THE RATES OF GROWTH IN A HYPERBOLIC GROUP

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Abstract. We study the countable set of rates of growth of a hyperbolic group with respect to all its finite generating sets. We prove that the set is well-ordered, and that every real number can be the rate of growth of at most finitely many generating sets up to automorphism of the group. We prove that the ordinal of the set of rates of growth is at least ω^ω, and in case the group is a limit group (e.g. free and surface groups) it is ω^ω.

We further study the rates of growth of all the finitely generated subgroups of a hyperbolic group with respect to all their finite generating sets. This set is proved to be well-ordered as well, and every real number can be the rate of growth of at most finitely many isomorphism classes of finite generating sets of subgroups of a given hyperbolic group. Finally, we strengthen our results to include rates of growth of all the finite generating sets of all the subsemigroups of a hyperbolic group.

1. Introduction

Growth of groups was studied extensively in the last decades. Finitely generated (f.g.) abelian groups have polynomial growth. This was generalized later to f.g. nilpotent groups [3]. F.g. solvable groups have either polynomial or exponential growth ([14],[24]), and the same holds for linear groups by the Tits alternative [21]. Gromov's celebrated theorem proves that a f.g. group has polynomial growth if and only if it is virtually nilpotent ([9],[22]). Grigorchuk constructed a group of intermediate growth [8], and by now there are known to be uncountably many such groups [11].

In this paper, we study the possible growth rates of groups of exponential growth, in particular, hyperbolic groups. We will be interested not in the growth rate with respect to a particular (finite) generating set, but with the countable set of rates of exponential growths with respect to all possible (finite) generating sets of the given hyperbolic

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group. By definition, the countable set of rates that we study is an invariant of the group, and we study its basic properties.

The motivation for our study comes partly from the work of Jorgensen and Thurston on the volumes of hyperbolic 3-manifolds [20]. By analyzing volumes of Dehn fillings of finite volume hyperbolic 3-manifolds with cusps, Jorgensen and Thurston proved that Dehn fillings have always strictly smaller volume than the cusped manifold, and deduced (using the Kazhdan-Margulis lemma and the thin-thick decomposition) that the set of volumes of hyperbolic 3-manifolds is well-ordered, and that the the ordinal of the (countable) set of volumes is ω^ω. They further proved that there are only finitely many hyperbolic 3-manifolds with the same volume.

We prove analogous results for the set of growth rates of a non-elementary hyperbolic group. We prove that the set of growth rates of such a group, with respect to its finite generating sets, is well-ordered (Theorem 2.2). We also prove that given a positive real number there are at most finitely many (finite) generating sets with this real number as a growth rate, up to the action of the automorphism group of the hyperbolic group (Theorem 3.1).

Since the set of growth rates of a hyperbolic group is well-ordered, we can associate a *growth* ordinal with every non-elementary hyperbolic group, the ordinal of the well-ordered set of growth rates. For limit groups (in particular, free and surface groups), we prove that the growth ordinal is ω^ω (Theorem 4.2). We conjecture that this is true for all non-elementary hyperbolic groups, and for all limit groups over hyperbolic groups. This conjecture turns out to be closely related to the existence of a Krull dimension for limit groups over hyperbolic groups, which is still open (the Krull dimension for limit groups (over free groups) is known to exist by a celebrated work of Louder [13]).

The well-ordering of the set of growth rates proves, in particular, that the set has a minimum. This answers a question of de la Harpe [5]. In his book, de la Harpe explains that the existence of a minimum for the set of growth rates, combined with a theorem of Arzhantseva and Lysenok [1], that claims that a hyperbolic proper quotient of a hyperbolic group has strictly smaller rate of growth, gives an alternative proof for the Hopf property of hyperbolic groups.

After we analyze rates of growth of hyperbolic groups, we study rates of growth of all the non-elementary f.g. subgroups of a given hyperbolic group with respect to all their finite generating sets. We manage to obtain the same results in this more general setting. The rates of growth of all the f.g. non-elementary subgroups of a given hyperbolic group, with respect to all their finite generating sets, is well-ordered

(Theorem 5.1). Every real number can be the growth rate of only finitely many isomorphism classes of pairs consisting of a subgroup of the hyperbolic group and its finite set of generators (Theorem 5.3).

To demonstrate the generality and the power of our techniques we continue further and study rates of growth of all the f.g. non-elementary subsemigroups of a given hyperbolic group, with respect to all their finite generating sets. We prove that the set of growth rates of all these subsemigroups and their finite generating sets is well-ordered as well (Theorem 6.1). In particular, we obtain that the ordinal of growth rates of all the f.g. subsemigroups of a free or a surface group (and more generally of a given limit group) with respect to all their finite generating sets is ω^ω (Corollary 6.8).

In his seminal work, Gromov analyzed groups with polynomial growth by using Gromov-Hausdorff convergence, obtaining a convergence of rescaled Cayley graphs into a manifold with an isometric group action, applying the solution of Hilbert's 5th problem to deduce linearity of the group in question, and finally referring to Tits alternative to conclude that these groups must be virtually nilpotent.

To analyze rates of growth of hyperbolic groups and their subgroups we use Gromov-Hausdorff convergence, via the Bestvina-Paulin method, and obtain convergence of larger and larger balls in rescaled Cayley graphs into a limit tree. This tree is not equipped with an action of the hyperbolic group in question, but rather with an isometric action of a limit group over that hyperbolic group. We then prove our results using the structure theory of limit groups over hyperbolic groups, and analyzing the action of these limit groups on limit trees.

Limit groups were originally defined and studied in order to understand the structure of varieties and first order formulas over certain classes of groups [16]. However, as can be seen in this paper, they provide a natural and powerful tool to study variational problems over groups, e.g., the existence of a minimum for a set of growth rates. Limit pairs, that were defined in [17] for studying varieties over free semigroups, play a similar role in studying variational problems in semigroups. We believe that limit algebras (see [18]), will eventually be used in a similar way in studying associative and non-commutative rings.

Throughout the paper we assume hyperbolicity of the ambient groups in question, but it is probably not necessary. We believe that it should be possible to prove most of our results under some weak acylindricity assumptions. Our basic study of the set of growth rates suggests quite a few natural problems, and we list several of them in the last section of the paper.

2. Well ordering of the set of growth rates

Let G be a finitely generated (f.g.) group with a finite generating set S. Let $B_n(G,S)$ be the set of elements in G whose word lengths are at most n with respect to the generating set S. Let $\beta_n(G,S) = |B_n(G,S)|$. The *exponential growth rate* of (G,S) is defined to be:

$$e(G,S) = \lim_{n\to\infty} \beta_n(G,S)^{\frac{1}{n}}$$

A f.g. group G has *exponential growth* if there exists a finite generating set S such that $e(G,S) > 1$. G has *uniform exponential growth* if there exists $c > 1$, such that for every finite generating set S, $e(G,S) > c > 1$.

Given a f.g. group G, We define:

$$e(G) = \inf_{|S|<\infty} e(G,S)$$

where the infimum is taken over all the finite generating sets S of G. Since there are f.g. groups that have exponential growth, but do not have uniform exponential growth [23], the infimum, $e(G)$, is not always obtained by a finite generating set of a f.g. group.

Remark 2.1. Note that $e(G,S)$ does not change if we make S symmetric, hence, for the rest of this paper we will always assume that our generating sets are symmetric.

Given a f.g. group G we further define the following set in $\mathbb{R}$:

$$\xi(G) = \{e(G,S) \mid |S| < \infty\}$$

where S runs over all the finite generating sets of G. The set $\xi(G)$ is always countable.

A non-elementary hyperbolic group contains a non-abelian free group, hence, it has exponential growth. In fact, a non-elementary hyperbolic group has uniform exponential growth [12]. Our main theorem proves that the set of growth rates of a non-elementary hyperbolic group is well-ordered (hence, in particular, has a minimum).

Theorem 2.2. *Let Γ be a non-elementary hyperbolic group. Then $\xi(\Gamma)$ is a well-ordered set.*

Proof. We need to prove that $\xi(\Gamma)$ does not contain a strictly decreasing convergent sequence. Suppose that there exists a sequence of finite generating sets $\{S_n\}$, such that $\{e(G,S_n)\}$ is a strictly decreasing sequence and $\lim_{n\to\infty} e(G,S_n) = d$, for some $d > 1$.

By [2], if $|S_n| = m$, then $e(G, S_n)$ has a lower bound that depends linearly only on m and on the hyperbolicity constant δ of Γ, so this lower bound grows to infinity with m. Hence, we may assume that the cardinality of the generating sets $|S_n|$ from the decreasing sequence is bounded, and by possibly passing to a subsequence we may assume that the cardinality of the generating sets is fixed, $|S_n| = \ell$.

Let $S_n = \{x_1^n, \cdots, x_\ell^n\}$. Let F_ℓ be the free group of rank ℓ with a free generating set: $S = \{s_1, \dots, s_\ell\}$. For each index n, we define a map: $g_n : F_\ell \to \Gamma$, by setting: $g_n(s_i) = x_i^n$. Since S_n are generating sets, the map g_n is an epimorphism for every n. Note that $e(\Gamma, S_n) = e(\Gamma, g_n(S))$.

We fix a Cayley graph X of Γ with respect to some finite generating set. Since Γ is a hyperbolic group, X is a δ-hyperbolic graph. Γ acts isometrically on its Cayley graph X by translations, hence, for each index n, F_ℓ acts on the Cayley graph X via the epimorphism: $g_n : F_\ell \to \Gamma$.

For $\gamma \in \Gamma$, let $|\gamma|$ denotes the word length. Since the sequence $e(\Gamma, S_n)$ is strictly decreasing, and in particular is not constant, the sequence:

$$\{\min_{\gamma \in \Gamma} \max_i |\gamma g_n(s_i) \gamma^{-1}|\}$$

is not bounded. Hence, we may pass to a subsequence for which the sequence converges to ∞. For each index n, we further replace the epimorphism g_n, by the epimorphism $\gamma_n g_n \gamma_n^{-1}$, where:

$$\max_i |\gamma_n g_n(s_i) \gamma_n^{-1}| = \min_{\gamma \in \Gamma} \max_i |\gamma g_n(s_i) \gamma^{-1}|$$

We still denote the conjugated epimorphisms $\{g_n\}$ (note that conjugating an epimorphism does not change the corresponding growth rate).

For each n, we set: $\rho_n = \max_i |g_n(s_i)|$, and denote by (X, d_n) the Cayley graph X with the metric obtained from the metric on X after multiplying it by $\frac{1}{\rho_n}$. From the sequence of actions of F_ℓ on the metric spaces (X, d_n) we extract (via the Bestvina-Paulin method) a subsequence that converges into a non-trivial action of F_ℓ on a real tree Y. The action of F_ℓ is not faithful, so we divide F_ℓ by the kernel of the action, i.e., by the normal subgroup of F_ℓ that acts trivially on Y, and get a faithful action of a *limit group* L on the real tree Y, where the limit group L is a limit group *over* the hyperbolic group Γ (for the definition of limit groups over hyperbolic groups and some of their basic properties see [15]).

Let: $\eta : F_\ell \to L$ be the associated quotient map. By Theorem 6.5 in [15] there exists some index n_0, such that for $n > n_0$, there exists an epimorphism $h_n : L \to \Gamma$ that satisfies: $g_n = h_n \circ \eta$. By passing to a subsequence we may assume that all the homomorphisms $\{g_n\}$ factor through the epimorphism: $\eta : F_\ell \to L$.

$$\begin{array}{ccc} (F_\ell, S) & & \\ \downarrow \eta & \overset{g_n}{\searrow} & \\ (L, \eta(S)) & \xrightarrow[h_n]{} & (\Gamma, g_n(S)) \end{array}$$

Since $g_n = h_n \circ \eta$, for every index n, $e(\Gamma, g_n(S)) \leq e(L, \eta(S))$. Our strategy to prove Theorem 2.2 is to show that:

$$\lim_{n\to\infty} e(\Gamma, g_n(S)) = e(L, \eta(S))$$

This will lead to a contradiction, since we assumed that the sequence $\{e(\Gamma, g_n(S))\}$ is strictly decreasing, hence, it can not converge to an upper bound of the sequence, $e(L, \eta(S))$.

Proposition 2.3. $\lim_{n\to\infty} e(\Gamma, g_n(S)) = e(L, \eta(S))$.

Proof. To prove the proposition we need to analyze the action of L on the limit tree Y. If the action of L on the real tree Y is free and simplicial, then L is free, and for large index n, the image, $h_n(L)$, is a free quasi-convex subgroup of infinite index in Γ. This is a contradiction since the homomorphisms $\{h_n\}$ are assumed to be epimorphisms for every n. Also, the rates of growth satisfy: $e(\Gamma, g_n(S)) = e(L, \eta(S))$ for large n (since h_n is injective for large n), and this contradicts our assumption that the sequence: $\{e(\Gamma, g_n(S))\}$ is strictly decreasing.

In general, the action of L on Y is faithful, but it need not be free nor simplicial. For presentation purposes, we start by assuming that the action of L on Y is free, but not necessarily simplicial. Note that in this case, L is necessarily torsion-free.

The case of a free action.

Let $T_1 \subset Y$ be the convex hull of the base point $y_0 \in Y$, and the images of the base point y_0 under the action of the elements in the set $\eta(S)$: $\eta(s_1)(y_0), \ldots, \eta(s_\ell)(y_0)$. We assumed that the homomorphisms g_n satisfy: $\max_i |g_n(s_i)| = \min_{\gamma\in\Gamma} \max_i |\gamma g_n(s_i)\gamma^{-1}|$. Hence, there are at least two distinct germs at y_0 in T_1. (By the way we chose y_0, it is not a root of T_1.)

Lemma 2.4. *Let $germ_1, germ_2$ be two distinct germs at y_0 in T_1. There are non-trivial elements $u_{i,j} \in L$, $i, j = 1, 2$, with the following properties:*

(1) *for every $i, j = 1, 2$, the segment: $[y_0, u_{i,j}(y_0)]$ starts with the germ $germ_i$ at y_0, and ends with the germ $germ_j$ at $u_{i,j}(y_0)$.*

(2) *$d_Y(y_0, u_{i,j}(y_0)) > 10$, for $i, j = 1, 2$.*
(3) *for every $w \in L$, and every two pairs: $(i_1, j_1), (i_2, j_2)$, $1 \leq i_1, j_1, i_2, j_2 \leq 2$, if the segment $[y_0, u_{i_1,j_1}(y_0)]$ intersects the segment $[w(y_0), wu_{i_2,j_2}(y_0)]$ non-trivially, then the length of the intersection is bounded by: $\frac{1}{10} d_Y(y_0, u_{i_1,j_1}(y_0))$ (if the pairs $(i_1, j_1), (i_2, j_2)$ are equal, we assume in addition that $w \neq 1$).*

We call property (3) the *small cancellation* property of *separators*.

Proof. Since, by construction, y_0 is a point that moves minimally by the generators $\eta(S)$, there are at least two distinct germs at y_0 in T_1. Let $germ_1, germ_2$ be two distinct germs at Y_0 in T_1. Let $s_1, s_2 \in S$ be generators in S for which: $[y_0, \eta(s_i)(y_0)]$ starts with the germ $germ_i$, $i = 1, 2$. Also, since Γ is non-elementary, there exist elements $w, z \in L$, such that:

(i) $< w, z > < L$ is a free subgroup.
(ii) $< w, s_i > < L$ and $< z, s_i > < L$, $i = 1, 2$, are free subgroups.

Given an element $v \in L$, we denote $\mu(v) = d_Y(y_0, v(y_0))$, and $tr(v)$ the displacement of v along its axis. We set the elements $u_{i,j} \in L$, $i, j = 1, 2$, to be elements of the form:

$$u_{i,j} = s_i^{\beta_i} w^{\alpha_1+i+3j} z w^{\alpha_2+i+3j} z \ldots w^{\alpha_{29}+i+3j} z w^{\alpha_{30}+i+3j} s_j^{-\beta_j}$$

where the parameters β_i, $i = 1, 2$, and α_k, $k = 1, \ldots, 30$, satisfy:

(1) $\beta_i tr(s_i) > 5\mu(s_i)$, and: $\beta_i tr(s_i) > 5\mu(w)$, $i = 1, 2$.
(2) $\alpha_1 tr(w) \geq \max(200\mu(w), 20(\beta_1\mu(s_1) + \beta_2\mu(s_2)), 20\mu(z), 1)$.
(3) $\alpha_k = \alpha_1 + 6k$, $k = 2, \ldots, 30$.

The conditions on the parameters β_i, $i = 1, 2$, and α_k, $k = 1, \ldots, 30$, guarantee that the lengths of the cancellations between consecutive intervals in the sequence: $[y_0, s_i^{\beta_i}(y_0)]$, $i = 1, 2$, $[y_0, w^{\alpha_1}(y_0)(y_0)]$, $[y_0, zw^{\alpha_k}(y_0)]$, $k = 2, \ldots, 30$, $[y_0, s_j^{-\beta_j}(y_0)]$, $j = 1, 2$, are limited to a small proportion of the lengths of these intervals. Hence, the interval $[y_0, u_{i,j}(y_0)]$ starts with the germ in which $[y_0, s_i(y_0)]$ starts, and terminates with the germ that $[y_0, s_j^{-1}(y_0)]$ terminates with, and we get part (1) of the lemma.

Part (2) of the lemma follows from the bound on the cancellations between consecutive intervals and condition (2) on $\alpha_1 tr(w)$. Part (3) of the lemma follows from the structure of the elements $u_{i,j}$ as products of high powers of an element w, separated by an element that does not commute with it, and the bound on the cancellations between consecutive intervals that correspond to these high powers.

□

A non-elementary (i.e., non virtually abelian) limit group over a hyperbolic group contains a non-abelian free subgroup, so it has exponential growth. Let $B_m(L,\eta(S))$ be the ball of radius m in the Cayley graph of L with respect to the generating set $\eta(S)$. Let $\beta_m = \beta_m(L,\eta(S))$ be the number of elements in the ball $B_m(L,\eta(S))$, and let: $C_L = e(L,\eta(S)) = \lim_{m\to\infty} {\beta_m}^{\frac{1}{m}}$ be the (exponential) rate of growth of L with the generating set $\eta(S)$.

Let b be the maximal length of the words $u_{i,j}$, $i,j=1,2$, that were constructed in lemma 2.4 (the length is with respect to the generating set $\eta(S)$). By the Gromov-Hausdorff convergence of the actions of L on the Cayley graph X of Γ via the epimorphisms $\{h_n\}$, for every fixed positive integer m, and every large enough n, there exists a bi-Lipschitz map from the ball: $B_{m+2b}(L,\eta(S))$, into the image of that ball under the epimorphism h_n: $h_n(B_{m+2b}(L,\eta(S))) \subset X$, where X is the fixed Cayley graph of Γ, and the ratios between the two bi-Lipschitz constants approaches 1 when n tends to infinity.

Given a non-trivial element $w \in B_m(L,\eta(S))$, the segment, $[y_0, w(y_0)] \subset Y$, starts and terminates in (an orbit of) a germ of y_0 in Y. For each pair of non-trivial elements, $w_1, w_2 \in B_m(L,\eta(S))$, we choose an element $u_{i,j}$, from the elements that were constructed in lemma 2.4, such that $u_{i,j}$ starts with a germ that w_1 does not end with, and $u_{i,j}$ ends with a germ that w_2 does not start with. By the Gromov-Hausdorff convergence, for large enough n, h_n maps the elements $w_1 u_{i,j} w_2$ in a bi-Lipschitz way into the Cayley graph X of Γ (with the same bi-Lipschitz constants as it maps the ball $B_{m+2b}(L,\eta(S))$.

Continuing inductively, let q be an arbitrary positive integer, and let $w_1, \ldots, w_q$ be a collection of non-trivial elements from $B_m(L,\eta(S))$. For each t, $1 \leq t \leq q-1$, we choose an element u_t from the collection $\{u_{i,j}\}$ that was constructed in lemma 2.4, such that u_t does not start with the germ that w_t ends with, and u_t does not end with the germ that w_{t+1} starts with. By the argument that we apply for a pair w_1, w_2, for large enough n, h_n maps in a bi-Lipschitz way all the elements of the form:

$$w_1 u_1 w_2 u_2 \ldots w_{q-1} u_{q-1} w_q$$

into the fixed Cayley graph X of Γ (with the same bi-Lipschitz constants as it maps the ball $B_{m+2b}(L,\eta(S))$. We call q the *length* of this form.

So far we know that all the elements that we constructed are mapped to non-trivial elements by the epimorphisms $\{h_n\}$. But the maps h_n may be not injective on these collections of elements. Hence, we take out some of the elements that we constructed, in order to guarantee that the remaining elements are mapped injectively by the epimorphisms h_n, for large enough n.

Definition 2.5 (Forbidden and feasible elements). We say that a non-trivial element $w_1 \in B_m(L, \eta(S))$ is *forbidden* if there exists an element $w_2 \in B_m(L, \eta(S))$, and an element $u_{i,j}$ that was constructed in lemma 2.4, such that:

(i) $[y_0, u_{i,j}(y_0)]$ does not start with the germ that $[y_0, w_1(y_0)]$ terminates with.
(ii) $d_Y(w_2(y_0), w_1 u_{i,j}(y_0)] \leq \frac{1}{5} d_Y(y_0, u_{i,j}(y_0))$.

An element $w_1 u_1 \dots w_{q-1} u_{q-1} w_q$ from the set that we constructed (where the elements $w_t \in B_m(L, \eta(S))$ and the elements u_t are elements that were constructed in lemma 2.4) is called *feasible* of *type* q, if all the elements w_t, $1 \leq t \leq q$, are not forbidden.

Feasible elements are mapped injectively by the epimorphisms h_n for large enough n.

Lemma 2.6. *Given m, for all large enough n and every fixed q, the epimorphisms h_n map the collections of feasible elements of type q to distinct elements in Γ.*

Proof. Suppose that h_n maps the two distinct feasible elements of type q: $w_1 u_1 \dots u_{q-1} w_q$ and $\hat{w}_1 \hat{u}_1 \dots \hat{u}_{q-1} \hat{w}_q$ to the same element of Γ.

If for every t, $1 \leq t \leq q$, $w_t = \hat{w}_t$, then by the small cancellation properties of the elements $u_{i,j}$ that were constructed in lemma 2.4 (part (3) of that lemma), it follows that $u_t = \hat{u}_t$ for every $1 \leq t \leq q-1$, and the two feasible elements are identical.

Hence, there exists an index t, for which $w_t \neq \hat{w}_t$. Let t_0 be the first such index t. Note that $1 \leq t_0 \leq q-1$, since if $t_0 = q$, h_n maps the two feasible elements to distinct elements in Γ. Since h_n maps the two feasible elements into the same element in Γ, the small cancellation properties of the elements, $u_{i,j}$, imply that for all $t < t_0$, $u_t = \hat{u}_t$.

Since h_n maps the two feasible elements to the same element in Γ, one of the two intervals, $[y_0, w_t u_t(y_0)]$ and $[y_0, \hat{w}_t \hat{u}_t(y_0)]$, is almost contained in the second one (which means that one of the intervals overlaps with the beginning of the second interval, possibly except for the last $\frac{1}{10}$ of its suffix $[w_t(y_0), w_t u_t(y_0)]$ or $[\hat{w}_t(y_0), \hat{w}_t \hat{u}_t(y_0)]$. This implies that either w_t or $\hat{w}_t$ are forbidden elements, which means that one of the two elements that were assumed to be mapped by h_n to the same element is not feasible.

□

Since for large n, h_n maps feasible elements injectively, to estimate from below the growth of Γ with the generating sets $\{S_n\}$, it is enough to count feasible elements.

Given m, recall that $\beta_m(L,\eta(S)) = |B_m(L,\eta(S)|$, which we denote for brevity β_m.

Lemma 2.7. *Given m, the following are lower bounds on the number of non-forbidden and feasible elements:*

(1) *The number of non-forbidden elements in the ball of radius m in L, $B_m(L,\eta(S))$, is at least $\frac{5}{6}|B_m(L,\eta(S))|$.*
(2) *For every positive q, the number of feasible elements of type q is at least: $(\frac{5}{6}\beta_m)^q$.*

Proof. Part (2) follows from part (1) since given m and q, feasible elements are built from all the possible q concatenations of non-forbidden elements in a ball of radius m in L (with respect to the generating set $\eta(S)$), with separators between the forbidden elements.

To prove part (1) we look at the convex hull of the images of the base point $y_0 \in Y$ under all the elements in the ball of radius m in L, $\{z(y_0) \mid z \in B_m(L,\eta(S))\}$. We denote this convex hull, which is a finite subtree of Y, T_m. By construction: $\max_{s_i \in S} d_Y(y_0, \eta(s_i)(y_0)) = 1$. Since: $T_1 \subset T_2 \subset \ldots \subset T_m$, every element in $B_m(L,\eta(S))$ adds at most 1 to the total length of the edges in T_m. Therefore, the sum of the lengths of the edges in the finite tree T_m is bounded by the number of elements in the ball of radius m, i.e., $\beta_m = |B_m(L,\eta(S))|$.

Now, let $w \in B_m(L,\eta(S))$ be a forbidden element. By definition, there exists an element $\hat{w} \in B_m(L,\eta(S))$, such that for some element $u_{i,j}$ that was constructed in lemma 2.4, $d_Y(wu_{i,j}(y_0), \hat{w}(y_0)) < \frac{1}{5}d_Y(y_0, u_{i,j}(y_0))$. Hence the interval: $[w(y_0), wu_{i,j}(y_0)]$ covers at least $\frac{4}{5}d_Y(y_0, u_{i,j}(y_0))$ from the total length of the edges in T_m.

The elements $\{u_{i,j}\}$ were constructed to satisfy a small cancellation property (part (3) in lemma 2.4). Hence, for two distinct forbidden elements w_1, w_2, the overlap between the intervals: $[w_1(y_0), w_1u^1_{i,j}(y_0)]$ and $[w_2(y_0), w_2u^2_{i,j}(y_0)]$, is bounded by $\frac{1}{10}d_Y(y_0, u^k_{i,j}(y_0)]$ for $k = 1, 2$. Therefore, with each forbidden $w \in B_m(L,\eta(S))$, it is possible to associate a subinterval I_w of length $\frac{6}{10}d_Y(y_0, u_{i,j}(y_0)]$ of the interval $[w(y_0), wu_{i,j}(y_0)]$ for which:

(i) the subinterval I_w starts after the first $\frac{1}{10}$ of the interval $[w(y_0), wu_{i,j}(y_0)]$, and ends at $\frac{7}{10}$ of that interval.
(ii) $I_w \subset T_m$.
(iii for distinct forbidden elements w_1, w_2, the intersection: $I_{w_1} \cap I_{w_2}$ is empty or degenerate.

Since in part (2) of lemma 2.4 we assumed that the length of an interval $[y_0, u_{i,j}(y_0)]$ is at least 10, it follows that the length of a subinterval I_w of a forbidden element w is at least 6. Hence, the collection of subintervals I_w, for all the forbidden elements w, cover a total length of 6 times the number of forbidden elements in $B_m(L,\eta)(S)$ in T_m. Since the total length of the edges in T_m is bounded by $|B_m(L,\eta(S))|$, the number of forbidden elements in $B_m(L,\eta(S))$ is bounded by: $\frac{1}{6}|B_m(L,\eta(S)|$, which gives the lower bound on the number of non-forbidden elements in part (1) of the lemma.

□

Recall that b is the maximal length of an element $u_{i,j}$ (that was constructed in lemma 2.4), with respect to the generating set $\eta(S)$ of L. At this stage we fix m, and look at the balls, $B_{q(m+b)}(\Gamma, g_n(S))$, where n is large enough, and q is an arbitrary positive integer. The ball, $B_{q(m+b)}(\Gamma, g_n(S))$, contains all the elements:

$$h_n(w_1u_1w_2u_2\dots w_{q-1}u_{q-1}w_q)$$

and in particular all such elements that are images of feasible elements. Since by lemma 2.6, h_n maps the feasible elements injectively, lemma 2.7 implies that (for large enough n, where large enough does not depend on q), the ball, $B_{q(m+b)}(\Gamma, g_n(S))$, contains at least the (distinct) images of feasible elements, hence:

$$(\frac{5}{6}\beta_m(L,\eta(S)))^q \leq |B_{q(m+b)}(\Gamma, g_n(S))|.$$

Therefore:

$$\log(e(L,\eta(S))) \geq \lim_{n\to\infty} \log(e(\Gamma, g_n(S))) \geq \lim_{m\to\infty}\lim_{q\to\infty} \frac{q\log(\beta_m)+q\log\frac{5}{6}}{q(m+b)} = \log(e(L,\eta(S))).$$

This finally proves proposition 2.3 in case the action of the limit group L on the limit tree Y is free.

The general case of possibly non-free actions

Suppose that the action of L on Y is faithful, but possibly with point stabilizers. We continue as in the free action case.

Recall that T_1 is a finite subtree in the limit tree Y, which is the convex hull of the images of the base point y_0 under the action of the generators $\eta(S)$. y_0 is a point that moves minimally by the set $\eta(S)$, hence, there are at least two distinct germs at y_0 in T_1.

Lemma 2.8. *There exist elements $u_{i,j} \in L$, $i,j = 1,2$ that satisfy the conditions that are listed in lemma 2.4, even if the action of L on the tree Y is not free.*

Proof. The argument is similar to the one that was used in lemma 2.4. Since $y_0 \in Y$ is, by construction, a point that moves minimally by the set of generators, $\eta(S)$, there are at least two distinct germs at y_0 in T_1. Let $germ_1, germ_2$ be two distinct germs at Y_0 in T_1.

L is not elementary, and the action of L on Y has virtually abelian segment stabilizers, hence, the limit tree Y has infinitely many ends. Since the action of L on Y is minimal (i.e., there is no proper invariant subtree), there exist elements $e_1, e_2 \in L$, that act hyperbolically on Y, and for which the interval: $[y_0, e_i(y_0)]$ starts with the germ $germ_i$ for $i = 1, 2$.

Furthermore, since Y has infinitely many ends, and L acts on Y cocompactly, there exist elements $w, z \in L$ that act hyperbolically on Y, such that:

(i) $< w, z > < L$ is a free subgroup.
(ii) $< w, e_i > < L$ and $< z, e_i > < L$, $i = 1, 2$, are free subgroups.

Since w, z, e_1, e_2 are elements that act hyperbolically on Y, and satisfy the same properties that the elements, w, z, s_1, s_2, do in the proof of lemma 2.4, the construction of the elements $u_{i,j}$, $i, j = 1, 2$, proceeds precisely as in their construction in lemma 2.4.

□

Let w_1, w_2 be a pair of elements of the limit group L. We look at the finite subtrees $w_1(T_1)$ and $w_2^{-1}(T_1)$, in the limit tree Y. The segment, $[y_0, w_1(y_0)] \subset Y$, terminates in a germ at the point $w_1(y_0) \in w_1(T_1)$. Similarly, the segment, $[y_0, w_2^{-1}(y_0)] \subset Y$, terminates in a germ at the point $w_2^{-1}(y_0) \in w_2^{-1}(T_1)$.

With the pair, w_1, w_2, we associate an element $u_{i,j}$ from the ones that were constructed in lemma 2.8. We choose that element $u_{i,j}$, to satisfy:

(i) $u_{i,j}$ starts with a germ in T_1, that is different than the germ that $[y_0, w_1(y_0]$ terminates in the tree $w_1(T_1)$ (in case w_1 fixes y_0 we can choose $u_{i,j}$ to start with any germ).
(ii) $u_{i,j}^{-1}$ starts with a germ in T_1, that is different than the germ that $[y_0, w_2^{-1}(y_0)]$ terminates in the tree $w_2^{-1}(T_1)$ (in case w_2 fixes y_0 we can choose $u_{i,j}^{-1}$ to start with any germ).

The elements that were constructed in lemma 2.8 start and terminate in all the combinations of two distinct germs of T_1 at y_0, so given the pair w_1, w_2 there exists at least one element that was constructed in lemma 2.8 and satisfies both (i) and (ii). For such an element $u_{i,j}$:

$$d_Y(y_0, w_1 u_{i,j} w_2(y_0)) = d_Y(y_0, w_1(y_0)) + d_Y(y_0, u_{i,j}(y_0)) + d_Y(y_0, w_2(y_0)).$$

As we argue in the case of a free action, by the Gromov-Hausdorff convergence, for large enough n, h_n maps the elements $w_1 u_{i,j} w_2$ in a bi-Lipschitz way into the Cayley graph X of Γ (with the same bi-Lipschitz constants as it maps the ball $B_{m+2b}(L, \eta(S))$ into X.

We continue inductively in the same way, similarly to what we did in the case of a free action. Let q be a positive integer, and let: $w_1, \dots, w_q$ be a collection of non-trivial elements from $B_m(L, \eta(S))$. We choose iteratively the elements u_t, $1 \leq t \leq q-1$, from the elements, $u_{i,j}$, that were constructed in lemma 2.8.

We choose u_1 to be an element that satisfies properties (i) and (ii) with respect to the pair: w_1, w_2. We choose u_t, $2 \leq t \leq q-1$, to be an element that satisfies properties (i) and (ii) with respect to the pair: $w_1 u_1 w_2 \dots u_{t-1} w_t, w_{t+1}$. By construction:

$$d_Y(y_0, w_1 u_1 w_2 \dots u_{q-1} w_q(y_0)) = d_Y(y_0, w_1(y_0)) + d_Y(y_0, u_1(y_0)) + d_Y(y_0, w_2(y_0)) + \dots$$

$$\dots + d_Y(y_0, u_{q-1}(y_0)) + d_Y(y_0, w_q(y_0))$$

As in the free action case, for large enough n, h_n maps in a quasi-isometric way all the elements of the form:

$$w_1 u_1 w_2 u_2 \dots w_{q-1} u_{q-1} w_q$$

into the fixed Cayley graph X of Γ (with the same quasi-isometric constants as it maps the ball $B_{m+2b}(L, \eta(S))$ into X).

We define forbidden and feasible elements precisely as we did in the case of the free action (Definition 2.5). The rest of the argument is identical to the one presented in the free case. This finally proves Proposition 2.3.

□

Proposition 2.3 proves that there is no strictly decreasing sequence of rates of growth, $\{e(\Gamma, S_n)\}$, hence, concludes the proof that the set of growth rates is well-ordered, and Theorem 2.2 follows.

□

Theorem 2.2 proves that the set of rates of growth of all the finite generating sets of a hyperbolic group is well ordered, so in particular it has a minimum. As pointed out by de la Harpe [5], the existence of a minimum for the set of growth rates gives an alternative proof for the Hopf property of hyperbolic groups.

Corollary 2.9. *(cf.* [19]*,* [15]*,* [5]*) Every hyperbolic group is Hopf.*

Proof. Let $\nu : \Gamma_1 \to \Gamma_2$ be a proper epimorphism between hyperbolic groups, and let S be a finite generating set of Γ_1. By [1]: $e(\Gamma_1, S) > e(\Gamma_2, \nu(S))$.

Now, suppose that there exists a proper epimorphism, $\tau : \Gamma \to \Gamma$, from a non-elementary hyperbolic group Γ onto itself. Let S be a finite generating set with a minimal possible rate of growth: $e(\Gamma) = e(\Gamma, S)$ (such a generating set exists by theorem 2.2). By [1]: $e(\Gamma) = e(\Gamma, S) > e(\Gamma, \tau(S))$, a contradiction to the minimality of the growth rate $e(\Gamma, S)$. □

3. Finiteness of equal growth generating sets

Jorgensen and Thurston proved that there are only finitely many hyperbolic 3-manifolds with the same volume. In this section we prove an analogous finiteness for generating sets of hyperbolic groups.

Theorem 3.1. *Let Γ be a non-elementary hyperbolic group, and let $r > 1$. Then up to the action of $Aut(\Gamma)$, there are at most finitely many finite generating sets $\{S_n\}$ of Γ, that satisfy: $e(\Gamma, S_n) = r$.*

Proof. Suppose that there are infinitely many finite sets of generators $\{S_n\}$ that satisfy: $e(G, S_n) = r$, and no pair of generating sets S_n is equivalent under the action of the automorphism group Aut (Γ). As in the proof of theorem 2.2, the cardinality of the generating sets $\{S_n\}$ is bounded, so we may pass to a subsequence that have a fixed cardinality ℓ. Hence, each generating set S_n corresponds to an epimorphism, $g_n : F_\ell \to \Gamma$, where S is a fixed free generating set of F_ℓ, and $g_n(S) = S_n$.

By passing to a further subsequence, we may assume that the sequence of epimorphisms $\{g_n\}$ converges into a faithful action of a limit group (over Γ) L on some real tree Y. Let $\eta : F_\ell \to L$ be the associated quotient map. As in the proof of theorem 2.2, by properties of limit groups over hyperbolic groups (Theorem 6.5 in [15]), for large n, $g_n = h_n \circ \eta$, where $h_n : L \to \Gamma$ is an epimorphism. In particular, $S_n = h_n(\eta(S))$. We pass to a further subsequence such that for every n, $g_n = h_n \circ \eta$.

If two of the epimorphisms h_{n_1}, h_{n_2} are isomorphisms, then the corresponding pair of generating sets: $S_{n_i} = g_{n_i}(S) = h_{n_i} \circ \eta(S)$, $i = 1, 2$, are equivalent under an automorphism of Γ. i.e., there exists an automorphism $\varphi \in Aut(\Gamma)$, $\varphi = h_{n_2} \circ h_{n_1}^{-1}$, that maps S_{n_1} to S_{n_2}. This contradicts our assumption that the generating sets $\{h_n\}$ are not equivalent under the action of $Aut(\Gamma)$. Hence, we may assume that none of the epimorphisms $\{h_n\}$ are isomorphisms.

The epimorphisms $\{h_n\}$ are proper epimorphisms from $(L, \eta(S))$ to (Γ, S_n). For each n, h_n has a non-trivial kernel. By basic properties of limit groups over hyperbolic groups [15], we may further pass to a subsequence, for which the kernel of the epimorphisms, $\{h_n\}$, contain no non-trivial torsion elements in L.

Since for every index n, h_n is an epimorphism from L onto Γ that maps $\eta(S)$ to $g_n(S)$, $e(\Gamma, g_n(S)) \leq e(L, \eta(S))$. By proposition 2.3, $\lim_{n\to\infty} e(\Gamma, S_n) = e(L, \eta(S))$. By our assumption, for every index n, $e(\Gamma, S_n) = r$. Hence, $e(L, \eta(S)) = r$. To obtain a contradiction to the existence of an infinite sequence of non-equivalent generating sets with the same rate of growth, and conclude the proof of the theorem, we prove the following:

Proposition 3.2. *For every index n, the generating sets $\{g_n(S)\}$ of Γ (from the remaining subsequence that factor through the limit group L) satisfy: $e(\Gamma, g_n(S)) < e(L, \eta(S))$.*

Proof. To prove the proposition, for any given index n, we construct collections of elements in larger and larger balls of the limit group L (with respect to the generating set $\eta(S)$), that grow with strictly bigger rate than the growth of the corresponding balls in Γ, with respect to the generating set $S_n = g_n(S)$. This contradicts the equality between the growth rates of Γ with respect to the generating sets $\{S_n\}$, and the growth rate of L with respect to $\eta(S)$.

As we did in the second section, for presentation purposes we first assume that the action of the limit group L on the limit tree Y is free.

The case of a free action.

We define the finite tree T_1 as we did in the second section, i.e., the convex hull in the limit tree Y of the points $\eta(s_i)(y_0)$, $s_i \in S$.

Lemma 3.3 (Separators)**.** *Let $germ_1, germ_2$ be two of the germs of y_0 in T_1, and fix an index n_0. There are non-trivial elements $v_{i,j} \in L$, $i,j = 1,2$, that satisfy the properties that the elements $u_{i,j}$, $i,j = 1,2$, satisfy in lemma 2.4 (with respect to the germs $germ_1, germ_2$), and in addition: $h_{n_0}(v_{i,j}) = 1$, for $i,j = 1,2$.*

Proof. First, since $h_{n_0} : L \to \Gamma$ is a proper epimorphism, there exists a non-trivial element $r \in L$, for which: $h_{n_0}(r) = 1$. Note that since r acts hyperbolically on Y, r is an element of infinite order in L. We construct the elements $v_{i,j}$, $i,j = 1,2$, as products of conjugates of r in a similar way to what we did in proving lemma 2.4.

Since Γ is not elementary, there are at least two distinct germs at y_0 in T_1. Let $germ_1, germ_2$ be two distinct germs at Y_0 in T_1. Let $s_1, s_2 \in S$ be generators in S for which: $[y_0, \eta(s_i)(y_0)]$ starts with the germ $germ_i$, $i = 1,2$. Also, since Γ is not elementary, there exist elements $z_1, \dots, z_{30} \in L$, such that:

(i) $< z_1, \dots, z_{30}, r > = < z_1 > * \dots < z_{30} > * < r >$.
(ii) $< z_t, s_i >$, are free subgroups for $i = 1,2$ and $t = 1,2,29,30$.

Let $f \in L$. We denote $\mu(f) = d_Y(y_0, f(y_0))$, and $tr(f)$ the displacement of f along its axis. We set the elements $v_{i,j} \in L$, $i,j = 1,2$, to be elements of the form:

$$v_{i,j} = s_i^{\beta_i} z_1 r^{\alpha_1+i+3j} z_1^{-1} s_i^{-\beta_i} z_2 r^{\alpha_2+i+3j} z_2^{-1} \dots$$
$$\dots z_{29} r^{\alpha_{29}+i+3j} z_{29}^{-1} s_j^{\beta_j} z_{30} r^{\alpha_{30}+i+3j} z_{30}^{-1} s_j^{-\beta_j}$$

where the parameters β_i, $i = 1,2$, and α_k, $k = 1,\dots,30$, satisfy:

(1) $\beta_i tr(s_i) > 5\mu(s_i)$, and: $\beta_i tr(s_i) > 5(\mu(r) + \mu(z_1) + \mu(z_2) + \mu(z_{29}) + \mu(z_{30}))$, $i = 1,2$.
(2) $\alpha_1 tr(r) \geq \max(200\mu(r), 10(\beta_1\mu(s_1)+\beta_2\mu(s_2)), 10\mu(z_1), \dots, 10\mu(z_{30}), 1)$.
(3) $\alpha_k = \alpha_1 + 6k$, $k = 2,\dots,30$.

The elements $v_{i,j}$ are products of conjugates of the element r, hence, $h_{n_0}(v_{i,j}) = 1$. As in the proof of lemma 2.4, the conditions on the parameters β_i, $i = 1,2$, and α_k $k = 1,\dots,30$, guarantee that the lengths of the cancellations between consecutive intervals in the sequence: $[y_0, s_i^{\beta_i}(y_0)]$, $i = 1,2$, $[y_0, z_1 r^{\alpha_1} z_1^{-1}(y_0)(y_0)]$, $[y_0, z_k r^{\alpha_k} z_k^{-1}(y_0)]$, $k = 2,\dots,30$, $[y_0, s_j^{-\beta_j}(y_0)]$, $j = 1,2$, are limited to a small proportion of the lengths of these intervals. Hence, the interval $[y_0, v_{i,j}(y_0)]$ starts with the germ in which $[y_0, s_i(y_0)]$ starts, and terminates with the germ that $[y_0, s_j^{-1}(y_0)]$ terminates with, and we get property (1) in the statement of lemma 2.4. The elements $v_{i,j}$ satisfy properties (2) and (3) in lemma 2.4, by the same arguments that were used to prove that the elements $u_{i,j}$ constructed in lemma 2.4 satisfy them.

□

We fix an index n_0. Since h_{n_0} is a bijection between the sets $\eta(S)$ and S_{n_0}, there is a canonical bijection between the set of words on $\eta(S)$ and the set of words on S_{n_0}.

At this point we need to construct collections of elements in larger and larger balls of the limit group L, that grow faster than corresponding balls in the hyperbolic group Γ with respect to the generating set S_{n_0}.

With the generating set $S_{n_0} = g_{n_0}(S)$ of Γ, we associate a finite automata that encodes geodesics in the Cayley graph of Γ with respect to the generating set S_{n_0}. The regular language that the automata produces, encodes the elements in Γ, i.e., with each element in Γ the language associates a unique element which is a geodesic in the corresponding Cayley graph of Γ.

We further fix a positive integer m (we will choose a specific value for m in the sequel), and look at all the subwords of length m in words in the regular language that the automata produces. By the properties of the automata, each such subword represents a geodesic of length m in the Cayley graph of Γ with respect to the generating set S_{n_0}. Because of the bijection between $\eta(S)$ and S_{n_0}, given by the epimorphism h_{n_0}, we can associate canonically with each such subword of length m in the generators S_{n_0}, a word of length m in L.

Let w be such a word of length m in L. Starting with the word w, we construct a collection of words in the limit group L. Given a positive integer k, $1 \leq k \leq m-1$, we separate the subword w into a prefix of length k, and a suffix of length $m-k$. The prefix corresponds to a non-trivial element in L that we denote w_p^k, and the suffix corresponds to a non-trivial element in L that we denote w_s^k.

The interval $[y_0, w_p^k(y_0)]$ in the real tree Y, terminates in a germ that is in the orbit of a germ of y_0 in Y. The interval $[y_0, w_s^k(y_0)]$ starts in a germ that is in the orbit of a germ of y_0 in Y. With the pair w_p^k, w_s^k we associate an element $v_{i,j}$, that was constructed in lemma 3.3, that does not start with the germ that $[y_0, w_p^k(y_0)]$ terminates with, and does not end with the germ that $[y_0, w_s^k(y_0)]$ starts with. With the pair w_p^k, w_s^k we associate the element in L: $w_p^k v_{i,j} w_s^k$.

The collection of words that we constructed in L from a given word $w \in L$ of length m, may contain elements that represent the same element in L. To prevent that, we take out from the collection that we constructed, a subcollection of *forbidden* words (in somewhat similar way to what we did in the second section).

Definition 3.4 (Forbidden words). Let $w \in L$ be an element that was produced from a subword of length m of a word in the regular language which is the output of the finite automata that is associated with (Γ, S_{n_0}). We say that a word $w_p^k v_{i,j} w_s^k$, from the collection that is built from w, is *forbidden* if there exists f, $1 \leq f \leq m$ such that:

$$d_Y(w_p^k v_{i,j}(y_0), w_p^f(y_0)) \leq \frac{1}{5} d_Y(y_0, v_{i,j}(y_0))$$

With a word $w \in L$ that is associated with a subword of length m of a word in the regular language that the automata that is associated with (Γ, S_{n_0}) produces, we have constructed $m-1$ words of the form $w_p^k v_{i,j} w_s^k$. It is further possible to bound the number of the forbidden words of that form.

Lemma 3.5. *Let $w \in L$ be associated with a subword of length m of a word in the regular language produced by the finite automata that is associated with (Γ, S_{n_0}). Then there are at most $\frac{1}{6}m$ forbidden words of the form: $w_p^k v_{i,j} w_s^k$ for $k = 1, \dots, m-1$.*

Proof. We argue in a similar way to the way we argued in the proof of lemma 2.7. Given w we look at the convex hull of the images of the base point $y_0 \in Y$ under all the prefixes w_p^k of w, where $k = 1, \dots, m-1$. We denote this convex hull, which is a finite subtree of Y, T_w.

By construction: $\max_{s_i \in S} d_Y(y_0, \eta(s_i)(y_0)) = 1$. w_p^{k+1} is obtained from w_p^k by multiplying w_p^k with one of the generators $\eta(s_i)$, $s_i \in S$. Hence, the segment: $[w_p^k(y_0), w_p^{k+1}(y_0)]$ is of length at most 1. Therefore, the total length of the edges in the finite tree T_w is bounded the length of the word w, i.e., bounded by m.

Now, let $w_p^k v_{i,j} w_s^k$ be a forbidden element. By definition, there exists an element w_p^f for some f, $1 \leq f \leq m$, such that:

$$d_Y(w_p^k v_{i,j}(y_0), w_p^f(y_0)) \leq \frac{1}{5} d_Y(y_0, v_{i,j}(y_0)).$$

Hence, the interval: $[w_p^k(y_0), w_p^k v_{i,j}(y_0)]$ covers at least $\frac{4}{5} d_Y(y_0, v_{i,j}(y_0))$ from the total length of the edges in T_w.

The elements $\{v_{i,j}\}$ were constructed to satisfy a small cancellation property (part (3) in lemma 2.4). Hence, for two distinct forbidden prefixes, $w_p^{k_1} v_{i,j}^1 w_s^{k_1}, w_p^{k_2} v_{i,j}^2 w_s^{k_2}$, $1 \leq k_1 < k_2 \leq m-1$, the overlap between the intervals: $[w_p^{k_1}(y_0), w_p^{k_1} v_{i,j}^1(y_0)]$ and $[w_p^{k_2}(y_0), w_p^{k_2} v_{i,j}^2(y_0)]$, is bounded by $\frac{1}{10}$ of the minimum of the lengths of these two intervals. Therefore, with each forbidden element: $w_p^k v_{i,j} w_s^k$, $1 \leq k \leq m-1$, it is possible to associate a subinterval I_k of length $\frac{6}{10} d_Y(y_0, v_{i,j}(y_0)]$ of the interval $[w_p^k(y_0), w_p^k v_{i,j}(y_0)]$ for which:

(i) the subinterval I_k starts after the first $\frac{1}{10}$ of the interval: $[w_p^k(y_0), w_p^k v_{i,j}(y_0)]$, and ends at $\frac{7}{10}$ of that interval.
(ii) $I_k \subset T_m$.
(iii for distinct forbidden elements: $w_p^{k_1} v_{i,j}^1 w_s^{k_1}, w_p^{k_2} v_{i,j}^2 w_s^{k_2}$, $1 \leq k_1 < k_2 \leq m-1$, the intersection: $I_{k_1} \cap I_{k_2}$ is empty or degenerate.

Since in part (2) of lemma 2.4 we assumed that the length of an interval $[y_0, v_{i,j}(y_0)]$ is at least 10, it follows that the length of a subinterval I_k of a forbidden element $w_p^k v_{i,j} w_s^k$ is at least 6. Hence, the collection of subintervals $\{I_k\}$, for all the forbidden elements: $w_p^k v_{i,j} w_s^k$, $1 \leq k \leq m-1$, cover a total length of 6 times the number of forbidden elements in the tree T_w. Since the total length of the edges in T_w is bounded by m, the number of forbidden elements that are associated with the word of length m, w, is bounded by $\frac{1}{6}m$.

□

We exclude forbidden words to guarantee that all the words that are constructed from a given element of length m, $w \in L$, are distinct in L.

Lemma 3.6. *Let $w \in L$ be an element that is associated with a subword of length m of a word in the regular language that is associated with (Γ, S_{n_0}).*

Then the non-forbidden words: $w_p^k v_{i,j} w_s^k$, for all k, $1 \leq k \leq m-1$, are distinct elements in L.

Proof. Suppose that for such element w of length m, and a pair: $1 \leq k_1 < k_2 \leq m-1$, two non-forbidden elements satisfy: $w_p^{k_1} v_{i,j}^1 w_s^{k_1} = w_p^{k_2} v_{i,j}^2 w_s^{k_2}$ in L. Since by part 3 of lemma 2.4, the elements $\{v_{i,j}\}$ satisfy a small cancellation property, it follows that there exists f for which either:

(i) $1 \leq f \leq k_1$ and:

$$d_Y(w_p^{k_2} v_{i,j}^2(y_0), w_p^f(y_0)) \leq \frac{1}{5} d_Y(y_0, v_{i,j}^2(y_0))$$

(ii) $k_1 < f \leq k_2$ and:

$$d_Y(w_p^{k_1} v_{i,j}^1(y_0), w_p^f(y_0)) \leq \frac{1}{5} d_Y(y_0, v_{i,j}^2(y_0))$$

In both cases one of the two elements that are assumed to represent the same element in L is forbidden, which contradicts the assumption of the lemma.

□

The non-forbidden words enable us to construct a collection of *feasible* words in L, that demonstrate that the growth of L with respect to the generating set $\eta(S)$ is strictly bigger than the growth of Γ with respect to S_{n_0}.

Definition 3.7 (Feasible words in L). Let q be a positive integer, and let $w \in L$ be an element that is associated with a word of length mq in the regular language that is associated with the automata that was constructed for (Γ, S_{n_0}). We present w as a concatenation of q subwords of length m: $w = w(1) \dots w(q)$.

With w, and any choice of integers: $k_1, \dots, k_q$, $1 \leq k_t \leq m-1$, $t = 1, \dots, q$, for which all the elements, $w(t)_p^{k_t} v_{i,j}^t w(t)_s^{k_t}$, are non-forbidden, we associate a *feasible* word (of type q) in L:

$$w(1)_p^{k_1} v_{i,j}^1 w(1)_s^{k_1} \, \hat{v}_{i,j}^1 \, w(2)_p^{k_2} v_{i,j}^2 w(2)_s^{k_2} \, \hat{v}_{i,j}^2 \, \dots w(q)_p^{k_q} v_{i,j}^q w(q)_s^{k_q}$$

where for each t, $1 \leq t \leq q-1$, $\hat{v}_{i,j}^t$ is one of the elements that were constructed in lemma 3.3, that starts with a germ that is different than the germ that $w(t)_s^{k_t}$ ends with, and ends with a germ that is different than the germ that $w(t+1)_p^{k_{t+1}}$ starts with.

Feasible words are all distinct:

Lemma 3.8. *Given a positive integer q, the feasible words that are associated with all the elements w that are associated with words of length mq in the regular language that is the output of the automata that is constructed for (Γ, S_{n_0}), are all distinct in L.*

Proof. Let $w_1, w_2 \in L$ be two distinct elements that are associated with (distinct) words of length mq in the regular language that is associated with the finite automata of (Γ, S_{n_0}). Since the words in the regular language are distinct, $h_{n_0}(w_1) \neq h_{n_0}(w_2)$, i.e., w_1 and w_2 are associated with distinct elements in Γ.

Let $\hat{w}_e \in L$, $e = 1, 2$, be feasible elements that are constructed from w_e, $e = 1, 2$, in correspondence. Since $h_{n_0}(v_{i,j}) = 1$ according to lemma 3.3, it follows that: $h_{n_0}(\hat{w}_e) = h_{n_0}(w_e)$. Since $h_{n_0}(w_1) \neq h_{n_0}(w_2)$, feasible elements that are constructed from distinct words in the regular language are distinct in L.

Let $w \in L$ be an element that is associated with a word of length mq in the regular language that is associated with (Γ, S_{n_0}). Let $k_1, \dots, k_q$, $k'_1, \dots, k'_q$, $1 \leq k_t, k'_t \leq m - 1$, be two distinct q-tuples of integers. Suppose that the corresponding elements that are constructed from the word w and each of the two tuples are feasible, and the two feasible elements represent the same element in L:

$$w(1)_p^{k_1} v_{i,j}^1 w(1)_s^{k_1} \hat{v}_{i,j}^1 \dots w(q)_p^{k_q} v_{i,j}^q w(q)_s^{k_q} = w(1)_p^{k'_1} {v'}_{i,j}^1 w(1)_s^{k'_1} \hat{v'}_{i,j}^1 \dots w(q)_p^{k'_q} {v'}_{i,j}^q w(q)_s^{k'_q}.$$

We argue with a similar argument to the one that was used in proving lemma 2.6. First, consider the case in which for every t, $1 \leq t \leq q$, $w(t)_p^{k_t} v_{i,j}^t w(t)_s^{k_t} = w(t)_p^{k'_t} {v'}_{i,j}^t w(t)_s^{k'_t}$ in L. Since the two elements that are associated with w and the two t-tuples are assumed to be feasible, the elements: $w(t)_p^{k_t} v_{i,j}^t w(t)_s^{k_t}$ and $w(t)_p^{k'_t} {v'}_{i,j}^t w(t)_s^{k'_t}$, are non-forbidden for every t, $1 \leq t \leq q$. By lemma 3.6 all the non-forbidden elements that are associated with the same word $w(t)$ represent distinct elements in L. Hence, for all t, $1 \leq t \leq q$, $k_t = k'_t$, and the two t-tuples that are associated with the two feasible elements are identical.

Furthermore, by the small cancellation properties of the elements $v_{i,j}$ (part (3) of lemma 2.4), it follows that: $\hat{v}_{i,j}^t = \hat{v'}_{i,j}^t$, for every $1 \leq t \leq q - 1$. Since the two t-tuples are identical and so are the separators, the two feasible elements are identical.

Next, assume that there exists an index t, $1 \leq t \leq q$, for which: $w(t)_p^{k_t} v_{i,j}^t w(t)_s^{k_t} \neq w(t)_p^{k'_t} {v'}_{i,j}^t w(t)_s^{k'_t}$ in L. Let t_0 be the first such index t. Note that $1 \leq t_0 \leq q - 1$, since if $t_0 = q$, the two feasible elements represent distinct elements in L.

Furthermore, for all $t < t_0$, $w(t)_p^{k_t} v_{i,j}^t w(t)_s^{k_t} = w(t)_p^{k'_t} {v'}_{i,j}^t w(t)_s^{k'_t}$, and the two feasible elements represent the same element in L. Hence, the small cancellation properties of the elements, $v_{i,j}$, imply that for every $t < t_0$, $\hat{v}^t_{i,j} = \hat{v'}^t_{i,j}$. Lemma 3.6 implies that in addition, for every $t < t_0$, $k_t = k'_t$.

since the two feasible elements represent the same element in L, and: $w(t_0)_p^{k_{t_0}} v_{i,j}^{t_0} w(t_0)_s^{k_{t_0}} \neq w(t_0)_p^{k'_{t_0}} v'^{t_0}_{i,j} w(t_0)_s^{k'_{t_0}}$, the small cancellation properties of the elements, $v_{i,j}$, imply that the segment: $[y_0, w(t_0)_p^{k_{t_0}} v_{i,j}^{t_0}(y_0)]$ is almost contained in the segment: $[y_0, w(t_0)_p^{k'_{t_0}} v'^{t_0}_{i,j}(y_0)]$, or vice versa. i.e., one of the two segments is contained in the second one possibly except for the last $\frac{1}{10}$ of its suffix: $[w(t_0)_p^{k_{t_0}}(y_0), w(t_0)_p^{k_{t_0}} v_{i,j}^{t_0}(y_0)]$ or $[w(t_0)_p^{k'_{t_0}}(y_0), w(t_0)_p^{k'_{t_0}} v'^{t_0}_{i,j}(y_0)]$.

By the argument that was used to prove lemma 3.6, this implies that at least one of the elements, $w(t_0)_p^{k_{t_0}} v_{i,j}^{t_0} w(t_0)_s^{k_{t_0}}$ or $w(t_0)_p^{k'_{t_0}} v'^{t_0}_{i,j} w(t_0)_s^{k'_{t_0}}$, is forbidden. A contradiction to our assumption that the two elements that we started with are feasible, and lemma 3.8 follows.

□

Let b be the maximal length of an element $v_{i,j}$ with respect to the generating set $\eta(S)$ of L. Given a positive integer q, the length of a feasible word (of type q) with respect to $\eta(S)$ is bounded by $q(m+2b)$.

Let $r = e(\Gamma, S_{n_0})$. Let $Sph_R(\Gamma, S_{n_0})$ be the collection of elements of distance R from the identity in the Cayley graph of Γ with respect to the generating set S_{n_0}. By [4], the number of elements in $Sph_R(\Gamma, S_{n_0})$ is bounded below by $c_1 r^R$ and bounded above by $c_2 r^R$, where $c_1 \leq c_2$ are two positive constants. In particular, $|Sph_{mq}(\Gamma, S_{n_0})| \geq c_1 r^{mq}$.

By lemma 3.8, the number of elements in a ball of radius $q(m+2b)$ in L, with respect to the generating set $\eta(S)$, is bounded below by the number of feasible elements in that ball. Notice that the number of feasible elements in that ball is at least:

$$|Sph_{mq}(\Gamma, S_{n_0})|(\frac{5}{6}m)^q.$$

Hence:

$$\log e(L, \eta(S)) \geq \lim_{q \to \infty} \frac{\log(|Sph_{mq}(\Gamma, S_{n_0})|(\frac{5}{6}m)^q)}{q(m+2b)}$$

$$= \lim_{q \to \infty} \frac{\log(c_1) + qm\log(r) + q(\log(m) + \log\frac{5}{6})}{q(m+2b)}$$

Therefore, if we choose m to satisfy:

$$\log m > 2b\log(r) - \log\frac{5}{6}$$

Then: $\log(e(L,\eta(S)) > \log(r)$, hence, $e(L,\eta(S)) > e(\Gamma, S_{n_0})$, and we get the conclusion of proposition 3.2 in case the action of L on the real tree Y is free.

The general case of possibly non-free actions

Suppose that the action of L on Y is faithful, but possibly with point stabilizers. In this general case, we modify the argument that was used in the free case, using similar modifications that was used in the generalization to general faithful actions in the proof of proposition 2.3.

Lemma 3.9. *There exist elements $v_{i,j} \in L$, $i,j = 1,2$ that satisfy the conditions that are listed in lemma 3.3, even if the action of L on the tree Y is not free.*

Proof. The argument is similar to the one that was used in lemma 3.3, with a modification to the non-free action case, that is similar to the modification that we used in lemma 2.8.

$y_0 \in Y$ is, by construction, a point that moves minimally by the set of generators, $\eta(S)$. Hence, there are at least two distinct germs at y_0 in T_1. Let $germ_1, germ_2$ be two distinct germs at Y_0 in T_1.

The limit tree Y has infinitely many ends, and the action of L on Y is minimal, hence, there exist elements $e_1, e_2 \in L$, that act hyperbolically on Y, and for which the interval: $[y_0, e_i(y_0)]$ starts with the germ $germ_i$ for $i = 1,2$.

Since $h_{n_0} : L \to \Gamma$ is a proper epimorphism, there exists a non-trivial element $\hat{r} \in L$, for which: $h_{n_0}(\hat{r}) = 1$. In selecting a subsequence of the epimorphisms, $\{h_n\}$, we used some basic properties of limit groups over hyperbolic groups, and passed to a subsequence of the proper epimorphisms $\{h_n\}$, so that the kernels of the epimorphisms $\{h_n\}$ contain no torsion elements. Hence, the element $\hat{r}$ must be of infinite order in L.

The limit tree Y has infinitely many ends, and L acts cocompactly and minimally on Y. Hence, there exist elements x_1, x_2 such that:

(i) $< x_1, x_2, r >=< x_1 > * < x_2 > * < r >$.
(ii) x_1 and x_2 act hyperbolically on Y.

Recall that $\mu(x) = d_Y(y_0, x(y_0))$, and $tr(x)$ is the displacement of x along its axis in Y. If we choose f to satisfy:

$$f \min(tr(x_1), tr(x_2)) \geq 10(\mu(x_1) + \mu(x_2) + \mu(\hat{r}))$$

Then the element $r = x_1^f \hat{r} x_1^{-f} x_2^f \hat{r} x_2^{-f}$ acts hyperbolically on Y, and $h_{n_0}(r) = 1$.

Having constructed elements e_1, e_2 that act hyperbolically on Y, for which $[y_0, e_i(y_0)]$ start with the germ $germ_i$, $i = 1, 2$, and an element r that acts hyperbolically on Y, and satisfies $h_{n_0}(r) = 1$, and observing that the limit tree Y has infinitely many ends, and the action of L on Y is minimal and cocompact, the proof of lemma 3.9 proceeds precisely as the rest of the proof of lemma 3.3.

□

Given lemma 3.9 we use the modification that we applied in the proof of proposition 2.3, and follow the argument that was used to prove proposition 3.2 in the free case.

With a pair, $w_1, w_2 \in L$, we associate an element $v_{i,j}$ from the ones that were constructed in lemma 3.9. We choose that element $v_{i,j}$, to satisfy:

(i) $v_{i,j}$ starts with a germ in T_1, that is different than the germ that $[y_0, w_1(y_0)]$ terminates in the tree $w_1(T_1)$ (in case w_1 fixes y_0 we can choose $v_{i,j}$ to start with any germ).

(ii) $v_{i,j}^{-1}$ starts with a germ in T_1, that is different than the germ that $[y_0, w_2^{-1}(y_0)]$ terminates in the tree $w_2^{-1}(T_1)$ (in case w_2 fixes y_0 we can choose $v_{i,j}^{-1}$ to start with any germ).

We define forbidden and feasible elements precisely as we did in the free action case (definitions 3.4 and 3.7). The lower bounds on the number of forbidden and feasible elements (lemma 3.5), and the fact they represent distinct elements in L (lemmas 3.6 and 3.8), remain valid in the general case by the same arguments that were used in the free case. Finally, the lower bound on the number of feasible elements implies the conclusion of proposition 3.2, precisely by the same argument that was used in the free case.

□

Proposition 3.2 contradicts our assumption that for all n, $r = e(L, \eta(S)) = e(\Gamma, g_n(S))$. Hence, proves that there can not be an infinite sequence of inequivalent generating sets of Γ with the same rate of growth, that finally proves theorem 3.1.

□

4. The growth ordinal

Theorem 2.2 proves that the set of growth rates of a non-elementary hyperbolic group is well-ordered. Hence, we can associate with this set an ordinal, that depends only on the group Γ, that we denote $\zeta_{GR}(\Gamma)$.

Jorgensen and Thurston proved that the ordinal that is associated with the well-ordered set of volumes of hyperbolic 3-manifolds is ω^ω. Although we conjecture that: $\zeta_{GR}(\Gamma) = \omega^\omega$ for all non-elementary hyperbolic groups, we were able to prove that only in the case of limit groups.

Theorem 4.1. *Let L be a non-abelian limit group (over a free group). Then the rates of growth of L, with respect to all its finite generating sets, is well ordered.*

Proof. We argue by contradiction. Let $\{S_n\}$ be a sequence of finite generating sets of the limit group L, such that the sequence of rates of growth, $\{e(L,S_n)\}$, is strictly decreasing. L is a limit group, hence, there exists a sequence of epimorphisms, $\{u_m : L \to F_2\}$, that converges into the limit group L.

Since F_2 is a hyperbolic group, by proposition 2.3, for each index n:

$$\lim_{m\to\infty} e(F_2, u_m(S_n)) = e(L, S_n)$$

Since the sequence $\{e(L,S_n)\}$ is strictly decreasing, for each index n, we can find an index $m(n)$, such that: $e(L,S_{n+1}) < e(F_2, u_{m(n)}(S_n)) \leq e(L,S_n)$. Therefore, the sequence $\{e(F_2, u_{m(n)}(S_n))\}$ is strictly decreasing, a contradiction to the well ordering of the rates of growth of F_2 (Theorem 2.2).

□

By theorem 4.1 the set of rates of growth of a non-abelian limit group L is well-ordered, hence, we can associate with it an ordinal, $\zeta_{GR}(L)$.

Also, given $r > 1$ we look at the set of rates of growth of a limit group L, that are bounded by r. This set is well ordered, hence, we can associate with it an ordinal that we denote, $\zeta^r_{GR}(L)$.

Theorem 4.2. *For every non-abelian limit group L, $\zeta_{GR}(L) = \omega^\omega$.*

Proof. Let $\{S_n\}$ be a sequence of generating sets of L, such that the sequence of rates of growth: $\{e(L,S_n)\}$ is strictly increasing and bounded. A non-abelian limit group L can be approximated by a sequence of epimorphisms: $\{u_m : L \to F_2\}$. Hence, the bound on the rates of growth of the sequence: $\{(L,S_n)\}$, bounds the rates of growth of the pairs: $(F_2, u_m(S_n))$, for all positive integers: m,n. Therefore, [2] implies that the cardinality of the generating sets $\{S_n\}$ is bounded. Hence, by passing to a subsequence, we may assume that it is fixed. As we did in proving theorem 2.2, with each generating set S_n we can associate an epimorphism: $g_n : F_\ell \to L$.

From the sequence of epimorphisms $\{g_n\}$ we can pass to a subsequence that converges into a limit group L_1 with a generating set U_1. Since a limit group is finitely presented [16], for large n, we get an epimorphism: $h_n : (L_1, U_1) \to (L, S_n)$. By proposition 2.3,

$\lim_{n\to\infty} e(L,S_n) = e(L_1,U_1)$. Note that for a large n, h_n is a proper epimorphism, since for every n, $e(L,S_n) < e(L_1,U_1)$.

$$\begin{array}{ccc} (F_\ell, S) & & \\ \eta \downarrow & \searrow^{\{g_n\}} & \\ (L_1,U_1) & \xrightarrow[\{h_n\}]{} & (L,S_n) \end{array}$$

So far we proved that with every convergent increasing sequence of rates of growth of L, we can associate (not uniquely) a pair, (L_1,U_1), where L is a proper quotient of L_1, and U_1 is a generating set of L_1:

$$F_\ell \to L_1 \to L.$$

If we repeat this construction, starting with a bounded increasing sequence of increasing sequences of rates of growth of L, we get a two step sequence: $L_2 \to L_1 \to L$ of proper epimorphisms of limit groups. Repeating the construction iteratively, for bounded iterated sequences of strictly increasing sequences of rates of growth, we get a sequence of proper epimorphisms:

$$F_\ell \to L_s \to L_{s-1} \to \ldots \to L_1 \to L.$$

By [2], given $r > 1$, there is a bound, denoted d_r, on the cardinality of a generating set S of L, for which: $e(L,S) \leq r$. By a celebrated theorem of L. Louder [13] limit groups have a *Krull dimension*. This means that given a limit group L, there is a uniform bound on the lengths of sequences of proper epimorphisms:

$$L = L_1 \to L_2 \to \ldots \to L_s$$

where all the L_i's are limit groups, and the bound depends only on the minimal number of generators of L.

We constructed sequences of proper epimorphisms of limit groups from bounded (iterations of) convergent sequences of convergent sequences of rates of growth of L. Given $r > 1$, a bound on the collection of rates of growth, the Krull dimension for limit groups implies that there is a uniform bound, depending only on d_r (that depends only on r), on the lengths of sequences of proper epimorphisms of limit groups of degree d_r. Hence, there is a bound, depending only on r, on the number of iterations of convergent sequences of convergent sequences of rates of growth of the limit group L, where all these rates are bounded by the given real number r. A bound on the number of iterations of increasing sequences, is identical to a bound on the degree of ω in the ordinal $\zeta^r_{GR}(L)$. Since $\zeta^r_{GR}(L)$ is a polynomial in ω for every $r > 1$, it follows that $\zeta_{GR}(L) \leq \omega^\omega$.

It remains to prove that: $\zeta_{GR}(L) \geq \omega^\omega$. Note that for every positive integer t, there exists a sequence of proper epimorphisms:

$$L_1 = L*F_t \to L_2 = L*F_{t-1} \to \ldots \to L_{t-1} = L*F_2 \to L_t = L*Z \to L_{t+1} = L.$$

For every index i, $1 \leq i \leq t$, let $v_n^i : L_i \to L_{i+1}$, be an approximating sequence of epimorphisms that converges into the limit group L_i.

We start with a finite generating set S of L, and extend it to a generating set S_1 of L_1, by adding to S a free basis of F_t. Suppose that $e(L_1, S_1) \leq r_1$. We continue with the approximating sequence of epimorphisms: $\{v_n^j\}$, $1 \leq j \leq t$. For each i, $1 \leq i \leq t$, we construct a (multi-index) sequence of generating sets of L_{i+1}:

$$\{(L_{i+1}, v_{n_i}^i \circ \ldots \circ v_{n_1}^1(S_1)\}$$

where $n_i^i, \ldots, n_1^1$ runs over all the possibilities for an i-tuple of positive integers.

The sequence of epimorphisms: $\{v_n^1 : L_1 \to L_2\}$ converges into L_1. Hence, by proposition 2.3, $\lim_{n\to\infty} e(L_2, v_n^1(S_1)) = e(L_1, S_1)$. The maps v_n^1 are proper epimorphisms, and they converge into L_1. Hence, the pairs $\{(L_2, v_n^1(S_1))\}$ belong to infinitely many distinct isomorphism classes of pairs (of a limit group and its finite set of generators). By passing to a subsequence we may assume that they all belong to distinct isomorphism classes of pairs.

By theorem 5.8 in the sequel, only finitely many isomorphism classes of pairs, $(L_2, v_n^1(S_1))$, can have the same growth rate. Hence, we can pass to a subsequence of the homomorphisms, $\{v_n^1\}$, such that the pairs, $(L_2, v_n^1(S_1))$, do all have different growth rates. Since: $\lim_{n\to\infty} e(L_2, v_n^1(S_1)) = e(L_1, S_1)$, and for each n: $e(L_2, v_n^1(S_1)) \leq e(L_1, S_1)$, we may pass to a further subsequence such that $\{e(L_2, v_n^1(S_1))\}$ is a strictly increasing sequence that converges to $e(L_1, S_1)$.

Now we fix an index n_1, and look at the sequence of pairs: $(L_3, v_n^2 \circ v_{n_1}^1(S_1))$. v_n^2 are proper epimorphisms that converge into L_2. Hence, by proposition 2.3: $lim_{n\to\infty} e(L_3, v_n^2 \circ v_{n_1}^1(S_1)) = e(L_2, v_{n_1}^1(S_1))$. Applying again the finiteness of non-isomorphic generating sets of a limit group with the same growth rate (theorem 5.8), there exists a subsequence of the homomorphisms $\{v_n^2\}$ such that the sequence of rates of growth, $\{e(L_3, v_n^2 \circ v_{n_1}^1(S_1))\}$ is a strictly increasing sequence that converges to $e(L_2, v_{n_1}^1(S_1))$.

So far from the two steps sequence of proper epimorphisms: $L_1 \to L_2 \to L_3$, we managed to construct a strictly increasing sequence of strictly increasing convergent sequences of rates of growth, $\{e(L_3, v_{n_2}^2 \circ v_{n_1}^1(S_1))\}$, that eventually converges to $e(L_1, S_1)$. Continuing iteratively with the same constructions for the t-steps sequence of proper epimorphisms: $L_1 \to L_2 \to \ldots \to L_{t+1} = L$, we can construct t-iterates of increasing sequence of increasing sequences of rates of growth: $\{(e(L = L_{t+1}, v^t_{n_t} \circ \ldots \circ v^1_{n_1}(S_1))\}$.

Therefore, $\zeta^{r_1}_{GR}(L) \geq \omega^t$. Since t was arbitrary, $\zeta_{GR}(L) \geq \omega^\omega$, so: $\zeta_{GR}(L) = \omega^\omega$.

□

Theorems 4.1 and 4.2 imply that in particular the sets of growth rates of all hyperbolic non-cyclic limit groups are well ordered, and their growth ordinals are ω^ω. We conjecture that the conclusion of Theorem 4.2 hold for all non-elementary hyperbolic groups and all non-virtually abelian limit groups over hyperbolic groups. This conjecture is related to the existence of a Krull dimension for limit groups over hyperbolic groups.

Proposition 4.3. *Let Γ be a non-elementary hyperbolic group. Then:*

$$\zeta_{GR}(\Gamma) \geq \omega^\omega.$$

Moreover, if limit groups over Γ have a Krull dimension, then:

$$\zeta_{GR}(\Gamma) = \omega^\omega$$

Proof. The argument in the second part of the proof of theorem 4.2, that proves a lower bound on the growth ordinal of a non-abelian limit group, generalizes to every non-elementary hyperbolic group, and implies that for every non-elementary hyperbolic group Γ, $\zeta_{GR}(\Gamma) \geq \omega^\omega$.

If limit groups over Γ have a Krull dimension, i.e., if for every limit group over Γ, L, there is a bound (depending only on L) on the length of a sequence of proper epimorphisms: $L = L_1 \to L_2 \to \ldots \to L_t$, then the first part of the proof of theorem 4.2, that proves an upper bound on the growth ordinal of a non-abelian limit group, generalizes to every non-elementary hyperbolic group, and implies that $\zeta_{GR}(\Gamma) \leq \omega^\omega$, so: $\zeta_{GR}(\Gamma) = \omega^\omega$.

□

Recall that a *resolution* over a non-elementary hyperbolic group Γ, is a sequence of proper quotients of limit groups over Γ: $L_1 \to L_2 \to \ldots \to L_s$, where to each of the limit groups over Γ, L_i, $1 \leq i \leq s-1$, one adds its virtually abelian JSJ decomposition and its associated modular group (see [15]). A resolution over Γ is called *strict*, if each epimorphism: $L_i \to L_{i+1}$ restricts to injective maps of the rigid vertex groups and the edge groups in the virtually abelian JSJ decomposition of L_i, and the image of every QH vertex group in the JSJ decomposition of L_i in L_{i+1} is non-elementary (for the definition and basic properties of strict resolutions see section 5 in [16]).

Proposition 4.4. *Let Γ be a non-elementary hyperbolic group. If limit groups over Γ do not have a Krull dimension for strict resolutions that encode epimorphisms onto Γ, i.e., if there exists a limit group L over Γ, with no bound on the lengths of strict resolutions that terminate in Γ: $L = L_0 \to L_1 \to \ldots \to L_s \to \Gamma$, where each of the L_i's is a limit group over Γ, then:*

$$\zeta_{GR}(\Gamma) > \omega^{\omega}$$

Proof. Let $\eta : L \to \hat{L}$ be a strict proper epimorphism, i.e., η maps each of the rigid vertex groups and each edge group in the virtually abelian JSJ decomposition of L monomorphically into $\hat{L}$, and maps every QH vertex group in this JSJ decomposition into a non-elementary subgroup in $\hat{L}$. Then it is possible to find a sequence of modular automorphisms of L, $\{\varphi_n\} \in \text{Mod}(L)$, such that the sequence $\{\eta \circ \varphi_n : L \to \hat{L}\}$ converges into L (see [15]).

If limit groups over Γ do not have a Krull dimension for strict resolutions that encode epimorphisms onto Γ, there exists a limit group L over Γ, with longer and longer strict resolutions: $L = L_1 \to L_2 \to \ldots \to L_t \to \Gamma$. Since the proper epimorphisms along the strict resolution are strict, there exist approximating sequences of proper epimorphisms: $\{v_n^i : L_i \to L_{i+1}\}$, $i = 1, \ldots, t$, i.e., the sequences $\{v_n^i\}$ converge into the limit groups L_i, for $i = 1, \ldots, t$.

Now we can fix a generating set S for L. Let $r = e(L, S)$. With each strict resolution $L = L_1 \to \ldots \to L_t$, there are associated sequences of approximate homomorphisms $\{v_n^i\}$, $i = 1, \ldots, t$. By the argument that was used to prove the lower bound on the growth ordinal of a limit group in the proof of theorem 4.2, the strict resolutions and the sequences of approximate homomorphisms imply that $\zeta_{GR}^r(\Gamma) \geq \omega^t$. Since we assumed that there is no bound on the length of a strict resolution of L, this implies that $\zeta_{GR}^r(\Gamma) \geq \omega^t$ for every positive integer t, so: $\zeta_{GR}^r(\Gamma) \geq \omega^{\omega}$.

By increasing the number of generators, there is a generating set $\hat{S}$ with $e(\Gamma, \hat{S}) > r$. It follows that: $\zeta_{GR}(\Gamma) \geq \omega^{\omega} + 1$, so: $\zeta_{GR}(\Gamma) > \omega^{\omega}$. □

5. Growth rates of subgroups of hyperbolic groups

In the previous sections we studied the rates of growth of hyperbolic groups with respect to all their generating sets. In this section we strengthen the results to include the rates of growth of all the f.g. subgroups of a given hyperbolic group. We prove that if Γ is a non-elementary hyperbolic group, then the set of growth rates of all the f.g. non-elementary subgroups of Γ with respect to all their finite generating sets is well ordered, strengthening theorem 2.2. Then we prove that every given real number can be obtained only finitely many times (up to a natural isomorphism) as the growth rate of a finite generating set of a non-elementary subgroup of Γ, strengthening theorem 3.1.

Let Γ be a hyperbolic group. Let $H < \Gamma$, be a non-elementary f.g. subgroup. Since H is a non-elementary subgroup in Γ it contains a free subgroup, so H has exponential growth. We set $e(H,S)$ to be the rate of the (exponential) growth of H with respect to the generating set S. We look at the following set in $\mathbb{R}$:

$$\Theta(\Gamma) = \{e(H,S) | H < \Gamma \, , \, |S| < \infty\}$$

where H runs over all the f.g. non-elementary subgroups in Γ, and S runs over all the finite generating sets of all such possible subgroups H. The set $\Theta(\Gamma)$ is a countable subset of $\mathbb{R}$, that contains the subset $\xi(\Gamma)$ that was studied in the second section, where $\xi(\Gamma)$ contains only growth rates of the ambient group Γ itself (and not of its non-elementary subgroups).

Theorem 5.1. *Let Γ be a non-elementary hyperbolic group. Then $\Theta(\Gamma)$ is a well-ordered set.*

Proof. The proof is essentially identical to the proof of theorem 2.2. Let $\{S_n\}$ be a sequence of finite generating sets of non-elementary subgroups, $\{H_n\}$, such that $\{e(H_n,S_n)\}$ is a strictly decreasing sequence and $\lim_{n\to\infty} e(H_n,S_n) = d$, for some $d > 1$ ($d > 1$ by a result of Koubi [12]).

By [7], there exists a lower bound on $e(H_n,S_n)$, that depends only on $|S_n|$ and the hyperbolicity constant δ of Γ, and this lower bound grows to infinity with $|S_n|$. Hence, $|S_n|$ is bounded for the entire strictly decreasing sequence. By passing to a subsequence we may assume that $|S_n|$ is fixed, $|S_n| = \ell$, for the entire sequence.

Let $S_n = \{x_1^n, \cdots, x_\ell^n\}$. Let F_ℓ be the free group of rank ℓ with a free generating set: $S = \{s_1, \dots, s_\ell\}$. For each index n, we define a map: $g_n : F_\ell \to \Gamma$, by setting: $g_n(s_i) = x_i^n$. By construction: $e(H_n,S_n) = e(H_n, g_n(S))$.

We fix a Cayley graph X of Γ with respect to some finite generating set. X is a δ-hyperbolic graph endowed with a Γ-action. Hence, for each n, F_ℓ acts on X via the homomorphism: $g_n : F_\ell \to \Gamma$.

Since the sequence $\{e(H_n,S_n)\}$ is strictly decreasing, the sequence:

$$\{\min_{\gamma\in\Gamma} \max_i |\gamma g_n(s_i)\gamma^{-1}|\}$$

is not bounded. Hence, we may pass to a subsequence for which the sequence converges to ∞. For each index n, we further replace the epimorphism g_n, by the homomorphism: $\gamma_n g_n \gamma_n^{-1}$, where:

$$\max_i |\gamma_n g_n(s_i)\gamma_n^{-1}| = \min_{\gamma \in \Gamma} \max_i |\gamma g_n(s_i)\gamma^{-1}| = \rho_n.$$

We denote by (X, d_n) the Cayley graph X with the metric obtained from the metric on X after multiplying it by $\frac{1}{\rho_n}$. From the sequence of actions of F_ℓ on the metric spaces (X, d_n) we extract a subsequence that converges into a non-trivial action of F_ℓ on a real tree Y. The action of F_ℓ is not faithful, so we divide F_ℓ by the kernel of the action, and get a faithful action of a limit group L on the real tree Y, where the limit group L is a limit group *over* the hyperbolic group Γ.

Let: $\eta : F_\ell \to L$ be the associated quotient map. By Theorem 6.5 in [15] there exists some index n_0, such that for every $n > n_0$ there exists an epimorphism $h_n : L \to H_n$ that satisfies: $g_n = h_n \circ \eta$. By passing to a subsequence we may assume that all the homomorphisms $\{g_n\}$ factor through the epimorphism: $\eta : F_\ell \to L$. Generalizing proposition 2.3 we have:

Proposition 5.2. $\lim_{n\to\infty} e(H_n, g_n(S)) = e(L, \eta(S))$.

Proof. The subgroups H_n are non-elementary subgroups of the hyperbolic group Γ. Hence, all of these subgroups, and the limit group L, contain non-cyclic free subgroups, and the limit tree Y has infinitely many ends. This is sufficient for constructing the elements $u_{i,j} \in L$, $i, j = 1, 2$, with the properties that are listed in lemma 2.4, for the action of L on the limit tree Y, using the construction that appears in the proof of lemma 2.8.

Once there exist elements $\{u_{i,j}\}$ with the properties that are listed in lemma 2.4, the definitions of forbidden and feasible elements and their properties in the current context, as well as the rest of the argument, are identical to the way they appear in the proof of proposition 2.3.

□

As in the proof of theorem 2.2, Proposition 5.2 proves that there is no strictly decreasing sequence of rates of growth, $\{e(H_n, S_n)\}$, since a strictly decreasing sequence can not approach its upper bound. Hence, it concludes the proof that the set of growth rates of all the f.g. non-elementary subgroups of a hyperbolic group with respect to all their finite generating sets, $\Theta(\Gamma)$, is well-ordered.

□

In the third section we have generalized another theorem of Jorgensen and Thurston, and proved that given a real number r, there are at most finitely many finite generating sets of a hyperbolic group Γ with growth rate r up to the action of $Aut(\Gamma)$ (theorem 3.1). In fact, it is possible to strengthen this finiteness theorem further, to include all the isomorphism classes of pairs of a non-elementary f.g. subgroup of Γ and a finite generating set of the f.g. subgroup.

Theorem 5.3. *Let Γ be a non-elementary hyperbolic group, and let $r > 1$. Then up to an isomorphism, there are at most finitely many non-elementary subgroups $\{H_n\}$ of Γ, each with a finite generating set S_n, with a growth rate r, i.e., finitely many isomorphism classes of pairs: (H_n, S_n), $H_n < \Gamma$, that satisfy: $e(H_n, S_n) = r$.*

Proof. The proof is a strengthening of the argument that was used to prove theorem 3.1. Let $r > 1$ and suppose that there are infinitely many isomorphism classes of pairs: (H_n, S_n), where $H_n < \Gamma$ is a non-elementary subgroup, and S_n is a finite generating set of H_n, for which: $e(H_n, S_n) = r$.

As in the proof of theorem 5.1, the cardinality of the generating sets $\{S_n\}$ is bounded, so we may pass to a subsequence that have a fixed cardinality ℓ. Hence, each generating set S_n corresponds to an epimorphism: $g_n : F_\ell \to H_n$, where S is a fixed free generating set of F_ℓ, and $g_n(S) = S_n$.

By passing to a further subsequence, we may assume that the sequence of homomorphisms $\{g_n\}$ converges into a faithful action of a limit group (over Γ) L on some real tree Y. Let $\eta : F_\ell \to L$ be the associated quotient map. Passing to a further subsequence, we may assume that for every n, $g_n = h_n \circ \eta$, where $h_n : L \to \Gamma$ is an epimorphism of L onto H_n.

If two of the homomorphisms h_{n_1}, h_{n_2} are isomorphisms of L onto H_{n_1} and H_{n_2} in correspondence, then the corresponding pairs of subgroups and generating sets: (H_{n_1}, S_{n_1}) and (H_{n_2}, S_{n_2}), are both isomorphic to the pair: $(L, \eta(S))$, so they are both in the same isomorphism class of pairs, a contradiction to our assumption. Hence, omitting at most one homomorphism h_n from the sequence, we may assume that for every n, the homomorphism h_n is not injective.

By proposition 5.2, $\lim_{n\to\infty} e(H_n, S_n) = e(L, \eta(S))$. By our assumption, for every index n, $e(H_n, S_n) = r$. Hence, $e(L, \eta(S)) = r$. As in the proof of theorem 3.1, to obtain a contradiction to the existence of an infinite sequence of non-isomorphic pairs of non-elementary subgroups of the hyperbolic group Γ and their finite generating sets with the same rate of growth, and conclude the proof of theorem 5.3, we prove the following:

Proposition 5.4. *For every index n, the pairs: $\{(H_n, S_n)\}$ from the convergence sequence, satisfy: $e(H_n, S_n) < e(L, \eta(S))$.*

Proof. To prove the proposition, we follow the proof of proposition 3.2, although unlike finite generating sets of the ambient hyperbolic group Γ, Cayley graphs of subgroups of Γ with respect to their finite generating sets are not guaranteed to have the Markov property. We fix an index n_0 and aim to prove that: $e(H_{n_0}, S_{n_0}) < e(L, \eta(S))$, using the limit action of L on the real tree Y, and the non-trivial kernel of the epimorphism: $h_{n_0} : L \to H_{n_0}$.

The subgroups $H_n < \Gamma$ are assumed to be non-elementary, hence, they contain a non-cyclic free subgroup, and so does the limit group (over Γ) L. As we noted in the proof of proposition 5.2, this suffices to construct elements $v_{i,j} \in L$, $i, j = 1, 2$, that satisfy the properties that are listed in lemma 3.3, using the construction that was used to prove lemma 3.9.

We fix n_0. With the generating set S_{n_0} and the subgroup H_{n_0} we can not associate a finite automata in general as we did in the proof of proposition 5.2. Let $X_{n_0} = X(H_{n_0}, S_{n_0})$ be the Cayley graph of H_{n_0} with respect to the generating set S_{n_0} (note that X_{n_0} is not a hyperbolic space in general). With each element in H_{n_0} we associate a geodesic from the identity to that element in the Cayley graph X_{n_0}. Note that there are several geodesics from the identity to each given vertex in X_{n_0}, and we choose a single one for every vertex arbitrarily (i.e., we choose an arbitrary combing by geodesics in the Cayley graph X_{n_0}).

We fix an integer m, and look at a geodesic of length m from the identity to an element of distance m from the identity in the Cayley graph X_{n_0}. Let w be a word that represents such a geodesic. w is a word in the generators S_{n_0}, so we can view it as a word of length m in the generators $\eta(S)$ in L. Given a positive integer k, $1 \leq k \leq m-1$, we separate the word into a prefix of length k and a suffix of length $m-k$. The prefix and suffix correspond to non-trivial elements in L (since they are mapped by h_{n_0} to nontrivial elements (of distances k and $m-k$ from the identity in X_{n_0}) in Γ). We denote the prefix w_p^k, and the suffix w_s^k (both are elements in L).

As we did in the proof of proposition 3.2, with the pair w_p^k, w_s^k we associate an element $v_{i,j}$ that was constructed in lemma 3.9, that does not start with the germ that $[y_0, w_p^k(y_0)]$ terminates with, and does not end with the germ that $[y_0, w_s^k(y_0)]$ starts with (in the tree Y).

Given the collection of elements in L that we constructed, we define *forbidden* elements precisely as we define them in definition 3.4. Lemma 3.5 that gives an upper bound on the number of forbidden elements, remains valid (by the same argument) for words w that represent geodesics in the Cayley graph X_{n_0}.

Let w_1, w_2 be words that represent geodesics between the identity and two distinct elements on the sphere of radius m in X_{n_0}. The words w_1 and w_2 (as words in $\eta(S)$) represent elements in L. Since h_{n_0} maps w_1 and w_2 to distinct elements in Γ, and $h_{n_0}(v_{i,j}) = 1$, $i,j = 1,2$, all the non-forbidden elements that are constructed from w_1, are distinct from the non-forbidden elements that are constructed from w_2. Furthermore, by the argument that proves lemma 3.6, all the non-forbidden elements that are constructed from a single word that represent a geodesic in X_{n_0}, w_1, are distinct.

As in the proof of theorem 3.2, the non-forbidden words enable us to construct a collection of *feasible* words in L.

(cf. definition 3.7) Let q be a positive integer, and let w be a word of length mq that represent a geodesic in the (fixed) geodesic combing of the Cayley graph X_{n_0} (i.e., a chosen geodesic from the identity to an element of distance mq from the identity in X_{n_0}). We present w as a concatenation of q subwords of length m: $w = w(1) \ldots w(q)$. Clearly, each subword $w(t)$, $1 \leq t \leq q$, represents a geodesic of length m in X_{n_0}.

We define the feasible elements that are constructed from w precisely as we defined them in definition 3.7. Given any choice of integers: $k_1, \ldots, k_q$, $1 \leq k_t \leq m-1$, $t = 1, \ldots, q$, for which all the elements: $w(t)_p^{k_t} v_{i,j}^t w(t)_s^{k_t}$, are non-forbidden, we associate a *feasible* word (of type q) in L:

$$w(1)_p^{k_1} v_{i,j}^1 w(1)_s^{k_1}\, \hat{v}_{i,j}^1\, w(2)_p^{k_2} v_{i,j}^2 w(2)_s^{k_2}\, \hat{v}_{i,j}^2\, \ldots w(q)_p^{k_q} v_{i,j}^q w(q)_s^{k_q}$$

where for each t, $1 \leq t \leq q-1$, $\hat{v}_{i,j}^t$ is one of the elements that were constructed in lemma 3.9, that starts with a germ that is different than the germ that:

$$[y_0, w(1)_p^{k_1} v_{i,j}^1 w(1)_s^{k_1}\, \hat{v}_{i,j}^1\, \ldots\, w(t)_p^{k_t} v_{i,j}^2 w(t)_s^{k_t}(y_0)]$$

ends with, and ends with a germ that is different than the germ that $[y_0, w(t+1)_p^{k_{t+1}} v_{i,j}^{t+1} w(t+1)_s^{k_{t+1}}(y_0)]$ starts with.

The argument that was used to prove lemma 3.8, proves that the constructed feasible elements from the geodesic combing of X_{n_0} are all distinct.

Lemma 5.5. *Given a positive integer q, the feasible words, that are constructed from all the geodesics of length mq in the geodesic combing of X_{n_0}, do all represent distinct elements in L.*

At this point, to complete the proof of proposition 5.4, we need to slightly modify the estimates that we used in section 3 because the geodesic combings of X_{n_0} that we use is not guaranteed to have the Markov property.

Let b be the maximal length of an element $v_{i,j}$ with respect to the generating set $\eta(S)$ of L. Given a positive integer q, the length of a feasible word (of type q) with respect to $\eta(S)$ is bounded by $q(m+2b)$. Let $Sph_{mq}(X_{n_0})$ be the sphere of radius mq in the Cayley graph X_{n_0} of H_{n_0}.

Let $r = e(H_{n_0}, S_{n_0})$. By lemma 3.8, the number of elements in a ball of radius $q(m+2b)$ in L, with respect to the generating set $\eta(S)$, is bounded below by the number of feasible elements in that ball.

Unlike the case of a Cayley graph of a hyperbolic group in section 3, for which we have a Markov type lower bound of $|Sph_{mq}(X_{n_0})|$, in the case of a general finite generating set of a subgroup of a hyperbolic group we only have an asymptotic estimate:

$$\lim_{q\to\infty} \frac{\log |Sph_{mq}(X_{n_0})|}{mq} = \log r,$$

Hence:

$$\log e(L, \eta(S)) \geq \lim_{q\to\infty} \frac{\log(|Sph_{mq}(X_{n_0})|(\frac{5}{6}m)^q)}{q(m+2b)} =$$

$$= \lim_{q\to\infty} \frac{\log(|Sph_{mq}(X_{n_0})|)}{q(m+2b)} + \frac{q(\log(m)+\log\frac{5}{6})}{q(m+2b)} = \log(r)\frac{m}{m+2b} + \frac{\log(m)+\log(\frac{5}{6})}{m+2b}$$

Therefore, if we choose m to satisfy:

$$\log m > 2b\log(r) - \log\frac{5}{6}$$

Then: $\log(e(L,\eta(S)) > \log(r)$, hence, $e(L,\eta(S)) > e(H_{n_0}, S_{n_0})$, and we get the conclusion of proposition 5.4.

□

Proposition 5.4 contradicts our assumption that for all n, $r = e(L,\eta(S)) = e(H_n, S_n)$. Hence, proves that there can not be an infinite sequence of non-isomorphic finite generating sets of non-elementary subgroups of Γ with the same rate of growth, that finally proves theorem 5.3.

□

Theorem 5.1 proves that the set of growth rates of all f.g. non-elementary subgroups of a given hyperbolic group Γ, with respect to all their finite generating sets, is well-ordered. This theorem has several immediate corollaries.

Corollary 5.6. *Let L be a non-elementary (i.e., non-virtually abelian) limit group over a hyperbolic group Γ. Then the rates of growth of all* *the non-elementary f.g. subgroups of L with respect to all their finite generating sets is well ordered.*

Proof. Every f.g. non-elementary subgroup of L is a limit group over Γ, and can be approximated by a sequence of epimorphisms of the subgroup of L onto non-elementary subgroups of Γ. Hence, by proposition 5.2, the rates of growth of all the non-elementary subgroups of L with respect to all their finite sets of generators, is a subset of the closure of the set of rates of growth of all the f.g. non-elementary subgroups of Γ with respect to all their finite sets of generators, i.e., a subset of the closure of $\Theta(\Gamma)$.

By theorem 5.1, $\Theta(\Gamma)$ is well-ordered. Hence, its closure is well-ordered. Every subset of a well-ordered set is well-ordered, and the corollary follows.

□

Corollary 5.7. *Let Γ be a hyperbolic group. The rates of growth of all the non-elementary limit groups over Γ, with respect to all their finite sets of generators is well ordered.*

Proof. Every non-elementary limit group over Γ can be approximated by a sequence of epimorphisms of the limit group onto non-elementary subgroups of Γ. By proposition 5.2 the rates of growth of an approximating sequence approaches the rate of growth of the limit group with its given set of generators.

Hence, the set of rates of growth of all the non-elementary Γ-limit groups with respect to all their finite generating sets, is the closure of $\Theta(\Gamma)$, i.e., the set of rates of growth of all the f.g. non-elementary subgroups of Γ with respect to all their finite generating sets. $\Theta(\Gamma)$ is well-ordered so its closure is well-ordered.

□

So far we have strengthened the well ordering that was proved in theorem 5.1 for the growth rates of all the finite generating sets of all the f.g. subgroups of a hyperbolic group Γ, to obtain well-ordering of the growth rates of all the finite generating sets of all the non-elementary limit groups over Γ (note that every f.g. subgroup of Γ is in particular a limit group over Γ).

The statement and the proof of theorem 5.3, about the finiteness of isomorphism classes of finite generating sets of subgroups of a hyperbolic group with the same growth rate, can be strengthened to include all the finite generating sets of all the limit groups over Γ. This strengthening plays an important role in studying the ordinal of growth rates in a hyperbolic group (see theorem 4.2).

Theorem 5.8. *Let Γ be a non-elementary hyperbolic group, and let $r > 1$. Then up to an isomorphism, there are at most finitely many non-elementary subgroups $\{H_n\}$, of all the limit groups over Γ, each with a finite generating set S_n, with a growth rate r, i.e., at most finitely many isomorphism classes of pairs: (H_n, S_n), H_n a non-elementary subgroup of a limit group over Γ, that satisfy: $e(H_n, S_n) = r$.*

Proof. The argument that was used to prove theorem 5.3, didn't really use the hyperbolicity of the ambient group Γ. Given a sequence of f.g. subgroups of a hyperbolic group Γ, and their finite generating sets, $\{(H_n, S_n\}$, that have all rate of growth r, we constructed a limit object, $(L, \eta(S))$, such that L acts minimally on a real tree Y, and a subsequence of the sequence of pairs, $\{H_n, S_n\}$, are proper quotients of the pair $(L, \eta(S))$.

Note that f.g. subgroups of limit groups over Γ are limit groups over Γ. Hence, we denote the subgroups H_n in the statement of the theorem, L_n. Given a sequence of non-isomorphic pairs, $\{(L_n, S_n)\}$, of limit groups over Γ, and their finite generating sets, all with rate of growth r, we can use the argument that was applied to study finite generating sets of subgroups of Γ with the same rate of growth in the proof of theorem 5.3, and extract a subsequence that converges into a pair, $(L, \eta(S))$, where L is a limit group over Γ, and $\eta(S)$ is its finite set of generators. Furthermore, by the same argument L is equipped with a minimal action on a real tree Y, and a subsequence of the non-isomorphic pairs, $\{(L_n, S_n)\}$, are proper quotients of the pair, $(L, \eta(S))$.

Having constructed the limit pair $(L, \eta(S))$ and its limit action on the limit tree Y, the rest of the proof follows precisely the proof of proposition 5.4 and theorem 5.3.

□

By theorem 5.1, the set of rates of growth of all f.g. non-elementary subgroups of a hyperbolic group Γ with respect to all their finite generating sets, $\Theta(\Gamma)$, is well-ordered. Hence, we can associate with this set an ordinal that we denote, $\theta_{GR}(\Gamma)$.

Furthermore, by corollary 5.7, the rates of growth of all the the non-elementary limit groups over a hyperbolic group Γ, with respect to all their finite sets of generators, is well ordered. Hence, we can associate with this set an ordinal, that depends only on the group Γ, that we denote $\lambda_{GR}(\Gamma)$.

We conjecture that for every non-elementary hyperbolic group Γ: $\theta_{GR}(\Gamma) = \lambda_{GR}(\Gamma) = \omega^\omega$, but as in section 4 and the ordinal $\zeta_{GR}(\Gamma)$, we are able to prove that only in the case of limit groups (over a free group).

Corollary 5.9. *For every non-abelian limit group (over a free group) L, $\theta_{GR}(L) = \lambda_{GR}(L) = \omega^{\omega}$.*

Proof. $\lambda_{GR}(L) \geq \theta_{GR}(L) \geq \zeta_{GR}(L) = \omega^{\omega}$. $\theta_{GR}(L) \leq \lambda_{GR}(L) \leq \omega^{\omega}$, by the same argument that was used to prove the upper bound: $\zeta_{GR}(L) \leq \omega^{\omega}$ in the proof of theorem 4.2.

□

We can't prove the generalization of the equality in corollary 5.9 to all hyperbolic groups, but as in proposition 4.3 we can prove a general inequality.

Corollary 5.10. *Let Γ be a non-elementary hyperbolic group. Then: $\lambda_{GR}(\Gamma) = \theta_{GR}(\Gamma) \geq \omega^{\omega}$.*

Moreover, if limit groups over Γ have a Krull dimension, then:

$$\theta_{GR}(\Gamma) = \lambda_{GR}(\Gamma) = \omega^{\omega}$$

Proof. $\omega^{\omega} \leq \zeta_{GR}(\Gamma)$ by proposition 4.3, and: $\zeta_{GR}(\Gamma) \leq \theta_{GR}(\Gamma) \leq \lambda_{GR}(\Gamma)$ since the associated well-ordered sets satisfies the corresponding inclusions. If limit groups over Γ have a Krull dimension, then all these ordinals are ω^{ω} by the proof of the upper bound: $\zeta_{GR} \leq omega^{\omega}$, in the proof of theorem 4.2.

The well-ordered set that is associated with $\lambda_{GR}(\Gamma)$ is the closure of the set that is associated with $\theta_{GR}(\Gamma)$. Hence, the difference between the two sets is a subset of the accumulation points of $\Theta_{GR}(\Gamma)$. Since the set of accumulation points of $\Theta_{GR}(\Gamma)$ is not bounded, it follows that: $\theta_{GR}(\Gamma) = \lambda_{GR}(\Gamma)$.

□

6. Growth rates of subsemigroups of a hyperbolic group

In the previous sections we studied the set of rates of growth of hyperbolic groups, their f.g. subgroups, and limit groups over hyperbolic groups, with respect to all their finite generating sets. The main tool that we used to study generating sets of subgroups was the structure of limit groups over hyperbolic groups and basic properties of the actions of these limit groups on real trees.

Limit groups were originally defined to study varieties and first order formulas over groups. In [17] the analysis of varieties over groups using limit groups is modified to analyze varieties over a free semigroup. In the case of semigroups, limit groups are replaced by limit pairs, (U, L), where L is a limit group (over a free group), and U is a f.g. subsemigroup that generates L as a group.

In this section we use these limit pairs (over a hyperbolic group). We modify the arguments that were used in previous sections to study rates of growth of subgroups, to study rates of growth of non-elementary f.g. subsemigroups of a given hyperbolic group, with respect to all their finite sets of generators. Such a modification demonstrates once again, that the use of limit objects (over groups, semigroups, algebras and so on), enables at time natural modifications of concepts, tools and objects that are used in studying questions in one algebraic category, to study analogous questions in other algebraic categories. This is a major principle in model theory, when the signature is changed, but basic properties of the corresponding theories do not.

Let U be a f.g. semigroup with a finite generating set S. The growth of the semigroup U with respect to the generating set S is defined precisely as in the group case. In case U has *exponential growth* with respect to S (hence, with respect to any other finite generating set), we define $e(U,S)$ to be the growth rate of U with respect to S, precisely as it is defined in the case of groups.

In this case of exponential growth, we further define the following set in $\mathbb{R}$:

$$\xi(U) = \{e(U,S) | |S| < \infty\}$$

where S runs over all the finite generating sets of U.

Now, let Γ be a hyperbolic group. We say that a subsemigroup U in Γ is non-elementary if the subgroup generated by U in Γ is non-elementary. Like subgroups, a f.g. subsemigroup of Γ has exponential growth if and only if it is non-elementary. In analogy with our study in section 4, we look at the following set in $\mathbb{R}$:

$$\Delta(\Gamma) = \{e(U,S) | U < \Gamma \, , \, |S| < \infty\}$$

where U runs over all the f.g. non-elementary subsemigroups in Γ, and S runs over all the finite generating sets of all such possible subsemigroups U. The set $\Delta(\Gamma)$ is a countable subset of $\mathbb{R}$, that contains the subset $\Theta(\Gamma)$ that was studied in section 4, and contains only growth rate of subgroups.

Theorem 6.1. *Let Γ be a non-elementary hyperbolic group. Then $\Delta(\Gamma)$ is a well-ordered set.*

Proof. The proof is a modification of the proofs of theorems 2.2 and 5.1, modified to the case of subsemigroups.

As in the proofs of theorems 2.2 and 5.1 we need to prove that $\Delta(\Gamma)$ does not contain a strictly decreasing convergent sequence. Suppose that there exists a sequence of non-elementary subsemigroups $\{U_n\}$, with finite generating sets $\{S_n\}$, such that $\{e(U_n, S_n)\}$ is a strictly decreasing sequence and $\lim_{n\to\infty} e(U_n, S_n) = d$, for some $d > 1$.

As in the case of groups, by [7] we may assume that the cardinality of the generating sets S_n from the decreasing sequence is bounded, and by possibly passing to a subsequence we may assume that the cardinality of the generating sets is fixed, $|S_n| = \ell$.

Let $S_n = \{x_1^n, \cdots, x_\ell^n\}$. Let FS_ℓ be the free semigroup of rank ℓ with a free generating set: $S = \{s_1, \ldots, s_\ell\}$. For each index n, we define a semigroup homomorphism: $g_n : FS_\ell \to \Gamma$, by setting: $g_n(s_i) = x_i^n$. By our assumptions, g_n is an epimorphism of FS_ℓ onto U_n.

Γ is a group, hence, every semigroup homomorphism: $g_n : FS_\ell \to \Gamma$ extends to a group homomorphism: $\hat{g}_n : F_\ell \to \Gamma$. Hence, following [17], we view every homomorphism g_n as a homomorphism of pairs (still denoted g_n): $g_n : (FS_\ell, F_\ell) \to (U_n, \Gamma)$. Note that unlike the convention in [17], in our current setting the free semigroup FS_ℓ generates F_ℓ as a group, but the subsemigroup U_n may not generate Γ (as it will be clear in the sequel, this change in the convention does not change the tools and the analysis).

We fix a Cayley graph X of Γ with respect to some finite generating set. Since Γ is a hyperbolic group, X is a δ-hyperbolic graph. For each index n, both the free semigroup FS_ℓ, and the free group F_ℓ that contains it, act isometrically on the Cayley graph X of Γ via the pair homomorphism g_n.

Since the sequence $e(U_n, S_n)$ is strictly decreasing, and in particular is not constant, the sequence:

$$\{\min_{\gamma \in \Gamma} \max_i |\gamma g_n(s_i^{\pm 1}) \gamma^{-1}|\}$$

is not bounded. Hence, we may pass to a subsequence for which the sequence converges to ∞. For each index n, we further replace the pair homomorphism g_n, by the pair homomorphism: $\gamma_n g_n \gamma_n^{-1}$, where:

$$\max_i |\gamma_n g_n(s_i^{\pm 1}) \gamma_n^{-1}| = \min_{\gamma \in \Gamma} \max_i |\gamma g_n(s_i^{\pm 1}) \gamma^{-1}|$$

We still denote the conjugated epimorphism $\{g_n\}$ (note that conjugating a pair epimorphism does not change the corresponding growth rate of the subsemigroup $U_n = g_n(FS_\ell)$).

For each n, we set:

$$\rho_n = \max_i |g_n(s_i^{\pm 1})|$$

and denote by (X, d_n) the Cayley graph X with the metric obtained from the metric on X after multiplying it by $\frac{1}{\rho_n}$. From the sequence of actions of the pair: (FS_ℓ, F_ℓ) on the metric spaces (X, d_n) we extract a subsequence that converges into a non-trivial action of the pair (FS_ℓ, F_ℓ) on a real tree Y. The action of F_ℓ is not faithful, so we divide the pair (FS_ℓ, F_ℓ) by the kernel of the action, i.e., by the normal subgroup of F_ℓ that acts trivially on Y. We get a faithful action of a *limit pair* (U, L) on the real tree Y, where the limit pair (U, L) is a limit pair *over* the hyperbolic group Γ. In particular, the limit group L is a limit group over Γ, and the subsemigroup U is the image of the free semigroup FS_ℓ in the limit group L.

Let: $\eta : (FS_\ell, F_\ell) \to (U, L)$ be the associated quotient map of pairs. Note that $U = \eta(FS_\ell)$. By Theorem 6.5 in [15] there exists some index n_0, such that for $n > n_0$, there exists an epimorphism (of pairs): $h_n : (U, L) \to (U_n, \Gamma)$ that satisfies: $g_n = h_n \circ \eta$. By passing to a subsequence, we may assume that all the homomorphisms of pairs $\{g_n\}$ factor through the epimorphism of pairs: $\eta : (FS_\ell, F_\ell) \to (U, L)$.

Since $g_n = h_n \circ \eta$, for every index n, $e(U_n, g_n(S)) \leq e(U, \eta(S))$. Following our strategy in the group case we prove:

Proposition 6.2. $\lim_{n\to\infty} e(U_n, g_n(S)) = e(U, \eta(S))$.

Proof. As we did in the group case, we start by constructing elements in the limit subsemigroup U with some small cancellation properties.

Lemma 6.3. *There exist non-trivial elements $z_1, z_2, z_3 \in U$, with the following properties:*

(1) *for every two elements $w_1, w_2 \in L$, there exists an index i, $i = 1, 2, 3$, such that:*
$$d_Y(y_0, w_1 z_i(y_0)) \geq d_Y(y_0, w_1(y_0)) + \frac{19}{20} d_Y(y_0, z_i(y_0))$$
and:
$$d_Y(y_0, z_i w_2(y_0)) \geq d_Y(y_0, w_2(y_0)) + \frac{19}{20} d_Y(y_0, z_i(y_0))$$
that implies:
$$d_Y(y_0, w_1 z_i w_2(y_0)) \geq d_Y(y_0, w_1(y_0)) + \frac{9}{10} d_Y(y_0, z_i(y_0)) + d_Y(y_0, w_2(y_0)).$$

(2) $d_Y(y_0, z_i(y_0)) > 20$, *for* $i = 1, 2, 3$.

(3) *for every $w \in L$, and every two indices: i, j, $1 \leq i, j \leq 3$, if the segment $[y_0, z_i(y_0)]$ intersects the segment $[w(y_0), wz_j(y_0)]$ non-trivially, then the length of the intersection is bounded by: $\frac{1}{20} d_Y(y_0, z_i(y_0))$ (if $i = j$ we assume in addition that $w \neq 1$).*

Proof. The semigroup U, is a subsemigroup of a limit group L over a hyperbolic group Γ, and L is not elementary. Furthermore, by construction the limit group L acts minimally and cocompactly on the limit tree Y, Y has infinitely many ends, and U generates L as a group.

For any $u \in U$, let $Fix(u) \subset Y$ be the fixed subset of u in Y. If for two generators of U, $\eta(s_1), \eta(s_2) \in \eta(S)$, the intersection $Fix(\eta(s_1)) \cap$

$Fix(\eta(s_2))$ is trivial, and both $\eta(s_1)$ and $\eta(s_2)$ act elliptically on Y, then $\eta(s_1)\eta(s_2)$ acts hyperbolically on Y.

If $\cap_{s \in S} Fix(\eta(s))$ is non-trivial, the semigroup U has a fixed point in Y, and since U generates L, L has a fixed point in Y, a contradiction to the minimality of the action of L on Y.

The fixed set of an element in U is a convex subset of Y, so by Helly's type theorem for convex subset of trees, if $\cap_{s \in S} Fix(\eta(s))$ is empty, there must exist $s_1, s_2 \in S$ for which $Fix(\eta(s_1)) \cap Fix(\eta(s_2))$ is empty. Hence, U contains an element that acts hyperbolically on Y.

Let $Y_U \subset Y$ be the convex hull of the images of the base point $y_0 \in Y$ under the action of the semigroup U. Since U contains a hyperbolic element, Y_U contains an infinite ray that starts at y_0. U generates L and L is not elementary, hence, Y_U must have infinitely many ends. This implies that there exist elements $u_1, u_2 \in U$ that act hyperbolically on Y, and $< u_1, u_2 >$ is a free subgroup in L.

Recall that $\mu(u) = d_Y(y_0, u(y_0))$ and $tr(u)$ is the displacement of u along its axis in Y, At this point we can construct the elements z_1, z_2, z_3:

$$z_i = u_1 u_2^{\alpha_1 + i} u_1 u_2^{\alpha_2 + i} \ldots u_1 u_2^{\alpha_{50} + i}$$

where $i = 1, 2, 3$, and the positive integers α_k, $k = 1, \ldots, 50$ satisfy:

(i) $\alpha_1 tr(u_2) \geq \max(20(\mu(u_1) + \mu(u_2)), 1)$
(ii) $\alpha_1 \geq 200$, $\alpha_k = \alpha_1 + 4(k-1)$, $k = 2, \ldots, 50$.

The elements z_i, $i = 1, 2, 3$, satisfy part (2) of the lemma by construction. The small cancellation requirement in part (3) of the lemma follows from the structure of the elements z_i. Given $w_1, w_2 \in L$, note that if:

$$d_Y(y_0, w_1 z_1(y_0)) < d_Y(y_0, w_1(y_0)) + \frac{19}{20} d_Y(y_0, z_1(y_0))$$

then for $i = 2, 3$:

$$d_Y(y_0, w_1 z_i(y_0)) \geq d_Y(y_0, w_1(y_0)) + \frac{19}{20} d_Y(y_0, z_i(y_0))$$

and if:

$$d_Y(y_0, z_1 w_2(y_0)) < d_Y(y_0, w_2(y_0)) + \frac{19}{20} d_Y(y_0, z_1(y_0))$$

then for $i = 2, 3$:

$$d_Y(y_0, z_i w_2(y_0)) \geq d_Y(y_0, w_2(y_0)) + \frac{19}{20} d_Y(y_0, z_i(y_0)).$$

Hence, a simple pigeonhole argument implies that at least for one of the elements z_i, $i = 1, 2, 3$, the inequalities in part (1) of the lemma must hold.

□

Let b be the maximal length of the words z_i, $i = 1, 2, 3$, that were constructed in lemma 6.3, as elements in the limit subsemigroup U with respect to the generating set $\eta(S)$. Let $B_m(U, \eta(S))$ be the ball of radius m in the Cayley graph of the semigroup U with respect to the generating set $\eta(S)$.

Let w_1, w_2 be two non-trivial elements in $B_m(U, \eta(S))$. Because of property (1) in lemma 6.3, there exists i, $1 \leq i \leq 3$, for which:

$$d_Y(y_0, w_1 z_i w_2(y_0)) \geq d_Y(y_0, w_1(y_0)) + \frac{9}{10} d_Y(y_0, z_i(y_0)) + d_Y(y_0, w_2(y_0))$$

We continue iteratively. Let q be an arbitrary positive integer, and let $w_1, \dots, w_q$ be a collection of non-trivial elements from $B_m(U, \eta(S))$. For each t, $1 \leq t \leq q-1$, we choose an element $z(t)$ from the collection z_1, z_2, z_3, that was constructed in lemma 6.3, such that $z(t)$ satisfies:

$$d_Y(y_0, w_1 z(1) w_2 z(2) \dots w_t z(t) w_{t+1}(y_0)) \geq$$

$$\geq d_Y(y_0, w_1 z(1) w_2 z(2) \dots w_t(y_0)) + \frac{9}{10} d_Y(y_0, z(t)(y_0)) + d_Y(y_0, w_{t+1}(y_0))$$

By the Gromov-Hausdorff convergence, for large enough n, h_n maps in a bi-Lipschitz way all the elements of the form:

$$w_1 z(1) w_2 z(2) \dots w_{q-1} z(q-1) w_q$$

into the fixed Cayley graph X of Γ.

As in the proofs of theorems 2.2 and 5.1, we know that all the elements that we constructed are mapped to non-trivial elements by the epimorphisms $\{h_n\}$, but the maps h_n may be not injective on these collections of elements. Hence, we need to exclude *forbidden* elements.

Definition 6.4. We say that a non-trivial element $w_1 \in B_m(U, \eta(S))$ is *forbidden* if there exists an element z_i ($i = 1, 2, 3$) that was constructed in lemma 6.3, and an element $w_2 \in B_m(U, \eta(S))$, such that:

(i)

$$d_Y(y_0, w_1 z_i(y_0)) \geq d_Y(y_0, w_1(y_0)) + \frac{19}{20} d_Y(y_0, z_i(y_0))$$

(ii) $d_Y(w_2(y_0), w_1 z_i(y_0)] \leq \frac{3}{20} d_Y(y_0, z_i(y_0))$.

As in definition 2.5, An element $w_1 z(1) \dots w_{q-1} z(q-1) w_q$ from the set that we constructed is called *feasible* of type q, if all the elements w_t, $1 \leq t \leq q$, are not forbidden.

Lemma 6.5. *Given m, for all large enough n and every fixed q, the semigroup homomorphisms h_n map the collections of feasible elements of type q to distinct elements in U_n.*

Proof. Identical to the proof of lemma 2.6.

□

As in the group case, the injectivity of h_n on the set of feasible elements of type q, enables us to estimate from below the number of elements in balls in the Cayley graph of the semigroups U_n, for large n.

Lemma 6.6. *The following are lower bounds on the numbers of non-forbidden and feasible elements:*

(1) *Given m, the number of non-forbidden elements (in the ball of radius m in U, $B_m(U,\eta(S))$) is at least $\frac{13}{14}|B_m(U,\eta(S))|$.*
(2) *Given m, we set $\beta_m = |B_m(U,\eta(S)|$. For every fixed m, and every positive q, the number of feasible elements of type q is at least: $(\frac{13}{14}\beta_m)^q$.*

Proof. As in the proof of lemma 2.7, part (2) follows from part (1) since given m and q, feasible elements are built from all the possible q concatenations of non-forbidden elements in a ball of radius m in U.

To prove (1) let $w \in B_m(U,\eta(S))$ be a forbidden element. By definition 6.4 this means that there exists an element z_i, $i = 1,2,3$, with the following properties:

(1) there exists a subinterval $J_w \subset Y$, such that: $[y_0, w_1 z_i(y_0)] = [y_0, w_1(y_0)] \cup J_w$ and this union is a disjoint union.
(2) $J_w \subset [w_1(y_0), w_1 z_i(y_0)]$. Hence, by part (3) of lemma 6.3, for distinct forbidden elements $w_1, w_2 \in B_m(U,\eta(S))$:
$$\text{length}(J_{w_1} \cap J_{w_2}) \leq \frac{1}{19}\min(\text{length}(J_{w_1}), \text{length}(J_{w_2})).$$
(3) let T_m be the convex hull of all the points: $\{u(y_0) \mid u \in B_m(U,\eta(S))\}$ in the limit tree Y. By part (ii) of definition 6.4:
$$\text{length}(J_w \cap T_m) \geq \frac{16}{19}\text{length}(J_w).$$

Therefore, with each forbidden $w \in B_m(U,\eta(S))$, it is possible to associate a subinterval $I_w \subset J_w$ of length: $\frac{15}{19} length(J_w)$, that satisfies similar properties to the ones that are listed in the proof of lemma 2.7:

(i) the subinterval I_w starts after the first $\frac{1}{19}$ of the length of the interval J_w, and ends at $\frac{16}{19}$ of the length of that interval.
(ii) $I_w \subset T_m$.
(iii) for distinct forbidden elements w_1, w_2, the intersection: $I_{w_1} \cap I_{w_2}$ is empty or degenerate.

In part (2) of lemma 6.3 we assumed that the length of an intervals $[y_0, z_i(y_0)]$, $i = 1, 2, 3$, is at least 20. Hence, the length of a subinterval I_w that is associated with a forbidden element w is at least 14. Since the interiors of the intervals I_w for different forbidden elements w are disjoint, the total length that the collection of subintervals, I_w, for all the forbidden elements $w \in B_m(U, \eta(S))$, cover in the tree $T_m \subset Y$, is at least 14 times the number of forbidden elements in $B_m(U, \eta(S))$. Since the total length of the edges in T_m is bounded by $|B_m(U, \eta(S))|$, the number of forbidden elements in $B_m(U, \eta(S))$ is bounded by: $\frac{1}{14}|B_m(U, \eta(S)|$, which gives the lower bound on the number of non-forbidden elements in part (1) of the lemma.

□

Given lemma 6.6, the proof of proposition 6.2 continues exactly as the proofs of propositions 2.3 and 5.2.

□

Proposition 6.2 proves that there is no strictly decreasing sequence of rates of growth, $\{e(U_n, S_n)\}$, hence, concludes the proof that the set of growth rates of all the f.g. subsemigroups of a hyperbolic group Γ, with respect to all their finite set of generators, is well-ordered (theorem 6.1).

□

As in the case of subgroups of a given hyperbolic group (theorem 5.1), theorem 6.1 has several immediate corollaries.

Corollary 6.7. *Let Γ be a hyperbolic group. The rates of growth of all the the non-elementary f.g. subsemigroups of limit groups over Γ, with respect to all their finite sets of generators is well ordered.*

Proof. Identical to the proof of corollary 5.7.

□

By theorem 6.1, the set of rates of growth of all the f.g. non-elementary subsemigroups of a hyperbolic group Γ with respect to all their finite generating sets, $\Delta(\Gamma)$, is well-ordered. Hence, we can associate with this set an ordinal that we denote, $\delta_{GR}(\Gamma)$. Furthermore, by corollary 6.7, the rates of growth of all the non-elementary f.g. subsemigroups of all the limit groups over a hyperbolic group Γ, with respect to all their finite sets of generators, is well ordered. Hence, we can associate with this set an ordinal, that depends only on the group Γ, that we denote $\tau_{GR}(\Gamma)$.

We conjecture that for every non-elementary hyperbolic group Γ: $\theta_{GR}(\Gamma) = \tau_{GR}(\Gamma) = \omega^\omega$, but as in sections 4 and 5, we are able to prove that only in the case of subsemigroups of limit groups (over a free group).

Corollary 6.8. *For every non-abelian limit group L, $\delta_{GR}(L) = \tau_{GR}(L) = \omega^\omega$.*

Proof. $\delta_{GR}(L) \leq \tau_{GR}(L) \leq \omega^\omega$ by the same argument that was used to prove the upper bound on the growth ordinal (of subgroups) of a limit group in the proof of theorem 4.2, i.e., by the existence of a Krull dimension for limit groups.

By theorem 4.2 $\omega^\omega = \zeta_{GR}(L)$. Hence: $\omega^\omega = \zeta_{GR}(L) \leq \delta_{GR}(L) \leq \tau_{GR}(L)$, since the set of rates of growth of the limit group L with respect to all its finite generating sets (as a group) is contained in the set of rates of growth of all the subsemigroups of L with respect to all their finite generating sets. Therefore: $\delta_{GR}(L) = \tau_{GR}(L) = \omega^\omega$.

□

As in corollary 5.10, for all hyperbolic groups, we can prove a general inequality.

Corollary 6.9. *Let Γ be a non-elementary hyperbolic group. Then: $\delta_{GR}(L) = \tau_{GR}(L) \geq \omega^\omega$.*

Moreover, if limit groups over Γ have a Krull dimension, then:

$$\delta_{GR}(\Gamma) = \tau_{GR}(\Gamma) = \omega^\omega.$$

Proof. $\omega^\omega \leq \theta_{GR}(\Gamma) \leq \lambda_{GR}(\Gamma)$ by corollary 5.10, and: $\theta_{GR}(\Gamma) \leq \delta_{GR}(\Gamma) \leq \tau_{GR}(\Gamma)$ by the inclusion of the corresponding sets. This proves the lower bound.

The set of rates of growth of all the non-elementary subsemigroups of limit groups over Γ, with respect to all their finite generating sets, is the closure of the set of rates of growth of all the non-elementary subsemigroups of Γ with respect to all their finite generating sets. Since both sets are well-ordered, and the first has an unbounded set of accumulation points, $\delta_{GR}(\Gamma) = \tau_{GR}(\Gamma)$.

If limit groups over hyperbolic groups have a Krull dimension then the argument that proves the upper bound on the growth ordinal of limit groups, $\zeta_{GR}(L) \leq \omega^\omega$, that was used in the proof of theorem 4.2, implies an upper bound: $\delta_{GR}(\Gamma) = \tau_{GR}(\Gamma) \leq \omega^\omega$, and the equality between the ordinals follows.

□

7. Some open problems

The growth ordinals of a hyperbolic group raise quite a few problems on rates of growth of particular hyperbolic groups, and on the set of rates of growth of other classes of groups.

Problem 7.1. Are the set of rates of growth well-ordered, or at least is there a minimum possible growth rate, for the following classes of groups:

- exponentially growing linear groups.
- lattices in (real and complex) Lie groups.
- acylindrically hyperbolic groups
- the mapping class groups $MCG(\Sigma)$.

In his Bourbaki seminar on the work of Jorgensen and Thurston [10], Gromov observed that covers of the same degree of a fixed hyperbolic manifold have the same volume, hence, there can not be a uniform bound on the number of hyperbolic manifolds with the same volume. He further asked if there is such a uniform bound if in addition we bound the volumes of the hyperbolic manifolds.

Problem 7.2. Let Γ be a non-elementary hyperbolic group. In theorem 3.1 we proved that only finitely many equivalence classes of generating sets (under the action of the automorphism group), can give the same rates of growth of Γ. Given $r_0 > 1$, is there a uniform bound b_0, such that for every $r < r_0$ there are at most b_0 equivalence classes of generating sets with growth rate r? is there such a uniform bound on the number of isomorphism classes of generating sets of subgroups of the hyperbolic group Γ that have the same rate of growth r, $r < r_0$?

Problem 7.3. We proved the finiteness of the number of isomorphism classes of finite generating sets of subgroups of a hyperbolic group with the same rate of growth only for subgroups (Theorem 5.3). We believe that the same finiteness should hold for subsemigroup generators of all the quasi-convex subgroups of a given hyperbolic group (since these have the Markov property). Does finiteness hold in the class of finite sets of generators of general subsemigroups of a hyperbolic group? for subsemigroups of the free semigroup?

Problem 7.4. Theorem 4.2 proves that the growth ordinal of a limit group (over a free group), and in particular, of a free or a surface group, is ω^ω. By theorem 4.2, given $r > 1$, $\zeta^r_{GR}(L)$ is a polynomial in ω. What can be said on the degree of these polynomials (as a function of r) for a free group or a surface group?

Problem 7.5. By theorem 3.1 the set of growth rates of a free group is well ordered. By theorem 3.1 there are only finitely many generating sets (up to the action of the automorphism group) with the same rate. The minimal growth rate of F_2 is 3. Denote by d_n the minimal growth of F_2 with a generating set of cardinality n. By [2], $\lim_n d_n = \infty$.

What can be said about d_n? About the generating sets that achieve the minimum for each n?

The following was already indicated in sections 4-6:

Problem 7.6. Is it true that: $\zeta_{GR}(\Gamma) = \theta_{GR}(\Gamma) = \delta_{GR}(\Gamma) = \omega^\omega$ for every non-elementary hyperbolic group Γ?

We conjecture that the answer to question 6 is positive. Hence, $\zeta_{GR}(\Gamma)$ does not carry any information about Γ, but the set of ordinals, $\zeta^r_{GR}(\Gamma)$, for every $r > 1$, does.

Problem 7.7. Given the set of ordinals, $\zeta^r_{GR}(\Gamma)$, for all reals $r > 1$, what can be said about the structure of Γ? suppose that two hyperbolic groups, Γ_1, Γ_2, satisfy: $\zeta^r_{GR}(\Gamma_1) \geq \zeta^r_{GR}(\Gamma_2)$. What can be said about the pair: Γ_1, Γ_2?

Hyperbolic 3-manifolds with small volumes have been studied extensively. One can ask similar questions regarding the rates of growth of their fundamental groups.

Problem 7.8. Is there a hyperbolic 3-manifold M, with a generating set S_M for its fundamental group $\pi_1(M)$, such that $e(\pi_1(M), S_M)$ is minimal among all rates of growth of fundamental groups of (closed) hyperbolic 3-manifolds? What can be said about this manifold and the minimizing generating set of its fundamental group?

Is the set of rates of growth of all the fundamental groups of (closed) hyperbolic 3-manifolds, with respect to all their finite generating sets, well ordered? If it is well ordered, is its ordinal ω^ω?

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