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DIOPHANTINE GEOMETRY OVER GROUPS VI: THE ELEMENTARY THEORY OF A FREE GROUP

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Abstract. This paper is the sixth in a sequence on the structure of sets of solutions to systems of equations in a free group, projections of such sets, and the structure of elementary sets defined over a free group. In the sixth paper we use the quantifier elimination procedure presented in the two parts of the fifth paper in the sequence, to answer some of A. Tarski's problems on the elementary theory of a free group, and to classify finitely generated (f.g.) groups that are elementarily equivalent to a non-abelian f.g. free group.

In the first 5 papers in the sequence on Diophantine geometry over groups we studied sets of solutions to systems of equations in a free group, and developed basic techniques and objects required for the analysis of sentences and elementary sets defined over a free group. The techniques we developed enabled us to present an iterative procedure that analyzes EAE sets defined over a free group (i.e. sets defined using 3 quantifiers), and shows that every such set is in the Boolean algebra of AE sets defined over a free group [Se6, 1.41].

In this paper we apply the tools and techniques presented in the previous 5 papers in the sequence to answer some of A. Tarski's problems on the elementary theory of a free group, and generalizations of these problems. We start by showing that every elementary set defined over a f.g. free group is in the Boolean algebra of AE sets, which is a direct corollary of our analysis of EAE sets presented in [Se5] and [Se6]. We continue by showing that for coefficient-free elementary sets, the quantifier elimination procedure can be done uniformly for all f.g. non-abelian free groups (Theorem 2). We then use this "uniform" quantifier elimination procedure, to prove the equivalence of the elementary theories of all non-abelian f.g. free groups (Theorem 3). A similar uniform quantifier elimination procedure

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can be used to show that if $2 \leq k \leq \ell$, the standard embedding of the free group F_k into the free group F_ℓ is an elementary embedding (Theorem 4).

We further analyze the entire collection of f.g. groups that are elementarily equivalent to a f.g. non-abelian free group. We prove that a f.g. group is elementarily equivalent to a non-abelian f.g. free group if and only if it is a non-elementary hyperbolic ω-residually free tower (such towers were presented in section 6 of [Se1]). At the end of the paper, we add an appendix that briefly summarizes the main constructions and results that are obtained in the previous papers in this sequence, constructions and results that are the key for proving some of the main results of this paper.

Theorems 3 and 4 of this paper, on the elementary equivalence of non-abelian free groups, and the elementary properties of the standard embeddings of f.g. free groups, answer questions of Tarski. In the next papers in this sequence, we generalize our results on the elementary theory of a free group to the elementary theory of an arbitrary (torsion-free) hyperbolic group, and then prove that the elementary theory of a (torsion-free) hyperbolic group, in particular a free group, is stable. Finally, we would like to thank M. Bestvina, F. Paulin, E. Rips, and the referees of this paper, whose comments helped us in improving its presentation.

In the two parts of the fifth paper of this sequence ([Se5] and [Se6]), we proved that the Boolean algebra of AE sets defined over a free group is invariant under projections, or equivalently, we proved that an EAE set is in the Boolean algebra of AE sets [Se6, 1.41]. This clearly implies that every elementary set defined over a free group is in the Boolean algebra of AE sets.

Theorem 1. *Let $F_k = \langle a_1, \ldots, a_k \rangle$ be a (non-abelian) free group, and let $Q(p)$ be a definable set over F_k. Then $Q(p)$ is in the Boolean algebra of AE sets over F_k.*

Proof. If $Q(p)$ is defined using not more than 3 quantifiers, then the theorem follows from Theorem 1.41 of [Se6]. Otherwise, the theorem follows by a standard induction on the number of quantifiers that defines the elementary set $Q(p)$. □

Theorem 1 proves that every definable set over a free group is in the Boolean algebra of AE sets. To show the equivalence of the elementary theories of free groups of various ranks, we need to show that for coefficient free predicates, our quantifier elimination procedure does not depend on the rank of the coefficient group.

Theorem 2. *Let $Q(p)$ be a set defined by a coefficient-free predicate over a group. Then there exists a set $L(p)$ defined by a coefficient-free predicate which is in the Boolean algebra of AE predicates, so that for every free group F_k, $k \geq 2$, the sets $Q(p)$ and $L(p)$ are equivalent.*

Proof. If a system of equations over a non-abelian free group is coefficient free, then by construction, its associated (canonical) Makanin–Razborov diagram is coefficient-free (see section 5 in [Se1]), and the diagram does not depend on the rank of the non-abelian free group. Similarly, the (multi-) graded Makanin–Razborov diagram associated with a coefficient-free parametric system of equations defined over a non-abelian free group, is coefficient free and it does not depend on the rank of the non-abelian free group (see section 10 in [Se1]). Furthermore, the bounds on the numbers of rigid solutions of the rigid limit groups in that diagram, and the bounds on the numbers of strictly solid families of solutions of the solid limit groups in the diagram, do not depend on the rank of the free group as well (see section 2 in [Se3]).

Let V be the variety associated with some coefficient-free system of equations defined over a non-abelian free group, and let

$$\forall y \in V \; \exists x \; \big(\Sigma_1(x,y) = 1 \wedge \Psi_1(x,y) \neq 1\big) \vee \cdots \vee \big(\Sigma_r(x,y) = 1 \wedge \Psi_r(x,y) \neq 1\big)$$

be a coefficient-free sentence. Since the Makanin–Razborov diagram associated with a coefficient-free system of equations defined over a non-abelian free group, does not depend on the rank of the free group, the formal Makanin–Razborov diagram associated with a coefficient-free sentence as above, is coefficient free and it does not depend on the rank of the free group (section 2 in [Se2]). Similarly, the graded formal Makanin–Razborov diagram associated with a coefficient-free parametric sentence defined over a non-abelian free group, is coefficient free and does not depend on the rank of the free group (section 3 in [Se2]).

Let $Res(t,v)$ be a (graded, multi-graded) coefficient-free well-separated resolution defined over some non-abelian free group. Then, by construction, the induced resolution, $IndRes(\langle v \rangle, Res(t,v))$, is coefficient free and it does not depend on the rank of the free group (section 3 in [Se4]). If $Res(t,v)$ is a (graded, multi-graded) coefficient-free well-separated resolution defined over some non-abelian free group, then the core resolution constructed by the procedure presented in section 4 of [Se5], is coefficient free and does not depend on the rank of the free group.

Suppose that $EAE(p)$ is an EAE set that is defined by a coefficient-free predicate over a non-abelian free group. The two iterative procedures used for the analysis of the set $EAE(p)$ in [Se5] and [Se6], the procedure for the construction of the tree of stratified sets, and the sieve procedure, are built from graded and multi-graded Makanin–Razborov resolutions, core and induced resolutions, and graded formal Makanin–Razborov diagrams. Hence, when these two iterative procedures are applied to the coefficient-free set $EAE(p)$, all the constructions made along these two iterative procedures are coefficient free, and they do not depend on the rank of the free group over which the set $EAE(p)$ is defined. Therefore, there exists a set $L(p)$ that is defined over a non-abelian free group by a coefficient-free predicate which is in the Boolean algebra of AE sets, and for every non-abelian free group $Q(p) = L(p)$. The standard finite induction used in the proof of Theorem 1, finally implies the claim of the theorem for an arbitrary coefficient-free set defined over a non-abelian free group. □

Theorem 2 proves that in handling coefficient-free predicates, our quantifier elimination procedure does not depend on the rank of the coefficient (free) group. This clearly implies an affirmative answer to Tarski's problem on the equivalence of the elementary theories of (f.g.) free groups.

Theorem 3. *The elementary theories of non-abelian f.g. free groups are equivalent.*

Proof. By Theorem 2, a coefficient-free elementary set defined over a non-abelian free group, is equivalent to a coefficient-free set that is in the Boolean algebra of AE sets, and the equivalence does not depend on the rank of the free group. Hence, every coefficient-free sentence defined over a non-abelian free group can be reduced, in a way that is independent of the rank of the free group, to a coefficient-free sentence that is in the Boolean algebra of AE sentences. Therefore, to prove the equivalence of the elementary theories of non-abelian f.g. free groups, it is enough to prove that a coefficient-free AE sentence defined over a non-abelian free group is true if and only if it is a true sentence over any other non-abelian f.g. free group.

The equivalence of the AE theories of non-abelian free groups was previously known [S], [H], but we chose to give an alternative argument using our machinery. Given a coefficient-free AE sentence defined over a non-abelian free group, we apply the procedure used for the construction of the tree of stratified sets, presented in section 2 of [Se5], to validate it. Note that unlike the general procedure for the construction of the tree of stratified sets, when this procedure is applied to validate an AE sentence, it is applied without using parameters. Since the sentence is coefficient free, all the constructions made along the procedure are coefficient free and independent of the rank of the free group. Hence, if a coefficient-free sentence is a true sentence over a given f.g. non-abelian free group it is a true sentence over every non-abelian f.g. free group. □

Arguments similar to the ones used to prove Theorems 2 and 3, enable us to answer affirmatively another question of Tarski's.

Theorem 4. *Let F_k, F_ℓ be free groups for $2 \leq k \leq \ell$. Then the standard embedding $F_k \to F_\ell$ is an elementary embedding.*

Proof. Let $F_k = \langle a_1, \dots, a_k \rangle$, and suppose that $\Sigma(y,a)$ is a system of equations over F_k. Instead of studying all the solutions to the system Σ in the coefficient group F_k, we study all its solutions in all free groups of the form $F_k * F$, where F is a (possibly trivial) f.g. free group. Our construction of the Makanin–Razborov diagram applies to study the entire collection of solutions of the system Σ in all free groups of the form $F_k * F$, hence, we are able to associate with this collection of solutions, a canonical Makanin–Razborov diagram.

The standard embedding $F_k \to F_\ell$ naturally enables us to interpret the system of equations Σ as a system of equations over the free group F_ℓ. Again, we study the entire set of solutions to the system Σ in all free groups of the form $F_\ell * F$, and associate with this set of solutions a canonical Makanin–Razborov diagram. Since the system of equations Σ involves only the (image of the) generators $a_1, \dots, a_k$ of F_ℓ, the (canonical) Makanin–Razborov diagrams associated with the system Σ over F_ℓ is obtained from the (canonical) Makanin–Razborov diagram of Σ over F_k, by replacing each (restricted) limit group L in the diagram associated with F_k with the limit group $L * F_{\ell-k}$ (i.e. only the factor containing the coefficient group is changed in all the restricted limit groups that appear in the diagrams), and in particular the structure of the two diagrams is essentially the same.

Using the same type of construction, if $\Sigma(y,p,a)$ is a parametric system of equations over F_k, then using the standard embedding $F_k \to F_\ell$, Σ can be viewed as a parametric system over the free group F_ℓ, and the graded Makanin–Razborov diagram of Σ over F_ℓ is obtained from the one associated with F_k, by replacing each of the graded limit groups L with $L * F_{\ell-k}$. Clearly, the same relation holds for multi-graded Makanin–Razborov diagrams associated with F_k and F_ℓ

Let V be the variety associated with some system of equations defined over F_k, and let

$$\forall y \in V \ \exists x \ \ \big(\Sigma_1(x,y,a) = 1 \wedge \Psi_1(x,y,a) \neq 1\big) \vee \dots \\ \vee \big(\Sigma_r(x,y,a) = 1 \wedge \Psi_r(x,y,a) \neq 1\big)$$

be a sentence defined over F_k. Using the standard embedding $F_k \to F_\ell$, both the variety V and the sentence can be interpreted as a variety and a sentence defined over the free group F_ℓ. Since the Makanin–Razborov diagram associated with the variety V over F_ℓ is obtained from the Makanin–Razborov diagram associated with V over F_k by replacing each limit group L with $L * F_{\ell-k}$, the formal Makanin–Razborov diagram associated with the above sentence over F_ℓ is obtained from the formal Makanin–Razborov diagram over F_k, by replacing each formal limit group FL with $FL * F_{\ell-k}$. Clearly, the same holds for the graded formal Makanin–Razborov diagram associated with a predicate defined over F_k.

Let $Res(t,v,a)$ be a (graded, multi-graded) well-separated resolution defined over F_k. Using the standard embedding $F_k \to F_\ell$, the resolution $Res(t,v,a)$ can be modified to a well-separated resolution $Res^\ell(t,v,a)$ by replacing each limit group L in $Res(t,v,a)$ by $L * F_{\ell-k}$. Then, by construction, the induced resolution, $IndRes(\langle v,a\rangle, Res^\ell(t,v,a))$, is obtained from the induced resolution $IndRes(\langle v,a\rangle, Res(t,v,a))$, by replacing each limit group L with $L * F_{\ell-k}$, and the same holds for the two core resolutions, $Core(\langle v,a\rangle, Res^\ell(t,v,a))$ and $Core(\langle v,a\rangle, Res(t,v,a))$, constructed by the procedure presented in section 4 of [Se5].

Suppose that $EAE(p)$ is an EAE set defined over F_k. By the standard embedding $F_k \to F_\ell$, the predicate defining the set $EAE(p)$ can be interpreted as a predicate defined over F_ℓ. The two iterative procedures used for the analysis of the set $EAE(p)$ in [Se5] and [Se6], the procedure for the construction of the tree of stratified sets, and the sieve procedure, are built from graded and multi-graded Makanin–Razborov resolutions, core and induced resolutions, and graded formal Makanin–Razborov diagrams. Hence, when these two iterative procedures are applied to the set $EAE(p)$ over F_k and F_ℓ, all the constructions made along these two iterative procedures are essentially similar. Hence, the set $EAE(p)$ defined over F_k, can be defined using a predicate in the Boolean algebra of AE sets over F_k, this predicate can be naturally interpreted as a predicate defined over F_ℓ, and when interpreted over F_ℓ it defines the set $EAE(p)$ over F_ℓ.

Using the standard embedding $F_k \to F_\ell$, every sentence defined over F_k can be naturally interpreted as a sentence defined over F_ℓ. Since our reduction procedures over F_k and F_ℓ are similar, a sentence defined over F_k is equivalent to a sentence in the Boolean algebra of AE sentences defined over F_k, and the reduction is valid when the two sentences are interpreted over F_ℓ as well.

The equivalence of an AE sentence defined over F_k to the same AE sentence interpreted as a sentence defined over F_ℓ was previously known [S], [H]. It also follows by applying the procedure used for the construction of the tree of stratified sets to validate it over F_k and F_ℓ. □

Note that although Theorems 1-4 are stated for f.g. free groups, the arguments that prove them do not really use the f.g. assumption, hence, the conclusions of these theorems remain valid for arbitrary non-abelian free groups and their non-abelian free factors.

Tarski's problems deal with the equivalence of the elementary theories of free groups of different ranks. Our next goal is to modify the tools used for quantifier elimination for predicates defined over a free group, in order to get a complete classification of all the f.g. groups that are elementarily equivalent to a (non-abelian) free group. We start by proving a necessary condition for a f.g. group to be elementarily equivalent to a non-abelian free group, and then we show that the necessary condition is also sufficient.

Non-abelian limit groups are known to be the f.g. groups that are universally equivalent to a non-abelian free group. A group that contains a maximal abelian subgroup that is isomorphic to a free non-cyclic abelian group, cannot be elementarily equivalent to a free group. This is true since one can count the number of classes of commuting elements mod 2, there are two such classes for an infinite cyclic group and at least four for every non-cyclic free abelian group. Hence, a f.g. group that is elementarily equivalent to a non-abelian free group, must be a limit group that contains no non-cyclic free abelian subgroups, therefore, it must be a non-elementary hyperbolic limit group. However, not every non-elementary hyperbolic limit group is elementarily equivalent to a free group.

Suppose that $G = F *_{\langle w \rangle} F = \langle b_1, b_2 \rangle *_{\langle w \rangle} \langle b_3, b_4 \rangle$ is a double of a free group of rank 2, suppose that w has no roots in F, and suppose that the given amalgamated product is the abelian JSJ decomposition of the group G. G is a limit group since it admits a strict Makanin–Razborov resolution (see [Se1, Theorem 5.12]), and it is hyperbolic since it contains no non-cyclic free abelian subgroups.

Claim 5. *The group $G = F *_{\langle w \rangle} F$ is not elementarily equivalent to the free group F.*

Proof. We look at a system of equations $\Sigma(y_1, \ldots, y_4) = 1$ that corresponds to the given presentation of G, i.e. a (coefficient-free) system of equations with four variables $y_1, \ldots, y_4$ and the equation $w(y_1, y_2) = w(y_3, y_4)$.

Clearly, there is a single limit group associated with this system, which is the limit group G.

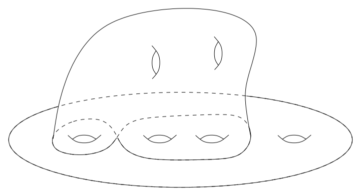
We first examine the system of equations Σ as a system of equations over a f.g. non-abelian free group. Since by our assumptions, the limit group G is not isomorphic to a free product of free groups and surface groups, the (canonical) Makanin–Razborov diagram of the limit group G is non-trivial, which implies that there exists a (canonical) finite collection of limit groups $G_1, \ldots, G_\ell$ that are all proper quotients of G, so that given an arbitrary solution of the system Σ in a non-abelian free group, we can act on the given solution with the modular group associated with G (the modular group of G is cyclic by our assumptions), so that the obtained solution factors through at least one of the limit groups $G_1, \ldots, G_\ell$.

We then view the system of equations Σ as a system of equations over the limit group G. Let $y_1^t, \ldots, y_4^t$ be the tautological solution of the system Σ over the limit group G, i.e. the solution obtained by substituting $y_i^t = b_i$ for $i = 1, \ldots, 4$. If we act by an arbitrary element of the modular group of G on the tautological solution $y_1^t, \ldots, y_4^t$, the subgroup generated by the obtained solution is still isomorphic to G, hence, the obtained solution does not factor through any of the subgroups $G_1, \ldots, G_\ell$, which are all proper quotients of G.

Since the action of the modular group of G on a given solution of Σ, gives rise to a conjugation of the specializations of y_3 and y_4 by an element that commutes with the specialization of w, the fact that every solution of Σ, viewed as a system of equations over a non-abelian free group, can be modified by an action of the modular group of G, to a solution that factors through at least one of the groups $G_1, \ldots, G_\ell$, can be translated to a coefficient-free AE sentence, which is a true sentence over a non-abelian free group. However, that coefficient-free AE sentence is not a true sentence over the limit group G. □

The argument used to prove Claim 5, demonstrates that hyperbolic limit groups with no QH vertex groups in their canonical JSJ decomposition are not elementarily equivalent to a non-abelian free group. In section 6 of [Se1] we presented ω-residually free towers, as an example for limit groups.

A hyperbolic ω-residually free tower is constructed in finitely many steps. In its first level there is a free product of (possibly none) (closed) surface groups and a (possibly trivial) free group, where each surface in this free product is a hyperbolic surface (i.e. with negative Euler characteristic), except the non-orientable surface of genus 2. In each additional level we add a punctured surface that is amalgamated to the group associated with the previous levels along its boundary components, and in addition there exists a retraction of the obtained group onto the group associated with the previous levels. The punctured surfaces are supposed to be of Euler characteristic bounded above by -2, or a punctured torus.



The argument used to prove Claim 5 can be generalized to show that if a hyperbolic limit group is elementarily equivalent to a non-abelian free group, then it must be a hyperbolic ω-residually free tower.

PROPOSITION 6. *Let G be a f.g. group that is elementarily equivalent to a non-abelian free group. Then G is a non-elementary hyperbolic ω-residually free tower.*

Proof. Let G be a f.g. group that is elementarily equivalent to a f.g. non-abelian free group. Since G is universally equivalent to a non-abelian free group it must be a limit group, and since it is AE equivalent to a non-abelian free group, it cannot contain a non-cyclic free abelian subgroup. Hence, G must be a non-elementary hyperbolic limit group.

If G is isomorphic to the free product of several surface groups and a free group, G is a hyperbolic ω-residually free tower, hence, we may assume that in the canonical (Grushko's) free decomposition of G, $G = B_1 * \cdots * B_m * F_u$, at least one of the factors B_i is not isomorphic to a surface group.

We analyze the structure of the various factors B_i that appear in the free decomposition of G, and are not isomorphic to a surface group, in parallel. Hence, we continue with a single such factor, that we denote B. Since G is a hyperbolic limit group, the factor B is a hyperbolic limit group as well. By section 2 of [Se1], since B is a freely-indecomposable hyperbolic limit group that is not isomorphic to a surface group, it admits a non-trivial abelian JSJ decomposition, and since B is hyperbolic, its abelian JSJ decomposition is a cyclic JSJ decomposition with no non-cyclic abelian vertex groups.

Let Λ be the cyclic JSJ decomposition of the factor B. If Λ contains no QH vertex groups, then the argument used to prove Claim 5, implies that the group G is not elementarily equivalent to a non-abelian free group. Hence, Λ contains QH vertex groups. At this point we show that there exists a retract in B which is the free product of the fundamental groups of disjoint subgraphs of Λ, so that at least one of the QH vertex groups in Λ is not a vertex group in any of these subgraphs.

Let $\Sigma(y) = 1$ be a system of equations that corresponds to a presentation of the factor $B = \langle b_1, \dots, b_s \rangle$. With the system $\Sigma(y) = 1$ we associate its Makanin–Razborov diagram over a non-abelian free group. Let $G_1, \dots, G_n$ be the limit groups that appear in the second level of this Makanin–Razborov diagram. By the construction of the diagram, $G_1, \dots, G_n$ are proper quotients of B, and every solution of the system $\Sigma(y) = 1$ can be modified by an element of the modular group of the factor B, to a specialization of at least one of the quotient limit groups $G_1, \dots, G_n$.

Given a non-abelian free group F, for every homomorphism $h : B \to F$, there exists a homomorphism $h' : B \to F$ that factors through one of the finitely many quotients $G_1, \dots, G_n$, and that maps each edge and vertex groups in the graph of groups Λ to a conjugate of its image under h. Furthermore, if the image under h of a QH vertex group in Λ is non-abelian, so is the image of that QH vertex group under h'. Since G is assumed to be elementarily equivalent to F, the same is true for every homomorphism $f : B \to G$. Hence, it applies to the inclusion $B \to G$. Therefore, there exists a homomorphism: $\mu : B \to G$ that has non-trivial kernel, for which each non-QH vertex group and each edge group in Λ is mapped to a conjugate in G (elementwise), and each QH vertex group in Λ is mapped to a non-abelian subgroup of G.

We modify the endomorphism $\mu : B \to G$ as follows. The image of the factor B, $\mu(B)$, inherits a free decomposition from the (Grushko's) free decomposition of G, $G = B_1 * \cdots * B_m * F_u$, $\mu(B) = D_1 * \cdots * D_r * F_\ell$, where each of the factors D_j can be conjugated into one of the factors B_i, and the image of each non-QH vertex group and each edge group in Λ can be conjugated into one of the factors D_j that is contained in a conjugate of the factor B.

Each QH vertex group Q in Λ inherits a cyclic decomposition Δ_Q from the Grushko decomposition of $\mu(B)$, $\mu(B) = D_1 * \cdots * D_r * F_\ell$, so that each edge group in the cyclic decomposition Δ_Q is mapped to the identity by μ, and each connected component that contains a boundary element in Δ_Q is mapped into a conjugate of B.

Given the homomorphism μ, we define an endomorphism $\mu_1 : B \to B$, by mapping each of the factors D_j that are subgroups of a conjugate of B, back into the factor B by a corresponding conjugation, and mapping each connected component in the cyclic decompositions Δ_Q that does not contain a boundary element of Q, into B. Since the homomorphism $\mu : B \to G$ has non-trivial kernel, the endomorphism $\mu_1 : B \to B$ has non-trivial kernel, and since μ maps each non-QH vertex group and each edge group in Λ into a conjugate in G, μ_1 maps each non-QH vertex group and each edge group in Λ into a conjugate in the factor B. Since μ maps each QH vertex group in Λ to a non-abelian subgroup, by choosing properly the elements that conjugates the factors D_j back into the factor B, and the maps from the connected components of the cyclic splittings Δ_Q that do not contain a boundary element into B, we can guarantee that every QH vertex group in Λ is mapped to a non-abelian subgroup by μ_1.

With each QH vertex group Q in Λ, we associate a (possibly trivial) maximal collection of simple closed curves (s.c.c.) that are mapped to the identity element by μ_1. Using the maximal collections of s.c.c. on the QH vertex groups in Λ, we construct a graph of groups $\hat{\Lambda}$, obtained from the graph of groups Λ by cutting the QH vertex groups along their associated collections of s.c.c. and filling each of the s.c.c. that we cut along with disks, and erase those pieces that are not connected to any of the boundary components of the original QH vertex groups in the graph of groups Λ. We set $\hat{B}$ to be the fundamental group of the graph of groups $\hat{\Lambda}$.

By construction there is a natural epimorphism, $\nu : B \to \hat{B}$. We define a homomorphism $\tau : \hat{B} \to B$, by defining τ on each factor of $\hat{B}$, where the factors of $\hat{B}$ are associated with the connected components in the graph of groups $\hat{\Lambda}$. On each such factor (connected component) we set τ to be the image under μ_1 of a preimage of that factor in B. Note that since the collections of s.c.c. on each of the QH vertex groups in Λ are assumed to be maximal collections of disjoint non-homotopic s.c.c. that are mapped to the identity by μ_1, no non-trivial s.c.c. on a QH vertex group in $\hat{\Lambda}$ is mapped by τ to the trivial element in B. We set the endomorphism $\mu_2 : B \to B$ to be $\mu_2 = \tau \circ \nu$. Since the endomorphism $\mu_1 : B \to B$ has non-trivial kernel, the endomorphism $\mu_2 : B \to B$ has non-trivial kernel, and since μ_1 maps each non-QH vertex group and each edge group in Λ onto a conjugate in B, μ_2 maps each non-QH vertex group and each edge group in Λ onto a conjugate in the factor B. Furthermore, the homomorphism μ_2 can be replaced by a homomorphism $\mu_2' : B \to B * F'$, where F' is a (possibly trivial) free group, so that μ_2 is obtained from μ_2' by composing it with the natural retraction, $B * F' \to B$, and the image of each QH vertex group in Λ under μ_2' is non-abelian. $\mu_2' = \tau' \circ \nu'$, where $\nu' : B \to \hat{B} * F'$ maps each of the connected components that do not contain boundary elements in the graphs of groups obtained from the QH vertex groups in Λ after decomposing them along the maximal collections of s.c.c. that are mapped to the identity by μ_1 onto (possibly trivial) factors of the free group F', $\tau'|_{\hat{B}} = \tau$ and $\tau'|_{F'}$ is the identity map.

The endomorphism μ_2 maps each non-QH vertex group and each edge group in Λ into a conjugate in B. Since the graph of groups $\hat{\Lambda}$ was obtained from Λ by cutting QH vertex groups along maximal collections of s.c.c. that are mapped to the identity by μ_1, and the endomorphism $\mu_2 = \tau \circ \nu$, by Lemma 1.4 in [Se4], if the image of a QH vertex group Q in Λ, $\mu_2(Q)$, intersects non-trivially a conjugate of a QH vertex group in Λ, then it intersects it in a group of finite index. By Lemma 1.3 in [Se4], if $\mu_2(Q)$ intersects a conjugate of a QH vertex group Q' in Λ in a subgroup of finite index, the topological complexity of Q', $(|\chi(Q')|, genus(Q'))$, is bounded by the topological complexity of Q, and if the topological complexities are equal, then Q is mapped by μ_2 isomorphically onto Q'.

If for every QH vertex group Q' in Λ, the image of μ_2 contains a conjugate of a finite index subgroup of Q', then each QH subgroup Q in Λ is mapped by μ_2 isomorphically onto a conjugate of a QH subgroup in Λ, and distinct QH subgroups in Λ are mapped onto conjugates of distinct QH vertex groups in Λ. Replacing μ_2 by a power of itself, we may assume that μ_2 preserves the conjugacy classes of all the vertex and edge groups in Λ. Since Λ is a cyclic splitting of the hyperbolic group B, hence, it can be assumed to be 2-acylindrical, if μ_2 maps each vertex group and each edge group in Λ to a conjugate, μ_2 maps the factor B onto itself, and since B is ω-residually free and ω-residually free groups are Hopf, μ_2 being an epimorphism is also a monomorphism, a contradiction since μ_2 was assumed to have non-trivial kernel. Therefore, there must exist QH vertex groups in Λ, so that $\mu_2(B)$ does not contain any finite index subgroups of their conjugates.

Let $Q_1, \dots, Q_\ell$ be all the QH vertex groups in Λ, for which $\mu_2(B)$ intersects conjugates of $Q_1, \dots, Q_\ell$ in subgroups of finite index. By construction, the image of $\mu_2(B)$ is generated by subgroups of the fundamental groups of connected components of the graph of groups obtained from Λ by erasing these QH vertex groups that are not in the list $Q_1, \dots, Q_\ell$.

First, suppose that the list $Q_1, \dots, Q_\ell$ is empty or that each of the QH vertex groups $Q_1, \dots, Q_\ell$ is mapped isomorphically onto a conjugate of one of the QH vertex groups $Q_1, \dots, Q_\ell$, and different QH vertex groups from this list are mapped to conjugates of distinct QH vertex groups in Λ, hence, μ_2 permutes the conjugacy classes of $Q_1, \dots, Q_\ell$. In this case a power of μ_2 preserves the conjugacy classes of $Q_1, \dots, Q_\ell$, hence, we can replace this power of μ_2, with a (proper) retraction from B onto the subgroup H generated by the connected components of the graph of groups obtained from Λ by erasing these QH vertex groups that are not in the list $Q_1, \dots, Q_\ell$, $r : B \to H$. Furthermore, in the same way we replaced $\mu_2 : B \to B$ by $\mu'_2 : B \to B * F'$, we may assume that the retraction r can be replaced by a map $u : B \to H * F'$, where F' is a (possibly trivial) free group, so that r is obtained from u by composing it with the natural retraction: $H * F' \to H$, and the image of each QH vertex group in Λ under u is non-abelian. Note that since the retract H is the fundamental group of a proper subgraph of the JSJ decomposition Λ of the hyperbolic factor B, H is a hyperbolic group as well.

Suppose that the list of QH vertex groups $Q_1, \dots, Q_\ell$ is not empty, and that μ_2 does not map $Q_1, \dots, Q_\ell$ isomorphically onto distinct conjugates of $Q_1 \dots, Q_\ell$. In this case we look at the endomorphism $\mu_2^2 : B \to B$. By (possibly) refining the cyclic decomposition $\hat{\Lambda}$ according to a maximal collection of s.c.c. on the various QH vertex groups in $\hat{\Lambda}$ that are mapped to the identity by μ_2^2, and use the construction of the map μ_2 from μ_1, we obtain a new map $\mu_3 : B \to B$, for which $\mu_3(B)$ intersects conjugates of a proper subset of the QH vertex groups $Q_1, \dots, Q_\ell$ in a subgroup of finite index, μ_3 maps each non-QH vertex group and each edge group in Λ into a conjugate, and μ_3 can be replaced by a map $v : B \to B * F'$, where F' is a (possibly trivial) free group, so that μ_3 is obtained from v by composing it with the natural retraction, $B * F' \to B$, and the image of each QH vertex group in Λ under v is non-abelian. Therefore, a finite induction proves the existence of a retraction r from B onto the subgroup generated by the connected components of the graph of groups obtained from Λ by erasing some (non-empty) subset of the QH vertex groups in Λ. Note that by Lemma 1.4 in [Se4] this subgroup (the image of r) is the free product of the fundamental groups of the corresponding connected components in Λ.

So far we have constructed a proper retraction of the original limit group G onto its proper subgroup H, which is a hyperbolic limit group like the ambient group G (since it is a f.g. subgroup of a hyperbolic limit group). We now continue in the same way with this proper subgroup H. We start with its (Grushko's) free decomposition, and then look at all the factors that are not surface (or free) groups. Since G is elementarily equivalent to a free group, the construction applied for the factor B, and the retraction from G onto H, enables one to show first that there exists a proper map from a given (non-surface, non-free) factor of H into itself that conjugates each non-QH vertex group and each edge group in its cyclic JSJ decomposition (elementwise). Furthermore the image of every QH vertex group under this map is non-abelian. Now applying the same construction that was used for the factor B, implies that H has a proper retraction onto a proper subgroup of itself with similar properties as the retraction from B to H. Since limit groups satisfy the descending chain condition [Se1, 5.1], this construction of proper retractions terminates after finitely many steps, and the limit group G is indeed a hyperbolic ω-residually free tower. □

Proposition 6 shows that a f.g. group that is elementarily equivalent to a free group must be a non-elementary hyperbolic ω-residually free tower. The techniques used in our quantifier elimination procedure can be modified to prove the converse, hence, we finally obtain a classification of the f.g. groups that are elementarily equivalent to a non-abelian f.g. free group.

Theorem 7. *A f.g. group is elementarily equivalent to a non-abelian free group if and only if it is a non-elementary hyperbolic ω-residually free tower.*

Proof. Proposition 6 proves that a f.g. group that is elementarily equivalent to a non-abelian free group must be a non-elementary hyperbolic ω-residually free tower. Hence, it is left to prove that if G is a non-elementary hyperbolic ω-residually free tower, then G is elementarily equivalent to a f.g. non-abelian free group.

Let G be a non-elementary hyperbolic ω-residually free tower. With the structure of the ω-residually free tower associated with G, we can naturally associate a (coefficient-free, strict) resolution $Res_G(b)$ over a non-abelian free group (where the groups that appear along the resolution are the groups associated with the various levels of the tower). By construction, the resolution $Res_G(b)$ is a completed resolution (i.e. it has the structure of a completion, see Definition 1.12 in [Se2]), and since G is assumed to be hyperbolic, the resolution $Res_G(b)$ contains no non-cyclic abelian vertex groups.

Let F be a f.g. non-abelian free group, and let $\Sigma(y) = 1$ be a (coefficient-free) system of equations. With the system $\Sigma(y) = 1$, interpreted as a system of equations over the free group F, we have associated a taut Makanin–Razborov diagram. With each resolution in this taut Makanin–Razborov diagram we have associated its completion. Let

$Comp(Res_1)(z,y),\dots,Comp(Res_d)(z,y)$ be the set of completions associated with the system Σ..

LEMMA 8. *Let y' be a solution of the system $\Sigma(y)=1$, interpreted as a system of equations over the ω-residually free tower G. Then there exist elements z' in G, so that the tuple (z',y') is a specialization of (at least) one of the completions $Comp(Res_1)(z,y),\dots,Comp(Res_d)(z,y)$, i.e. it is the image of a homomorphism from the limit group associated with one of these completions into the group G.*

Proof. We look at a test sequence of the (coefficient-free) resolution, $Res_G(b)$, in the free group F (test sequences are presented in Definition 1.20 in [Se2]). Let $y'_n \in F$ be the sequence of specializations of the elements $y' \in G$ along the given test sequence. Since $\Sigma(y')=1$ in G, the sequence y'_n satisfies $\Sigma(y'_n)=1$ in F, for every index n. Hence, for every index n, there exist elements $z'_n \in F$, so that the tuple (z'_n,y'_n) is a specialization of one of the completions: $Comp(Res_1)(z,y),\dots,Comp(Res_d)(z,y)$. If for every n we choose the shortest such z'_n, then the techniques used for the construction of a formal solution [Se2, 1.18]) prove that there exists a subsequence of tuples (z'_n,y'_n) that are all specializations of the same completion, $Comp(Res_i)(z,y)$, that converge into a tuple $(z',y') \in G$. Hence, there is a homomorphism from the completion, $Comp(Res_i)(z,y)$, into the tower G, that sends the elements y to y' and the elements z to z'. □

Clearly, the same argument used to prove Lemma 8 applies to given covering closures of the resolutions in the taut Makanin–Razborov diagram, and to graded and multi-graded systems of equations, and the completions and covering closures of the resolutions in their associated graded and multi-graded taut Makanin–Razborov diagrams.

Let $\Theta(y)=1$ be a coefficient-free system of equations, and let

$$(\forall y)\ \big(\Theta(y)=1\big)\ \exists x\ \Sigma(x,y)=1 \wedge \Psi(x,y)\neq 1$$

be a coefficient-free sentence. Suppose that the sentence is a true sentence over the (non-abelian) free group F. By Theorem 1.18 of [Se2], with each resolution $Res(y)$ in the taut Makanin–Razborov diagram associated with the system $\Theta(y)=1$, it is possible to associate a covering closure, $Cl_1(Res)(s,z,y),\dots,Cl_e(Res)(s,z,y)$, and formal solutions $x_i=x_i(s,z,y)$ defined over the closures, $Cl_i(Res)(s,z,y)$, for which

(i) $\Sigma(x_i(s,z,y),y)=1$ in the limit group corresponding to the closure, $Cl_i(Res)(s,z,y)$.

(ii) There exists a specialization (s_0, z_0, y_0) (in the free group F), of the closure, $Cl_i(Res)(s, z, y)$, for which $\Psi(x_i(s_0, z_0, y_0), y_0) \neq 1$.

By Lemma 8, if y' is a solution of the system $\Theta(y) = 1$, interpreted as a system of equations over the hyperbolic ω-residually free tower G, then there exists a resolution $Res(y)$ in the taut Makanin–Razborov diagram associated with the system $\Theta(y) = 1$ over a non-abelian free group F, a closure $Cl(Res)(s, z, y)$ from a given covering closure of the resolution $Res(y)$, and elements s' and z' in G, so that the tuple (s', z', y') is a specialization of the closure $Cl(Res)(s, z, y)$. Hence, there exists a formal solution, $x_i(s, z, y)$, for which: $\Sigma(x_i(s', z', y'), y') = 1$. Clearly, the graded and multi-graded categories are completely analogous.

PROPOSITION 9. *Let*

$$\forall y \; \exists x \; \big(\Sigma_1(x, y) = 1 \wedge \Psi_1(x, y) \neq 1\big) \vee \cdots \vee \big(\Sigma_r(x, y) = 1 \wedge \Psi_r(x, y) \neq 1\big)$$

be a coefficient-free AE sentence. Then the AE sentence is a true sentence over the f.g. non-abelian free group F if and only if it is a true sentence over the non-elementary hyperbolic ω-residually free tower G. i.e. the AE theories of non-abelian free groups and non-elementary hyperbolic ω-residually free towers are equivalent.

Proof. Suppose that

$$\forall y \; \exists x \; \big(\Sigma_1(x, y) = 1 \wedge \Psi_1(x, y) \neq 1\big) \vee \cdots \vee \big(\Sigma_r(x, y) = 1 \wedge \Psi_r(x, y) \neq 1\big)$$

is a coefficient-free sentence. If the sentence is a true sentence over a f.g. non-abelian free group F, then it is possible to apply the iterative procedure presented in section 4 of [Se4], and associate iteratively with the sentence a (finite) sequence of (coefficient-free) anvils, developing resolutions and formal solutions defined over them, that prove the validity of the sentence over the free group F. By the arguments given above, the proof given by the sequence of (coefficient-free) anvils, developing resolutions, and formal solutions defined over them, is valid over the non-elementary hyperbolic ω-residually free tower G.

Suppose that the given sentence is false over the free group F. By applying the iterative procedure for validation of a sentence presented in section 4 of [Se4], there exists a coefficient free resolution $Res(z, y)$, so that there exists a test sequence of specializations of this resolution, for which for the corresponding specializations of the variables y, there is no specialization of the variables x (in the free group F) for which

$$\big(\Sigma_1(x, y) = 1 \wedge \Psi_1(x, y) \neq 1\big) \vee \cdots \vee \big(\Sigma_r(x, y) = 1 \wedge \Psi_r(x, y) \neq 1\big) .$$

We want to show that the given (coefficient-free) sentence is false over the non-elementary hyperbolic ω-residually free tower G. G is, in particular, a non-elementary hyperbolic group, hence, the free group F admits a quasi-isometric embedding into G. Let F_G be the (quasi-isometric) image of that embedding in G. By our assumptions, the resolution $Res(z,y)$ has a test sequence of specializations in the free group F, so that for the corresponding sequence of specializations of the variables y, there are no specializations of the variables x in F, for which

$$\big(\Sigma_1(x,y)=1 \wedge \Psi_1(x,y)\neq 1\big) \vee \dots \vee \big(\Sigma_r(x,y)=1 \wedge \Psi_r(x,y)\neq 1\big)\,.$$

The free group F is isomorphic (and quasi-isometric) to the subgroup F_G of G, hence, we can naturally interpret the given test sequence of specializations of the resolution $Res(z,y)$ in F, as a test sequence of specializations of $Res(z,y)$ in its isomorphic image F_G. Suppose that our given sentence is a true sentence over the group G. Then for each specialization of the variables y from our given test sequence, there exists a specialization of the variables x in G for which

$$\big(\Sigma_1(x,y)=1 \wedge \Psi_1(x,y)\neq 1\big) \vee \dots \vee \big(\Sigma_r(x,y)=1 \wedge \Psi_r(x,y)\neq 1\big)$$

where the equalities and inequalities are in the hyperbolic group G. Let X be the Cayley graph of the hyperbolic group G. For each specialization of the variables y from the given test sequence, we pick the specializations of the variables x in G, which are the shortest possible for which the above disjunction of conjunctions of equalities and inequalities hold. Since the Cayley graph X of G is δ-hyperbolic, we can extract a subsequence of specializations of the variables (x,z,y) from our given sequence, that converges in the Gromov–Hausdorff topology on metric spaces (after rescaling). Since the specializations z,y form a test sequence, and since the specializations of the variables x were taken to be the shortest possible, the subsequence of specializations (x,z,y) converges into a faithful action of a limit group of the form $L=\langle x,z,y\rangle = Cl(Res)(s,z,y) * M$ on a real tree Y (see the argument used to prove Theorem 1.18 in [Se2] for the structure of L). In the limit group L,

$$\big(\Sigma_1(x,y)=1 \wedge \Psi_1(x,y)\neq 1\big) \vee \dots \vee \big(\Sigma_r(x,y)=1 \wedge \Psi_r(x,y)\neq 1\big)\,.$$

The original test sequence of specializations in the non-abelian free group F factors through the closure, $Cl(Res)(s,z,y)$, hence, from the original test sequence of specializations of the resolution $Res(z,y)$ in the free group F, it is possible to extract a subsequence, for which for the corresponding specializations of the variables y, there exist specializations of the variables x

(in F), obtained from homomorphisms from the limit group $h{:}L(x,z,y)\to F$, for which

$$\big(\Sigma_1(x,y)=1\wedge\Psi_1(x,y)\neq 1\big)\vee\cdots\vee\big(\Sigma_r(x,y)=1\wedge\Psi_r(x,y)\neq 1\big)$$

in the free group F, and we get a contradiction. □

Proposition 9 proves that the AE theories of non-abelian free groups and non-elementary hyperbolic ω-residually free towers are equivalent. To prove the equivalence of the elementary theories of these groups, it remains to prove that, for coefficient-free predicates, the quantifier elimination procedure constructed in [Se5] and [Se6], for predicates defined over a free group, works for all non-elementary hyperbolic ω-residually free towers. The quantifier elimination procedure presented in [Se5] and [Se6] is composed of two parts, the procedure for the construction of the tree of stratified sets (section 2 in [Se5]), and the sieve procedure ([Se6]). We start by showing that the procedure for the construction of the tree of stratified sets remains valid over the hyperbolic ω-residually free tower G.

The procedure for the construction of the tree of stratified sets (section 2 in [Se5]) analyzes the remaining set of y's using multi-graded resolutions, and their associated developing resolutions and anvils, and then we associate with each developing resolution, its entire collection of formal solutions, which are encoded in the graded formal Makanin–Razborov diagram. Since the predicates we consider are all coefficient free, all the constructions along the procedure for the construction of the tree of stratified sets are coefficient free as well.

Lemma 8 proves that the set of the remaining y's over the group G can be completed to specializations that factor through the completions (and closures) of the multi-graded resolutions at each step of the procedure for the construction of the tree of stratified sets. We have also argued that if p_0 is in the definable set of the given predicate, interpreted as a predicate over G, then with each ungraded resolution associated with the (graded) developing resolution and p_0, there exists a formal solution (over G) that satisfies the properties of Theorem 1.18 of [Se2].

The ungraded resolution defined over G can be approximated by a sequence of ungraded resolutions over the free group F. From the existence of a formal solution defined over the given ungraded resolution over F, it follows that all the approximating resolutions over F admit formal solutions that approximate the formal solution over G. Hence, the given formal solution defined over G, factors through the completion of at least one of the graded formal resolutions associated with the given developing resolution over the free group F.

The iterative procedure for validation of an AE sentence, shows that if an AE sentence over a free group F is a true sentence, then it has a proof using a finite collection of formal solutions. By the proof of Lemma 8, the same is true for sentences defined over G.

Let $L(w,p)$ be a coefficient-free AE predicate. Let $Q_F(w,p)$ be the set defined by $L(w,p)$ over the free group F, and $Q_G(w,p)$ be the set defined by $L(w,p)$ over the hyperbolic ω-residually free tower G. The tree of stratified sets constructed over the free group F, shows that if $(w_0,p_0) \in Q_F(w,p)$, then the sentence corresponding to the specialization (w_0,p_0) can be proved using a sequence of formal solutions according to one of the proof systems given by the tree of stratified sets over F. The argument given above, shows that if $(w_0,p_0) \in Q_G(w,p)$, then the sentence corresponding to the specialization (w_0,p_0) can be proved using a sequence of formal solutions encoded by one of the proof systems (over F) given by the same (coefficient-free) tree of stratified sets (see Definition 1.20 in [Se5] for a proof system).

Furthermore, if $Rgd(x,p)$ $(Sld(x,p))$ is a (coefficient-free) rigid (solid) limit group, then the maximal number of rigid (families of strictly solid) specializations of $Rgd(x,p)$ $(Sld(x,p))$ for a specialization of the defining parameter p is identical over the free group F and the hyperbolic ω-residually free tower F. Hence, the collection of proof systems associated with the tree of stratified sets is identical over F and over G, i.e. for the two sets $Q_F(w,p)$ and $Q_G(w,p)$.

Since the tree of stratified sets over the free group F is coefficient free, so is the collections of varieties and resolutions associated with the sieve procedure over F (presented in [Se6]). The tree of stratified sets over F is coefficient free and is also the tree of stratified sets defined over G. The collection of proof systems associated with the tree of stratified sets over F and over G are identical. Any valid PS statement over G (Definition 1.23 in [Se5]) can be approximated by a sequence of valid PS statements over F. Furthermore, any valid PS statement over G does not (extend to a specialization that) factor through any of the Non-Rigid, Non-Solid, Root or Left PS resolutions constructed along the sieve procedure (over F), and if it factors through an extra PS resolution (constructed along the sieve procedure over F), it has to factor through either one of the collapse extra PS limit groups associated with it, or one of the generic collapse extra PS resolutions associated with it. Hence, any valid PS statement over G extends to a specialization that belongs to at least one of the $TSPS(p)$ sets constructed along the sieve procedure (over the free group F. See

Definition 1.34 in [Se5] for the set $TSPS(p)$). Therefore, the reduction of a coefficient-free predicate to a (coefficient-free) predicate in the Boolean algebra of AE sets can be done uniformly, for all non-elementary hyperbolic ω-residually free towers (by applying the corresponding reduction over the a non-abelian free group F). Since the AE theories of these towers are equivalent by Proposition 9, the elementary theories of non-elementary hyperbolic residually free towers are equivalent, which implies that they are all elementarily equivalent to a non-abelian free group. □

Appendix. A Brief Survey of Previous Results

In proving most of the theorems in this paper, we needed to use not only the quantifier elimination over a free group [Se6, 1.41], but also some of the constructions and results that were presented in the previous papers in this sequence, and in particular the structures of the iterative procedure for validation of an AE sentence, and the quantifier elimination process.

For the benefit of the reader, we summarize some of the required results and procedures in this short appendix. However, for the precise constructions, the detailed procedures, and the actual results, that are usually technically involved, the interested reader can check the indicated references.

In the first paper in the sequence on Diophantine geometry over groups, we studied sets of solutions to systems of equations defined over a free group and parametric families of such sets, and associated a canonical Makanin–Razborov diagram that encodes the entire set of solutions to the system [Se1, 5]. Later on we studied systems of equations with parameters, and with each such system we associated a (canonical) graded Makanin–Razborov diagram, that encodes the Makanin–Razborov diagrams of the systems of equations associated with each specialization of the defining parameters [Se1, 10].

In the second paper we generalized Merzlyakov's theorem on the existence of a formal solution associated with a positive sentence. We first constructed a formal solution for a general AE sentence which is known to be true over some variety [Se2, 1.18], and then presented formal limit groups and graded formal limit groups that enable us to collect and analyze the collection of all such formal solutions [Se2, 2-3].

In the third paper we studied the structure of exceptional solutions of a parametric system of equations (see Definitions 10.5 in [Se1] and 1.5 in [Se3] for these exceptional solutions). We proved the existence of a global bound (independent of the specialization of the defining parameters) on the number of rigid solutions of a rigid limit group [Se3, 2.5], and a global bound on the number of strictly solid families of solutions of a solid limit group [Se3, 2.9]. Using these bounds we studied the stratification of the "base" of the "bundle" associated with the set of solutions of a parametric system of equations in a free group, and showed that the set of specializations of the defining parameters in each stratum is in the Boolean algebra of AE sets [Se3, 3].

In the fourth paper we applied the structural results obtained in the first two papers in the sequence, to analyze AE sentences. Given a truth sentence of the form,

$$\forall y \; \exists x \; \Sigma(x,y,a) = 1 \; \wedge \; \Psi(x,y,a) \neq 1 \,,$$

we presented an iterative procedure, that produces a sequence of varieties and formal solutions defined over them that together prove the validity of the given sentence. The procedure uses a trial and error approach. It starts with a formal solution that proves the validity of the given sentence in a generic point of the affine set associated with the corresponding universal (y) variables. If we substitute the given formal solution into the system of equations Σ they hold over the whole affine set. However, the inequalities Ψ fail on some proper subvariety V. Hence, in the second step of the iterative procedure, we apply Theorem 1.18 in [Se3], and get formal solutions that prove the validity of the given sentence in generic points of (closures of) completions of resolutions in the Makanin–Razborov diagram of the variety V. Again, if we substitute the given formal solutions into the inequalities Ψ, they fail to hold on some proper subvarieties of the varieties associated with the (closures of) completions of resolutions in the Makanin–Razborov diagrams associated with the variety V. We continue iteratively by constructing formal solutions over each of (the closures) of the completions of the resolutions associated with varieties that collect the set of the remaining y's at each step of the procedure.

Since in order to define the completions of the resolutions associated with a variety, and the closures of these completions, additional variables are required, the varieties produced along the iterative procedure are determined by larger and larger sets of variables, and so are the formal solutions defined over them. Still, by carefully analyzing these varieties and the Diophantine sets associated with them, using induced resolutions [Se4, 3], certain fiber products that we call anvils, and their associated developing resolutions [Se4, 4] and properly measuring the complexity of Diophantine sets associated with them, we were able to show that certain complexity of the varieties produced along the procedure strictly decreases, which finally forces the iterative procedure to terminate after finitely many steps. It should be noted, that even at this stage we don't know any conceptual reason that explains the ability to find such a terminating procedure, except for its existence.

The outcome of the terminating iterative procedure is a collection of varieties, together with a collection of formal solutions defined over them. The varieties are determined by the original universal variables y, and extra (auxiliary) variables. The collection of varieties gives a partition of the initial domain of the universal variables y, which is a power of the original free group of coefficients, into sets which are in the Boolean algebra of universal sets, so that on each such set the sentence can be validated using a finite family of formal solutions. Hence, the outcome of the iterative procedure can be viewed as a *stratification theorem* that generalizes Merzlyakov's theorem from positive sentences to general AE ones.

In the two papers on quantifier elimination we apply the tools and techniques presented in the previous 4 papers in the sequence, to prove quantifier elimination in the elementary theory of a free group. In order to prove quantifier elimination we show that the Boolean algebra of AE sets is invariant under projections. The projection of a set that is in the Boolean algebra of AE sets, is naturally an EAE set, hence, to show that the Boolean algebra of AE sets is invariant under projections, we need to show that a general EAE set is in the Boolean algebra of AE sets [Se6, 1.41].

To prove that an EAE set is in the Boolean algebra of AE sets we use a couple of terminating iterative procedures that are based on the procedure for validation of an AE sentence presented in the fourth paper. Given an EAE set, the first (terminating) iterative procedure is devoted to uniformization of proofs, i.e. it produces finitely many (graded) families of formal solutions together with (graded) varieties on which these formal solutions are defined, so that each AE sentence associated with a specialization of the defining parameters and a specialization of the first existential variables, which is a truth sentence, can be proved using part of the constructed families of formal solutions, in a similar way to our validation of a (single) AE sentence obtained in the fourth paper.

Each step of this procedure [Se5, 2] is divided into two parts. In the first part we collect all the formal solutions defined over the (finitely many, graded) varieties that collect the set of those values of the universal variables, for which the corresponding AE sentence is yet to be proved. The second part uses the constructed formal solutions to get a proof for a subset of the relevant values of the universal variables, and collect those values for which the proof is yet incomplete. We call the outcome of this procedure, i.e. the families of formal solutions and the varieties on which they are defined, the *tree of stratified sets*. Both its construction and its termination are uniformizations of the procedure for validation of a single AE sentence, presented in [Se4, 4].

The procedure for uniformization of proofs constructs the tree of stratified sets, that leaves us with finitely many forms of proof, i.e. possible (finite) subsets of the families of formal solutions encoded by this tree, for all the truth AE sentences associated with the set EAE. We call each such form of proof a *proof system* [Se5, 1.20].

To analyze an EAE set we start with the Zariski closures of all the *valid proof statements* associated with each of the (finitely many) proof systems [Se5, 1.23]. The second terminating iterative procedure that we call the *sieve* procedure, presented in [Se6], starts with each of these Zariski closures and constructs a (finite) sequence of bundles of (virtual) proof statements that are supposed to "testify" that a given specialization of the defining parameters is in the set EAE. This finite sequence of bundles reduces the question of the existence of a possible *witness* (i.e. a value of the first existential variables) with a valid proof statement [Se5, 1.19] for any given specialization of the defining parameters, to the structure of the bases of these bundles of proof statements. Since by section 3 of [Se3] it is possible to stratify the base of such a bundle, and the existence of a witness for a given specialization of the defining parameters depends only on the stratum (and not on the specific specialization), the set EAE is the union of finitely many strata in the stratifications of the constructed bundles. Since every stratum in the stratification is in the Boolean algebra of AE sets [Se3, 3], we are finally able to conclude that the original EAE set is in the Boolean algebra of AE sets.

We should note that like the procedure for validation of a sentence, presented in the fourth paper, we still do not know a conceptual reason for obtaining quantifier elimination, and for the ability to construct a terminating procedure like our sieve procedure, apart from its existence. Indeed, the construction of the sieve procedure and its termination are technically the heaviest part of our work, and require techniques and methods to handle Diophantine sets.

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