D_1$, we look at the entire collection of specializations:

$$(x_1, \dots , x_s, y_1, \dots , y_m, r_1, \dots , r_s, p, q, a)$$

for which:

(i) for the tuple $p, q$ there exist precisely $s$ rigid (strictly solid families of) specializations of the rigid (solid) limit group, $Rgd(x, p, q, a)$ ($Sld(x, p, q, a)$), and at least (total number of) $m$ distinct rigid and strictly solid families of specializations of the terminal (rigid and solid) limit groups of the closures: $GCl_1(z,x,p,q,a),\dots,GCl_t(z,x,p,q,a)$.

(ii) in case the terminal limit groups of $GRes$ is rigid, the $x_i$'s are the distinct rigid specializations of $Rgd(x,p,q,a)$. In case the terminal limit group of $GRes$ is solid, the $x_i$'s belong to the $s$ distinct strictly solid families of $Sld(x,p,q,a)$.

(iii) the $y_j$'s are either distinct rigid specializations or belong to distinct strictly solid families of specializations of the terminal limit groups of the closures: $GCl_1,\dots,GCl_t$.

(iv) the $r_i$'s are variables that are added only in case the terminal limit group of $GRes$ is solid. In this case the $r_i$'s demonstrate that the (ungraded) resolutions associated with the given closures and the specializations $y_1,\dots,y_m$, form a covering closure of the (ungraded) resolutions associated with the resolution $GRes$ and the specializations $x_1,\dots,x_s$. These include primitive roots of the specializations of all the non-cyclic abelian groups, and edge groups, in the abelian decomposition associated with the solid terminal limit group of $GRes$, $Sld(x,p,q,a)$, and variables that demonstrate that multiples of these primitive roots up to the least common multiples of the indices of the finite index subgroups associated with the graded closures, $GCl_1,\dots,GCl_t$, factor through the ungraded resolutions associated with the specializations $y_1,\dots,y_m$ and their corresponding closures (cf. section 1 of [Se5] in which we added similar variables to form valid proof statements, that initialize the sieve procedure).

We look at the collection of such specializations for all the possible values of $s$ and $m$ (note that $s$ and $m$ are bounded, since the number of rigid specializations of a rigid limit group and the number of strictly solid families of specializations of a solid limit group associated with a given specialization of the defining parameters are globally bounded by theorems 2.5, 2.9 and 2.13 in [Se3]).

With this collection we associate a canonical finite collection of maximal limit groups, that we view as graded with respect to the parameter subgroup $<q>$. With these graded limit groups we associate the (graded) resolutions that appear in their strict graded Makanin-Razborov diagrams, and the resolutions associated with the various strata in the singular loci of the diagrams. Since we assumed that the terminal limit group $Rgd(x,p,q,a)$ ($Sld(x,p,q,a)$) are of minimal (graded) rank, all the resolutions in these graded Makanin-Razborov diagrams are of minimal (graded) rank.

Given a (graded) completion, $Comp(x_1,\dots,x_s,y_1,\dots,y_m,r_1,\dots,r_s,p,z,q,a)$, of one of these graded resolutions, we look at all the specializations that factor through that completion, and for which either:

(1) (at least) one of the specializations $x_1,\dots,x_s,y_1,\dots,y_m$ that is supposed to be rigid or strictly solid is not rigid or not strictly solid.

(2) two of these specializations that are supposed to be rigid and distinct coincide, or two strictly solid specializations that are supposed to belong to distinct strictly solid families belong to the same strictly solid family. Note that if a given specialization is not strictly solid, or if two specializations belong to the same strictly solid family, then the (ambient) specialization of the given completion has to satisfy at least one of finitely many (prescribed)

Diophantine conditions that are associated with the finitely many solid limit groups that are associated with the given specializations (see definition 1.5 in [Se3]).

(3) a specialization of what is supposed to be a primitive root, $r_i$, has a root of order that is not co-prime to the least common multiple of the indices of the finite index subgroups that are associated with the corresponding graded closures $GCl_1, \dots, GCl_t$. Note that these specializations satisfy (at least) one of finitely many Diophantine conditions.

(4) there exists an extra rigid (strictly solid family of) specialization(s) of the rigid limit group $Rgd(x,p,q,a)$ ($Sld(x,p,q,a)$), in addition to those specified by the specializations $x_1, \dots, x_s$.

With this collection of specializations together with specializations of extra variables that are being added to demonstrate the Diophantine conditions they satisfy, or the extra rigid or strictly solid specialization (case (4)), we canonically associate a finite collection of graded limit groups, and the finite collection of graded (minimal rank) resolutions associated with these graded limit groups according to the iterative procedure for the analysis of minimal rank quotient resolutions presented in section 1 of [Se5]. By this iterative procedure, the complexity of each of the obtained resolutions is bounded above by the complexity of the completion we started with, and in case of equality in complexities, an obtained resolution has to be a graded closure of the completion we started with (see theorem 1.17 in [Se5]).

Furthermore, as we did in the proofs of theorems 1.2 and 1.4, with the completion, $Comp(x_1, \dots, x_s, y_1, \dots, y_m, p, z, q, a)$, we associate the collection of all the sequences:

$$\{(f(n), x_1(n), \dots, x_s(n), y_1(n), \dots, y_m(n), p(n), z(n), q(n), a)\}_{n=1}^{\infty}$$

so that the restricted sequences, $\{(x_1(n), \dots, x_s(n), y_1(n), \dots, y_m(n), p(n), z(n), q(n), a)\}_{n=1}^{\infty}$, are test sequences with respect to the given completion (see definition 1.20 in [Se2]), and the specializations $f(n)$ were added to demonstrate that the corresponding specialization of the given completion satisfies one of the finitely many (Diophantine) conditions (1)-(4).

By the techniques that were used to analyze graded formal limit groups, presented in section 3 of [Se2], with this collection of sequences it is possible to canonically associate a finite collection of (graded) closures of the completion, $Comp(x_1, \dots, x_s, y_1, \dots, y_m, p, z, q, a)$, through which they factor. We will denote these graded closures:

$$Cl_i(x_1, \dots, x_s, y_1, \dots, y_m, p, f, z, q, a)$$

(note that these closures are graded with respect to the parameter subgroup $< q >$).

With each of the closures, $Cl_i(x_1, \dots, x_s, y_1, \dots, y_m, p, f, z, q, a)$, that is constructed from a test sequence of specializations that satisfy case (4), i.e., a test sequence of specializations for which there exists an extra rigid (strictly solid family of) specialization(s) of the rigid (solid) limit group $Rgd(x,p,q,a)$ ($Sld(x,p,q,a)$), we further collect all the specializations and the test sequences of specializations that factor through that closure, $Cl_i$, and for which the corresponding extra rigid or strictly solid specialization is either flexible or it coincides with one of the rigid specializations $x_1, \dots, x_s$ (in the rigid case), or it belongs to one of the strictly solid

families associated with the specializations $x_1, \dots, x_s$ (in the solid case). Note that these collapses of the extra specialization can be enforced by one of finitely many Diophantine conditions, as we did in cases (1) and (2).
As we did earlier, with the collection of specializations and test sequences of specializations of $Cl_i$ for which the corresponding extra rigid or strictly solid specialization collapses, we can associate a finite collection of graded (minimal rank) resolutions (graded with respect to the parameter subgroup $< q >$), so that their complexity is bounded above by the complexity of the closure $Cl_i$ we have started with, and in case of equality, the structure of such resolution is similar to structure of the closure $Cl_i$.

We set the Diophantine set $D_1$ to be the disjunction of all the graded resolutions that are obtained through this process that have strictly smaller complexity than the completion they are associated with, and those closures that are associated with the finite set of completions and are proper closures of the completion they are associated with. By the same argument that was used in proving theorem 1.3, the set $Cov(p,q) \cup D_1$ is equational.

As in analyzing rigid and solid limit groups, we continue by analyzing the set $Cov(p,q) \cap D_1$. The rest of the argument is identical to that used in proving theorems 1.2 and 1.4.

□

Proving that (minimal rank) Diophantine sets are equational, that (in minimal (graded) rank) sets for which a rigid or solid limit group have at least $s$ rigid (strictly solid families of) specializations, are in the Boolean algebra of equational sets, and that the set of specializations of the defining parameters for which a given set of closures forms a covering closure of a given graded resolution (assuming its terminating rigid or solid limit group is of minimal (graded) rank), is in the Boolean algebra of equational sets, we are finally ready to prove the main theorem of this section, i.e., that minimal rank definable sets are in the Boolean algebra of equational sets.

**Theorem 1.6.** *Let $F_k =< a_1, \dots, a_k >$ be a non-abelian free group, and let $L(p,q)$ be an EAE set:*

$$L(p,q) \;=\; \exists w \; \forall y \; \exists x \; (\Sigma_1(x,y,w,p,q,a) = 1 \wedge \Psi_1(x,y,w,p,q,a) \neq 1) \; \vee \dots$$
$$\dots \vee \; (\Sigma_r(x,y,w,p,q,a) = 1 \wedge \Psi_r(x,y,w,p,q,a) \neq 1)$$

*With each system $\Sigma_i(x,y,w,p,q,a)$ we can formally associate a finitely presented group, $G(\Sigma_i)$, which is generated by $< x,y,w,p,q,a >$ and its relations are precisely the equations in the system $\Sigma_i$. Assume that for all the finitely presented groups $G(\Sigma_i)(x,y,w,p,q,a)$, if $G(\Sigma_i)$ admits an epimorphism onto a free group $F_k * F$, where the subgroup $< a >$ is mapped naturally onto the coefficient group $F_k$, and the subgroup $< q >$ is mapped into $F_k$, then the subgroup $< p >$ is mapped into $F_k$ as well (in this case we consider the EAE set $L(p,q)$ to be of minimal (graded) rank).*

*Then the EAE set $L(p,q)$ is in the Boolean algebra of equational sets.*

*Proof:* To analyze the minimal (graded) rank set $L(p,q)$, we use the precise description of a definable set that was obtained using the sieve procedure for quantifier elimination presented in [Se6]. Recall that with the set $L(p,q)$ the sieve procedure associates a finite collection of graded (PS) resolutions that terminate in rigid and solid limit groups (with respect to the parameter subgroup $< p, q >$), and with each such graded resolution it associates a finite collection of graded closures that are composed from Non-Rigid, Non-Solid, Left, Root, Extra PS, and collapse extra PS resolutions (see definitions 1.25-1.30 of [Se5] for the exact definitions).

By the construction of the sieve procedure, and the procedure for the construction of the tree of stratified sets (presented in section 2 of [Se5]) that precedes it, since the set $L(p, q)$ is assumed to be of minimal (graded) rank, all the terminating rigid and solid limit groups of the PS resolutions associated with $L(p, q)$ are of minimal (graded) rank as well. By the sieve procedure, that eventually leads to quantifier elimination over a free group, the definable set $L(p, q)$ is equivalent to those specializations of the terminal rigid and solid limit groups of the PS resolutions constructed along the sieve procedure, for which the collection of Non-Rigid, Non-Solid, Left, Root and extra PS resolutions (minus the specializations that factor through the associated collapse extra PS resolutions) do not form a covering closure.

Therefore, using the output of the sieve procedure and the resolutions it constructs, with each terminating rigid or solid limit group $Term$ of a PS resolution along it we associate finitely many sets:

(1) $B_1(Term)$ - the set of specializations of $< p, q >$ for which the terminal rigid or solid limit group $Term$ admits rigid or strictly solid specializations.
(2) $B_2(Term)$ - the set of specializations of $< p, q >$ for which the associated Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through the associated collapse extra PS resolutions), associated with the PS resolution that terminates in $Term$, form a covering closure of all the (ungraded) PS resolutions associated with the rigid or strictly solid specializations that are associated with the given specialization of $< p, q >$.
(3) $B_3(Term)$ - the set of specializations of $< p, q >$ for which the Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through associated collapse extra PS resolutions) form a covering closure of all the (ungraded) PS resolutions associated with a given specialization of $< p, q >$ and with PS resolutions that extend the PS resolution that terminates in $Term$, and for which there exist rigid or strictly solid specializations of $Term$ with respect to that covering closure.
(4) $B_4(Term)$ - the set of specializations of $< p, q >$ in $B_3(Term)$, for which the associated Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through the associated collapse extra PS resolutions), associated with the PS resolution that terminates in $Term$, form a covering closure of all the (ungraded) PS resolutions associated with the rigid or strictly solid specializations of $Term$, where these rigid and strictly solid specializations are associated with a given specialization of $< p, q >$, and are taken with respect to the covering closure of all the PS resolutions that extend the PS resolution that terminates in $Term$, that is associated with their collections of Non-Rigid, Non-Solid, Left, Root and extra PS resolutions.

Let $Term$ be one of the terminal rigid or solid limit groups of the PS resolutions constructed along the sieve procedure associated with the definable set $L(p, q)$,

$Term_1, \dots, Term_m$. By theorems 1.4 and 1.5, $B_1(Term)$ and $B_2(Term)$ are in the Boolean algebra of equational sets. Furthermore, precisely the same argument and procedures that were used in proving theorem 1.5, i.e., in proving that the collection of rigid or strictly solid specializations of a given rigid or solid limit group, for which a given collection of closures forms a covering closure, is in the Boolean algebra of equational sets, imply that the sets $B_3(Term)$ and $B_4(Term)$ are in the Boolean algebra of equational sets.

By the structure of the sieve procedure [Se6], if $Term_1, \dots, Term_m$ are the terminal rigid or solid limit groups of the PS resolutions constructed along the sieve procedure associated with the definable set $L(p,q)$, then $L(p,q)$ is the finite union:

$$L(p,q) \ = \cup_{i=1}^{m} (B_1(Term_i) \setminus B_2(Term_i)) \cup (B_3(Term_i) \setminus B_4(Term_i)).$$

Therefore, since the sets: $B_1(Term_i), \dots, B_4(Term_i)$, $1 \leq i \leq m$, are in the Boolean algebra of equational sets, so is the minimal (graded) rank definable set $L(p,q)$.

## §2. Diophantine Sets

Our first step in approaching the stability of free (and hyperbolic) groups, is proving that Diophantine sets are equational. This was proved in theorem 1.1 in the minimal rank case, and is more involved though still valid in general.

**Theorem 2.1.** *Let $F_k = < a_1, \dots, a_k >$ be a non-abelian free group, and let*

$$D(p,q) \ = \ \{ \, (p,q) \ \exists x \ \Sigma(x,p,q,a) = 1 \, \}$$

*be a Diophantine set defined over $F_k$. Then $D(p,q)$ is equational.*

*Proof:* With the system of equations $\Sigma(x,p,q,a) = 1$ we associate its graded Makanin-Razborov diagram (with respect to the parameter subgroup $< p,q >$), and we look at the (finite) collection of rigid limit groups, $Rgd(x,p,q,a)$, and solid limit groups, $Sld(x,p,q,a)$, along the diagram. By the properties of the graded diagram, the Diophantine set $D(p,q)$ is precisely the collection of specializations of the parameter subgroup $< p,q >$, for which at least one of the rigid or solid limit groups along the graded diagram of $\Sigma(x,p,q,a)$ admits a rigid or a strictly solid solution.

To prove the equationality of a general Diophantine set $D(p,q)$, we associate with it a finite diagram, similar but somewhat different to the one we associated with a minimal rank Diophantine set in proving theorem 1.1. To prove the termination of the iterative procedure that is used for the construction of the diagram, we apply the techniques that were used in proving the termination of the sieve procedure that was used in obtaining quantifier elimination in [Se6].

We start the construction of the diagram by collecting all the specializations of the form $(x,p,q,a)$ that are rigid or strictly solid specializations of one of the rigid or solid limit groups that appear along the graded Makanin-Razborov diagram of the system $\Sigma(x,p,q,a)$. This collection factors through a canonical finite collection of maximal limit groups, that we denote $L_i(x,p,q,a)$. With a limit group $L_i(x,p,q,a)$, viewed as a graded limit group with respect to the parameter subgroup $< q >$, we associate its taut graded Makanin-Razborov diagram. With each resolution in the taut Makanin-Razborov diagram, we further associate its singular locus, and the graded resolutions associated with each of the strata in the singular locus. We conclude the first step of the construction of the diagram, by associating the (graded) completion with each of the graded resolutions in our finite collection, that we denote, $Comp(z,x,p,q,a)$.

We continue to the construction of the second step of the diagram with each of the completions $Comp(z,x,p,q,a)$ in parallel. With each such completion we associate the collection of specializations $(x_2,z,x_1,p,q_1,q_2,a)$ for which:

(1) $(z,x_1,p,q_1,a)$ factors through the completion, $Comp(z,x,p,q,a)$, and $(x_1,p,q_1,a)$ is rigid or strictly solid with respect to one of the rigid or solid limit groups in the graded diagram of $\Sigma(x,p,q,a)$.

(2) $(x_2,p,q_2,a)$ is a rigid or a strictly solid specialization of one of the rigid or solid limit groups in the graded diagram of $\Sigma(x,p,q,a)$.

With the completion, $Comp(z,x,p,q,a)$, we associate the collection of all the sequences:

$$\{(x_2(n),z(n),x_1(n),p(n),q_1(n),q_2(n),a)\}_{n=1}^{\infty}$$

so that for each $n$, the corresponding specialization satisfies conditions (1) and (2), and the sequence: $\{(z(n),x_1(n),p(n),q_1(n),a)\}_{n=1}^{\infty}$ forms a (graded) test sequence with respect to the given (graded) completion $Comp(z,x,p,q,a)$. By the techniques used to analyze graded formal limit groups, presented in section 3 of [Se2], with this collection of sequences it is possible to canonically associate a finite collection of (graded) closures, through which they factor. We will denote these graded closures, $Cl_i(x_2,z,x_1,p,q_1,q_2,a)$ (note that these closures are graded with respect to the parameter subgroup $<q_1,q_2>$). We continue to the next (third) step of the iterative procedure only with those closures that are proper closures of the completion, $Comp(z,x,p,q,a)$, we have started this branch of the second step with.

We further look at all the specializations, $(x_2,z,x_1,p,q_1,q_2,a)$, that satisfy conditions (1) and (2), and do not factor through any of the (finite set of) closures, $Cl_i(x_2,z,x_1,p,q_1,q_2,a)$. With this collection of specializations we can associate a canonical finite collection of maximal limit groups, $M_j(x_2,z,x_1,p,q_1,q_2)$, which we view as graded limit groups with respect to the parameter subgroup $<q_1,q_2>$.
Using the iterative procedure for the construction of resolutions associated with a Diophantine set, used in each step of the sieve method for quantifier elimination and presented in [Se6], we associate with this collection of specializations finitely many graded resolutions with respect to the defining parameters $<q_1,q_2>$, and with each such graded resolution we associate its finitely many (multi-graded) core resolutions, developing resolutions, anvils, and (possibly) sculpted resolutions and carriers (see (the first step in) [Se6] for a detailed description of the iterative construction of the graded resolutions and the resolutions attached to them).

We continue iteratively. With each resolution constructed at the $s$-th step, we associate its sequence of core resolutions, a developing resolution, (possible) sculpted resolutions, (possible) carriers, and an anvil, precisely as we did in the sieve procedure (see (the general step in) [Se6] for a detailed description of these associated objects).
Given a developing resolution that is associated with an anvil that was constructed in the $s$-th step of the procedure, we look at the collection of sequences of special-

izations:

$$(x_{s+1}(n), w(n), p(n), q_1(n), \ldots, q_{s+1}(n), a)$$

for which the specializations $(x_{s+1}(n), p(n), q_{s+1}(n))$ are rigid or strictly solid with respect to one of the (finitely many) rigid or solid limit groups associated with our given Diophantine set, and the corresponding restricted specializations, $(w(n), p(n), q_1(n), \ldots, q_s(n), a)$ form a test sequence with respect to the given developing resolution. Given all these sequences, we apply the techniques presented in section 3 of [Se2] and associate with the developing resolution a finite set of (graded) closures. As in the second step of the procedure, we continue to the next step only with those closures that are proper closures of the developing resolution we have started this branch of the $(s+1)$ step of the procedure.

At this point we look at all the specializations:

$$(x_{s+1}(n), t(n), p(n), q_1(n), \ldots, q_{s+1}(n), a)$$

for which the specialization $(x_{s+1}(n), p(n), q_{s+1}(n), a)$ are rigid or strictly solid with respect to one of the (finitely many) rigid or solid limit groups associated with our given Diophantine set, and the specialization, $(t(n), p(n), q_1(n), \ldots, q_s(n), a)$ factors through an anvil that was constructed in the $s$-th step of the procedure. We further assume that the specialization $(x_{s+1}(n), p(n), q_{s+1}(n), a)$ can not be completed to a specialization that factors through one of the closures associated with the developing resolution that is associated with the corresponding anvil.
By our standard techniques, that were presented in section 5 of [Se1], with this collection of specializations we associate its Zariski closure, that is associated with a (canonical) finite collection of (graded) limit groups. Given these limit groups, and the given (finite) collection of anvils, core resolutions, developing resolutions, (possible) sculpted resolutions and carriers, constructed in the previous steps of the iterative procedure, we apply the construction that was used in the general step of the sieve procedure and presented in [Se6], to construct a finite collection of multi-graded resolutions, with which there are associated core resolutions, developing resolutions, (possible) sculpted resolution and carriers, and anvils.

**Proposition 2.2.** *The iterative procedure associated with a Diophantine set terminates after finitely many steps.*

*Proof:* To prove termination we use essentially the same argument that was used to prove the termination of the sieve procedure in ([Se6],22). Unfortunately, the sieve procedure is long and technical, hence, we can not repeat even the definitions of the objects that are constructed along it, and are used in proving the termination. Therefore, for the rest of the proof we will assume that the reader is familiar with the structure of the sieve procedure, the objects that are constructed along it, and the proof of its termination, that are all presented in [Se6].

Since our procedure is a locally finite branching process, if it doesn't terminate it must contain an infinite path. Since the construction of the resolutions and the anvils we use is identical to the construction used in the sieve procedure, proposition 26 in [Se6] remains valid, i.e., given an infinite path of our procedure, for each positive integer $m$, there exists a step $n_m$ and width $d_m$, so that the sculpted and penetrated sculpted resolutions of width $d_m$ at step $n_m$ are all eventual (i.e., they do not change along the rest of the infinite path), and the number of $n_m$ sculpted resolutions of width $d_m$, $sc(n_m, d_m)$, satisfies: $sc(n_m, d_m) = m$.

Therefore, as in theorem 27 in [Se6], to conclude the proof of theorem 2.2, i.e., to prove the termination of the procedure for the construction of the diagram associated with a Diophantine set after finitely many steps, we need to show the existence of a global bound on the number of eventual sculpted resolutions of the same width that are associated with the anvils along an infinite path of the procedure.

Our approach towards obtaining a bound on the number of eventual sculpted resolutions with the same width along an infinite path of the procedure, is essentially identical to the one used to prove theorem 27 in [Se6], and is based on the argument that was used to obtain a bound on the number of rigid and strictly solid families of solutions (with respect to a given covering closure) of rigid and solid limit groups, presented in the first two sections of [Se3] (theorems 2.5,2.9 and 2.13 in [Se3]).

Recall that in proving theorem 27 in [Se6], we argued that if there is no bound (independent of the width) on the number of eventual sculpted resolutions of the same width associated with an anvil along a given infinite path of the sieve procedure, then there must be two sequences of specializations of the same rigid or solid limit group: $\{(x_i(n), w(n), p(n), a)\}$ and $\{(x_j(n), w(n), p(n), a)\}$, for some $j > i$, so that for every $n$: $x_i(n) = x_j(n)$ in the rigid case, and $(x_i(n), w(n), p(n), a)$ belongs to the same family of $(x_j(n), w(n), p(n), a)$ in the solid case. However, the specializations $\{(x_j(n), w(n), p(n), a)\}$ are assumed to be either (extra) rigid or strictly solid, and the specializations $\{(x_i(n), w(n), p(n), a)\}$ are assumed to be flexible, and we got a contradiction, hence, we obtained a bound on the number of eventual sculpted resolutions of the same width.

Assume that our procedure for the construction of the diagram associated with a Diophantine set contains an infinite path, and along this path there is no bound (independent of the width) on the number of eventual sculpted resolutions of the same width that are associated with an anvil along this given infinite path of the procedure.

First, we observe that in a test sequence of each of the developing and sculpted resolutions associated with the anvils along our iterative procedure, we may assume that the lengths of the parts of the variables $x_i(n)$, and the parameters $p(n)$, that do not belong to the bottom level of the graded sculpted resolutions, are much bigger than the lengths of the specializations $q_i(n)$ (that are assumed to be part of the bottom level). Therefore, in applying the argument that was used in proving theorem 27 in [Se6] to the sculpted resolutions that were constructed along an infinite path of our procedure, in obtaining a sequence of compatible JSJ decompositions that are used in analyzing the sequences $\{(x_i(n), p(n), q_i(n), a)\}$ (see theorem 36 in [Se6]), which is a restriction of a test sequence of some developing resolution along the infinite path, the subgroup $< q_i >$ remains elliptic, until we approach the bottom level of the (eventual) sculpted resolution in question.

Hence, by applying the same argument that was used to prove theorem 27 in [Se6], we obtain two sequences of rigid or strictly solid specializations of the same rigid or solid limit group: $\{(x_i(n), p(n), q_i(n), a)\}$ and $\{(x_j(n), p(n), q_j(n), a)\}$, for some $j > i$, along a given infinite path, that are compatible in all the levels except, perhaps, the bottom level. This contradicts the assumption that along our iterative procedure, we collected specializations that do not factor through the closures associated with developing resolutions constructed in the previous steps of the procedure. Therefore, we obtained a global bound on the number of eventual sculpted resolutions of the same width along an infinite path of our procedure, so we got a contradiction to the existence of an infinite path in the procedure for the construction of the diagram associated with a Diophantine set, that implies that the procedure terminates after finitely many steps.

□

By theorems 2.5, 2.9 and 2.13 in [Se3] the number of rigid or strictly solid families of specializations associated with a given specialization of the defining parameters of a rigid or solid limit group is bounded by a bound that depends only on the rigid or solid limit group. This global bound on the number of rigid and strictly solid families, together with the finiteness of the constructed diagram, and the finiteness of the number of closures dropped along each step of the construction, imply the equationality of Diophantine predicates.

□

## §3. Duo Limit Groups

In section 1 we have shown that in the minimal rank case Diophantine sets are equational, and then used it to show that the sets $NR_s$ ($NS_s$), that indicate those values of the parameter set $< p, q >$, for which a minimal rank rigid limit group $Rgd(x, p, q, a)$ ($Sld(x, p, q, a)$) admits at least $s$ rigid (strictly solid families of) specializations, is in the Boolean algebra generated by equational sets (theorems 1.2,1.4).

In the previous section, we have shown that general Diophantine sets are equational. Unlike the minimal rank case, as we will see in the sequel (section 6), the sets $NR_s$ and $NS_s$, associated with general rigid and solid limit groups, are not always contained in the Boolean algebra generated by equational sets. However, as we show in the next section, sets of the form $NR_s$ and $NS_s$ are stable.

In this section we present the main tool needed for proving the stability of the sets $NR_s$ and $NS_s$ (and afterwards the stability of general definable sets over a free group), that we call *duo limit groups*. We first present duo limit groups associated with configuration limit groups of rigid and solid limit groups, and then prove the existence of a finite collection of duo limit groups associated with a configuration limit group that "covers" all the other duo limit groups associated with a rigid limit group (we leave this property for duo limit groups associated with a solid limit group as an open question).

**Definition 3.1.** *Let $F_k$ be a non-abelian free group, and let $Rgd(x, p, q, a)$ ($Sld(x, p, q, a)$) be a rigid (solid) limit group with respect to the parameter subgroup $< p, q >$. Let $s$ be a (fixed) positive integer, and let $Conf(x_1, \dots, x_s, p, q, a)$ be a configuration limit group associated with the limit group $Rgd(x, p, q, a)$ ($Sld(x, p, q, a)$) (see definition 4.1 in [Se3] for configuration limit groups). Recall that a configuration limit group is obtained as a limit of a convergent sequence of specializations $(x_1(n), \dots, x_s(n), p(n), q(n), a)$, called configuration homomorphisms ([Se3],4.1), in which each of the specializations $(x_i(n), p(n), q(n), a)$ is rigid (strictly solid) and $x_i(n) \neq x_j(n)$ for $i \neq j$ (belong to distinct strictly solid families). See section 4 of [Se3] for a detailed discussion of these groups.*

*A duo limit group, $Duo(d_1, p, d_2, q, d_0, a)$, is a limit group with the following properties:*

(1) *with Duo there exists an associated map:*

$$\eta : Conf(x_1, \dots, x_s, p, q, a) \to Duo.$$

*For brevity, we denote* $\eta(p), \eta(q), \eta(a)$ *by* $p, q, a$ *in correspondence.*

(2) $Duo = < d_1 > *_{<d_0>} < d_2 >$, $\eta(F_k) = \eta(< a >) << d_0 >$, $\eta(< p >) << d_1 >$, *and* $\eta(< q >) << d_2 >$.

(3) $Duo = Comp(d_1, p, a) *_{<d_0>} Comp(d_2, q, a)$, *where* $Comp(d_1, p, a)$ *and* $Comp(d_2, q, a)$, *are (graded) completions with respect to the parameter subgroup* $< d_0 >$, *that terminate in the subgroup* $< d_0 >$.

(4) *there exists a specialization* $(x_1, \ldots, x_s, p, q, a)$ *of the configuration limit group* $Conf$, *for which the corresponding elements* $(x_i, p, q, a)$ *are distinct and rigid specializations of the rigid limit group,* $Rgd(x, p, q, a)$ *(strictly solid and belong to distinct strictly solid families), that can be extended to a specialization that factors through the duo limit group* $Duo$ *(i.e., there exists a configuration homomorphism that can be extended to a specialization of* $Duo$*).*

*Given a duo limit group,* $Duo(d_1, p, d_2, q, d_0, a)$, *and a specialization of the variables* $d_0$, *we call the set of specializations that factor through* $Duo$ *for which the specialization of the variables* $d_0$ *is identical to the given one, a duo-family. We say that a duo family associated with a duo limit group* $Duo$ *is covered by the duo limit groups* $Duo_1, \ldots, Duo_t$, *if there exists a finite collection of duo families associated with the duo limit groups,* $Duo_1, \ldots, Duo_t$, *and a covering closure of the duo family, so that each configuration homomorphism that can be extended to a specialization of a closure in the covering closure of the given duo family, can also be extended to a specialization that factors through one of the members of the finite collection of duo families of the duo limit groups* $Duo_1, \ldots, Duo_t$ *(see definition 1.16 in [Se2] for a covering closure).*

The procedure that was used to prove the equationality of Diophantine sets in the previous section, enables one to prove the existence of a finite collection of duo limit groups, that cover all the duo families associated with a duo limit group that is associated with a given rigid limit group.

**Theorem 3.2.** *Let* $F_k$ *be a non-abelian free group, let* $s$ *be a positive integer, and let* $Rgd(x, p, q, a)$ *be a rigid limit group defined over* $F_k$. *There exists a finite collection of duo limit groups associated with configuration homomorphisms of* $s$ *distinct rigid homomorphisms of* $Rgd$, $Duo_1, \ldots, Duo_t$, *and some global bound* $b$, *so that every duo family that is associated with a duo limit group* $Duo$, *that is associated with configuration homomorphisms of* $s$ *distinct rigid homomorphisms of* $Rgd$, *is covered by the given finite collection* $Duo_1, \ldots, Duo_t$. *Furthermore, every duo family that is associated with an arbitrary duo limit group* $Duo$, *is covered by at most* $b$ *duo families that are associated with the given finite collection,* $Duo_1, \ldots, Duo_t$.

*Proof:* To construct the (finite) universal collection of duo limit groups, we apply the iterative procedure that was used to prove the equationality of Diophantine sets (theorem 2.1).

First, we associate with the given rigid limit group $Rgd(x, p, q, a)$ and the given positive integer $s$, a finite collection of configuration limit groups (as we did in section 4 of [Se3]). To do that we collect all the specializations of the form $(x_1, \ldots, x_s, p, q, a)$, for which each specialization $(x_i, p, q, a)$ is a rigid specialization of the rigid limit group $Rgd(x, p, q, a)$, with respect to the parameter subgroup $< p, q >$, and so that for each $i, j$, $1 \leq i < j \leq s$, the rigid specializations $(x_i, p, q, a)$ and $(x_j, p, q, a)$ are distinct. By the standard arguments presented in section 5 of [Se1], with this collection of specializations $\{(x_1,\ldots,x_s,p,q,a)\}$, we can canonically associate a finite collection of (configuration) limit groups, $Conf_i(x_1,\ldots,x_s,p,q,a)$, $1\leq i\leq m$.

With each of the configuration limit groups $Conf_i(x_1,\ldots,x_s,p,q,a)$, viewed as a graded limit group with respect to the parameter subgroup $<q>$, we associate its taut graded Makanin-Razborov diagram. With each resolution in the taut Makanin-Razborov diagram, we further associate its singular locus, and the graded resolutions associated with each of the strata in the singular locus. We conclude the first step of the construction of the diagram, by associating the (graded) completion with each of the graded resolutions in our finite collection, that we denote, $Comp(z,p,q,a)$.

We continue to the construction of the second step of the diagram with each of the completions $Comp(z,p,q,a)$ in parallel. With each such completion we associate the collection of specializations $(y_1,\ldots,y_s,z,p,q_1,q_2,a)$ for which:

(1) $(z,p,q_1,a)$ factors through the completion, $Comp(z,p,q,a)$, each of the associated specializations $(x_i,p,q_1,a)$, $1\leq i\leq s$, is rigid with respect to the given rigid limit group $Rgd(x,p,q,a)$, and any two rigid specializations, $(x_i,p,q,a)$ and $(x_j,p,q,a)$, are distinct for $1\leq i<j\leq s$.
(2) each of the specializations $(y_i,p,q_2,a)$, $1\leq i\leq s$, is a rigid specialization of the rigid limit group $Rgd(x,p,q,a)$, and any two rigid specializations, $(y_i,p,q,a)$ and $(y_j,p,q,a)$, are distinct for $1\leq i<j\leq s$.

With the completion, $Comp(z,p,q,a)$, we associate the collection of all the sequences:

$$\{(y_1(n),\ldots,y_s(n),z(n),p(n),q_1(n),q_2(n),a)\}_{n=1}^{\infty}$$

so that for each $n$, the corresponding specialization satisfies conditions (1) and (2), and the (restricted) sequence $\{(z(n),p(n),q_1(n),a)\}_{n=1}^{\infty}$ form a (graded) test sequence with respect to the given (graded) completion $Comp(z,p,q,a)$. By the techniques used to analyze graded formal limit groups, presented in section 3 of [Se2], with this collection of sequences it is possible to canonically associate a finite collection of (graded) closures of the completion, $Comp(z,p,q,a)$. We will denote these graded closures, $Cl_i(s,z,p,q_1,q_2,a)$ (note that these closures are graded with respect to the parameter subgroup $<q_1,q_2>$).

We further look at all the specializations $(y_1,\ldots,y_s,z,p,q_1,q_2,a)$ that satisfy conditions (1) and (2), and do not factor through any of the (finite) closures, $Cl_i(s,z,p,q_1,q_2,a)$. With this collection of specializations we can associate a canonical finite collection of maximal limit groups, $M_j(y_1,\ldots,y_s,z,p,q_1,q_2)$, which we view as graded limit groups with respect to the parameter subgroup $<q_1,q_2>$.

Using the iterative procedure for the construction of resolutions associated with a Diophantine set, used in each step of the sieve method for quantifier elimination and presented in [Se6], we associate with this collection of specializations finitely many multi-graded resolutions, and with each such multi-graded resolution we associate its (multi-graded) core resolutions, developing resolutions, anvils, and (possibly) sculpted resolutions and carriers (see [Se6] for a detailed description of the iterative construction of the multi-graded resolutions and the resolutions attached to them).

We continue iteratively, precisely as we did in proving the equationality for Diophantine sets (theorem 2.1). Finally we get:

**Proposition 3.3.** *The iterative procedure presented above terminates after finitely* *many steps.*

*Proof:* Identical to the proof of proposition 2.2.

□

To define the universal set of duo limit groups, claimed in theorem 3.2, we start with the collection of closures constructed along the terminating iterative procedure, and with each such closure we associate a finite collection of duo limit groups. Given a (graded) closure, $Cl_i$, constructed along the diagram, we associate with it finitely many duo limit groups. To construct these duo limit groups, we look at the entire collection of graded test sequences that factor through the given graded closure, $Cl_i$, and can be extended to $s$ distinct rigid specializations of the given rigid limit group, $Rgd(x,p,q,a)$. With this entire collection of graded test sequences, we associate a graded Makanin-Razborov diagram, precisely as we did in constructing the formal graded Makanin-Razborov diagram in section 3 of [Se2]. As in the formal Makanin-Razborov diagram, each resolution in the diagram we construct terminates with a (graded) closure of the given closure, $Cl_i$, we have started with, amalgamated with another group along its base (which is the terminal rigid or solid limit group).

By construction, a completion of a resolution in one of the constructed graded diagrams is a duo limit group. We take the completions of the resolutions that appear in the entire collection of diagrams associated with the various closures, $Cl_i$, to be the finite collection of (universal) duo limit groups, $Duo_1, \dots, Duo_t$

Let $Duo$ be a duo limit group that is associated with the given rigid limit group, and suppose that we are given a duo family, *duo*, that factors through it, i.e., a family that is associated with a given specialization of the variables $d_0$ in the duo limit group $Duo$. We need to show that the given duo family, *duo*, that factors through the duo limit groups $Duo$, is covered by a bounded collection of duo families that factor through the (universal) finite collection of duo limit groups $Duo_1, \dots, Duo_t$. Note that the bound on the number of duo families of $Duo_1, \dots, Duo_t$ is global and does not depend on the duo limit group $Duo$.

Given the duo family, *duo*, we start with a specialization $q_1$ of the parameters $q$, that can be extended to a specialization of the duo family, *duo*, $(d_1, d_0, d_2)$, that restricts to a *configuration homomorphism* of the corresponding configuration limit group (see definition 4.1 in [Se3]), i.e., a specialization of the duo family that satisfies property (4) of a duo limit group (definition 3.1). With this specialization $q_1$ of the parameters $q$ we associate the boundedly many fibers that are associated with it in the initial level of the diagram that was constructed iteratively from the sets of configuration homomorphisms.

We look at all the duo test sequences that factor through the given duo family, *duo*, that is covered by the duo limit group, $Duo$, and restrict to configuration homomorphisms. i.e., sequences of specializations that belong to the given duo family, satisfy property (4) in definition 3.1, and form a test sequence to both completions, $Comp(d_1,p,a)$ and $Comp(d_2,q,a)$, that are subgroups of the duo limit group $Duo$ (see definition 3.1).
With this collection of test sequences of the given duo family, we look at those test sequences for which their restrictions to configuration homomorphisms ([Se3],4.1) can be extended to specializations of one of the (boundedly many) duo families, that are duo families of the duo limit groups $Duo_1, \dots, Duo_t$ that were constructed from our universal diagram, and are associated with the collection of boundedly

many fibers that are associated with the specialization $q_1$ in the initial level of the universal diagram.

Given this collection of test sequence of specializations of the given duo family, we extended each of their restrictions to configuration homomorphisms, to the shortest possible homomorphism of the (boundedly many) duo families, that are associated with the duo limit groups $Duo_1, \ldots, Duo_t$, and the boundedly many fibers that are associated with $q_1$ in the initial level of the universal diagram.
By the techniques presented in section 3 of [Se2] (that constructs graded formal limit groups), with this collection of sequences, we can associate finitely many limit groups, that are all *duo closures* of the given duo limit group (associated with the duo family), *duo*, i.e., limit groups that form a closure of both completions, $Comp(d_1, p, a)$ and $Comp(d_2, q, a)$, that are associated with the duo family, *duo*. Furthermore, with each such duo closure, there is an associated map from one of the boundedly many duo families that are associated with the duo limit groups, $Duo_1, \ldots, Duo_t$, into the duo closure.

Since the graded limit group we started with, $Rgd(x, p, q, a)$, is rigid, and the test sequences of the duo family, *duo*, that we consider, restrict to configuration homomorphisms, the collection of test sequences of the duo family, *duo*, that restrict to configuration homomorphisms that can be extended to specializations of the (boundedly many) duo families of the duo limit groups, $Duo_1, \ldots, Duo_t$, that are associated with the fibers that are associated with $q_1$ in the initial level of the universal diagram, has a product structure. i.e., if $\{(d_1(n), p(n), d_2(n), q(n), a)\}$ and $\{(\hat{d}_1(n), \hat{p}(n), \hat{d}_2(n), \hat{q}(n), a)\}$ are such test sequences of *duo*, for which their corresponding (restricted) configuration homomorphisms can be extended to specializations of the same duo family that is associated with $Duo_1, \ldots, Duo_t$, then so is the restrictions to configuration homomorphisms of the test sequence: $\{(\hat{d}_1(n), \hat{p}(n), d_2(n), q(n), a)\}$.

Suppose that the given (finite) set of closures of the given duo family, *duo*, is not yet a covering closure. For each subcollection of the boundedly many duo families associated with $Duo_1, \ldots, Duo_t$ that were chosen previously (i.e., those associated with $q_1$ in the initial level of the universal diagram), we choose a specialization of the parameters $q$ that satisfies the following properties (if such exists):

(i) there exists a specialization of the variables $d_2$ so that $(d_2, q)$ can be extended to a specialization that factors through the duo family, *duo*, and restricts to a configuration homomorphism.

(ii) let $\{(d_1(n), p(n), a)\}$ be an arbitrary test sequence of specializations of $Comp(d_1, p, a)$ (which is a subgroup of the given duo limit group $Duo$), for which the sequence: $\{(d_1(n), p(n), d_2, q, a)\}$ is contained in the duo family, *duo*, and restricts to configuration homomorphisms, and for which no (infinite) subsequence of the sequence, $\{(d_1(n), p(n), a)\}$, can be extended to a test sequence of specializations of the duo family, *duo*, that extend to specializations that factor through one of the closures of *duo* that are associated with the given subcollection of the (boundedly many) duo families that are associated with $Duo_1, \ldots, Duo_t$ (and with $q_1$).

Then no (infinite) subsequence of the sequence $\{d_1(n), p(n), d_2, q, a)\}$ (which is a sequence of specializations of *duo*) restricts to configuration homomorphisms that can be extended to specializations of the given subcollection of the (boundedly many) duo families that are associated with $Duo_1, \ldots, Duo_t$ and with the fibers associated with $q_1$ in the initial level of the universal diagram.

Since so far we have associated with the given duo family, *duo*, only boundedly many duo families that are associated with $Duo_1, \dots, Duo_t$, these have only boundedly many subcollections. We continue only with those subcollections for which there exists a specialization of the parameters $q$, that satisfies properties (i) and (ii).

We continue with each such subcollection in parallel. Given a specialization $q$ that satisfies properties (i) and (ii) with respect to a subcollection, we continue from each of the boundedly many fibers in the initial level of the diagram constructed from $Rgd(x,p,a)$ (and with which we associated the duo families we have started with), to the (boundedly many) fibers that are associated with the specialization $q$ in the next (second) level of that diagram. Since we have started with boundedly many fibers in the initial level of the diagram, we continued with boundedly many specializations of the parameters $q$, and by the finiteness of the diagram associated with $Rgd(x,p,a)$, and the global bounds on the number of rigid specializations of a rigid limit group, and strictly solid families of specializations of a solid limit group obtained in [Se3], the number of new fibers we obtain in the next (second) level of the diagram is globally bounded.

With the boundedly many fibers in the second level of the diagram associated with $Rgd(x,p,a)$, we associate boundedly many duo families that are all associated with the duo limit groups $Duo_1, \dots, Duo_t$. Having the (bounded) collection of duo families, we repeat what we did with the previous collection. First, we look at the collection of test sequences of the given duo family, *duo*, that can be extended to specializations of the new duo families, and construct a finite collection of closures of the duo family, *duo*, where for each of these closures there exists a map from one of the (boundedly many) new duo families into the closure.

Next, we look at all the subcollections of the (bounded) collection of previous and new duo families, for which there exists a specialization $q$ that satisfies properties (i) and (ii). We continue in parallel with these specializations $q$ (that are associated with the boundedly many subcollections of duo families that were associated with the first two levels of the diagram) to the next (third) level of the diagram associated with $Rgd(x,p,a)$, and construct a new additional (bounded) collection of duo families (that are all covered by the universal collection of duo limit groups $Duo_1, \dots, Duo_t$).

Since the diagram associated with $Rgd(x,p,a)$ is finite, the process of enlarging the collection of duo families terminates after finitely many steps, and when the process terminates we are left with a bounded collection of duo families that are all associated with the universal collection of duo limit groups, $Duo_1, \dots, Duo_t$. From the universality of the diagram associated with $Rgd(x,p,a)$, we obtain the covering property, that concludes the proof of theorem 3.2.

**Proposition 3.4.** *The bounded collection of duo families of the universal duo limit groups $Duo_1, \dots, Duo_t$, that were constructed iteratively by going through the levels of the universal diagram and choosing a bounded collection of specializations $q_i$, covers the given duo family duo.*

*Proof:* Each duo family that is associated with one of the universal duo limit groups, $Duo_1, \dots, Duo_t$, covers a (possibly empty) collection of duo test sequences of the given duo family, *duo*, that has a product structure (see the argument above for this product structure). Hence, given a subcollection of these duo families we can naturally divide the duo test sequences of *duo* into finitely many cosets of test sequences of the two completions from which the given duo limit group, *Duo*, is composed: $Comp(d_1,p,d_0,a)$ and $Comp(d_2,q,d_0,a)$.

We start by examining the duo families that are associated with the initial level of the universal diagram. First, we look at a coset of test sequences of $Comp(d_1,p,d_0,a)$ that can be extended to duo test sequences of *duo*, that are covered by the duo families associated with the initial level. In that case we look at the subcollection of duo families that are associated with the initial level, and cover extensions of the given coset of test sequences of $Comp(d_1,p,d_0,a)$ to duo test sequences of *duo*. If the collection of duo test sequences of *duo* associated with this subcollection form a covering for all the test sequences of the completion, $Comp(d_2,q,d_0,a)$, all the extensions to duo sequences of *duo* of the given coset of test sequences of $Comp(d_1,p,d_0,a)$ are covered by the duo families associated with the initial level of the universal diagram. Otherwise, there must exist a specialization $q_2$ of the variables $q$ that satisfies properties (i) and (ii) with respect to the given subcollection. In this case the (boundedly many) fibers associated with $q_2$ in the second level of the universal diagram cover the given coset of test sequences of the completion, $Comp(d_1,p,d_0,a)$.

Suppose that a coset of test sequences of $Comp(d_1,p,d_0,a)$ can not be extended to duo test sequences of *duo*, that are covered by the duo families associated with the initial level. We look at the entire collection of duo families that are associated with the initial level. In this case, there must exist a specialization $q_2$ of the variables $q$ that satisfies properties (i) and (ii) with respect to the given collection, and the (boundedly many) fibers associated with $q_2$ in the second level of the universal diagram cover the given coset of test sequences of the completion, $Comp(d_1,p,d_0,a)$.

By applying this argument iteratively for the subcollections of duo families that are associated with the various levels of the universal diagram, and using the universality of the diagram, the entire (bounded) collection of duo families that were associated iteratively with the various levels of the universal diagram, and with the given duo family, *duo*, covers all the duo test sequences of duo, hence, this bounded collection of duo families cover the duo family, *duo*.

□

Question: In case the group we start with is a rigid limit group, $Rgd(x,p,q,a)$, theorem 3.2 proves the existence of finitely many universal duo limit groups, so that every duo family that is associated with $Rgd(x,p,q,a)$, is covered by boundedly many duo families of the universal duo limit groups. Does the same statement remain valid for solid limit groups $Sld(x,p,q,a)$? note that in the solid case it is not difficult to prove the existence of a finite collection of universal duo limit groups, so that every duo family that is associated with the given solid limit group is covered by finitely many duo families of the universal duo limit groups. The main difficulty is the ability to replace finitely many duo families by boundedly many ones.

## §4. Rigid and Solid Specializations

In section 1 we have shown that in the minimal (graded) rank case Diophantine sets are equational, and then used it to show that the sets $NR_s$ ($NS_s$), that indicate those values of the parameter set $< p,q >$, for which a minimal (graded) rank rigid

(solid) limit group $Rgd(x,p,q,a)$ ($Sld(x,p,q,a)$) admits at least $s$ rigid (strictly solid families of) specializations, are in the Boolean algebra generated by equational sets (theorem 1.2).
In section 2 we have shown that Diophantine sets are equational in the general case, omitting the minimal (graded) rank assumption. As we will see in the sequel (section 6), unlike the minimal (graded) rank case, the sets $NR_s$ and $NS_s$ are not in the Boolean algebra of equational sets in general. Still, in this section we combine the equationality of general Diophantine sets with the concept of duo limit groups presented in the previous section, to show that the sets $NR_s$ and $NS_s$ associated with general rigid and solid limit groups are stable.

**Theorem 4.1.** *Let $F_k = < a_1, \dots, a_k >$ be a non-abelian free group, and let $Rgd(x,p,q,a)$ ($Sld(x,p,a)$) be a rigid (solid) limit group, with respect to the parameter subgroup $< p,q >$. Let $s$ be a positive integer, and let $NR_s$ ($NS_s$) be the set of specializations of the defining parameters $< p,q >$ for which the rigid (solid) limit group, $Rgd(x,p,q,a)$ ($Sld(x,p,a)$), has at least $s$ rigid (strictly solid families of) specializations. Then the set $NR_s$ ($NS_s$) is stable.*

*Proof:* To prove the stability of the set $NR_s$ ($NS_s$), we use Shelah's criterion, and give a bound on the length of a sequence of couples $(p_1,q_1),\dots,(p_n,q_n)$ for which the predicate associated with the set $NR_s$ ($NS_s$) defines a linear order, i.e., for which $(p_i,q_j) \in NR_s$ if and only if $i<j$.

We start with the construction of the diagrams that are needed in order to get the bound on the lengths of linearly ordered sequences of couples. First, we associate with the set $NR_s$ ($NS_s$) the finite diagram constructed in proving theorem 3.2. In each step of the diagram we collect all the specializations

$$(\{x_1^i,\dots,x_s^i\}_{i=1}^{\ell}, p, q_1, \dots, q_\ell, a)$$

for which for all indices $i$, $1 \leq i \leq \ell$, the specializations: $(x_1^i,\dots,x_s^i,p,q_i,a)$ are rigid (strictly solid) and distinct (belong to distinct strictly solid families). With this diagram we associate the finite collection of "universal" duo limit groups: $Duo_1,\dots,Duo_t$ (see the proof of theorem 3.2).

The diagram associated with $NR_s$ ($NS_s$) is a directed graph for which in each vertex we attach an anvil with the resolutions attached to it. We set $depth_{NR_s}$ ($depth_{NS_s}$) to be the depth of the directed graph associated with the diagram. We further set $width_{NR_s}$ ($width_{NS_s}$) to be the maximal number of fibers to which one continues to along the constructed diagram, from a given fiber and a given specialization of the parameters $q$.
By the existence of a bound on the numbers of rigid specializations of a rigid limit group, and strictly solid families of a solid limit group ([Se3],2.5 and 2.9), there exists a bound on the maximal number of duo families that are associated with one of the duo limit groups $Duo_1,\dots,Duo_t$ and a fixed fiber in the diagram associated with $Rgd(x,p,q,a)$. We set $df_{NR_s}$ to be this bound.

Let $Duo$ be one of the (universal) duo limit groups $Duo_1,\dots,Duo_t$ that are associated with $NR_s$ ($NS_s$), and suppose that $Duo = < d_1,p > *_{< d_0 >} < d_2,q >$. We view $Duo$ as a graded limit group with respect to the parameter subgroup $< d_2,q >$. With each specialization of $Duo$ there exists a specialization of the associated configuration limit group $Conf$, $(x_1,\dots,x_s,p,q,a)$ (see definition 3.1). With $Duo$ we further associate the Diophantine condition that forces the associated specialization of the configuration limit group, $Conf$, not to be a configuration homomorphism. i.e., either one of the specializations $(x_i, p, q, a)$ is flexible (not strictly solid), or two rigid specializations $(x_i, p, q, a)$ and $(x_j, p, q, a)$, $i < j$, coincide (belong to the same strictly solid family. See definition 1.5 in [Se3] for this Diophantine condition).

Starting with the duo limit group $Duo$ (and the variety associated with it), and iteratively apply the prescribed Diophantine condition, we construct a finite diagram, using the construction of the diagram associated with a Diophantine set presented in proving theorem 2.1 (and used for proving the equationality of Diophantine sets).

By the finiteness of the constructed diagram associated with the duo limit group $Duo$, there exists a global bound on the length of a sequence of specializations $d_2(1), \ldots, d_2(u)$ of the variables $d_2$, for which the intersections of the Diophantine sets (of specializations of $d_1$) associated with the prefixes $d_1(1), \ldots, d_1(m)$ strictly decreases for $1 \leq m \leq u$ (note that the existence of this bound also follows from the equationality of Diophantine sets proved in theorem 2.1). Let $length_{NR_S}$ ($length_{NS_S}$) be the maximum of these bounds for all the universal duo limit groups $Duo_1, \ldots, Duo_t$.

**Proposition 4.2.** *With the notation of theorem 4.1, let:* $(p_1, q_1), \ldots, (p_n, q_n)$ *be a sequence of couples of specializations of the defining parameters* $p, q$ *for which* $(p_i, q_j) \in NR_s$ *if and only if* $i < j$*. Then* $n < M$ *where:*

$$M = (1 + width_{NR_s})^{1+depth_{NR_s} \cdot (t \cdot df_{NR_s} \cdot length_{NR_s} + 3)} + 1$$

*and the same holds for the sets* $NS_s$.

*Proof:* We prove the proposition for a set $NR_s$ (associated with a rigid limit group). The proof for the sets $NS_s$ (associated with sold limit groups) is identical. Let $n \geq M$ and: $(p_1, q_1), \ldots, (p_n, q_n)$ be a sequence of specializations of the parameters $p, q$, for which $(p_i, q_j) \in NR_s$ if and only if $i < j$. By the definition of the set $NR_s$, for every $i < j$, there exists an $s$-tuple: $x^{i,j} = (x_1^{i,j}, \ldots, x_s^{i,j})$, so that for every $1 \leq m \leq s$, $(x_m^{i,j}, p_i, q_j, q)$ is a rigid specialization of the given rigid limit group $Rgd(x, p, q, a)$, and for $1 \leq m_1 < m_2 \leq s$, the corresponding rigid specializations are distinct. For the rest of the argument, with each couple $(p_i, q_j)$, $i < j$, we further associate such a specialization $x^{i,j}$.

We iteratively filter the tuples $(x^{i,j}, p_i, q_j)$, and then apply a simple pigeon-hole principle. We start with $q_n$. By the construction of the diagram associated with the rigid limit group, $Rgd(x, p, q, a)$, we have started with (the diagram constructed in proving theorem 3.2), at least $\frac{1}{width_{NR_s}}$ of the specializations, $\{(x^{i,n}, p_i, q_n, a)\}_{i=1}^{n-1}$, belong to the same fiber associated with $q_n$ in the initial level of the diagram. We proceed only with those indices $i$ for which the specializations, $(x^{i,n}, p_i, q_n, a)$, belong to that fiber.

We continue with the largest index $i$, $i < n$, for which the tuple, $(x^{i,n}, p_i, q_n, a)$, belongs to that fiber. We denote that index $i$, $u_2$. By the structure of the diagram associated with $Rgd(x, p, q, a)$, at least $\frac{1}{1+width_{NR_s}}$ of the specializations $\{(x^{i,u_2}, p_i, q_{u_2}, a)\}$, for those indices $i < u_2$ that remained after the first filtration, belong to either the same fiber in the initial level, or to one of the fibers in the second level of the constructed diagram. We proceed only with those indices $i$ for which the specialization, $(x^{i,u_2}, p_i, q_{u_2}, a)$, belongs either to the initial fiber or to the same fiber in the second level.

We proceed this filtration process iteratively. Since the diagram is finite and has depth, $depth_{NR_s}$, and since at each step we remain with at least $\frac{1}{1+width_{NR_s}}$ of the specializations we have started the step with, and since $n$, the number of couples we started with satisfies $n \geq M$, along the filtration process we must obtain a subsequence $\{(p_{u_m}, q_{u_m})\}_{m=c}^{c+t\cdot df_{NR_s}\cdot length_{NR_s}+2}$ for which:

(1) $(p_{u_{m_1}}, q_{u_{m_2}}) \in NR_s$ for $c \leq m_1 < m_2 \leq c + t \cdot df_{NR_s} \cdot length_{NR_s} + 2$.

(2) $(x^{u_{m_1},u_{m_2}}, p_{u_{m_1}}, q_{u_{m_2}}, a)$ belongs to the same fiber in the diagram associated with $Rgd(x,p,q,a)$, for: $c \leq m_1 < m_2 \leq c + t \cdot df_{NR_s} \cdot length_{NR_s} + 2$.

Since there are $t$ duo limit groups associated with the diagram constructed from $Rgd(x,p,a)$, and there are at most $df_{NR_s}$ duo families associated with a given fiber in the universal diagram associated with $NR_s$ and the universal duo families, $Duo_1, \dots, Duo_t$, we obtained a strictly decreasing sequence of Diophantine sets of length $length_{NR_s} + 1$ associated with at least one of the duo limit groups, $Duo_1, \dots, Duo_t$, where $length_{NR_s}$ is supposed to be a bound on such a strictly decreasing sequence, a contradiction, and the proposition follows.

□

Proposition 4.2 proves the stability of the sets $NR_s$ and $NS_s$.

□

Theorem 4.1 proves the stability of the sets $NR_s$ and $NS_s$, i.e. sets for which a rigid or solid limit group have at least $s$ rigid or strictly solid families of specializations are stable. Since stable sets are closed under Boolean operations, this proves that sets for which there are precisely $s$ rigid or strictly solid families of specializations (of a given rigid or solid limit group) are stable. As we did in the minimal rank case (theorem 1.5), in order to prove that the theory of a free group is stable, i.e. that a general definable set over a free group is stable, we need to analyze the (definable) set of specializations of the defining parameters for which a given (finite) collection of covers of a graded resolution forms a covering closure (see definition 1.16 in [Se2] for a covering closure).

**Theorem 4.3 (cf. theorem 1.5).** *Let $F_k = < a_1, \dots, a_k >$ be a non-abelian free group, let $G(x,p,q,a)$ be a graded limit group (with respect to the parameter subgroup $< p, q >$), and let $GRes(x,p,q,a)$ be a well-structured graded resolution of $G(x,p,q,a)$ that terminates in the rigid (solid) limit group, $Rgd(x,p,q,a)$ ($Sld(x,p,q,a)$).*

*Let $GCl_1(z,x,p,q,a), \dots, GCl_v(z,x,p,q,a)$ be a given set of graded closures of $GRes(x,p,q,a)$. Then the set of specializations of the parameters $< p, q >$ for which the given set of closures forms a covering closure of the graded resolution $GRes(x,p,q,a)$, $Cov(p,q)$, is stable.*

*Proof:* The proof is based on the arguments that were used to prove theorems 1.5 and 4.1. We start with the construction of the diagrams that are needed in order to get the bound on the lengths of linearly ordered sequences of couples for $Cov(p,q)$. We begin with the construction of a diagram that is similar in nature to the diagram constructed in analyzing the sets $NR_s$ and $NS_s$ (in proving theorem 4.1), that starts with the same collection of specializations as we did in analyzing the sets $Cov(p,q)$ in the minimal (graded) rank case (theorem 1.5).

We look at the entire collection of specializations:

$$(x_1, \dots, x_s, y_1, \dots, y_m, r_1, \dots, r_s, p, q, a)$$

for which (cf. the proof of theorem 1.5):

(i) for the tuple $p,q$ there exist precisely $s$ rigid (strictly solid families of) specializations of the rigid (solid) limit group, $Rgd(x,p,q,a)$ ($Sld(x,p,q,a)$), and at least (total number of) $m$ distinct rigid and strictly solid families of specializations of the terminal (rigid and solid) limit groups of the closures: $GCl_1(z,x,p,q,a),\dots,GCl_v(z,x,p,q,a)$.

(ii) in case the terminal limit groups of $GRes$ is rigid, the $x_i$'s are the distinct rigid specializations of $Rgd(x,p,q,a)$. In case the terminal limit group of $GRes$ is solid, the $x_i$'s belong to the $s$ distinct strictly solid families of $Sld(x,p,q,a)$.

(iii) the $y_j$'s are either distinct rigid specializations or belong to distinct strictly solid families of specializations of the terminal limit groups of the closures: $GCl_1,\dots,GCl_v$.

(iv) the $r_i$'s are variables that are added only in case the terminal limit group of $GRes$ is solid. In this case the $r_i$'s demonstrate that the (ungraded) resolutions associated with the given closures and the specializations $y_1,\dots,y_m$, form a covering closure of the (ungraded) resolutions associated with the resolution $GRes$ and the specializations $x_1,\dots,x_s$. These include primitive roots of the specializations of all the non-cyclic abelian groups, and edge groups, in the abelian decomposition associated with the solid terminal limit group of $GRes$, $Sld(x,p,q,a)$, and variables that demonstrate that multiples of these primitive roots up to the least common multiples of the indices of the finite index subgroups associated with the graded closures, $GCl_1,\dots,GCl_v$, factor through the ungraded resolutions associated with the specializations $y_1,\dots,y_m$ and their corresponding closures (cf. section 1 of [Se5] in which we added similar variables to form valid proof statements, that initialize the sieve procedure).

We look at the collection of such specializations for all the possible values of $s$ and $m$ (note that $s$ and $m$ are bounded, since the number of rigid specializations of a rigid limit group and the number of strictly solid families of specializations of a solid limit group associated with a given specialization of the defining parameters are globally bounded by theorems 2.5 and 2.9 in [Se3]).

For each fixed $s$ and $m$ we associate with the collection of the specializations that satisfy properties (i)-(iv) its Zariski closure, i.e. a canonical finite collection of maximal limit groups, that we view as graded with respect to the parameter subgroup $<q>$. With these graded limit groups we associate the (graded) resolutions that appear in their strict graded Makanin-Razborov diagrams, and the resolutions associated with the various strata in the singular loci of the diagrams. Given the resolutions in the collections of strict Makanin-Razborov diagrams for all the possible values of $s$ and $m$, we iteratively construct a diagram in a similar way to the construction of the diagrams associated with the sets $NR_s$ and $NS_s$, in proving theorem 4.1. This construction terminates after finitely many steps (for precisely the same reasons the construction of the diagram associated with $NR_s$ and $NS_s$ terminates after finitely many steps), that finally gives us the first diagram associated with the set $Cov(p,q)$. With this first diagram we associate its finite collection of associated "universal" duo limit groups, that we denote: $Duo^1_1,\dots,Duo^1_{t_1}$.

As we did in proving theorem 4.1, we set $depth^1_{Cov}$ to be the depth of the directed graph associated with the first diagram, and $width^1_{Cov}$ to be the maximal number of fibers to which one continues to along the first diagram, from a given fiber and a given specialization of the parameters $q$.

Given each (graded) limit group that appears along the diagram, we associate with it its collection of (universal) duo limit groups. Given each associated duo limit group $Duo = < d_1, p > *_{<d_0>} < d_2, q >$, we view it both as a graded limit group with respect to the parameter subgroup $< p, d_0, d_2 >$, and as a graded limit group with respect to the parameter subgroup $< d_1, d_0, q >$. With $Duo$ we associate its two graded Makanin-Razborov diagrams (with respect to the two subgroups of parameters). By theorems 2.5 and 2.9 in [Se3], there exist global bounds on the number of rigid and strictly solid families of solutions (having the same specialization of the parameter subgroup), for each of the rigid and solid limit groups in these graded Makanin-Razborov diagrams. For each bundle (graded limit group) that appears in the first diagram, we sum these bounds for all the rigid and solid limit groups that appear in the graded diagrams of its (finitely many) associated duo limit groups (where with each duo limit group we associate two Makanin-Razborov diagrams with respect to the two subgroups of parameters). We set $Sum^1_{Cov}$ to be the maximum of these sums.

By construction, given a bundle (graded limit group) along the first diagram associated with $Cov(p, q)$, there are finitely many (universal) duo limit groups associated with it. Given a fiber in such a bundle, there are boundedly many duo families of these duo limit groups that are associated with the given fiber (i.e., boundedly many values of the parameters $d_0$ that are associated with the given fiber). We set $Fam^1_{Cov}$ to be the maximal number of such duo families that are associated with a fiber along the first diagram. We further set (the capacity) $Cap^1_{Cov}$ to be: $Cap^1_{Cov} = Sum^1_{Cov} \cdot Fam^1_{Cov}$.

Let $Duo$ be one of the (universal) duo limit groups $Duo^1_1, \ldots, Duo^1_{t_1}$ that are associated with $Cov(p, q)$, and suppose that $Duo = < d_1, p > *_{<d_0>} < d_2, q >$. We view $Duo$ as a graded limit group with respect to the parameter subgroup $< d_2, q >$. With each specialization of $Duo$ there exists a restricted specialization: $(x_1, \ldots, x_s, y_1, \ldots, y_m, r_1, \ldots, r_s, p, q, a)$ (see the construction of the first diagram associated with $Cov(p, q)$). With $Duo$ we further associate the Diophantine condition that forces one of the conditions (ii)-(iv) that are imposed on the collection of associated specializations, to collapse (to be non-valid), i.e., either one of the specializations $(x_i, p, q, a)$ or $(y_i, p, q, a)$ is flexible (not strictly solid), or two rigid (strictly solid) specializations $(x_i, p, q, a)$ and $(x_j, p, q, a)$, $i < j$, coincide (belong to the same strictly solid family), or correspondingly for specializations $(y_i, p, q, a)$ and $(y_j, p, q, a)$, or one of the specializations $r_i$ has a root with an order that divides the least common multiple of the indices of the finite index subgroups that are attached to the closures $GCl_1, \ldots, GCl_v$.
Starting with the duo limit group $Duo$ (and the variety associated with it), and iteratively applying the prescribed Diophantine condition, we construct a finite diagram, using the construction of the diagram associated with a Diophantine set presented in proving theorem 2.1 (and used for proving the equationality of Diophantine sets).

By the finiteness of the constructed diagram associated with the duo limit group $Duo$, there exists a global bound on the length of a sequence of specializations $d_2(1), \ldots, d_2(u)$ of the variables $d_2$, for which the intersections of the Diophantine sets (of specializations of $d_1$) associated with the prefixes $d_1(1), \ldots, d_1(m)$ strictly decreases for $1 \leq m \leq u$ (note that the existence of this bound also follows from the equationality of Diophantine sets proved in theorem 2.1). Let $length^1_{Cov}$ be the maximum of these bounds for all the universal duo limit groups $Duo^1_1,\dots,Duo^1_{t_1}$.

After constructing the first diagram associated with $Cov(p,q)$, we continue with each of the (universal) duo limit groups, $Duo^1_i$, associated with $Cov(p,q)$, and construct a second diagram, that is similar to the first one. Given a (universal) duo limit group, $Duo^1_i$, that is associated with one of the bundles that appears along the first diagram associated with $Cov(p,q)$, we continue to the second step of the construction by collecting all the tuples:

$$(x_0,p_0,q_0,d_1,d_0,d_2)$$

for which:

(1) the specialization: $(d_1,d_0,d_2)$ factors through the duo limit group $Duo^1_i$. The specialization $d_2$ restricts to $q_0$, and the specialization $d_1$ restricts to $p_0$.
(2) the specialization $(d_1,d_0,d_2)$ restricts to a specialization:

$$(x_1,\dots,x_s,y_1,\dots,y_m,r_1,\dots,r_s,p_0,q_0,a).$$

This (restricted) specialization satisfies properties (ii)-(iv) that are listed in the construction of the first diagram associated with $Cov(p,q)$.
(3) the specialization: $(x_0,p_0,q_0,a)$ is a rigid or strictly solid specialization of the terminal rigid or solid limit group of the graded resolution $GRes$ we have started with, and it is an extra rigid or solid specialization, i.e., it does not coincide with any rigid specialization (does not belong to the strictly solid family of one of the strictly solid specializations) that are part of the specialization $(x_1,\dots,x_s,p_0,q_0,a)$.

We continue to the next steps of the construction by collecting specializations in the same form, precisely like in the construction of the first diagram associated with $Cov(p,q)$, where we let $< d_1 >$ play the role of the parameter subgroup $< p >$, and $< d_2 >$ play the role of the parameter subgroup $< q >$ in the construction of the first diagram associated with $Cov(p,q)$. By the same arguments that imply the termination of the construction of the first diagram associated with $Cov(p,q)$, the constructions of the diagrams associated with the various duo limit groups, $Duo^1_i$, terminate after finitely many steps. With the finite collection of diagrams associated with the various bundles (graded limit groups) that appear in the first diagram, we associate their finite collection of associated "universal" duo limit groups, that we denote: $Duo^2_1,\dots,Duo^2_{t_2}$ (see the proof of theorem 3.2 for the associated universal duo limit groups).

As we did with the first diagram associated with $Cov(p,q)$, we set $depth^2_{Cov}$ to be the maximal depth of the directed graphs associated with the constructed (second) diagrams, and $width^2_{Cov}$ to be the maximal number of fibers to which one continues to along (one of) the second diagrams, from a given fiber and a given specialization of the parameters $q$.

By the existence of a bound on the numbers of rigid specializations of a rigid limit group, and strictly solid families of a solid limit group ([Se3],2.5 and 2.9), there exists a bound on the maximal number of duo families that are associated with one of the duo limit groups $Duo^2_1,\dots,Duo^2_{t_2}$ and a fixed fiber in the diagrams associated with $Duo^1_1,\dots,Duo^1_{t_1}$. We set $df_2$ to be this bound.

Let $Duo$ be one of the (universal) duo limit groups $Duo^2_1, \dots, Duo^2_{t_2}$ that are associated with $Cov(p,q)$, and suppose that $Duo = <e_1, d_1> *_{<e_0,d_0>} <e_2, d_2>$. We view $Duo$ as a graded limit group with respect to the parameter subgroup $<e_2, d_2>$. With each specialization of $Duo$ there exists an associated extra specialization of the (terminal) rigid or solid limit group, $Rgd(x,p,q,a)$ or $Sld(x,p,q,a)$. With $Duo$ we further associate the Diophantine condition that forces the associated extra specialization of the rigid or solid limit group to be either flexible (not rigid or not strictly solid), or to coincide with one of the rigid specializations, or to belong to one of the strictly solid families of specializations, that appear in the corresponding (restricted) specialization: $(x_1, \dots, x_s, y_1, \dots, y_m, r_1, \dots, r_s, p_0, q_0, a)$, i.e. we add a Diophantine condition that forces the collection of specializations not to satisfy property (3) in the definition of the collection of specializations that are collected in each step of the construction of the second diagram.
Starting with the duo limit group $Duo$ (and the variety associated with it), and iteratively applying the prescribed Diophantine condition, we construct a finite diagram, using the construction of the diagram associated with a Diophantine set presented in proving theorem 2.1 (and used for proving the equationality of Diophantine sets). By the finiteness of the constructed diagram associated with the duo limit group $Duo$, there exists a global bound on the length of a sequence of specializations $e_2(1), \dots, e_2(u)$ of the variables $e_2$, for which the intersections of the Diophantine sets (of specializations of $e_1$) associated with the prefixes $e_2(1), \dots, e_2(m)$ strictly decreases for $1 \leq m \leq u$ (note that the existence of this bound also follows from the equationality of Diophantine sets proved in theorem 2.1). Let $length^2_{Cov}$ be the maximum of these bounds for all the universal duo limit groups $Duo^2_1, \dots, Duo^2_{t_2}$.

**Proposition 4.4.** *With the notation of theorem 4.3, let:* $(p_1, q_1), \dots, (p_n, q_n)$ *be a sequence of couples of specializations of the defining parameters* $p, q$ *for which* $(p_i, q_j) \in Cov(p,q)$ *if and only if* $i < j$*. Then* $n < M$ *where:*

$$M = (1 + width^1_{Cov})^{(depth^1_{Cov} \cdot (t_1^{(t_1 \cdot S)}))} + 1$$

$$S = (Cap^1_{Cov})^{((Cap^1_{Cov})^{(2^L)})}$$

*and:*

$$L = length^1_{Cov} + (1 + width^2_{Cov})^{(depth^2_{Cov} \cdot (t_2 \cdot df_2 \cdot length^2_{Cov} + 2))} + 3$$

*Proof:* The argument we use is a strengthening of the argument that was used to prove proposition 4.2. Let $n \geq M$ and: $(p_1, q_1), \dots, (p_n, q_n)$ be a sequence of specializations of the parameters $p, q$, for which $(p_i, q_j) \in Cov(p,q)$ if and only if $i < j$. By the definition of the set $Cov(p,q)$, for every $i < j$, there exists a tuple:

$$(x_1^{i,j}, \dots, x_s^{i,j}, y_1^{i,j}, \dots, y_m^{i,j}, r_1^{i,j}, \dots, r_s^{i,j}, p_i, q_j, a)$$

that satisfies properties (i)-(iv), that the specializations from which we construct the first diagram associated with $Cov(p,q)$ have to satisfy.

We iteratively filter the tuples associated with the couples $(p_i, q_j)$, for $i < j$, in a similar way to what we did in proving proposition 4.2. We start with $q_n$. By the construction of the first diagram associated with $Cov(p,q)$, at least $\frac{1}{width^1_{Cov}}$ of the specializations associated with the couples $(p_i, q_n)$, $1 \leq i \leq n-1$, belong to the same fiber associated with $q_n$ in the initial level of the diagram. We proceed only with those indices $i$ for which the specializations associated with $(p_i, q_n)$ belong to that fiber.

We proceed as in the proof of proposition 4.2. We continue with the largest index $i$, $i < n$, for which the tuple associated with $(p_i, q_n)$ belongs to that fiber. We denote that index $i$, $u_2$. By the structure of the first diagram associated with $Cov(p,q)$, at least $\frac{1}{1+width^1_{Cov}}$ of the specializations associated with $(p_i, q_{u_2})$, for those indices $i < u_2$ that remained after the first filtration, belong to either the same fiber in the initial level, or to one of the fibers in the second level of the constructed diagram. We proceed only with those indices $i$ for which the specializations associated with $(p_i, q_{u_2})$ belong either to the initial fiber or to the same fiber in the second level.

We proceed this filtration process iteratively (as in the proof of proposition 4.2). Since the diagram is finite and has depth, $depth^1_{Cov}$, and since at each step we remain with at least $\frac{1}{1+width^1_{Cov}}$ of the specializations we have started the step with, and since $n$, the number of couples we started with satisfies $n \geq M$, along the filtration process we must obtain a consecutive subsequence: $\{(p_{u_m}, q_{u_m})\}_{m=c}^{c+R}$ for which:

(1) $R = t_1^{t_1 \cdot S}$ (where $S$ is defined in the statement of the proposition).
(2) for $m_1, m_2$ that satisfy, $c \leq m_i \leq c+R$, $(p_{u_{m_1}}, q_{u_{m_2}}) \in Cov(p,q)$ if and only if $m_1 < m_2$.
(3) the specializations:

$$(x_1^{u_{m_1},u_{m_2}}, \ldots, x_s^{u_{m_1},u_{m_2}}, y_1^{u_{m_1},u_{m_2}}, \ldots, y_m^{u_{m_1},u_{m_2}}, r_1^{u_{m_1},u_{m_2}}, \ldots, r_s^{u_{m_1},u_{m_2}}, p_{u_{m_1}}, q_{u_{m_2}}, a)$$

that satisfy properties (i)-(iv) and testify that the couple $(p_{u_{m_1}}, q_{u_{m_2}}) \in Cov(p,q)$ for: $c \leq m_1 < m_2 \leq c+R$, can be extended to specializations that factor through the same fiber in the first diagram associated with $Cov(p,q)$, i.e. they factor through the duo families associated with that fiber.

For the rest of the argument we continue with this set of specializations: $\{(p_{u_m}, q_{u_m})\}_{m=c}^{c+R}$. With the fiber in the first diagram associated with $Cov(p,q)$, through which the associated specializations factor, there exist finitely many associated (universal) duo limit groups. This number of duo limit groups that are associated with the given fiber does not exceed $t_1$, which is the total number of duo limit groups that are associated with the various bundles that are constructed along the first diagram.

Hence, by iteratively applying the pigeon hole principle, we can choose a subsequence of the specializations $\{(p_{u_m}, q_{u_m})\}_{m=c}^{c+R}$, that we denote: $\{(p_{u_\ell}, q_{u_\ell})\}_{\ell=1}^{S}$ (where the constant $S$ is defined in the statement of proposition 4.4), that satisfies property (2) and for which the specializations:

$$(x_1^{u_{\ell_1},u_{\ell_2}}, \ldots, x_s^{u_{\ell_1},u_{\ell_2}}, y_1^{u_{\ell_1},u_{\ell_2}}, \ldots, y_m^{u_{\ell_1},u_{\ell_2}}, r_1^{u_{\ell_1},u_{\ell_2}}, \ldots, r_s^{u_{\ell_1},u_{\ell_2}}, p_{u_{\ell_1}}, q_{u_{\ell_2}}, a)$$

that satisfy properties (i)-(iv) and testify that the couple $(p_{u_{\ell_1}}, q_{u_{\ell_2}}) \in Cov(p,q)$ for: $1 \leq \ell_1 < \ell_2 \leq S$, can be extended to specializations $(d_1(\ell_1,\ell_2), d_0(\ell_1,\ell_2), d_2(\ell_1,\ell_2))$ that factor through the same fiber in the first diagram associated with $Cov(p,q)$, and do all factor through the same duo limit group associated with that fiber. For the rest of the argument we continue with this subsequence of specializations, and denote the corresponding duo limit group, $Duo^1$.

Recall that by our notation, when we view the duo limit group, $Duo^1 =< d_1, p > *_{<d_0>} < d_2, q >$, as a graded limit group with respect to the enlarged parameter subgroups, $< p, d_0, d_2 >$ or $< d_1, d_0, q >$, then the sums of the numbers of rigid and strictly solid families of specializations (having the same value of the enlarged parameter subgroup) of the rigid and solid limit groups that appear in the graded Makanin-Razborov diagrams (with respect to the two enlarged parameter subgroups) of $Duo^1$, are bounded by $Sum^1_{Cov}$. Furthermore, the number of duo families of $Duo^1$ that are associated with the given fiber through which the sequence of specializations: $(d_1(\ell_1,\ell_2), d_0(\ell_1,\ell_2), d_2(\ell_1,\ell_2))$ factor, is bounded by $Fam^1_{Cov}$. Recall that $Cap^1_{Cov}$ was set to be $Sum^1_{Cov} \cdot Fam^1_{Cov}$.

Therefore, by iteratively using the pigeon hole principle, we may pass to a further subsequence that contains at least $2^L$ couples, $\{(p_{i_f}, q_{i_f})\}_{f=1}^{2^L}$, so that for any couple of indices $i < j$ from this subsequence, the specializations:

$$(x_1^{i,j}, \ldots, x_s^{i,j}, y_1^{i,j}, \ldots, y_m^{i,j}, r_1^{i,j}, \ldots, r_s^{i,j}, p_i, q_j, a)$$

that testify that $(p_i, q_j) \in Cov(p,q)$, can be extended to specializations, $(d_1(i), d_0, d_2(j))$, that factor through a fixed duo family of the duo limit group, $Duo^1$ (note that the specializations $d_1$ and $d_2$ depend only on the indices $i$ and $j$, respectively, and $d_0$ is fixed for the entire subsequence).

By our assumption on the original sequence of specializations, for every $i \geq j$, the couple $(p_i, q_j) \notin Cov(p,q)$. Hence, by passing to a further subsequence $\{(p_{i_v}, q_{i_v})\}_{v=1}^{L}$, we may assume that for every index $j$ from the chosen subsequence, either:

(1) for every index $i$, $i > j$, from the subsequence, the specialization $(d_1(i), d_0, d_2(j))$ that is assumed to factor through the duo limit group $Duo^1$, restricts to a specialization:

$$(x_1^{i,j}, \ldots, x_s^{i,j}, y_1^{i,j}, \ldots, y_m^{i,j}, r_1^{i,j}, \ldots, r_s^{i,j}, p_i, q_j, a)$$

which is degenerate, i.e. it satisfies some Diophantine condition that demonstrates that at least one of the properties (ii)-(iv) that are needed in order to testify that $(p_i, q_j) \in Cov(p,q)$, does not hold.

(2) for every index $i$, $i > j$ from the subsequence, the specialization for which the specialization, $(d_1(i), d_0, d_2(j))$ that is assumed to factor through the duo limit group $Duo^1$, restricts to a specialization:

$$(x_1^{i,j}, \ldots, x_s^{i,j}, y_1^{i,j}, \ldots, y_m^{i,j}, r_1^{i,j}, \ldots, r_s^{i,j}, p_i, q_j, a)$$

which is not degenerate, i.e. properties (ii)-(iv) hold for it. Therefore, for every $i \geq j$ property (i) does not hold for the corresponding (restricted) specialization, i.e. there exists some extra rigid or strictly solid specialization $(x_0^{i,j}, p_i, q_j, a)$ of the corresponding rigid or solid limit group, which is the terminal limit group of the graded resolution $GRes$ we have started with, and this extra rigid or solid specialization does not coincide with a rigid specialization and does not belong to any strictly solid family which is a part of the restricted specialization.

As we argued in proving proposition 4.2, by equationality of Diophantine sets (theorem 2.1), or alternatively, by the termination of the construction of the diagram that starts with $Duo^1$ and forces the Diophantine condition that prevents the (restricted) specializations to be degenerate, i.e. not to satisfy at least one of the properties (ii)-(iv), part (1) can hold for at most $length^1_{Cov}$ indices $j$. Hence, for at least $L - length^1_{Cov}$ indices $j$ part (2) holds, and we continue with the subsequence containing these indices.

We proceed with the same argument that was used to prove proposition 4.2, going in the inverse direction, i.e. over the indices $i > j$ for which $(p_i, q_j) \notin Cov(p,q)$ because part (2) holds, hence, because of the existence of an extra rigid or strictly solid family of solutions.

We start with at least $L - length^1_{Cov}$ couples $(p_i, q_j)$, so that $(p_i, q_j) \in Cov(p,q)$ iff $i < j$, and for which:

(1) the associated specializations $(d_1(i), d_0, d_2(j))$ factor through the duo limit group $Duo^1$, for every $i$ and $j$.
(2) $(d_1(i), d_0, d_2(j))$ restricts to a specialization that testifies that $(p_i, q_j) \in Cov(p,q)$ when $i < j$.
(3) $(d_1(i), d_0, d_2(j))$ restricts to a specialization that satisfies properties (ii)-(iv), but not property (i) if $i > j$, because part (2) holds (i.e., because there exists an extra rigid or strictly solid specialization).

We iteratively filter the tuples associated with the couples $(p_i, q_j)$, for $i > j$, as in the beginning of the proof of proposition 4.4. We start with $d_2(1)$, the specialization of $Duo^1$ associated with $(p_1, q_1)$. By the construction of the second diagram associated with the duo limit group $Duo^1$, at least $\frac{1}{width^2_{Cov}}$ of the specializations associated with the couples $(p_i, q_1)$, $1 < i \leq L - length^1_{Cov}$, belong to the same fiber associated with $d_2(1)$ in the initial level of the diagram. We proceed only with those indices $i$ for which the specializations associated with $(p_i, q_1)$ belong to that fiber.

We continue with the smallest index $i$, $1 < i < L - length^1_{Cov}$, for which the tuple associated with $(p_i, q_1)$ belongs to that fiber. We denote that index $i$, $u_2$. By the structure of the second diagram associated with $Duo^1$, at least $\frac{1}{1+width^2_{Cov}}$ of the specializations associated with $(p_i, q_{u_2})$, for those indices $i > u_2$ that remained after the first filtration, belong to either the same fiber in the initial level, or to one of the fibers in the second level of the second diagram associated with $Duo^1$. We proceed only with those indices $i$ for which the specializations associated with $(p_i, q_{u_2})$ belong either to the initial fiber or to the same fiber in the second level of the second diagram associated with $Duo^1$.

We proceed this filtration process iteratively (as in the beginning of the proof of proposition 4.4). Since the second diagram associated with $Duo^1$ is finite and has depth, $depth^2_{Cov}$, and since at each step we remain with at least $\frac{1}{1+width^2_{Cov}}$ of the specializations we have started the step with, we must obtain a subsequence $\{(p_{u_m}, q_{u_m})\}_{m=c}^{c+B}$ for which:

(1) $B = t_2 \cdot df_2 \cdot length^2_{Cov} + 1$.
(2) for $m_1, m_2$ that satisfy, $c \leq m_i \leq c + B$, $(p_{u_{m_1}}, q_{u_{m_2}}) \in Cov(p,q)$ if and only if $m_1 < m_2$.
(3) for $m_1 > m_2$, the specializations $(d_1(u_{m_1}), d_0, d_2(u_{m_2}))$ that factor through $Duo^1$ can be extended to specializations that factor through the same fiber in the second diagram associated with $Duo^1$, i.e. they factor through the duo families associated with that fiber.

For the rest of the argument we continue with this set of specializations: $\{(p_{u_m}, q_{u_m})\}_{m=c}^{c+B}$. With the fiber in the second diagram associated with $Duo^1$, through which the associated specializations factor, there exist finitely many associated (universal) duo limit groups. This number of duo limit groups that are associated with the given fiber does not exceed $t_2$, which is the total number of duo limit groups that are associated with the various bundles that are constructed along the second diagrams that start with the duo limit groups that are associated with the first diagram associated with $Cov(p,q)$. By our assumptions the number of duo families that are associated with a fiber in the second diagram and the duo limit groups that are associated with the fiber does not exceed $df_2$.

Like in the argument that was used to finish the proof of proposition 4.2, by going through the $B+1$ specializations associated with $\{(p_{u_m},q_{u_m})\}_{m=c}^{c+B}$, from the index $c+B$ to $c$, we obtain a strictly decreasing sequence of $B$ Diophantine sets. Since there are at most $t_2$ duo limit groups associated with the fiber through which the specializations associated with $\{(p_{u_m},q_{u_m})\}_{m=c}^{c+B}$ factor, and at most $df_2$ duo families associated with that fiber and the duo limit groups, $Duo_1^2,\dots,Duo_{t_2}^2$, we obtained a strictly decreasing sequence of Diophantine sets of length $length_{Cov}^2+1$ associated with at least one of the duo limit groups, $Duo_1^2,\dots,Duo_{t_2}^2$, where $length_{Cov}^2$ is supposed to be a bound on such a strictly decreasing sequence, a contradiction, and the proposition follows.

□

Proposition 4.4 proves the stability of the set $Cov(p,q)$.

□

## §5. Stability

In the previous section we have shown that the sets $NR_s$, $NS_s$, and $Cov(p,q)$, that indicate those values of the parameter set $<p,q>$, for which a rigid limit group $Rgd(x,p,q,a)$ admits at least $s$ rigid specializations, a solid limit group admits at least $s$ strictly solid families of specializations (theorem 4.1), and a given finite set of (graded) closures forms a covering closure of a given graded resolution (theorem 4.3), are stable. In this section we combine these theorems, with the arguments that were used in proving theorems 4.3 and 1.6 to prove that a general definable set over a free group is stable.

**Theorem 5.1.** *The elementary theory of a non-abelian free group is stable.*

*Proof:* The argument we use is a rather straightforward modification of the argument that was used in the minimal (graded) rank case (theorem 1.6). Let $L(p,q)$ be a definable set over a non-abelian free group $F_k$. Recall that with the definable set $L(p,q)$ the sieve procedure [Se6] associates a finite collection of graded (PS) resolutions that terminate in rigid and solid limit groups (with respect to the parameter subgroup $<p,q>$), and with each such graded resolution it associates a finite collection of graded closures that are composed from Non-Rigid, Non-Solid, Left, Root, Extra PS, and collapse extra PS resolutions (see definitions 1.25-1.30 of [Se5] for the exact definitions).

By the sieve procedure, that eventually leads to quantifier elimination over a free group, the definable set $L(p,q)$ is equivalent to those specializations of the terminal rigid and solid limit groups of the PS resolutions constructed along the sieve procedure, for which the collection of Non-Rigid, Non-Solid, Left, Root and extra PS resolutions (minus the specializations that factor through the associated collapse extra PS resolutions) do not form a covering closure.

Therefore, as in the minimal (graded) rank case, using the output of the sieve procedure and the resolutions it constructs, with each terminating rigid or solid limit group $Term$ of a PS resolution along the sieve procedure we associate finitely many sets:

item"(1)" $B_1(Term)$ - the set of specializations of $< p,q >$ for which the terminal rigid or solid limit group $Term$ admits rigid or strictly solid specializations.

(2) $B_2(Term)$ - the set of specializations of $< p,q >$ for which the associated Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through the associated collapse extra PS resolutions), associated with the PS resolution that terminates in $Term$, form a covering closure of all the (ungraded) PS resolutions associated with the rigid or strictly solid specializations that are associated with the given specialization of $< p,q >$.

(3) $B_3(Term)$ - the set of specializations of $< p,q >$ for which the Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through associated collapse extra PS resolutions) form a covering closure of all the (ungraded) PS resolutions associated with a given specialization of $< p,q >$ and with PS resolutions that extend the PS resolution that terminates in $Term$, and for which there exist rigid or strictly solid specializations of $Term$ with respect to that covering closure.

(4) $B_4(Term)$ - the set of specializations of $< p,q >$ in $B_3(Term)$, for which the associated Non-Rigid, Non-Solid, Left, Root, and extra PS resolutions (minus the specializations that factor through the associated collapse extra PS resolutions), associated with the PS resolution that terminates in $Term$, form a covering closure of all the (ungraded) PS resolutions associated with the rigid or strictly solid specializations of $Term$, where these rigid and strictly solid specializations are associated with a given specialization of $< p,q >$, and are taken with respect to the covering closure of all the PS resolutions that extend the PS resolution that terminates in $Term$, that is associated with their collections of Non-Rigid, Non-Solid, Left, Root and extra PS resolutions.

Let $Term$ be one of the terminal rigid or solid limit groups of the PS resolutions constructed along the sieve procedure associated with the definable set $L(p,q)$, $Term_1,\ldots,Term_m$. By theorems 4.1 and 4.3, $B_1(Term)$ and $B_2(Term)$ are stable. Furthermore, precisely the same argument and procedures that were used in proving theorem 4.3, i.e., in proving that the collection of rigid or strictly solid specializations of a given rigid or solid limit group, for which a given collection of closures forms a covering closure, is stable, imply that the sets $B_3(Term)$ and $B_4(Term)$ are stable.

By the construction of the sieve procedure [Se6], if $Term_1,\ldots,Term_m$ are the terminal rigid or solid limit groups of the PS resolutions constructed along the sieve procedure associated with the definable set $L(p,q)$, then $L(p,q)$ is the finite union:

$$L(p,q) \;=\; \Cup_{i=1}^{m} \left(B_1(Term_i) \setminus B_2(Term_i)\right) \cup \left(B_3(Term_i) \setminus B_4(Term_i)\right).$$

Therefore, since the sets: $B_1(Term_i),\ldots,B_4(Term_i)$, $1 \leq i \leq m$, are stable, and stability is preserved under Boolean operations, the definable set $L(p,q)$ is stable. $\square$

According to [Se8], a definable set over a non-elementary, torsion-free hyperbolic group can be analyzed using the same sieve procedure as the one constructed over a free group in [Se8]. In particular, every definable set over such a group is in the Boolean algebra generated by AE sets, and every such definable set is a finite union of sets of rigid and solid specializations of some (finite collection of) rigid and solid limit groups, for which a given collection of closures do not form a covering closure. Furthermore, with every system of equations over a non-elementary, torsion-free hyperbolic group it is possible to associate a Makanin-Razborov diagram, the analysis of parametric system of equations over such groups is similar to the corresponding analysis over a free group [Se3], the generalized form of Merzlyakov's theorem [Se2], and the techniques to analyze the entire collection of formal solutions over a free group presented in [Se2], remain valid over a non-elementary, torsion-free hyperbolic group.

Therefore, Diophantine sets over a non-elementary, torsion-free hyperbolic groups are equational (theorem 2.1). Duo limit groups can be defined over such groups in the same way they are defined over a free group (definition 3.1), and their basic properties remain valid (theorem 3.2). The same arguments that were used in proving theorems 4.1 and 4.3 over a free group, prove that sets of the form $NR_s$ and $NS_s$, and sets of rigid and solid specializations of a given rigid or solid limit group for which a given finite set of closures form a covering closure, are stable over a non-elementary, torsion-free hyperbolic group. Finally, as over a free group, the arguments that prove theorems 4.1 and 4.3 together with the proof of theorem 5.1, prove the stability of such groups.

**Theorem 5.2.** *The elementary theory of a non-elementary (torsion-free) hyperbolic group is stable.*

## §6. (Non) Equationality

In section 1 we showed that a minimal rank definable set is in the Boolean algebra of equational sets. In section 2 we showed that general Diophantine sets are equational, and later used it together with the collection of duo limit groups and their properties to prove stability for the first order theories of free and non-elementary, torsion-free hyperbolic groups (theorems 5.1 and 5.2).

In this section we give an example of a definable set over a free group (and correspondingly over a non-elementary, torsion-free hyperbolic group) which is not in the Boolean algebra generated by equational sets, hence, disprove the equationality of the elementary theory of a free group. We note that the question of the existence of a theory which is stable but not equational was raised by Pillay and Srour [Pi-Sr], and such examples were constructed by Hrushovski and Srour [Hr-Sr] and Baudisch and Pillay [Ba-Pi].

**Theorem 6.1.** *The elementary theory of a non-abelian free group is not equational.*

*Proof:* Let $F_k$ be a non-abelian free group, and let $NE(p,q)$ be the existential set:

$$NE(p,q) \ = \ \exists x_1,x_2 \ \ qp=x_1^{10}x_2^{-9} \ \wedge \ [x_1,x_2] \neq 1.$$

With the set $NE(p,q)$ we associate the following natural duo limit group:

$$Duo = < d_1 > *_{<d_0>} < d_2 > = < t,u > *_{<u>} < u,s > ; \ q=s^{10}u \, ; \ p=u^{-1}t^{-9}.$$

Note that if we fix a value of the variable $u$, i.e., we fix a duo-family associated with the duo limit group $Duo$, then for every non-trivial value of the variable $s$, a couple $(q(s), p(t))$ is in the set $NE(p,q)$ for every generic value of the variable $t$, and correspondingly a non-trivial value of $t$ and a generic value of $s$.

**Lemma 6.2.** *The set $NE(p,q)$ is not equational.*

*Proof:* If we fix a value $u_0$ for the variable $u$, and a non-trivial value $s_0$ for the variable $s$, so that $q(s_0) = s_0^{10}u_0$, then for all values of the variable $t$ for which $[t, s_0] \neq 1$, the couple $(q(s_0), p(t)) \in NE(p,q)$ where $p(t) = u_0^{-1}t^{-9}$. A work of Lyndon and Schutzenberger [Ly-Sch], shows that for $m, n, p \geq 2$, the solutions of the equation $x^m y^n z^p = 1$ in a free group, generate a cyclic subgroup. Hence, for all but at most 2 values $t_0$ of the variable $t$, for which $[s_0, t_0] = 1$, $(q(s_0), q(t_0)) \notin NE(p,q)$. Hence, if we consider a sequence, $s_1, s_2, \dots$, where $[s_{i_1}, s_{i_2}] \neq 1$ for $i_1 < i_2$, then the sequence of intersections: $\{\Cap_{i=1}^{j} NE(p, q(s_i))\}_{j=1}^{\infty}$ is a strictly decreasing sequence. Therefore, $NE(p,q)$ is not equational.

□

By lemma 6.2 the set $NE(p,q)$ is not a finite union nor a finite intersection of equational sets since such sets are equational.

**Lemma 6.3.** *The set $NE(p,q)$ is not co-equational (i.e., it is not the complement of an equational set).*

*Proof:* The (infinite) union of the sets $NE(p, q_i)$, for all possible values of $q_i$, is the entire coefficient group $F_k$. Every finite subunion of such sets is a proper subset of $F_k$. Hence, $NE(p,q)$ can not be co-equational.

□

To prove that the theory of a free group is not equational, we need to show that the set $NE(p,q)$ is not a union nor an intersection of an equational and a co-equational sets.

**Lemma 6.4.** *The set $NE(p,q)$ is not a union of an equational and a co-equational sets.*

*Proof:* Suppose that $NE(p,q) = E_1 \cup (E_2)^c$ where $E_1, E_2$ are equational sets. Let $N_1, N_2$ be the bounds on the lengths of strictly decreasing sequences: $\{E_1(p, q_i)\}$ and $\{E_2(p, q_j)\}$, and let $N = \max(N_1, N_2)$. We look at a sequence of specializations: $\{q_j^i = (s_j^i)^{10} u_i\}_{i,j=1}^{N+1}$, and $\{p_j^i = u_i^{-1} (s_j^i)^{-9}\}_{i,j=1}^{N+1}$, where $\{s_j^i\}$ and $\{u_i\}$ are distinct elements in a given test sequence (see theorem 1.1 and definition 1.20 in [Se2] for a test sequence. Elements in a test sequence should be considered "generic" elements).

Since $E_2$ is equational, a strictly decreasing sequence of sets of the form $E_2(p, q_j^i)$ has length bounded by $N$, and the elements $s_j^i$ and $u_i$ were chosen to be elements from a given test sequence ("generic"), there must exist some index $i_0$, $1 \leq i_0 \leq N+1$, for which for every $1 \leq j, j' \leq N+1$: $p_j^{i_0} \in E_2(p, q_{j'}^{i_0})$.
Hence, the sequence of intersections: $\{\cap_{j=1}^{m} E_1(p, q_j^{i_0})\}_{m=1}^{N+1}$ is a strictly decreasing sequence, a contradiction to the assumption that $N$ is the global bound on the length of such a strictly decreasing sequence.

□

Using a somewhat similar argument one can show that $NE(p,q)$ is not the intersection of an equational and a co-equational sets.

**Lemma 6.5.** *The set $NE(p,q)$ is not an intersection between an equational and a co-equational sets.*

*Proof:* Suppose that $NE(p,q) = E_1 \cap (E_2)^c$ where $E_1, E_2$ are equational sets. Let $N_1, N_2$ be the bounds on the lengths of strictly decreasing sequences: $\{E_1(p,q_i)\}$ and $\{E_2(p,q_j)\}$, and let $N = \max(N_1, N_2)$. We look at a sequence of specializations: $\{s_i = (v_i)^{10}(w_i)^{-9}\}_{i=1}^{N+1}$, $\{q_j^i = s_i^{(10^j)}\}_{i,j=1}^{N+1}$, where $\{v_i\}$ and $\{w_i\}$ are distinct elements in a given test sequence ("generic" elements).

Since $E_1$ is equational, a strictly decreasing sequence of sets of the form $E_1(p,q_j^i)$ has length bounded by $N$, and the elements $s_i$ were chosen to be elements from a given test sequence ("generic"), there must exist some index $i_0$, $1 \leq i_0 \leq N+1$, for which for every $1 \leq j \leq N+1$:

$$\{p = t^9 \mid t \in F_k \ \wedge \ t \neq 1\} \ \subset \ E_1(p, q_j^{i_0})$$

Hence, the sequence of intersections: $\{\cap_{j=1}^{m} E_2(p,q_j^{i_0})\}_{m=1}^{N+1}$ is a strictly decreasing sequence, a contradiction to the assumption that $N$ is the global bound on the length of such a strictly decreasing sequence.

□

So far we proved that the set $NE(p,q)$ is not an intersection nor a union of an equational and a co-equational sets. Since a finite union and a finite intersection of equational sets are equational, to prove that the set $NE(p,q)$ is not in the Boolean algebra of equational sets it is enough to prove that $NE(p,q)$ is not a finite union of intersections between an equational and a co-equational sets. Unlike the previous lemmas, to prove that we use the structure of a definable set over a free group, that was used in proving the stability of the theory, and is the outcome of the quantifier elimination procedure [Se6].

**Proposition 6.6.** *The set $NE(p,q)$ is not a finite union of intersections between an equational and a co-equational sets.*

*Proof:* Suppose that:

$$NE(p,q) \ = \ (E_1^1 \ \cap \ (E_2^1)^c) \ \cup \ \ldots \ \cup \ (E_1^\ell \ \cap \ (E_2^\ell)^c)$$

Let $M_{i_1,\ldots,i_h}(p,q) \ = \ E_2^{i_1} \ \cap \ldots \cap \ E_2^{i_h}$, for $1 \leq h \leq \ell$, and $1 \leq i_1 < \ldots < i_h \leq \ell$. Since the sets $E_2^i$ are equational, so are the sets $M_{i_1,\ldots,i_h}$. Let $N$ be a bound on the lengths of strictly decreasing sequences: $\{\cap_{j=1}^{t} M_{i_1,\ldots,i_h}(p,q_j)\}_{t=1}^{r}$, for all $1 \leq h \leq \ell$ and $1 \leq i_1 < \ldots < i_h \leq \ell$.

We look at the collection of all the test sequences $\{v_j\}_{j=1}^{\infty}$ of elements in the (coefficient) free group $F_k$ (recall that elements in a test sequence should be considered "generic" elements). For each index $j$, and a couple of non-zero integers $m_j^1, m_j^2$, we set $s_j = v_j^{10} v_{j+1}^{-9}$, $t_j = v_{j+2}^{10} v_{j+3}^{-9}$, and: $q_j = s_j^{10 m_j^1}$, $p_j = t_j^{-9 m_j^2}$. By construction, for every test sequence $\{v_j\}$, every index $j$, and every couple of non-zero integers $m_j^1, m_j^2$: $(p_j, q_j) \in NE(p,q)$.

By our assumptions, for each $(p,q) \in NE(p,q)$, there exists some index $i$, $1 \leq i \leq \ell$, for which $(p,q) \in \ E_1^i \cap (E_2^i)^c$, hence, in particular, $(p,q) \in E_1^i$. For each index $i$, $1 \leq i \leq \ell$, we look at the collection of all the sequences $\{(p_j, q_j)\}_{j=1}^{\infty}$, that were defined above, for which for every index $j$, $(p_j, q_j) \in E_1^i$.

Recall that by the sieve procedure for quantifier elimination [Se6], a definable set over a free group is the finite union of sets of specializations of the defining parameters for which there exists a rigid (strictly solid) specialization of a given rigid (solid) limit group, so that a given collection of closures of a given graded resolution that terminates in the rigid (solid) limit group does not form a covering closure.

Therefore, by the techniques that were used to analyze all the formal solutions to a given AE sentence ([Se2],sections 2 and 3), with each index $i$, $1 \leq i \leq \ell$, for which there exists a sequence $\{(p_j, q_j)\} \subset E_1^i$, there exists a finite collection of limit groups: $U_1^i, \dots, U_{r_i}^i$ so that from each sequence $\{(p_j, q_j)\} \subset E_i$ there exists a subsequence that can be extended to specializations of one of the limit groups $U_d^i$, for some $d$, $1 \leq d \leq r_i$.

Furthermore, using the same techniques, for each index $i$, and each index $d$, $1 \leq d \leq r_i$, there exists a finite collection of limit groups, $V_1^{i,d}, \dots, V_{f_{i,d}}^{i,d}$, so that if a sequence of specializations $\{(p_j, q_j\}$ can be extended to specializations that factor through $U_d^i$, and for every $j$, $(p_j, q_j) \notin E_i$, then there exists a subsequence of the sequence $\{(p_j, q_j)\}$ that can be extended to specializations of one of the limit groups, $V_e^{i,d}$, for some $e$, $1 \leq e \leq f_{i,d}$.

By the existence of the collections of limit groups: $U_1^i, \dots, U_{r_i}^i$, and $V_1^{i,d}, \dots, V_{f_{i,d}}^{i,d}$, $1 \leq i \leq \ell$, $1 \leq d \leq r_i$, into which all the sequences $(p_j, q_j) \in NE(p, q)$ subconverge, and using the equationality of the sets $E_1^1, \dots, E_1^{\ell}$, there must exist two sequences: $q_1, \dots, q_{N+1}$ and $p_1, \dots, p_{N+1}$, for which:

1. $s = v_1^{10} v_2^{-9}$, where $v_1, v_2$ are non-trivial, non-commuting elements, that do not have non-trivial roots.
2. $q_j = s^{10m_j^1}$, $p_j = s^{-9m_j^2}$, $1 \leq j \leq N+1$.
3. $q_j p_j = s$, $1 \leq j \leq N+1$. Hence, for every couple of indices $j, j'$, $1 \leq j, j' \leq N+1$, $(p_j, q_{j'}) \in NE(p, q)$ if and only if $j = j'$.
4. there exists a positive integer $h$, $1 \leq h \leq \ell$, and a tuple of indices $1 \leq i_1 < \dots < i_h \leq \ell$, so that for every couple of indices: $j, j'$, $1 \leq j, j' \leq N+1$, $(p_j, q_{j'}) \in E_1^i$ if and only if $i = i_1, \dots, i_h$.

Let $M_{i_1, \dots, i_h}(p, q) = E_2^{i_1} \cap \dots \cap E_2^{i_h}$. By (3) and (4) for every couple of indices $j, j'$, $1 \leq j, j' \leq N+1$, $(p_j, q_{j'}) \in M_{i_1, \dots, i_h}$ if and only if $j \neq j'$. Hence, the sequence, $\{\Cap_{j=1}^{t} M_{i_1, \dots, i_h}(p, q_j)\}_{t=1}^{N+1}$, is a strictly decreasing sequence, which clearly contradicts our assumption that $N$ bounds the length of all such strictly decreasing sequences.

□

Theorem 6.1 proves that free groups are not equational. The results of [Se8] allow one to modify the argument that was used to generalize theorem 6.1 to every non-elementary, torsion-free hyperbolic group.

**Theorem 6.7.** *The elementary theory of a non-elementary, torsion-free hyperbolic group is not equational.*

*Proof:* To modify the argument that was used in the free group case, we only need to modify the proof of proposition 6.6. By the results of [Se8], the same sieve procedure that was used for quantifier elimination over a free group can be used to obtain quantifier elimination over a general torsion-free, non-elementary hyperbolic group. Hence, the description of a definable set as the union of finitely many sets of the defining parameters for which there exists a rigid (strictly solid) specialization of a given rigid (solid) limit group, so that a given collection of closures of a given graded resolution that terminates in the rigid (solid) limit group does not form a covering closure, remains valid over a general (non-elementary, torsion-free) hyperbolic group. The techniques that were applied to collect all the formal solutions of a given AE sentence over some variety presented in [Se2], generalize to a hyperbolic groups as well [Se8].

Since any non-elementary hyperbolic group contains a quasi-isometric embedding of a f.g. free group, to generalize the argument that was used to prove proposition 6.6 to non-elementary, torsion-free hyperbolic groups, the only result that still needs to be generalized to hyperbolic groups is the Lyndon-Schutzenberger theorem [Ly-Sch], that shows that for $m, n, p \geq 2$, the solutions of the equation $x^m y^n z^p = 1$ in a free group, generate a cyclic subgroup. For the purposes of generalizing the proof of proposition 6.6 we need a slightly weaker result that can be proved easily using Gromov-Hausdorff convergence.

**Lemma 6.8.** *Let $\Gamma$ be a non-elementary, torsion-free hyperbolic group. There exists some constant $b > 0$, so that for every $m > b$, the solutions of the system $x^1 0 y^{-9} z^m = 1$ in $\Gamma$, generate a cyclic subgroup in $\Gamma$.*

Lemma 6.8 and the results of [Se8] allows one to generalize the proof of proposition 6.6 to non-elementary, torsion-free hyperbolic groups, hence, to prove that such groups are not equational.

□

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