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DIOPHANTINE GEOMETRY OVER GROUPS $\mathbf{V}_1$: QUANTIFIER ELIMINATION I

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ABSTRACT

This paper is the first part (out of two) of the fifth paper in a sequence on the structure of sets of solutions to systems of equations in a free group, projections of such sets, and the structure of elementary sets defined over a free group. In the two papers on quantifier elimination we use the iterative procedure that validates the correctness of an AE sentence defined over a free group, presented in the fourth paper, to show that the Boolean algebra of AE sets defined over a free group is invariant under projections, and hence show that every elementary set defined over a free group is in the Boolean algebra of AE sets. The procedures we use for quantifier elimination, presented in this paper and its successor, enable us to answer affirmatively some of Tarski's questions on the elementary theory of a free group in the sixth paper of this sequence.

Introduction

In the first four papers in the sequence on Diophantine geometry over groups we studied sets of solutions to systems of equations in a free group, and developed basic techniques and objects required for the analysis of sentences and elementary sets defined over a free group.

In the first paper in this sequence we studied sets of solutions to systems of equations defined over a free group and parametric families of such sets, and associated a canonical Makanin–Razborov diagram that encodes the entire set of solutions to the system. Later on we studied systems of equations

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with parameters, and with each such system we associated a (canonical) graded Makanin–Razborov diagram that encodes the Makanin–Razborov diagrams of the systems of equations associated with each specialization of the defining parameters.

In the second paper we generalized Merzlyakov's theorem on the existence of a formal solution associated with a positive sentence [Me]. We first constructed a formal solution for a general AE sentence which is known to be true over some variety, and then presented formal limit groups and graded formal limit groups that enable us to collect and analyze the collection of all such formal solutions.

In the third paper we studied the structure of exceptional solutions of a parametric system of equations. We proved the existence of a global bound (independent of the specialization of the defining parameters) on the number of rigid solutions of a rigid limit group, and a global bound on the number of strictly solid families of solutions of a solid limit group. Using these bounds we studied the stratification of the "base" of the "bundle" associated with the set of solutions of a parametric system of equations in a free group, and showed that the set of specializations of the defining parameters in each of the strata is in the Boolean algebra of AE sets.

In the fourth paper, we applied the structural results obtained in the first two papers in the sequence to analyze AE sentences. Given a true sentence of the form

$$\forall y \quad \exists x \quad \Sigma(x,y,a) = 1 \wedge \Psi(x,y,a) \neq 1$$

we presented an iterative procedure that produces a sequence of varieties and formal solutions defined over them. Since in order to define the completions of a variety, and the closures of these completions, additional variables are required, the varieties produced along the iterative procedure are determined by larger and larger sets of variables, and so are the formal solutions defined over them. Still, by carefully analyzing these varieties, and properly measuring the complexity of Diophantine sets associated with them, we were able to show that certain complexity of the varieties produced along the iterative procedure strictly decreases, which finally forces the iterative procedure to terminate after finitely many steps.

The outcome of the terminating iterative procedure is a collection of varieties, together with a collection of formal solutions defined over them. The varieties are determined by the original universal variables y, and extra (auxiliary) variables. The collection of varieties gives a partition of the initial domain of the universal variables y, which is a power of the original free group of coefficients, into sets which are in the Boolean algebra of universal sets, so that on each such set the sentence can be validated using a finite family of formal solutions. Hence, the outcome of the iterative procedure can be viewed as a *stratification theorem* that generalizes Merzlyakov's theorem from positive sentences to general AE ones.

In the two papers on quantifier elimination we apply the tools and techniques presented in the previous four papers in the sequence, to prove quantifier elimination for the elementary theory of a free group. In order to prove quantifier elimination we show that the Boolean algebra of AE sets is invariant under projections. The projection of a set that is in the Boolean algebra of AE sets is naturally an EAE set, hence to show that the Boolean algebra of AE sets is invariant under projections, we need to show that a general EAE set is in the Boolean algebra of AE sets. Let

$$\begin{aligned} EAE(p) =& \exists w\ \forall y\ \exists x\ (\Sigma_1(x,y,w,p,a)=1 \wedge \Psi_1(x,y,w,p,a)\neq 1) \vee \cdots \\ & \cdots \vee (\Sigma_r(x,y,w,p,a)=1 \wedge \Psi_r(x,y,w,p,a)\neq 1) \end{aligned}$$

be a general EAE set. The set $EAE(p)$ is, by definition, the set of all specializations p_0 of the defining parameters p, for which the corresponding sentence

$$\begin{aligned} &\exists w\ \forall y\ \exists x\ (\Sigma_1(x,y,w,p_0,a)=1 \wedge \Psi_1(x,y,w,p_0,a)\neq 1) \vee \cdots \\ &\qquad \cdots \vee (\Sigma_r(x,y,w,p_0,a)=1 \wedge \Psi_r(x,y,w,p_0,a)\neq 1) \end{aligned}$$

is a true sentence. Hence, we start the analysis of the set $EAE(p)$ by analyzing the set of tuples (w_0,p_0) for which the corresponding sentence

$$\begin{aligned} &\forall y\ \exists x\ (\Sigma_1(x,y,w_0,p_0,a)=1 \wedge \Psi_1(x,y,w_0,p_0,a)\neq 1) \vee \cdots \\ &\qquad \cdots \vee (\Sigma_r(x,y,w_0,p_0,a)=1 \wedge \Psi_r(x,y,w_0,p_0,a)\neq 1) \end{aligned}$$

is a true sentence.

By the procedure for validation of an AE sentence, presented in the fourth paper in this sequence, if (w_0,p_0) is a tuple for which the corresponding sentence is a true sentence, then there exists a (finite) collection of varieties and formal solutions defined over these varieties that prove the validity of the sentence. However, the finite collection of varieties and formal solutions may depend on the particular specialization (w_0,p_0). Therefore, our first goal in analyzing the collection of tuples (w_0,p_0) that are in the true set is obtaining a "uniformization of proof systems". To get such "uniformization", we present an iterative procedure that is based on the iterative procedure for validation of an AE sentence, that starts with our given EAE predicate, and produces a finite tree for which in each vertex we place a variety that is graded with respect to the parameter subgroup $<w,p>$, and a family of formal solutions defined over this variety, so that if (w_0,p_0) is a specialization for which the corresponding sentence

$$\forall y\ \exists x\ (\Sigma_1(x,y,w_0,p_0,a)=1 \wedge \Psi_1(x,y,w_0,p_0,a)\neq 1) \vee \cdots$$
$$\cdots \vee (\Sigma_r(x,y,w_0,p_0,a)=1 \wedge \Psi_r(x,y,w_0,p_0,a)\neq 1)$$

is a true sentence, then there exists a proof of the sentence that goes along a finite collection of paths in the finite tree associated with the set $EAE(p)$. Hence, the finite tree associated with the set $EAE(p)$ encodes proofs of the corresponding AE sentences for all the tuples (w_0,p_0) that are in the true set. We call this finite tree associated with the set $EAE(p)$ the *tree of stratified sets*.

Having constructed such a finite tree, to analyze the set $EAE(p)$ we need to look for all the specializations p_0 of the defining parameters p for which there exists a specialization w_0 of the (existential) variables w, so that the AE sentence corresponding to the tuple (w_0,p_0) has a proof that goes along a collection of paths in the constructed finite tree. To carry out the analysis of this set of specializations p_0 of the defining parameters p, we present an iterative procedure which we view as a "sieve procedure", that produces an increasing sequence of sets of specializations of the defining parameters p that are all in the Boolean algebra of AE sets, and are all approximations of the set $EAE(p)$, and in particular are all contained in the set $EAE(p)$. The sieve procedure we present terminates after finitely many steps, and the set it produces when it terminates is equal to the set $EAE(p)$, and is in the Boolean algebra of AE sets, hence the sieve procedure finally enables us to show that the set $EAE(p)$ is in the Boolean algebra of AE sets.

Since the iterative procedures are rather involved, we prefered to present them first in a special case, which is conceptually and technically simpler, but it already demonstrates some of the principles used in the general case. Hence, in the first section we present the two iterative procedures used for the analysis of an EAE set in the minimal rank (rank 0) case, i.e., for predicates for which the limit groups involved in their analysis are all of minimal possible rank, i.e., limit groups that do not admit an epimorphism onto a free group so that the coefficient group is mapped onto a proper factor. In the second section we present the procedure for the construction of the tree of stratified sets, i.e., the tree associated with a general EAE set that encodes proofs for all tuples (w_0,p_0) for which the corresponding AE sentence is a true sentence. In the third section we analyze a few special cases, in which it is possible to slightly modify the procedure used for the construction of the tree of stratified sets to get a sieve procedure, and hence prove quantifier elimination. In the fourth section we present core resolutions and some of their basic properties. Core resolutions seem to be a basic tool in analyzing projections of varieties defined over a free group (Diophantine sets), and they play an essential role in the (general) sieve procedure that is finally presented in the next paper, and concludes our quantifier elimination procedure.

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1. The minimal (graded) rank case

To obtain quantifier elimination for elementary predicates over a free group, our goal is to show that the Boolean algebra of AE sets is invariant under projections. For presentation purposes, we will first present our approach to the analysis of the projection of the Boolean algebra of AE sets assuming the (graded) limit groups that appear in our procedure are of minimal (graded) rank (graded rank 0), and then analyze the general case. We start with the following immediate fact.

LEMMA 1.1: *Let $F_k = < a_1, \dots, a_k >$ be a free group, and let the EA set $EA(w,p)$ be defined as*

$$EA(w,p) = \exists y\ \forall x\ (\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1) \vee \cdots$$
$$\cdots \vee (\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1).$$

Then the projection of $EA(w,p)$, i.e., the set

$$\pi EA(p) = \exists w\ (w,p) \in EA(w,p),$$

is an EA set.

Proof: $\pi EA(p)$ is defined by the EA predicate

$$\pi EA(p) = \exists w, y\ \forall x\ (\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1) \vee \cdots$$
$$\cdots \vee (\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1). \quad \blacksquare$$

Unlike Lemma 1.1, the analysis of projections of a general set in the Boolean algebra of AE sets requires the techniques and results obtained in our previous papers, graded limit groups and their Makanin–Razborov diagrams, the construction of formal solutions and graded formal limit groups and the analysis of iterated quotients of completed resolutions, similar to the one presented in analyzing AE sentences presented in [Se4]. We start with the following reduction of our main goal from general sets in the Boolean algebra of AE sets to AE sets.

LEMMA 1.2: *Let $F_k =< a_1, \dots, a_k >$ be a free group, and let $V(w,p)$ be a set in the Boolean algebra of AE sets over F_k. If the projection of every AE set defined over F_k is in the Boolean algebra of AE sets over F_k, then the projection of $V(w,p)$ is in the Boolean algebra of AE sets.*

Proof: By ([Se3], 3.6) the set $V(w,p)$ is the (finite) union of some AE sets, EA sets and sets which are the intersection of an AE and an EA set. The projection of a union of sets is the union of the projections of the individual sets. By Lemma 1.1 the projection of an EA set is in the Boolean algebra of EA sets and by our assumptions so is the projection of every AE set. Therefore, to prove the lemma we only need to show that under our assumption on the projections of AE sets, the projection of the intersection of an EA set and an AE set is in the Boolean algebra of AE sets.

Let $V(w,p)$ be the intersection between an AE and an EA set. $V(w,p)$ is defined by the predicate

$$\begin{aligned} V(w,p) = \exists w \ (\forall y \ \exists x \ (\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1) \vee \cdots \\ \cdots \vee (\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1))\wedge \\ \wedge(\exists t \ \forall u \ (\Sigma'_1(u,t,w,p,a) = 1 \wedge \Psi'_1(u,t,w,p,a) \neq 1) \vee \cdots \\ \cdots \vee (\Sigma'_{r'}(u,t,w,p,a) = 1 \wedge \Psi'_{r'}(u,t,w,p,a) \neq 1)). \end{aligned}$$

Equivalently

$$\begin{aligned} V(w,p) = \exists w \ \exists t \ \forall y \ \forall u \ \exists x \ \Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1) \vee \cdots \\ \cdots \vee (\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1)\wedge \\ \wedge(\Sigma'_1(u,t,w,p,a) = 1 \wedge \Psi'_1(u,t,w,p,a) \neq 1) \vee \cdots \\ \cdots \vee (\Sigma'_{r'}(u,t,w,p,a) = 1 \wedge \Psi'_{r'}(u,t,w,p,a) \neq 1). \end{aligned}$$

So, $V(w,p)$ is the projection of an AE set, which under our assumption implies that $V(w,p)$ is in the Boolean algebra of AE sets. ∎

Given Lemmas 1.1 and 1.2, the analysis of the projection of the Boolean algebra of AE sets reduces to the analysis of the projection of AE sets.

THEOREM 1.3: *Let $F_k =< a_1, \ldots, a_k >$ be a free group, and let the AE set $AE(w,p)$ be defined as*

$$AE(w,p) = \forall y \; \exists x \; (\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1) \vee \cdots \\ \cdots \vee (\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1).$$

Then the projection of the AE set $AE(w,p)$, i.e., the set

$$EAE(p) = \exists w \; \forall y \; \exists x \; (\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1) \vee \cdots \\ \cdots \vee (\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1),$$

is in the Boolean algebra of AE sets.

For presentation purposes we will first show that the projection of an AE set, $EAE(p)$, is in the Boolean algebra of AE sets in the case $r = 1$. The generalization to arbitrary r is fairly straightforward, and is presented afterwards.

THEOREM 1.4: *Let $F_k =< a_1, \ldots, a_k >$ be a free group, and let $EAE(p)$ be a set defined by the predicate*

$$EAE(p) = \exists w \; \forall y \; \exists x \; \Sigma(x,y,w,p,a) = 1 \wedge \Psi(x,y,w,p,a) \neq 1.$$

Then $EAE(p)$ is in the Boolean algebra of AE sets.

Let F_y be the free group $F_y =< y_1, \ldots, y_\ell >$, and let

$$\psi_1(x,y,w,p,a) = 1, \ldots, \psi_q(x,y,w,p,a) = 1$$

be the defining equations of the system $\Psi(x,y,w,p,a) = 1$. By ([Se2], 1.2), for every $p_0 \in EAE(p)$ there exists some w_0 and a formal solution $x = x_{w_0,p_0}(y,a)$ so that the words corresponding to the defining equations of the system $\Sigma(x_{w_0,p_0}(y,a), y, w_0, p_0, a) = 1$ are trivial in the free group $F_y * F_k =< y, a >$, and the sentence

$$\exists y \; \psi_1(x_{w_0,p_0}(y,a), y, w_0, p_0, a) \neq 1 \wedge \cdots \wedge \psi_q(x_{w_0,p_0}(y,a), y, w_0, p_0, a) \neq 1$$

is a true sentence in F_k.

By the construction of graded formal limit groups presented in section 3 of [Se2], viewing $WP =< w,p >$ as the parameter subgroup, one can associate with the free group F_y and the system of equations $\Sigma(x,y,w,p,a) = 1$ a (canonical) finite collection of graded formal limit groups

$$GFL_1(x,y,w,p,a), \ldots, GFL_r(x,y,w,p,a)$$

so that every formal solution $x = x_{w_0,p_0}(y,a)$ of the system $\Sigma(x,y,w,p,a) = 1$ factors through one of the resolutions of the graded formal Makanin–Razborov diagram of one of the graded formal limit groups

$$GFL_1(x,y,w,p,a), \ldots, GFL_r(x,y,w,p,a).$$

Viewing $WP =< w,p >$ as the parameter subgroup, each graded formal resolution in the graded formal Makanin–Razborov diagrams of the graded formal limit groups $GFL_1(x,y,w,p,a), \ldots, GFL_r(x,y,w,p,a)$ terminates in either a rigid formal limit group of the form $WPRgd(h_R,w,p,a) * F_y$, where $WPRgd(h_R,w,p,a)$ is a rigid (not formal!) graded limit group (with respect to WP), or in a solid formal limit group of the form $WPSld(h_S,w,p,a) * F_y$, where $WPSld(h_S,w,p,a)$ is a solid (not formal!) graded limit group. Note that by ([Se3], 2.5), for each specialization (w_0,p_0) there exists a global bound (independent of the particular specialization (w_0,p_0)) on the number of rigid solutions of the form (h_R,w_0,p_0,a) of any of the rigid graded limit groups $WPRgd(h_R,w,p,a)$, and by ([Se3], 2.9), for each specialization (w_0,p_0) there exists a global bound on the number of strictly solid families of solutions of the form (h_S,w_0,p_0,a) of any of the solid graded limit groups $WPSld(h_S,w,p,a)$.

Let $WPRgd(h_R,w,p,a) * F_y$ be one of the terminating limit groups in the formal graded Makanin–Razborov diagram with respect to the defining parameters $WP =< w,p >$, where the limit group $WPRgd(h_R,w,p,a)$ is rigid (see section 3 of [Se2] for the structure of the graded formal Makanin–Razborov diagram and its resolutions). The tower of modular groups associated with each ungraded formal resolution that terminates in a rigid specialization of the rigid graded limit group $WPRgd(h_R,w,p,a)$, that lies outside its singular locus, is compatible with the tower of modular groups associated with the graded formal resolution that covers this ungraded formal resolution, i.e., the graded formal resolution that terminates in the rigid graded formal limit group $WPRgd(h_R,w,p,a) * F_y$. Therefore, using the tower of modular groups that lie "above" each of the rigid formal graded limit groups $WPRgd(h_R,w,p,a) * F_y$, we can associate a (usually infinite) system of equations (in the variables (h_R,y,w,p) and coefficients in F_k) corresponding to each of the equations in the system $\Psi(x,y,w,p,a) = 1$. By Guba's theorem [Gu], each such infinite system of equations is equivalent to a finite system of equations $\lambda_R(h_R,y,w,p,a) = 1$. Similarly, with each terminating solid formal graded limit group $WPSld(h_S,w,p,a) * F_y$ we can associate a system of equations $\lambda_S(h_S,y,w,p,a) = 1$.

From now on we work with each of the graded formal resolutions $WPGFRes_i(x,y,w,p,a)$ that appears in the graded formal Makanin–Razborov diagrams of $GFL_1,\dots,GFL_r$, and its terminating rigid formal graded limit group $WPRgd_i(h_R,w,p,a) * F_y$, or solid graded formal limit group $WPSld_i(h_S,w,p,a) * F_y$, in parallel, so we may restrict our attention to one of these graded formal resolutions, omit its index and denote it $WPGFRes(x,y,w,p,a)$. Note that each of these formal resolutions is with respect to the parameter subgroup $WP =< w,p >$.

Suppose that the graded formal resolution $WPGFRes(x,y,w,p,a)$ terminates in the rigid graded formal limit group $WPRgd(h_R,w,p,a) * F_y$. Let

$$\lambda WPRGL_1(h_R,y,w,p,a),\dots,\lambda WPRGL_d(h_R,y,w,p,a)$$

be the canonical collection of maximal graded limit groups (with respect to the parameter subgroup $WP =< w,p >$) corresponding to the set of specializations (h_R,y,w,p,a) for which (h_R,w,p,a) is a rigid specialization of $WPRgd(h_R,w,p,a)$, (h_R,y,w,p,a) factors through the graded formal resolution $WPGFRes(x,y,w,p,a)$, and through one of the systems of equations, $\lambda_R(h_R,y,w,p,a) = 1$, associated with the various equations in the system $\Psi(x,y,w,p,a) = 1$.

Similarly, if the graded formal resolution $WPGFRes(x,y,w,p,a)$ terminates in the solid formal graded limit group $WPSld(h_S,w,p,a)*F_y$, we associate with the solid graded limit group $WPSld(h_S,w,p,a)$ and the (finitely many) systems of equations $\lambda_S(h_S,y,w,p,a)$ the canonical collection of maximal graded limit groups:

$$\lambda WPSGL_1(h_S,y,w,p,a),\dots,\lambda WPSGL_d(h_S,y,w,p,a),$$

corresponding to the set of specializations (h_S,y,w,p,a), for which (h_S,w,p,a) is a strictly solid specialization of $WPSld(h_S,w,p,a)$, (h_S,y,w,p,a) factors through the graded formal resolution $WPGFRes(x,y,w,p,a)$, and

$$\lambda_S(h_S,y,w,p,a) = 1$$

(for one of the systems associated with the equations in the system $\Psi(x,y,w,p,a) = 1$).

At this point we need to collect the "remaining" set of specializations of the variables y for each value of our parameters (w,p). Suppose that the terminating graded limit group of the formal graded resolution $WPGFRes(x,y,w,p,a)$ is the rigid formal graded limit group $WPRgd(h_R,w,p,a) * F_y$ or the solid formal graded limit group $WPSld(h_S,w,p,a) * F_y$. With each of the graded limit groups

$$\lambda WPRGL_1(h_R,y,w,p,a),\dots,\lambda WPRGL_d(h_R,y,w,p,a)$$

or

$$\lambda WPRGL_1(h_S,y,w,p,a),\dots,\lambda WPRGL_d(h_S,y,w,p,a)$$

(depending on whether the graded formal resolution $WPGFRes(x,y,w,p,a)$ terminates in a rigid or solid limit group with respect to $WP=<w,p>$) we associate its taut graded Makanin–Razborov diagram with respect to the parameter subgroup $<h_R,w,p>$ or $<h_S,w,p>$ in correspondence. Each graded resolution $\lambda WPGRes(y,h_R,w,p,a)$ (or $\lambda WPGRes(y,h_S,w,p,a)$) in one of the taut graded diagrams of the graded limit groups

$$\lambda WPRGL_1(h_R,y,w,p,a),\dots,\lambda WPRGL_d(h_R,y,w,p,a)$$

(or

$$\lambda WPRGL_1(h_S,y,w,p,a),\dots,\lambda WPRGL_d(h_S,y,w,p,a))$$

terminates in either a rigid graded limit group (with respect to $<h_R,w,p>$ or $<h_S,w,p>$) which we denote

$$\lambda WPRgd(g_R,h_R,w,p,a) \quad (\text{or } \lambda WPRgd(g_R,h_S,w,p,a)),$$

or a solid graded limit group which we denote

$$\lambda WPSld(g_S,h_R,w,p,a) \quad (\text{or } \lambda WPSld(g_S,h_S,w,p,a)).$$

Before continuing with our iterative procedure for analyzing the structure of an EAE set, we use the limit groups constructed in the first step of the procedure to give a first approximation of an EAE set, an approximation which is an EA set. To get the approximation we need to use the entire collection of limit groups $WPRgd_i(h_R,w,p,a),WPSld_i(h_S,w,p,a)$ and

$$\lambda WPRgd_j(g_R,h_R,w,p,a),\lambda WPSld_j(g_S,h_R,w,p,a),\lambda WPRgd_j(g_R,h_S,w,p,a),$$
$$\lambda WPSld_j(g_S,h_S,w,p,a).$$

Lemma 1.5: *Let the set $U_1(p)$ be defined as the union of two sets.*

(1) $p_0\in U_1^R(p)$ *if for some w_0, some index i and some hr_0 (which is a given specialization of the tuple of variables h_R), (hr_0,w_0,p_0) is a rigid specialization of the rigid limit group $WPRgd_i(h_R,w,p,a)$, and for every index j, there exists no specialization gr_0 for which (gr_0,hr_0,w_0,p_0) is a specialization of $\lambda WPRgd_j(g_R,h_R,w,p,a)$ and no specialization gs_0 for which (gs_0,hr_0,w_0,p_0) is a specialization of $WPSld_j(g_S,h_R,w,p,a)$.*

(2) $p_0\in U_1^S(p)$ *if for some w_0, some index i and some hs_0 (which is a given specialization of the tuple of variables h_S), (hs_0,w_0,p_0) is a strictly solid*

specialization of the solid limit group $WPSld_i(h_S,w,p,a)$*, and for every index* j*, there exists no specialization* gr_0 *for which* (gr_0,hs_0,w_0,p_0) *is a specialization of* $\lambda WPRgd_j(g_R,h_S,w,p,a)$ *and no specialization* gs_0 *for which* (gs_0,hs_0,w_0,p_0) *is a specialization of* $WPSld_j(g_S,h_S,w,p,a)$.

(3) *We define* $U_1(p)$ *to be* $U_1(p)=U_1^R(p)\cup U_1^S(p)$.

Then:

(i) $U_1(p)\subset EAE(p)$.

(ii) $U_1(p)$ *is an* EA *set.*

Proof: Both conditions (1) and (2) can be stated by an EA predicate (see section 3 of [Se3]), so $U_1(p)$ is indeed an EA set. If w_0 satisfies either (1) or (2), then the sentence

$$\forall y\ \exists x\ \Sigma(x,y,w_0,p_0,a)=1\wedge\Psi(x,y,w_0,p_0,a)\neq 1$$

is a true sentence by construction, so $U_1(p)\subset EAE(p)$. ∎

To construct the graded formal resolutions $WPGFRes(x,y,w,p,a)$, we have collected all the formal solutions $x_{(w,p)}(y,a)$, for which all the words corresponding to the equations in $\Sigma(x_{(w,p)}(y,a),y,w,p,a)$ represent the trivial words in $F_{a,y}=<a,y>$. By theorem 1.18 of [Se2], if $p_0\in EAE(p)$ then there must exist some "witness" w_0 and a formal solution $x_{(w_0,p_0)}(y,a)$ so that the maximal limit groups corresponding to each of the equations in the system $\Psi(x_{(w_0,p_0)}(y,a),y,w,p,a)=1$ are all proper quotients of the free group $<a,y>=F_k*F_y$. Hence, for every $p_0\in EAE(p)$ there must exist some witness w_0, and a rigid specialization (hr_0,w_0,p_0) of one of the rigid limit groups $WPRgd(h_R,w,p,a)$, or a strictly solid specialization (hs_0,w_0,p_0) of one of the solid limit groups $WPSld(h_S,w,p,a)$, so that every ungraded resolution $\lambda WPGRes(y,hr_0,w_0,p_0,a)$ (or $\lambda WPGRes(y,hs_0,w_0,p_0,a)$) does not correspond to the entire set of y's but rather to a resolution of a limit group which is a proper quotient of the free group $<a,y>=F_k*F_y$.

Therefore, the outcome of the first step of our "trial and error" procedure is a decrease in the complexity (definitions 1.14 and 3.2 in [Se4]) of the ungraded resolutions of (the remaining) y's associated with each $p_0\in EAE(p)$, at least for one rigid or strictly solid specialization (hr_0,w_0,p_0), or (hs_0,w_0,p_0). Each of the next steps of the procedure is meant to sequentially decrease the complexity of the ungraded resolutions of the remaining y's. Once the iterative procedure terminates, we present a second iterative procedure that uses the outcome of the first iterative procedure to sequentially approximate the set $EAE(p)$ by sets which are all in the Boolean algebra of AE sets. Finally, we show that the approximations we construct in the second iterative procedure become identical with the set $EAE(p)$ after finitely many steps. Since the approximations are all in the Boolean algebra of AE sets, this will imply that an EAE set is indeed in the Boolean algebra of AE sets, which finally proves Theorem 1.4.

In this section we present the iterative "trial and error" procedure, and the proof of Theorem 1.4 under the minimal (graded) rank (graded rank 0) assumption, i.e., from now on we will assume that if any of the graded limit groups $\lambda WPGL_j(y,h_R,w,p,a)$ or $\lambda WPGL_j(y,h_S,w,p,a)$ admits an epimorphism τ onto a free group $F_k * F$ where $\tau(<p>) < F_k$, then F is the trivial group. Under the minimal graded rank assumption, the termination of the "trial and error" procedure is based on the analysis of iterative quotients of completed resolutions in the minimal rank case, presented in section 1 of [Se4]. In the next sections we will use the analysis of iterative quotients of completed resolutions in the general case presented in section 4 of [Se4] to get a "trial and error" procedure, hence a proof of Theorem 1.4, omitting the minimal graded rank assumption.

For the continuation of the iterative procedure we will denote (for brevity) each of the limit groups $WPRgd(h_R,w,p,a)$ or $WPSld(h_S,w,p,a)$ as $WPH(h,w,p,a)$, and each of the limit groups

$$\lambda WPRgd(g_R,h_R,w,p,a), \lambda WPRgd(g_R,h_S,w,p,a), \lambda WPSld(g_S,h_R,w,p,a), \lambda WPSld(g_S,h_S,w,p,a)$$

as $WPHG(g,h,w,p,a)$. Our treatment of these limit groups will be conducted in parallel, so we don't keep the indices associated with each of these (finite collection of) limit groups. Also, the rest of our "trial and error" procedure does not depend in an essential way on the type (rigid or solid) of the terminating graded limit groups in the first two steps, hence we do not keep notation for the type of each of these terminating limit groups.

For each tuple (p_0,w_0,h_0,g_0), which is either a rigid or a strictly solid specialization of the terminating limit group $WPHG$, there is an ungraded (well-structured) resolution associated. The associated ungraded resolution depends only on the strictly solid family of the specialization in case the corresponding terminating limit group of the graded formal resolution $WPGFRes$ is solid (i.e., it is the same ungraded resolution for all the specializations that belong to a given strictly solid family). Also, the ungraded resolution may be degenerate, so we separate the finitely many possible types of ungraded resolutions associated with a rigid or strictly solid specialization (p_0, w_0, h_0, g_0) of $WPHG$ according to the stratum of the corresponding singular locus (see section 12 of [Se1]), and continue with each singular stratum separately. With the associated ungraded well-structured resolution, we may associate its completion. Given a tuple (p_0, w_0, h_0, g_0), which is a rigid or a strictly solid specialization of $WPHG$, we collect all the formal solutions $\{x_{(p_0,w_0,h_0,g_0)}(s, z, y, a)\}$ for which the words corresponding to the equations in the system $\Sigma(x_{(p_0,w_0,h_0,g_0)}(s, z, y, a), y, w_0, p_0, a) = 1$ are the trivial words in some closure of the completion of the ungraded resolution associated with the given specialization. Using the construction presented in section 3 of [Se2], and viewing the subgroup $< p, w, h, g >$ as parameters, from the entire collection of formal solutions for all possible specializations (p_0, w_0, h_0, g_0) which are rigid or strictly solid specializations of $WPHG$, we can construct a graded formal Makanin–Razborov diagram, so that any formal solution defined over a closure of (a completion of) an ungraded resolution associated with a rigid or strictly solid specialization of $WPHG$ factors through one of the graded formal Makanin–Razborov resolutions.

Note that by the construction of formal graded limit groups and their associated graded formal Makanin–Razborov diagrams, the collection of maximal graded formal limit groups and their graded formal Makanin–Razborov diagrams associated with a given terminating limit group $WPHG$, depends only on the strictly solid family from which the specialization of $WPHG$ is taken, and not on the particular specialization taken from this strictly solid family.

Let $GFL_1(x, z, y, g_1, h_1, w, p, a), \ldots, GFL_r(x, z, y, g_1, h_1, w, p, a)$ be the maximal graded formal limit groups constructed from the collection of formal solutions associated with the graded limit group $WPHG$. By section 3 of [Se2], with each of the graded formal limit groups there is an associated graded formal Makanin–Razborov diagram with respect to the parameter subgroup $< g_1, h_1, w, p >$. By theorems 3.7 and 3.8 in [Se2], each of the graded formal resolutions in the graded formal Makanin–Razborov diagram associated with each of the graded formal limit groups

$$GFL_1(x, z, y, g_1, h_1, w, p, a), \ldots, GFL_r(x, z, y, g_1, h_1, w, p, a)$$

(with respect to the subgroup $< g_1, h_1, w, p >$) terminates in a group of the form

$$WPHGRgd(h_2^R, g_1, h_1, w, p, a) *_{Term(\hat{s}, g_1, h_1, w, p, a)} GFCl(s, z, y, g_1, h_1, w, p, a),$$

where $WPHGRgd(h_2^R, g_1, h_1, w, p, a)$ is a graded (not formal!) limit group which is rigid with respect to the parameter subgroup $< g_1, h_1, w, p >$, and

$$GFCl(s, z, y, g_1, h_1, w, p, a)$$

is a graded formal closure of the graded resolution $\lambda WPGRes(y, g_1, h_1, w, p, a)$ associated with the graded limit group $WPHG$. We will denote a terminating rigid graded limit group $WPHGRgd(h_2^R, g_1, h_1, w, p, a)$ by $WPHGH^R$.

Alternatively, the terminating graded formal limit group of a graded formal resolution in one of the graded formal Makanin–Razborov diagrams associated with the graded formal limit groups

$$GFL_1(x, z, y, g_1, h_1, w, p, a), \ldots, GFL_r(x, z, y, g_1, h_1, w, p, a)$$

is of the form

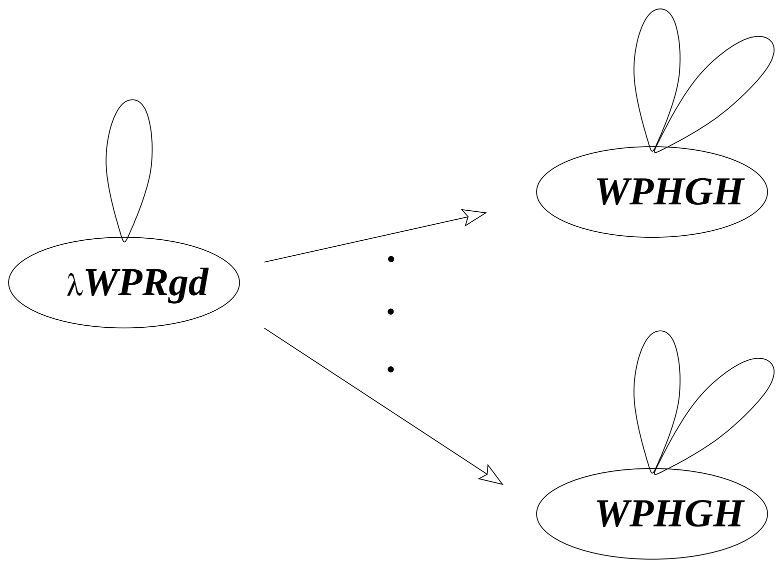
$$WPHGSld(h_2^S, g_1, h_1, w, p, a) *_{Term(\hat{s}, g_1, h_1, w, p, a)} GFCl(s, z, y, g_1, h_1, w, p, a),$$

where $WPHGSld(h_2^S, g_1, h_1, w, p, a)$ is a graded (not formal!) limit group which is solid with respect to the parameter subgroup $< g_1, h_1, w, p >$, and

$$GFCl(s, z, y, g_1, h_1, w, p, a)$$

is a graded formal closure of the graded resolution $\lambda WPGRes(y, g_1, h_1, w, p, a)$ associated with the graded limit group $WPHG$. We will denote the terminating solid graded limit group $WPHGSld(h_2^S, g_1, h_1, w, p, a)$ by $WPHGH^S$.

At this point we continue as in the first step, i.e., we analyze the set of y's for which for all the formal solutions $x = x(s, z, y, h_2, g_1, h_1, w, p, a)$, at least one of the equations in the system $\Psi(x(s, z, y, h_2, g_1, h_1, w, p, a), y, w, p, a) = 1$ holds. Since we have assumed that all the limit groups $WPHG$ are of minimal (graded) rank, we will be able to use the analysis of quotient resolutions presented in section 1 of [Se4], to analyze the resolutions containing the entire set of the remaining y's. In the next section we will modify this part of the iterative procedure in order to prove Theorem 1.4 in the general case, using the analysis of quotient resolutions that appear in section 4 of [Se4].



Let

$$WPHGRgd(h_2^R, g_1, h_1, w, p, a) *_{Term(\hat{s}, g_1, h_1, w, p, a)} GFCl(s, z, y, g_1, h_1, w, p, a)$$

or

$$WPHGSld(h_2^S, g_1, h_1, w, p, a) *_{Term(\hat{s}, g_1, h_1, w, p, a)} GFCl(s, z, y, g_1, h_1, w, p, a)$$

be the terminating rigid or solid limit group (with respect to the parameter subgroup $< g_1, h_1, w, p >$) of the graded formal resolution

$$WPHGFRes(x, s, z, y, g_1, h_1, w, p, a).$$

As we pointed out in the first step, the (formal) modular groups associated with each ungraded formal resolution that is covered by the graded formal resolution

$$WPHGFRes(x, s, z, y, g_1, h_1, w, p, a)$$

that terminates in the rigid or solid graded limit group $WPHGH$, so that the rigid or strictly solid specialization of $WPHGH$ in which the ungraded resolution terminates does not belong to the singular locus, are compatible with the graded formal modular groups of the graded formal resolution $WPHGFRes$. Therefore, using the "tower" of graded formal modular groups associated with the graded formal resolution $WPHGFRes$, a "tower" that lies "above" the terminating rigid or solid graded limit group $WPHGH$, we can associate a (usually infinite) system of equations (in the variables (s,z,y,h_2,g_1,h_1,w,p) and coefficients in F_k) corresponding to each of the equations in the system $\Psi(x,y,w,p,a)=1$, imposed on all specializations of the tuple (x,y,w,p,a) in the same (formal) modular block as a given specialization of the tuple (h_2,g_1,h_1,w,p,a). By Guba's theorem [Gu] this infinite system of equations is equivalent to a finite system of equations $\beta(s,z,y,h_2,g_1,h_1,w,p,a)=1$.

At this stage we need to collect all the remaining y's, i.e., those values of the variables y that satisfy one of the systems of equations $\beta(s,z,y,h_2,g_1,h_1,w,p,a)$ $=1$. As we did in our iterative procedure for validation of a sentence in section 1 of [Se4], our aim is to collect all the remaining y's for all $p_0 \in EAE(p)$, and all possible tuples (w_0,p_0), in finitely many graded resolutions which will be of complexity not bigger than the complexity of the corresponding graded resolution $WPHGRes(y,h_1,g_1,w,p,a)$ with which we have started the second step. To be able to collect all the remaining y's in graded resolutions of complexity bounded by the complexity of the resolution $WPHGRes$ with which we have started the second step, we need to apply the same techniques used for that purpose in section 1 of [Se4], modified slightly to be suitable for the graded set-up.

The collection of all specializations of the variables y that can be extended to a specialization that factors through the graded formal resolution

$$WPHGFRes(x,s,z,y,g_1,h_1,w,p,a)$$

and satisfies the system of equations $\beta(s,z,y,h_2,g_1,h_1,w,p,a)=1$ factors through a canonical collection of maximal graded limit groups

$$QGlim_1(s,z,y,h_2,g_1,h_1,w,p,a),\ldots,QGlim_q(s,z,y,h_2,g_1,h_1,w,p,a).$$

Let $WPHGRes(y,h_1,g_1,w,p,a)$ be the graded resolution (with respect to the parameter subgroup $<h_1,g_1,w,p,a>$) of y's that remained after the first step with which we have started the second step. Our analysis of the well-structured resolutions of the graded limit groups, $QGlim_i$, is conducted in parallel, hence we will omit the index and denote the quotient graded limit group under consideration $QGlim(s,z,y,h_2,g_1,h_1,w,p,a)$.

Let z_{base} be a generating set of the limit group associated with all the levels of the graded formal closure $WPHGFCl(s,z,y,g_1,h_1,w,p,a)$ except the top level (i.e., the distinguished vertex group in the abelian decomposition associated with the top level of the graded formal closure $WPHGFCl$). We will call this set of generators the **basis** of the graded formal closure

$$WPHGFCl(s,z,y,g_1,h_1,p,a).$$

Following the construction of the strict Makanin–Razborov diagram ([Se2], 1.10), we construct the (canonical) strict graded Makanin–Razborov diagram of the graded limit group $QGlim(s,z,y,h_2,g_1,h_1,w,p,a)$ viewed as a graded limit group with respect to the parameter subgroup $< z_{base},h_2,g_1,h_1,w,p >$. Let

$$WPHGHRes_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a),\ldots,$$
$$WPHGHRes_v(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

be the well-structured graded Makanin–Razborov resolutions that appear in the strict graded Makanin–Razborov diagram of the (graded) limit group

$$QRlim(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

with respect to the parameter subgroup $< z_{base},h_2,g_1,h_1,w,p >$, where each graded resolution is terminating in either a rigid or a solid graded limit group (with respect to the parameter subgroup $< z_{base},h_2,g_1,h_1,w,p >$).

We will treat the graded resolutions

$$WPHGHRes_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a),\ldots,$$
$$WPHGHRes_v(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

in parallel, so for the continuation we will restrict ourselves to one of them which we denote $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ for brevity. Let

$$WPHGHRlim(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

be the graded limit group corresponding to the graded resolution

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a).$$

Let $Glim_j(s,z,y,z_{base},h_2,g_2,h_1,w,p,a)$ be the graded limit group that appears in the j-th level of the graded resolution

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a).$$

Let $WPHGRlim(y,h_1,g_1,w,p,a)$ be the graded limit group associated with the graded resolution $WPHGRes(y,g_1,h_1,w,p,a)$ with which we have started the second step. Naturally, there exists a canonical map

$$\tau_j\colon WPHGRlim(y,h_1,g_1,w,p,a) \to Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a).$$

Let Λ_j be the graded quadratic decomposition of

$$Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a),$$

i.e., the graded cyclic decomposition of $Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ obtained from the graded abelian JSJ decomposition of

$$Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

by collapsing all the edges connecting two non-QH subgroups. Let Q be a quadratically hanging subgroup in the graded abelian JSJ decomposition of $WPHGRlim(y,h_1,g_1,w,p,a)$, and let S be the corresponding (punctured) surface. Since the boundary elements of Q are mapped by τ_j to either the trivial element or elliptic elements in Λ_j, the (possibly trivial) cyclic decomposition inherited by $\tau_j(Q)$ from the cyclic decomposition Λ_j can be lifted to a (possibly trivial) cyclic decomposition of the QH subgroup Q of $WPHGRlim(y,h_1,g_1,w,p,a)$, which corresponds to some decomposition of the (punctured) surface S along a (possibly trivial) collection of disjoint non-homotopic s.c.c. Let $\Gamma_j(Q)$ be the corresponding cyclic decomposition of the QH subgroup Q, and let $\Gamma_j(S)$ be a maximal associated collection of non-homotopic essential s.c.c. on S. Note that, by construction, every s.c.c. from the defining collection of $\Gamma_j(S)$ is mapped by τ_j to either a trivial element or to an elliptic element in Λ_j.

Lemma 1.6:

(i) *Every s.c.c. from the collection $\Gamma_j(S)$ is mapped by τ_j to either the trivial element or to a non-trivial elliptic element in Λ_j.*

(ii) *Every non-separating s.c.c. on the surface S is mapped to a non-trivial element by the homomorphism τ_j.*

(iii) *Let Q' be a quadratically hanging subgroup in Λ_j and let S' be the corresponding (punctured) surface. If τ_j maps non-trivially a connected subsurface of $S \setminus \Gamma_j(S)$ into Q', then $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$. Furthermore, in this case τ_j maps the fundamental group of a subsurface of S into a finite index subgroup of Q'.*

Proof: Identical to the proof of lemma 1.3 in [Se4]. ∎

Part (iii) of Lemma 1.6 bounds the topological complexity of those QH subgroups Q' that appear in the graded abelian JSJ decompositions associated with the various levels of the graded resolution

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

into which a QH subgroup Q that appears in the graded abelian JSJ decomposition of $WPHGRlim(y,h_1,g_1,w,p,a)$ is mapped non-trivially. To show that Lemma 1.6 can be applied to bound the topological complexity of all the QH subgroups Q' that appear in the graded abelian decompositions associated with the various levels of $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$, we need the following proposition.

PROPOSITION 1.7: *Let $Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ be the graded limit group that appears in the j-th level of the graded resolution*

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a),$$

and let Q' be a QH subgroup that appears in the graded JSJ decomposition associated with $Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ and S' be its corresponding punctured surface. Let τ_j be the natural map

$$\tau_j\colon WPHGRlim(y,h_1,g_1,w,p,a) \to Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a).$$

Then there exists a QH subgroup Q in the graded abelian JSJ decomposition of $WPHGRlim(y,h_1,g_1,w,p,a)$ with corresponding punctured surface S, so that a subsurface S_1 of the punctured surface S is mapped by τ_j into a finite index subgroup of a conjugate of Q'. In particular, $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$.

Proof: Identical with the proof of proposition 1.5 in [Se4]. ∎

Definition 1.8: Let Q be a quadratically hanging subgroup in the graded abelian JSJ decomposition of $WPHGRlim(y,h_1,g_1,w,p,a)$ and let S be its corresponding (punctured) surface. The QH subgroup Q (and the corresponding surface S) is called **surviving** if for some level j, there exists some quadratically hanging subgroup Q' in Λ_j, the graded abelian JSJ decomposition of $Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$, with corresponding surface S', so that τ_j maps Q non-trivially into Q', $genus(S') = genus(S)$ and $\chi(S') = \chi(S)$.

By definition, if Q is a non-surviving QH subgroup in the graded abelian JSJ decomposition of $WPHGRlim(y,h_1,g_1,w,p,a)$, then every QH subgroup Q' in every level of the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ into which a subsurface of Q is mapped non-trivially has either a strictly lower genus or a strictly smaller (absolute value of the) Euler characteristic than that of the QH subgroup Q. This would eventually "force" the complexity of the graded resolution

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

to be bounded by the complexity of the resolution $WPHGRes(y,h_1,g_1,w,p,a)$ once one is able to "isolate" the surviving surfaces. This is the purpose of the following theorem.

Theorem 1.9: *Let $Q_1,\dots,Q_r$ be the surviving QH subgroups in the graded abelian JSJ decomposition of $WPHGRlim(y,h_1,w,p,a)$. Then the graded resolution*

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

can be replaced by finitely many graded resolutions, each composed from two consecutive parts. The first part is a graded resolution of

$$QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$$

with respect to the parameter subgroup $< z_{base},h_2,g_1,h_1,w,p,Q_1,\dots,Q_r >$, which we denote $WPHGHRes(s,z,y,(z_{base},h_2,g_1,h_1,w,p,Q_1,\dots,Q_r),a)$. The second part is a one-step resolution that maps the rigid (solid) terminal graded limit group of $WPHGHRes(s,z,y,(z_{base},h_2,g_1,h_1,w,p,Q_1,\dots,Q_r),a)$ to the rigid (solid) terminal graded limit group of the resolution

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a).$$

The two consecutive parts of the graded resolution have the following properties:

1. *The graded decomposition corresponding to the second part of the resolution contains a vertex stabilized by the terminal rigid (solid) graded limit group of the resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ connected to r' surviving QH subgroups $Q_{i_1},\dots,Q_{i_{r'}}$, for some $r'\leq r$, and $1\leq i_1<i_2<\dots<i_{r'}\leq r$.*
2. *If the terminal graded limit group of the graded resolution*

$$WPHGHRes(s,z,y,(z_{base},h_2,g_1,h_1,w,p,Q_1,\dots,Q_r),a)$$

is rigid (solid), so is the terminal graded limit group of the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$. Furthermore, if they *are both solid, their principal graded decompositions are in one-to-one correspondence, i.e., the decomposition differ only in the stabilizer of one vertex, the vertex stabilized by* $< z_{base}, h_2, g_1, h_1, w, p, Q_1, \dots, Q_r >$ *in the first part, and by* $< z_{base}, h_2, g_1, h_1, w, p >$ *in the second part.*

Proof: Identical to the proof of theorem 1.7 of [Se4]. ∎

We continue the analysis of the limit group $QGlim(s,z,y,h_2,g_1,h_1,w,p,a)$ by reducing the set of defining parameters sequentially. Recall that in order to obtain the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ we used the subgroup $< z_{base} >$ as the parameter subgroup, where $< z_{base} >$ is the limit group associated with all the levels of the graded formal closure $WPHGHFCl(s,z,y,h_2,g_1,h_1,w,p,a)$ except the highest level. For the next step of the analysis we take z^2_{base} as the defining parameters, where z^2_{base} generate the limit group associated with all the levels of the graded formal closure $WPHGHFCl(s,z,y,h_2,g_1,h_1,w,p,a)$ except for the two highest levels.

Let $T_1(s,z,y,h_2,g_1,h_1,w,p,a)$ be the terminal rigid or solid graded limit group in the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ with respect to the parameter subgroup $< z_{base} >$. From the collection of rigid (strictly solid) specializations of $T_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$, that are obtained (using our shortening procedure) from specializations that factor through the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$, we construct the graded strict Makanin–Razborov diagram of $T_1(s,z,y,h_2,g_1,h_1,w,p,a)$, viewed as a graded limit group with respect to the parameter subgroup $< z^2_{base} >$. Let

$$WPHGHRes_1(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a), \dots, \quad WPHGHRes_m(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

be the resolutions that appear in the strict graded Makanin–Razborov diagram of the (graded) limit group $T_1(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ with respect to the parameter subgroup $< z^2_{base} >$, where each graded resolution terminates in either a rigid or a solid graded limit group (with respect to the parameter subgroup $< z^2_{base} >$).

We will treat the graded resolutions

$$WPHGHRes_i(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

in parallel, so we will restrict ourselves to one of them which we denote

$$WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

for brevity. Let $Glim(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ be the graded limit group corresponding to the graded resolution

$$WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a).$$

If the subgroup $Glim(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ is a proper quotient of the subgroup $T_1(s,z,y,h_2,g_1,h_1,w,p,a)$, we need to modify the graded resolution

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

so that it becomes a strict graded Makanin–Razborov resolution terminating with the limit group $Glim(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$. Note that by modifying the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$, we may need to replace the quotient limit group $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$ or one of the groups $Glim_j(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ that appear in one of the levels of the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ by a proper quotient of itself.

Let $Glim_j(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ be a graded limit group that appears in the j-th level of the graded resolution

$$WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a).$$

Let $Zlim(z_{base})$ be the limit group generated by z_{base} in the graded formal closure $WPHGHFCl(s,z,y,h_2,g_1,h_1,w,p,a)$. Naturally, there exists a canonical map

$$\tau_j\colon Zlim(z_{base}) \to Glim_j(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a).$$

Let Λ_j be the graded quadratic decomposition of

$$Glim_j(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a),$$

i.e., the graded cyclic decomposition of $Glim_j(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ obtained from the abelian graded JSJ decomposition of

$$Glim_j(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

by collapsing all the edges connecting two non-QH subgroups. Let Q be a quadratically hanging subgroup in the abelian graded JSJ decomposition of $Zlim(z_{base})$ and let S be the corresponding (punctured) surface. Since the boundary elements of Q are mapped by τ_j to elliptic elements in Λ_j, the (possibly trivial) cyclic decomposition inherited by $\tau_j(Q)$ from the cyclic decomposition

Λ_j can be lifted to a (possibly trivial) cyclic decomposition of the QH subgroup Q of $Zlim(z_{base})$, which corresponds to some decomposition of the (punctured) surface S along a (possibly trivial) collection of disjoint non-homotopic s.c.c. Let $\Gamma_j(Q)$ be the corresponding cyclic decomposition of the QH subgroup Q, and let $\Gamma_j(S)$ be a maximal associated collection of non-homotopic essential s.c.c. on S. Note that by construction, every s.c.c. from the defining collection of $\Gamma_j(S)$ is mapped by τ_j to either a trivial element or to an elliptic element in Λ_j.

LEMMA 1.10:

(i) *Every s.c.c. from the collection $\Gamma_j(S)$ is mapped by τ_j to either the trivial element or to a non-trivial elliptic element in Λ_j.*

(ii) *Every non-separating s.c.c. on the surface S is mapped to a non-trivial element by the homomorphism τ_j.*

(iii) *Let Q' be a quadratically hanging subgroup in Λ_j and let S' be the corresponding (punctured) surface. If τ_j maps non-trivially a connected subsurface of $S \setminus \Gamma_j(S)$ into Q', then $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$. Furthermore, in this case τ_j maps the fundamental group of a subsurface of S into a finite index subgroup of Q'.*

Proof: Identical to the proof of lemma 1.3 in [Se4]. ∎

The precise statement of Proposition 1.7 is not valid for the graded resolution $WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$. If Q' is a QH subgroup that appears in an abelian decomposition associated with one of the levels of the graded resolution $WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$, then it is not true that there exists some QH subgroup Q in the principal graded JSJ decomposition of $Zlim(z_{base})$, so that the fundamental group of a subsurface of the punctured surface corresponding to Q is mapped non-trivially into Q'. However, we can still define **surviving surfaces**.

Definition 1.11: Let Q be a quadratically hanging subgroup in the JSJ decomposition of $Zlim(z_{base})$ and let S be its corresponding (punctured) surface. The QH subgroup Q (and the corresponding surface S) is called **surviving** if for some level j there exists some quadratically hanging subgroup Q' in Λ_j, the JSJ decomposition of $Glim_j(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$, with corresponding surface S', so that τ_j maps Q non-trivially into Q', $genus(S') = genus(S)$ and $\chi(S') = \chi(S)$.

To control the “complexity” of the graded resolution

$$WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

we need to “isolate” the surviving QH subgroups. This can be done in a similar way to Theorem 1.9.

THEOREM 1.12: *Let $Q_1,\ldots,Q_r$ be the surviving QH subgroups in the JSJ decomposition of $Zlim(z_{base})$. Then the graded resolution*

$$WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

can be replaced by finitely many graded resolutions, each composed from two consecutive parts. The first part is a graded resolution of $T_1(s,z,y,a)$ with respect to the parameter subgroup $< z^2_{base},Q_1,\ldots,Q_r >$, which we denote

$$WPHGHRes(s,z,y,(z^2_{base},h_2,g_1,h_1,w,p,Q_1,\ldots,Q_r),a).$$

The second part is a one step resolution that maps the rigid (solid) terminal graded limit group of

$$WPHGHRes(s,z,y,(z^2_{base},h_2,g_1,h_1,w,p,Q_1,\ldots,Q_r),a)$$

to the rigid (solid) terminal graded limit group of the resolution

$$WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a).$$

The two consecutive parts of the graded resolution have the following properties:

1. *The graded decomposition corresponding to the second part of the resolution contains a vertex stabilized by the terminal rigid (solid) graded limit group of the resolution $WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ connected to the r' surviving QH subgroups $Q_{i_1},\ldots,Q_{i_{r'}}$, for some $r' \leq r$, and $1 \leq i_1 < i_2 < \cdots < i_{r'} \leq r$.*
2. *If the terminal graded limit group of the graded resolution*

$$WPHGHRes(s,z,y,(z^2_{base},h_2,g_1,h_1,w,p,Q_1,\ldots,Q_r),a)$$

is rigid (solid), so is the terminal graded limit group of the graded resolution $GRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$. Furthermore, if they are both solid, their principal graded decompositions are in one-to-one correspondence, i.e., the decomposition differs only in the stabilizer of one vertex, *the vertex stabilized by $< z_{base}^2, Q_1, \ldots, Q_r >$ in the first part and by the subgroup $< z_{base}^2 >$ in the second part.*

Proof: Identical to the proof of theorem 1.7 in [Se4]. ∎

Given Lemma 1.10 and Theorem 1.12, to complete the analysis of the structure of the graded resolution $WPHGHRes(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$ we still need an appropriate analogue of Proposition 1.7, i.e., we need to associate every QH subgroup that appears in one of the abelian graded JSJ decompositions of the different levels of the graded resolution

$$WPHGHRes(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$$

with a QH subgroup of either the graded JSJ decomposition of $Zlim(z_{base})$ or a QH subgroup that appears in the abelian graded JSJ decomposition of the terminal subgroup $T_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ of the graded resolution

$$WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

in case $T_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ is solid. We divide the final analysis of $WPHGHRes(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$ into two cases depending on

$$T_1(s,z,y,h_2,g_1,h_1,w,p,a)$$

being rigid or solid.

Proposition 1.13: *Suppose that the terminal subgroup*

$$T_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

of the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ is rigid. Let Q' be a QH subgroup that appears in the j-th level abelian graded JSJ decomposition of the graded resolution

$$WPHGHRes(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$$

and let S' be its corresponding surface. Then:

(i) *There exists a QH subgroup Q in the graded abelian JSJ decomposition of $Zlim(z_{base})$, with an associated surface S, so that the map τ_j: $Zlim(z_{base}) \to Glim_j(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$ maps the fundamental group of a subsurface of S onto a subgroup of finite index of Q'.*

(ii) *$genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$.*

(iii) *genus*(S') = *genus*(S) *and* $|\chi(S')| = |\chi(S)|$ *if and only if* Q *is a surviving* QH *subgroup. In this last case,* j *is the bottom level of the graded resolution* $GRes(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$, *and* Q *is mapped to the vertex stabilized by* $< z_{base}^2 >$ *by the maps*

$$\tau_j\colon Zlim(z_{base}) \to Glim_j(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$$

in all the levels above the bottom one.

Proof: Identical to the proof of proposition 1.11 in [Se4]. ∎

Proposition 1.14: *Suppose that the terminal subgroup*

$$T_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$$

of the graded resolution $WPHGHRes(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$ *is solid. Let* Q' *be a* QH *subgroup that appears in the* j*-th level abelian graded JSJ decomposition of the graded resolution*

$$WPHGHRes(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a).$$

Then:

(i) *There exists a* QH *subgroup* Q *with an associated surface* S *which is either:*

   (1) *a* QH *subgroup in the (graded) JSJ decomposition of* $Zlim(z_{base})$, *so that the map*

$$\tau_j\colon Zlim(z_{base}) \to Glim_j(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$$

   maps the fundamental group of a subsurface of S *onto a subgroup of finite index of* Q'*; or*

   (2) *a* QH *subgroup in the (graded) JSJ decomposition of (the solid graded limit group)* $T_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a)$, *so that the map*

$$\eta_j\colon T_1(s,z,y,z_{base},h_2,g_1,h_1,w,p,a) \to Glim_j(s,z,y,z_{base}^2,h_2,g_1,h_1,w,p,a)$$

   maps the fundamental group of a subsurface of S *onto a subgroup of finite index of* Q'.

(ii) *genus*$(S') \leq$ *genus*(S) *and* $|\chi(S')| \leq |\chi(S)|$.

(iii) *If Q is a QH subgroup in the JSJ decomposition of $Zlim(z_{base})$ and $genus(S') = genus(S)$ and $|\chi(S')| = |\chi(S)|$, then Q is a surviving QH subgroup. In this last case, j is the bottom level of the graded resolution*

$$WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

and Q is mapped to the vertex stabilized by $< z^2_{base} >$ by the maps

$$\tau_j \colon Zlim(z_{base}) \to Glim_j(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

in all the levels above the bottom one.

Proof: Identical to the proof of proposition 1.12 in [Se4]. ∎

We continue the analysis of the limit group $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$ by further reducing the group of parameters sequentially. Recall that in order to obtain the graded resolution $WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ we first used the subgroup $< z_{base} >$ as the parameter subgroup, where z_{base} is a generating set of the limit group associated with all the levels of the graded formal closure $WPHGHFCl(s,z,y,h_2,g_1,h_1,w,p,a)$ except the limit group associated with the highest level, and then used the subgroup $< z^2_{base} >$ as the parameter subgroup, where z^2_{base} generates the limit group associated with all the levels of the graded formal closure $WPHGHFCl(s,z,y,h_2,g_1,h_1,w,p,a)$ except for the two highest levels. To continue the analysis of the resolutions of the limit group $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$, we set z^{ℓ}_{base} to be a generating set of the limit group associated with all levels of the graded formal closure $WPHGHFCl(s,z,y,h_2,g_1,h_1,w,p,a)$ except for the ℓ highest levels.

Let $T_2(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ be the terminal rigid or solid graded limit group in the graded resolution $WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ with respect to the parameter subgroup $< z^2_{base} >$. We continue the graded resolution $WPHGHRes(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$ by viewing

$$T_2(s,z,y,z^2_{base},h_2,g_1,h_1,w,p,a)$$

as a graded limit group with respect to the parameter subgroup $< z^3_{base} >$, the obtained terminal rigid or solid graded limit group as a graded limit group with respect to the parameter subgroup $< z^4_{base} >$ and so on, until we exclude all the limit groups associated with the various levels of the graded formal closure $WPHGHFCl(s,z,y,h_2,g_1,h_1,w,p,a)$, except the bottom (base) level. Clearly, Lemma 1.10 and Theorem 1.12 remain valid for all the steps of the procedure.

Note that the final resolution we obtain terminates in a graded limit group which is rigid or solid with respect to the parameter subgroup $< h_2, g_1, h_1, w, p >$. We denote this terminating limit group $T(s, z, y, h_2, g_1, h_1, w, p, a)$. Also, note that the described procedure produces (canonically) finitely many such resolutions of the limit group $QRlim(s, z, y, h_2, g_1, h_1, w, p, a)$ or of a proper quotient of it.

To get the graded resolutions $WPHGHRes(s, z, y, h_2, g_1, h_1, w, p, a)$ we have considered the subgroup $< h_2, g_1, h_1, w, p >$ as the parameter subgroup. To complete the analysis of the set of the remaining y's we need to further decrease the parameter subgroup to be the subgroup $WP =< w, p >$. We do that by continuing the resolution $WPHGH$ with each of the graded resolutions that appear in the graded Makanin–Razborov diagram of the terminal limit group of the resolution $WPHGH$ with respect to the parameter subgroup $WP =< w, p >$. This last technical change of the group of parameters is done for convenience (and is not really necessary), and it does not change the ungraded resolutions that are "covered" by the obtained graded resolution. In particular, it does not change the complexity of the obtained graded resolutions.

The canonical (finite) collection of graded resolutions

$$WPHGHRes(s, z, y, h_2, g_1, h_1, w, p, a)$$

produced by the procedure described above contain the entire set of the remaining y's. To be able to continue with our iterative procedure and collect all the formal solutions defined over closures of the various multi-graded resolutions we have constructed, we need these multi-graded resolutions to be well-structured.

PROPOSITION 1.15: *The graded resolutions*

$$WPHGHRes(s, z, y, h_2, g_1, h_1, w, p, a)$$

constructed by our procedure are well-structured (graded) resolutions.

Proof: Identical with the proof of theorem 1.13 of [Se4]. ∎

Having constructed the graded resolutions $WPHGH(s, z, y, h_2, g_1, h_1, w, p, a)$ that contain all the "remaining" specializations of the variables y, we are ready to define their **complexity**.

Definition 1.16: Let $WPHGHRes(s, z, y, h_2, g_1, h_1, w, p, a)$ be one of the (well-structured) graded resolutions constructed by the procedure for the collection of the remaining y's. By construction, the complexities of the completions of all non-degenerate ungraded resolutions covered by the graded resolution

$$WPHGHRes(s, z, y, h_2, g_1, h_1, w, p, a)$$

are identical (see definition 1.14 of [Se4] for the definition of the complexity of an ungraded completed resolution). We set the **complexity** of the graded resolution

$$WPHGHRes(s,z,y,h_2,g_1,h_1,w,p,a),$$

denoted $Cmplx(WPHGHRes(s,z,y,h_2,g_1,h_1,w,p,a))$, to be the complexity of the completion of a non-degenerate ungraded resolution covered by the multi-graded resolution $WPHGHRes(s,z,y,h_2,g_1,h_1,w,p,a)$.

In order to ensure the termination of our "trial and error" procedure, we need the complexities of the graded resolutions obtained in its different steps to decrease.

THEOREM 1.17: *Let $WPHRes(y,g_1,h_1,w,p,a)$ be one of the resolutions obtained in the first step of our iterative procedure, and suppose that the graded resolution $WPHRes(y,h_1,g_1,w,p,a)$ does not correspond to the entire free group F_y we have started with. Let*

$$WPHGHRes(s,z,y,h_2,g_1,h_1,w,p,a)$$

be one of the multi-graded resolutions obtained in the second step of our iterative procedure staring with the graded resolution $WPHRes(y,g_1,h_1,w,p,a)$. Then

$$Cmplx(WPHGHRes(s,z,y,h_2,g_1,h_1,w,p,a)) \\ \leq Cmplx(WPHRes(y,g_1,h_1,w,p,a))$$

and in case of equality the two graded resolutions are compatible, i.e., the abelian decompositions that are associated with the two resolutions along their various levels are in one-to-one correspondence, and the vertex groups in the abelian decompositions associated with the resolution $WPHRes$ are mapped into corresponding vertex groups in the abelian decompositions associated with the obtained resolution $WPHGHRes$.

Furthermore, suppose that a specialization p_0 of the defining parameter p satisfies $p_0 \in EAE(p)$, and w_0 is a specialization of the variables w for which the sentence

$$\forall y \quad \exists x \quad \Sigma(x,y,w_0,p_0,a) = 1 \wedge \Psi(x,y,w_0,p_0,a) \neq 1$$

is a true sentence over the free group F_k. Then there exists a specialization h_0^1 of the variables h_1 for which:

(i) *For every (rigid or strictly solid) specialization g_0^1 of the variables g_1, its corresponding ungraded resolution $WPHGRes(y,g_0^1,h_0^1,w_0,p_0,a)$, that* *is covered by one of the graded resolutions* $WPHGRes(y,g_1,h_1,w,p,a)$, *does not correspond to the entire free group* F_y.

(ii) *Let* $WPHG(g_1,h_1,w,p,a)$ *be the (rigid or solid) terminal graded limit group of one of the graded resolutions* $WPHGRes(y,g_1,h_1,w,p,a)$, *and let* (g_0^1,h_0^1,w_0,p_0,a) *be a rigid or strictly solid specialization of* $WPHG(g_1,h_1,w,p,a)$. *Then there exists a specialization* h_0^2 *of the variables* h_2 *so that every ungraded resolution*

$$WPHGH(s,z,y,h_0^2,g_0^1,h_0^1,w_0,p_0,a)$$

satisfies

$$Cmplx(WPHGHRes(s,z,y,h_0^2,g_0^1,h_0^1,w_0,p_0,a)) < Cmplx(WPHGRes(y,g_0^1,h_0^1,w_0,p_0,a)).$$

Proof: By the proof of theorem 1.15 in [Se4]

$$Cmplx(WPHGHRes(s,z,y,h_1,w,p,\hat{h}_2,\hat{g}_1,a)) \leq Cmplx(WPHGRes(y,g_1,h_1,w,p,a)),$$

and in case of equality the two graded resolutions are compatible, i.e., the abelian decompositions that are associated with the two resolutions along their various levels are in one-to-one correspondence, and the vertex groups in the abelian decompositions associated with the resolution $WPHRes$ are mapped into corresponding vertex groups in the abelian decompositions associated with the obtained resolution $WPHGHRes$.

Suppose that the specialization p_0 of the defining parameter p satisfies $p_0 \in EAE(p)$, and w_0 is a specialization of the variables w for which the sentence

$$\forall y \quad \exists x \quad \Sigma(x,y,w_0,p_0,a)=1 \wedge \Psi(x,y,w_0,p_0,a) \neq 1$$

is a true sentence over the free group F_k. By the construction of the formal resolutions $WPFRes(x,y,w,p,a)$ constructed in the first step of our iterative procedure, every formal solution $x=x_{(w_0,p_0)}(y,a)$ (for which

$$\Sigma(x_{(w_0,p_0)}(y,a),y,w,p,a)=1$$

in the free group $F_k * F_y$) factors through at least one of the formal resolutions $WPFRes(x,y,w,p,a)$. By theorem 1.2 of [Se2], there exists a formal solution $x=x_{(w_0,p_0)}(y,a)$ for which $\Sigma(x_{(w_0,p_0)}(y,a),y,w,p,a)=1$ in the free group $F_k * F_y$ and there exists some specialization y_0 of the variables y for which

$$\Psi(x_{(w_0,p_0)}(y_0,a),y_0,w_0,p_0,a) \neq 1.$$

Suppose that this last formal solution factors through the formal resolution $WPFRes(x,y,w,p,a)$ that terminates in the rigid or solid graded limit group (with respect to the parameter subgroup $< w,p >$) $WPH(h_1,w,p,a)$, and let (h_0^1,w_0,p_0) be the rigid or strictly solid specialization corresponding to the ungraded formal resolution containing the formal solution $x = x_{(w_0,p_0)}(y,a)$. Since there exists a specialization y_0 of the variables y for which

$$\Psi(x_{(w_0,p_0)}(y_0,a),y_0,w_0,p_0,a) \neq 1,$$

every ungraded resolution $WPHRes(y,g_0^1,h_0^1,w_0,p_0,a)$ covered by one of the graded resolutions $WPHRes(y,g_1,h_1,w,p,a)$ does not correspond to the entire free group F_y, and we get part (i) of the theorem.

Let $WPHRes(y,g_0^1,h_0^1,w_0,p_0,a)$ be an ungraded resolution that is covered by one of the graded resolutions $WPHRes(y,g_1,h_1,w,p,a)$ with rigid or solid terminal graded limit group $WPHG(g_1,h_1,w,p,a)$, and suppose (g_0^1,h_0^1,w_0,p_0) is the specialization of $WPHG(g_1,h_1,w,p,a)$ corresponding to the ungraded resolution $WPHRes(y,g_0^1,h_0^1,w_0,p_0,a)$. By the construction of the formal resolutions $WPHGFRes(x,s,z,y,g_1,h_1,w,p,a)$ constructed in the second step of our iterative procedure, every formal solution $x = x_{(g_0^1,h_0^1,w_0,p_0)}(s,z,y,a)$, for which $\Sigma(x_{(g_0^1,h_0^1,w_0,p_0)}(s,z,y,a),y,w,p,a) = 1$ in some closure of the ungraded resolution $WPHRes(y,g_0^1,h_0^1,w_0,p_0,a)$, corresponding to the specialization (g_0^1,h_0^1,w_0,p_0), factors through at least one of the formal resolutions $WPHGFRes(x,s,z,y,g_1,h_1,w,p,a)$. By theorem 1.18 of [Se2], there exists a formal solution $x = x_{(g_0^1,h_0^1,w_0,p_0)}(s,z,y,a)$ for which

$$\Sigma(x_{(g_0^1,h_0^1,w_0,p_0)}(s,z,y,a),y,w,p,a) = 1$$

over the corresponding closure of the ungraded resolution

$$WPHRes(y,g_0^1,h_0^1,w_0,p_0,a)$$

that corresponds to the specialization (g_0^1,h_0^1,w_0,p_0) of $WPHG(g_1,h_1,w,p,a)$, and for some specialization $(s_0,z_0,y_0,g_0^1,h_0^1,w_0,p_0)$ that factors through the closure $Cl(WPHGRes)(s,z,y,g_0^1,h_0^1,w_0,p_0,a)$:

$$\Psi(x_{(g_0^1,h_0^1,w_0,p_0)}(s_0,z_0,y_0,a),y_0,w_0,p_0,a) \neq 1.$$

Suppose that this last formal solution factors through the graded formal resolution

$$WPHGFRes(x,y,g_1,h_1,w,p,a)$$

that terminates in the rigid or solid graded limit group (with respect to the parameter subgroup $< g_1, h_1, w, p >$) $WPHGH(h_2, g_1, h_1, w, p, a)$, and let $(h_0^2, g_0^1, h_0^1, w_0, p_0)$ be the rigid or strictly solid specialization corresponding to the ungraded formal resolution containing the formal solution

$$x = x_{(g_0^1, h_0^1, w_0, p_0)}(s, z, y, a).$$

Since there exists a specialization $(s_0, z_0, y_0, g_0^1, h_0^1, w_0, p_0)$ that factors through the closure $Cl(WPHRes)(s, z, y, g_0^1, h_0^1, w_0, p_0, a)$ for which

$$\Psi(x_{(g_0^1, h_0^1, w_0, p_0)}(s_0, z_0, y_0, w_0, p_0, a) \neq 1,$$

theorem 1.15 of [Se4] implies that every ungraded resolution

$$WPHGHRes(s, z, y, h_0^2, g_0^1, h_0^1, w_0, p_0, a)$$

covered by one of the graded resolutions $WPHGHRes(s, z, y, h_2, g_1, h_1, w, p, a)$ satisfies

$$Cmplx(WPHGHRes(s, z, y, h_0^2, g_0^1, h_0^1, w_0, p_0, a)) \\ < Cmplx(WPHGRes(y, g_0^1, h_0^1, w_0, p_0, a))$$

which proves part (ii) of the theorem. ∎

Given the rigid or solid limit groups $WPHG(g_1, h_1, w, p, a)$ and the graded resolutions associated with each of their rigid or strictly solid specializations with corresponding non-degenerate ungraded resolutions, we have collected all the formal solutions, defined over some closures of these graded resolutions, in a finite collection of graded formal resolutions $WPHGFRes(x, y, g_1, h_1, w, p, a)$, each terminating in a rigid or solid graded limit group

$$WPHGH(h_2, g_1, h_1, w, p, a).$$

Given the set of rigid or strictly solid specializations of

$$WPHGH(h_2, g_1, h_1, w, p, a)$$

with corresponding non-degenerate ungraded formal closures, we used the "towers" of graded formal modular groups associated with the graded formal resolution $WPHGHFRes(x, z, y, h_2, g_1, h_1, w, p, a)$ to get a system of equations imposed on the remaining modular block of (the remaining) y's associated with each of the rigid or strictly solid specializations of the limit group

$$WPHGH(h_2, g_1, h_1, w, p, a).$$

Given this system of equations, we have used an iterative procedure to construct a finite set of (well-structured) graded resolutions

$$WPHGHRes(s,z,y,h_2,g_1,h_1,w,p,a)$$

each terminating in a rigid or solid graded limit group

$$WPHGHG(g_2,h_2,g_1,h_1,w,p,a).$$

These graded resolutions contain the entire set of the remaining y's for each specialization of the tuple (w,p). By Theorem 1.17, the complexity of each of the multi-graded resolutions $WPHGHRes(s,z,y,h_2,g_1,h_1,w,p,a)$ is bounded above by the complexity of the graded resolution $WPHGRes(y,h_1,g_1,w,p,a)$ from which it was constructed, and if $p_0 \in EAE(p)$ there must exist some specialization w_0 and a specialization h_0^1 of the variables h_1 so that for every strictly solid or rigid specialization (g_0^1,h_0^1,w_0,p_0,a) of $WPHG(g_1,h_1,w,p,a)$ there exists a specialization h_0^2 of the variables h_2, so that the complexity of every ungraded resolution $WPHGH(s,z,y,h_0^2,g_0^1,h_0^1,w_0,p_0,a)$ has strictly smaller complexity than the complexity of the ungraded resolution

$$WPHGRes(y,g_0^1,h_0^1,w_0,p_0,a).$$

The continuation of our "trial and error" procedure is defined iteratively, where at each step we continue only with those graded resolutions that have strictly smaller complexity than the graded resolutions they are associated with in the previous step. Given a graded resolution $WP(HG)^{n-1}HRes$ with terminating rigid or solid graded limit group $WP(HG)^n$, and the set of rigid or strictly solid specializations of a limit group $WP(HG)^n$ with non-degenerate associated ungraded resolutions, we collect all the formal solutions defined over a closure of an ungraded resolution associated with each of the rigid or strictly solid specializations of the limit group $WP(HG)^n$, and obtain finitely many formal multi-graded resolutions $WP(HG)^nFRes$, each terminating in a rigid or solid graded limit group $WP(HG)^nH$.

Given one of the graded formal resolutions $WP(HG)^nFRes$ and its terminating rigid or solid graded limit group $WP(HG)^nH$, we use the "tower" of graded formal modular groups associated with the graded formal resolution to get a system of equations imposed on the remaining set of y's associated with each of the specializations of the terminating limit group $WP(HG)^nH$. Given this system of equations we use the iterative procedure presented above to construct a finite set of (well-structured) graded resolutions (which we denote $WP(HG)^nHRes$) that terminate in a rigid or solid graded limit group $WP(HG)^{n+1}$, that contain the set of y's remaining after the first $n+1$ steps of our "trial and error" procedure. Theorem 1.17 remains valid for all the steps of the iterative procedure, so the complexity of each of the graded resolutions $WP(HG)^nHRes$ is bounded above by the complexity of the graded resolution $WP(HG)^{n-1}HRes$ it was constructed from, and if $p_0 \in EAE(p)$ there must exist some specialization w_0 for which there exists a strict reduction in the corresponding complexities. Hence, for the purpose of our "trial and error" procedure for the analysis of the set $EAE(p)$, we need to continue only with graded resolutions $WP(HG)^nHRes$ that have strictly smaller complexity than the graded resolution $WPHG^{n-1}HRes$ they are associated with.

Like in the analysis of an AE sentence, the (strict) reduction in the complexity of the graded resolutions containing the sets of the remaining y's stated in Theorem 1.17 guarantees the termination of our "trail and error" procedure in the minimal (graded) rank case.

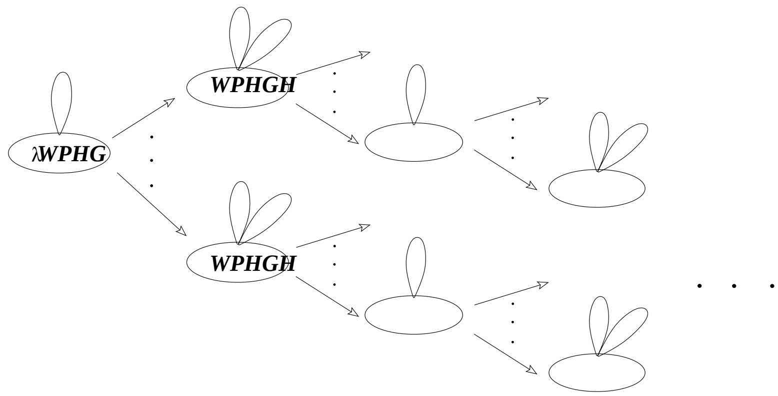
Theorem 1.18: *Suppose that if an initial graded limit group*

$$\lambda WPGL_j(y, h_1, w, p, a)$$

admits an epimorphism τ onto a free group $F_k * F$ where $\tau(<p>) < F_k$, then F is the trivial group. Then the iterative "trial and error" procedure presented above terminates after finitely many steps.

Proof: Identical to the proof of theorem 1.18 of [Se4]. ∎

The outcome of the "trial and error" procedure gives us a finite diagram constructed along the various steps of the iterative procedure, a diagram which is a directed tree in which on every vertex we place a rigid or solid limit group of the form $WP(HG)^n$ or $WP(HG)^nH$ which is the basis of a bundle of the set of the remaining y's or the set of formal solutions defined over the bundle of the remaining y's analyzed along the iterative procedure, which we call the **tree of stratified sets**. This tree encodes all the (finitely many) possible sequences of forms of formal solutions that are needed in order to validate that a certain specialization p_0 of the defining parameters p is indeed in the set $EAE(p)$. This tree and the stratification associated with its various rigid and solid limit groups is the basis for our analysis of the structure of the set $EAE(p)$.



To explain the motivation for the way we analyze the structure of the set $EAE(p)$, we need the natural notion of a **witness**.

Definition 1.19: Let $F_k =< a_1, \dots, a_k >$ be a free group and let $EAE(p)$ be the set defined by the predicate

$$EAE(p) = \exists w\ \forall y\ \exists x\ \Sigma(x, y, w, p, a) = 1 \wedge \Psi(x, y, w, p, a) \neq 1.$$

A specialization w_0 of the variables w is said to be a **witness** for a specialization p_0 of the defining parameters p, if the following sentence:

$$\forall y\ \exists x\ \Sigma(x, y, w_0, p_0, a) = 1 \wedge \Psi(x, y, w_0, p_0, a) \neq 1$$

is a true sentence. Clearly, if there exists a witness for a specialization p_0 then $p_0 \in EAE(p)$, and every $p_0 \in EAE(p)$ has a witness.

By definition, in order to show that a specialization p_0 of the defining parameters p is in the set $EAE(p)$, we need to find a witness w_0 for the specialization p_0. By the construction of the tree of stratified sets, given a witness w_0 for a specialization p_0, it is possible to prove the validity of the AE sentence corresponding to the couple (w_0, p_0), using a proof that is encoded by a subtree of the tree of stratified sets, i.e., a proof built from a finite sequence of (families of) formal solutions, constructed along a (finite) collection of paths in the tree of stratified sets. By the finiteness of the tree of stratified sets there are only finitely many possibilities for such a collection of paths (a subtree). Hence, there are only finitely many possibilities for the structure of a proof encoded by the tree of stratified sets, and these finitely many structures of proofs are sufficient for proving the validity of the AE sentences corresponding to all couples (w_0, p_0), where $p_0 \in EAE(p)$ and w_0 is a witness for p_0. In the sequel, we call each possibility for the structure of a proof encoded by the tree of stratified sets a **proof system**.

Given $p_0 \in EAE(p)$, we are not able to say much about a possible witness for p_0 using the information we have collected so far. With each "proof system" encoded by a subtree of the tree of stratified sets associated with the set $EAE(p)$, we can naturally associate a subset of $EAE(p)$. This subset is defined to be all the specializations $p_0 \in EAE(p)$ for which there exists a witness w_0 so that the validity of the AE sentence corresponding to the couple (w_0, p_0) can be proved using a proof with the structure of the given "proof system". Our main goal in proving Theorem 1.4, i.e., in proving that every EAE set defined over a free group is in the Boolean algebra of AE sets, will be to show that the subset of $EAE(p)$ associated with a given proof system is in the Boolean algebra of AE sets. Since there are only finitely many proof systems encoded by the tree of stratified sets associated with an EAE set, and the EAE set itself is a union of the subsets associated with its (finitely many) proof systems, this implies that the set $EAE(p)$ is in the Boolean algebra of AE sets.

Definition 1.20: Let $p_0 \in EAE(p)$ be a specialization of the defining parameters p and let w_0 be a witness for p_0. By the construction of the tree of stratified sets, the validity of the AE sentence corresponding to the couple (w_0, p_0) can be proved using a proof encoded by some subtree of the tree of stratified sets associated with the set $EAE(p)$. We call such a subtree, that encodes the structure of a proof, a **proof system**. Note that given a couple (w_0, p_0) there may be several proof systems associated with it, but the construction of the tree of stratified sets (i.e., its finiteness) guarantees that the number of possible proof systems is bounded.

We will say that a given proof system associated with the couple (w_0, p_0) is of **depth** d, if all the paths associated with the proof system terminate after d steps (levels) of the tree of stratified sets (i.e., if the subtree associated with the proof system is of depth d).

For presentation purposes, we will start demonstrating our approach for the analysis of the set $EAE(p)$ by analyzing those specializations of the defining parameters p that have witnesses with proof systems of depth 1. We will continue by analyzing the specializations of the defining parameters p for which there are witnesses with proof systems of depth at most 2, and then present the analysis of the entire set $EAE(p)$ under the minimal (graded) rank assumption.

LEMMA 1.21: *Let $T_1(p) \subset EAE(p)$ be the subset of all specializations $p_0 \in EAE(p)$ of the defining parameters p that have witnesses with proof system of depth 1. Then $T_1(p)$ is an EA set.*

Proof: $T_1(p)$ is precisely the set $U_1(p)$, which is proven to be an EA set in Lemma 1.5. ∎

Lemma 1.21 proves that the set of specializations p_0 of the defining parameters p that have a witness with a proof system of depth 1 is an EA set. Before analyzing the entire set $EAE(p)$, we analyze the set of specializations p_0 that have witnesses with a proof system of depth 2. The analysis of specializations $p_0 \in EAE(p)$ that have witnesses with proof systems of depth 2 is much more complicated than the analysis of witnesses with proof systems of depth 1, and will be presented in this section under the minimal (graded) rank assumption. In the next sections we will use the general approach for validation of a sentence presented in [Se4] to drop the minimal rank assumption.

THEOREM 1.22: *Let $T_2(p) \subset EAE(p)$ be the subset of all specializations $p_0 \in EAE(p)$ of the defining parameters p that have witnesses with proof system of depth 2. Then $T_2(p)$ is in the Boolean algebra of AE sets.*

Proof: In this section we present a proof of Theorem 1.22 under the minimal (graded) rank assumption, i.e., we will assume that if any of the graded limit groups $\lambda WPGL(y, h_R, w, p, a)$ or $\lambda WPGL(y, h_S, w, p, a)$ admit an epimorphism τ onto a free group $F_k * F$ where $\tau(< p >) < F_k$, then F is the trivial group. Before we start with the proof of Theorem 1.22 under the minimal (graded) rank assumption, we need the notion of a **valid PS statement**.

Definition 1.23: Suppose that a specialization $p_0 \in EAE(p)$ has a witness w_0 with a proof system of depth 2 (i.e., $p_0 \in T_2(p)$). The structure of the tree of stratified sets guarantees the existence of a rigid or a strictly solid family of specializations (h_0^1, w_0, p_0) of one of the rigid or solid limit groups $WPH(h, w, p, a)$ with the following properties:

(i) For every rigid or solid limit group $WPHG(g_1, h_1, w, p, a)$ there are at most (globally) boundedly many rigid or strictly solid families of specializations of the form $(g_0^1, h_0^1, w_0, p_0, a)$ of $WPHG(g_1, h_1, w, p, a)$, where the strictly solid families are with respect to the given set of closures associated with (some of the other (deeper in the Makanin–Razborov diagram)) limit groups $WPHG(g_1, h_1, w, p, a)$, and their associated limit groups (their successors in the tree of stratified sets) $WPHGH(h_2, g_1, h_1, w, p, a)$ (strictly solid families with respect to a given set of closures are presented in definition 2.12 in [Se3]). The elements (g_0^1, h_0^1, w_0, p_0) that appear in a proof statement are representatives for all the boundedly many classes in the various strictly solid families that do not factor through the covering closure associated with (some of) the other (deeper) limit groups $WPHG$.

(ii) The specialization (h_0^1, w_0, p_0, a) is a rigid or a strictly solid specialization of the corresponding limit group WPH, and it cannot be extended to a specialization that factors through a graded resolution associated with one of the limit groups $\lambda WPGL(y, h, w, p, a)$, so that (a fiber of) this graded resolution corresponds to the entire free group F_y.

(iii) For each of the (boundedly many) rigid or strictly solid families of specializations $(g_0^1, h_0^1, w_0, p_0, a)$ there exists a finite collection of rigid or strictly solid families of specializations $(h_0^2, g_0^1, h_0^1, w_0, p_0, a)$ of the rigid or solid limit groups $WPHGH(h_2, g_1, h_1, w, p, a)$, so that the (ungraded) resolutions corresponding to the specializations $(h_0^2, g_0^1, h_0^1, w_0, p_0, a)$ form a covering closure of the (ungraded) resolution corresponding to the specialization $(g_0^1, h_0^1, w_0, p_0, a)$.

(iv) For each of the (boundedly many) rigid or strictly solid families of specializations $(h_0^2, g_0^1, h_0^1, w_0, p_0, a)$ there exists no specialization g_0^2 of the variables g_2 so that the specialization $(g_0^2, h_0^2, g_0^1, h_0^1, w_0, p_0, a)$ factors through one of the (rigid or solid) limit groups $WPHGHG(g_2, h_1, g_1, h_1, w, p, a)$.

We call a specialization of the form

$$((h_1^2, g_1^1), \ldots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

that satisfies conditions (i)–(iv) above, where the integer $d(ps)$ depends on the fixed proof system, a **valid PS statement**.

The tree of stratified sets guarantees that there are finitely many proof systems of depth 2. Once we fix a proof system of depth 2, we have fixed the rigid or solid limit group $WPH(h, w, p, a)$, the number of rigid or strictly solid families of specializations of each of the limit groups $WPHG(g_1, h_1, w, p, a)$ (with respect to the associated set of closures), and the number of rigid or strictly solid families of specializations of each of the limit groups $WPHGH(h_2, g_1, h_1, w, p, a)$. We start the analysis of the set $T_2(p)$ by enumerating all the possible proof systems, and for each proof system we collect all possible (configurations of) valid PS statements $((h_1^2, g_1^1), \ldots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$ (the integer $d(ps)$ depends on the fixed proof system).

In the tree of stratified sets, each graded limit group $WPHGH$ is a successor of a graded limit group $WPHG$. With each graded limit group $WPHG$ there is an associated graded resolution, and with its successor $WPHGH$ there is an associated graded formal closure of that graded resolution (graded formal closures are presented in definition 3.4 in [Se2]). In general, with a (graded) closure we associate the closure domain (definition 1.16 in [Se2]), which is a coset of a finite index subgroup of the direct sum of the (pegged) abelian groups that appear along the various levels of the closure. Note that by construction, the closure domain is specified by a system of (integer) Diophantine equations that are associated with the closure ([Se2], 1.16).

To each valid PS statement $((h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$ we add specializations that extend the specializations of the limit groups $WPHGH$ in the valid PS statement, by adding specializations of primitive roots of edge groups and pegged abelian vertex groups in the graded abelian JSJ decompositions of the limit groups $WPHG$ that occur along the given proof system (i.e., we add specializations of primitive roots of a fixed set of elements in the valid PS statement). We further add specializations that demonstrate how all the multiples of these primitive roots, multiples up to the least common multiple of the indices of the finite index subgroups associated with the closure domains associated with the various groups $WPHGH$, can be extended to specializations that factor through the finite set of (graded formal) closures specified by the valid PS statement (in fact, these closures are specified by the proof system, not just by the proof statement), i.e., the closures associated with the various limit groups $WPHGH$. This is equivalent to demonstrating that the multiples of the primitive roots do belong to the union of the closure domains associated with the (integer) Diophantine systems of equations associated with the closures specified by the proof system, hence it is equivalent to showing that the given set of closures (associated with the specializations of the groups $WPHGH$) is a covering closure for the ungraded resolutions associated with the specializations (specified by the proof statement) of the groups $WPHG$.

For brevity, in the sequel we still call such extended specializations valid PS statements and denote them $(r,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$. By the standard arguments presented in section 5 of [Se1], the entire collection of (extended) valid PS statements factor through a (canonical) collection of maximal limit groups $PSHGH_1,\dots,PSHGH_m$, which we call PS (proof system) **limit groups**.

By construction, for each $p_0 \in T_2(p)$ there exists some witness w_0 and a proof system, so that a specialization of the form

$$(r,(h_1^2,g_1^1),\ldots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

that is associated with the specialization p_0, the witness w_0 and the proof system, is a valid PS statement (i.e., it satisfies conditions (i)–(iv) of Definition 1.23), and factors through a PS limit group $PSHGH_j$. Naturally, we will try to understand the set of valid PS statements

$$(r,(h_1^2,g_1^1),\ldots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

that factor through a given PS limit group $PSHGH_j$. Our main goal will be to show that these valid PS statements are "generic" in some Diophantine sets that are iteratively associated with each of the PS limit groups $PSHGH$. Before we start with the (technically involved) analysis in the general case, we analyze those PS limit groups which are rigid or solid with respect to the parameter subgroup $P=<p>$.

Proposition 1.24: *Suppose that a PS limit group $PSHGH$ is rigid or solid with respect to the parameter subgroup $P=<p>$, and if it is solid suppose that the subgroup $WP=<w,p>$ is a subgroup of the distinguished vertex group in the graded JSJ decomposition of $PSHGH$ (i.e., the vertex stabilized by the subgroup $AP=<a,p>$). The set of specializations p_0 that have a witness w_0, and a rigid or strictly solid specialization*

$$(r,(h_1^2,g_1^1),\ldots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

of $PSHGH$ which is a valid PS statement, which we denote $PS(p)$, is in the Boolean algebra of AE sets.

Proof: Since we have assumed that the PS limit group $PSHGH$ is either rigid or solid with respect to the parameter subgroup $P=<p>$, with each specialization p_0 of the defining parameters P there exist boundedly many rigid or strictly solid families of specializations

$$(r,(h_1^2,g_1^1),\ldots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

of the limit group $PSHGH$. If the PS limit group is solid, we assume that the subgroup $WP=<w,p>$ is elliptic in its graded JSJ decomposition, hence a PS statement in a strictly solid family is a valid PS statement if and only if every other PS statement that belongs to the same strictly solid family is a valid PS statement.

By section 3 of [Se3] we already know that the set of specializations of the defining parameters P for which there are precisely s rigid or strictly solid specializations of the PS limit group $PSHGH$ is in the Boolean algebra of AE sets. We define the set $PS(p)$ by a predicate which is a disjunction of conjunctions of an EA and an AE predicate.

Since by theorems 2.5 and 2.9 in [Se3] there is a bound on the number of rigid and strictly solid families of specializations of the PS limit group $PSHGH$ for every possible value of the defining parameters p, to define our predicate we count on s, the number of either rigid or strictly solid families of specializations of the given PS limit group $PSHGH$, and for each s we define a predicate which is a conjunction of an EA and an AE predicate as follows.

(1) The EA predicate verifies that there exist (at least) s rigid or strictly solid families of specializations

$$(r,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

of the PS limit group $PSHGH$.

(2) The AE predicate verifies that there exist at most s rigid or strictly solid families of specializations

$$(r,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

of the PS limit group $PSHGH$.

(3) The AE predicate verifies that at least one of the s PS statements is a valid PS statement. It verifies that the (restricted) specialization (h_0^1,w_0,p_0) (which is part of the PS statement) is a rigid or strictly solid specialization of the rigid or solid limit group WPH, that each specialization (g_j^1,h_0^1,w_0,p_0,a) is a rigid or strictly solid specialization of the corresponding rigid or solid limit group $WPHG(g_1,h_1,w,p,a)$ (with respect to the given set of closures — see definition 2.12 in [Se3]), and that each couple of rigid or strictly solid specializations (g_j^1,h_0^1,w_0,p_0,a) are different or belong to distinct strictly solid families, in correspondence. It further verifies that the specializations $(h_j^2,g_j^1,h_0^1,w_0,p_0)$ are rigid or strictly solid specializations of the corresponding (rigid or solid) limit groups $WPHGH$, and that the corresponding (ungraded) resolutions form a covering closure of the resolutions associated with the specializations (g_i^1,h_0^1,w_0,p_0).

Finally, it verifies that there are no extra rigid or strictly solid specializations (g_0^1,h_0^1,w_0,p_0) (with respect to the given set of closures) that are not specified by the PS statement, and that for no specialization g_0^2 the combined specialization $(g_0^2, h_j^2, g_j^1, h_0^1, w_0, p_0)$ factors through any of the limit groups $WPHGHG$. ∎

Proposition 1.24 proves Theorem 1.22 in case the PS limit groups $PSHGH$ are rigid or solid with no flexible quotients, and the subgroup $WP = < w, p >$ is a subgroup of the distinguished vertex group in the graded JSJ decomposition of $PSHGH$. In this special case, the number of possible witnesses w_0 associated with each specialization p_0 of the defining parameter p is finite and globally bounded. In the general case, the number of possible witnesses associated with each specialization p_0 of the defining parameters p is infinite, hence there is no direct way to present the set $T_2(p)$ using a predicate which is in the Boolean algebra of AE sets. Our goal in the analysis of the set $T_2(p)$ in the general case is to find iteratively finitely many Diophantine sets associated with each of the PS limit groups $PSHGH$, so that if a specialization p_0 of the defining parameters p is in the set $T_2(p)$, then a generic specialization of the variables w in at least one of the Diophantine sets is a witness for the corresponding specialization p_0 of the defining parameters P. To achieve this goal, i.e., to find the finitely many Diophantine sets associated with each of the PS limit groups $PSHGH$, we present a "trial and error" procedure, similar to the one used to construct the tree of stratified sets. The output of the iterative "trial and error", i.e., the finitely many Diophantine sets associated with each PS limit group $PSHGH$, is later used to derive a predicate in the Boolean algebra of AE predicates that describes the set $T_2(p)$.

Let $P = < p >$ be the group of defining parameters. With each of the limit groups $PSHGH_i$ we associate its canonical graded taut Makanin–Razborov diagram (with respect to the parameter subgroup P), which contains finitely many graded resolutions which we denote $PSHGHRes_j$, and each graded resolution $PSHGHRes_j$ is defined over the rigid or solid limit group $PT_j(t, p, a)$. We will treat the limit groups $PSHGH_i$ and their graded resolutions $PSHGHRes_j$ and terminal rigid or solid limit groups $PT_j(t, p, a)$ in parallel, hence we will omit the indices of the limit group and its graded resolution. In the sequel, we will treat each stratum in the singular locus of the graded resolutions $PSHGHRes$ separately, and do it in parallel.

We start our analysis with the definition of **Non-Rigid** and **Non-Solid** PS **limit groups**.

Definition 1.25: Let

$$Comp(PSHGHRes)(v, r, (h_1^2, g_1^1), \ldots, (h_{d(ps)}^2, g_{d(ps)}^1), h_1, w, p, a)$$

be the completion of the graded resolution $PSHGHRes$. We look at the entire collection of all test sequences that factor through the graded completion $Comp(PSHGHRes)$:

$$\{v_n, r_n, (h_1^2(n), g_1^1(n)), \ldots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_n, a)\}$$

(see definitions 1.20 and 3.1 in [Se2] for the notion of a test sequence of a completed graded resolution).

We start by looking at those graded test sequences for which the specialization $(h_1(n), w_n, p_n, a)$ is supposed to be rigid according to the proof system, but is in fact flexible, or for some index j for which the specialization $(g_j^1(n), h_1(n), w_n, p_n, a)$ or the specialization $(h_j^2(n), g_j^1(n), h_1(n), w_n, p_n, a)$ is supposed to be rigid according to our fixed proof system, but these specializations are flexible, for every index n. The collection of all these "non-rigid PS" (graded) test sequences factor through a (canonical) collection of **maximal Non-Rigid PS limit groups** $NRgdPS_1, \ldots, NRgdPS_q$. The analysis of graded formal limit groups presented in section 3 of [Se2] associates (canonically) with each Non-Rigid PS limit group, $NRgdPS_i$, a graded formal Makanin–Razborov diagram, and each such graded formal resolution is in fact a one-level graded formal resolution, which is a graded formal closure of the graded resolution $PSHGHRes$, $GFCl(PSHGHRes)$ (graded formal closures are presented in definition 3.4 in [Se2]). Clearly, no specialization

$$(r, (h_1^2, g_1^1), \ldots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

that factors through the resolution $PSHGH$, and which is a valid PS statement with respect to our fixed proof system, factors through one of the $NRgdPS$ limit groups $NRgdPS_1, \ldots, NRgdPS_q$.

After defining Non-Rigid PS limit groups, we continue by defining **Non-Solid PS limit groups**. We look at those graded test sequences for which the specialization $(h_1(n), w_n, p_n, a)$, or for some index j the specialization $(g_j^1(n), h_1(n), w_n, p_n, a)$, or the specialization $(h_j^2(n), g_j^1(n), h_1(n), w_n, p_n, a)$, is supposed to be strictly solid (with respect to the given covering closures) according to our fixed proof system, but these specializations are not strictly solid with respect to the given set of closures, for every index n. For each specialization

$$(r, (h_1^2, g_1^1), \ldots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

in such a "non-solid PS" (graded) test sequence, we add variables that "demonstrate" that the specific specialization is indeed not strictly solid, i.e., it factors through a closure associated with some flexible quotient of the corresponding solid limit group. The collection of all these “non-solid PS” (graded) test sequences, together with the extra variables that “demonstrate” they are indeed “non-solid PS” test sequences, factor through a (canonical) collection of **maximal Non-Solid PS limit groups** $NSldPS_1, \dots, NSldPS_r$. The analysis of graded formal limit groups presented in section 3 of [Se2] associates (canonically) with each Non-Solid PS limit group $NSldPS_i$ a graded formal Makanin–Razborov diagram, and each such graded formal resolution terminates with a graded formal closure of the graded resolution $PSHGHRes$, $GFCl(PSHGHRes)$. Clearly, no specialization

$$(r, (h_1^2, g_1^1), \dots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

that factors through the resolution $PSHGH$, and which is a valid PS statement with respect to our fixed proof system, factors through any of the graded formal closures $GFCl(PSHGHRes)$ that are associated with one of the non-solid limit groups $NSldPS_1, \dots, NSldPS_r$.

After analyzing the collection of test sequences in which parts of virtual proofs (i.e., specializations of the PS limit groups) that are supposed to be either rigid or strictly solid specializations of rigid and solid limit groups constructed along the tree of stratified sets are in fact non-rigid or non-strictly-solid, we need to collect all the test sequences that factor through the PS resolutions $PSHGHRes$, and for which for at least one of the tuples $(h_j^2(n), g_j^1(n), h_1(n), w_n, p_n, a)$ there exists some specialization $g_j^2(n)$ so that the (combined) specialization

$$(g_j^2(n), h_j^2(n), g_j^1(n), h_1(n), w_n, p_n, a)$$

factors through (at least) one of the limit groups $WP(HG)^2 = WPHGHG$.

Definition 1.26: Let

$$Comp(PSHGHRes)(v, r, (h_1^2, g_1^1), \dots, (h_{d(ps)}^2, g_{d(ps)}^1), h_1, w, p, a)$$

be the completion of the graded resolution $PSHGHRes$. We look at the entire collection of all test sequences that factor through the graded completion $Comp(PSHGHRes)$:

$$\{v_n, r_n, (h_1^2(n), g_1^1(n)), \dots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_n, a)\}.$$

We look at those graded test sequences for which for some index j there exists some specialization $g_j^2(n)$ so that the combined specialization

$$(g_j^2(n), h_j^2(n), g_j^1(n), h_1(n), w_n, p_n)$$

that is not supposed to factor through any of the limit groups $WP(HG)^2 = WPHGHG$ according to our fixed proof system (which has depth 2), factors through at least one of these limit groups. The collection of all these (graded) test sequences factor through a (canonical) collection of **maximal Left PS limit groups** $LeftPS_1, \ldots, LeftPS_m$. The analysis of graded formal limit groups presented in section 3 of [Se2] associates (canonically) with each Left PS limit group $LeftPS_i$ a graded formal Makanin–Razborov diagram, and each such graded formal resolution is in fact a one-level graded resolution, which is a graded formal closure of the graded resolution $PSHGHRes$, $GFCl(PSHGHRes)$, that we denote $LeftPSRes$ and call Left PS resolution. Clearly, no specialization $(r,(h_1^2,g_1^1),\ldots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$ that factors through the resolution $PSHGH$, and which is a valid PS statement with respect to our fixed proof system, factors through one of the $LeftPS$ limit groups $LeftPS_1, \ldots, LeftPS_m$, and their associated Left PS resolutions.

The graded formal closures associated with the collection of non-rigid and non-solid and Left PS limit groups determine those "generic" specializations that factor through the various PS resolutions $PSHGHRes$ but fail to be valid PS statements with respect to the (fixed) proof system. "Generic" specializations that factor through the PS resolutions $PSHGHRes$ can fail to be valid PS statements in other ways as well.

To a valid PS statement we have added additional variables, so that their specializations are supposed to be primitive roots of the specializations of pegs of abelian groups that appear in the graded formal closures associated with the groups $WPHGH$, in order to demonstrate that the given sets of closures (specified by the proof system) form a covering closure (for the specializations given by the proof statement). This demonstration remains valid if the orders of the specializations of the variables that are supposed to be primitive roots are prime to the indices of the finite index subgroups associated with the (finitely many) closures. The demonstration may fail to be valid if the orders of these specializations are not prime to the order of the finite index subgroups. To check if this failure occurs for a generic specialization of a PS resolution, $PSHGHRes$, we construct *Root PS* limit groups and resolutions.

Definition 1.27: Let

$$Comp(PSHGHRes)(v,r,(h_1^2,g_1^1),\ldots,(h_{d(ps)}^2,g_{d(ps)}^1),h_1,w,p,a)$$

be the completion of the graded resolution $PSHGHRes$. We look at the entire collection of all test sequences that factor through the graded completion

$Comp(PSHGHRes)$:

$$\{v_n, r_n, (h_1^2(n), g_1^1(n)), \dots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_n, a)\}.$$

We look at those graded test sequences for which, for some index j, there exists a specialization of a variable that is supposed to be a primitive root of a specialization of a peg of some abelian group that appears in the graded formal closure associated with the corresponding group $WPHGH$, but in fact the specialization of this variable has a root of order that divides the least common multiple of the indices of the finite index subgroups associated with the (finitely many) graded formal closures (that are associated with the various limit groups $WPHGH$).

The collection of all these (graded) test sequences factor through a (canonical) collection of **maximal Root PS limit groups** $RootPS_1, \dots, RootPS_m$. The analysis of graded formal limit groups presented in section 3 of [Se2] associates (canonically) with each Root PS limit group $RootPS_i$ a graded formal Makanin–Razborov diagram, and each such graded formal resolution is in fact a one-level graded resolution, which is a graded formal closure of the graded resolution $PSHGHRes$, $GFCl(PSHGHRes)$, that we denote $RootPSRes$ and call Root PS resolution. Clearly, no specialization

$$(r, (h_1^2, g_1^1), \dots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

that factors through the resolution $PSHGH$, and which is a valid PS statement with respect to our fixed proof system, factors through one of the $RootPS$ limit groups $RootPS_1, \dots, RootPS_m$, and their associated Root PS resolutions.

So far we constructed *auxiliary* bundles, for which if in a given fiber a "generic" PS statement fails to be a valid PS statement, then any PS statement in that fiber fails to be a valid PS statement, i.e., the whole fiber can be avoided. The next *auxiliary* bundle that we construct using "generic" specializations that fail to be valid PS statements has the same structure as the previous ones; however, in this bundle it may be that even though "generic" PS statements in a given fiber fail to be valid PS statements, the fiber may contain (non-"generic") valid PS statements.

This next bundle we construct collects all the generic PS statements for which some of the limit groups $WPHG$ have additional rigid or strictly solid specializations (with respect to the given set of closures) that are not specified by the given "generic" PS statements. The "generic" specializations for which there exists a "surplus" in rigid or strictly solid specializations are collected in *Extra PS* (graded) *limit groups* and graded resolutions.

Definition 1.28: Let

$$Comp(PSHGHRes)(v,r,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_1,w,p,a)$$

be the completion of the graded resolution $PSHGHRes$. We look at the entire collection of all test sequences that factor through the graded completion $Comp(PSHGHRes)$

$$\{v_n,r_n,(h_1^2(n),g_1^1(n)),\dots,(h_{d(ps)}^2(n),g_{d(ps)}^1(n)),h_1(n),w_n,p_n,a)\}.$$

By theorems 2.5 and 2.13 of [Se3], for each specialization (h_0^1,w_0,p_0,a) there exist boundedly many rigid or strictly solid families of specializations (g_0^1,h_0^1,w_0,p_0) of each of the rigid or solid limit groups $WPHG(g_1,h_1,w,p,a)$ (strictly solid with respect to the given set of closures — see definition 2.12 in [Se3]). We look at those graded test sequences for which, for every index n, there are rigid or strictly solid families of specializations of the form $(g^1(n),h_1(n),w_n,p_n)$ which are not in the rigid or strictly solid families of each of the specializations specified by the PS statement $(g_j^1(n),h_1(n),w_n,p_n)$, for every index j. To each specialization

$$(v_n,r_n,(h_1^2(n),g_1^1(n)),\dots,(h_{d(ps)}^2(n),g_{d(ps)}^1(n)),h_1(n),w_n,p_n,a)),$$

we add variables for the additional rigid specializations or for almost shortest specializations (see definition 2.8 in [Se3]) of the additional strictly solid families of specializations. Since by theorems 2.5 and 2.13 of [Se3] there exists a global bound on the number of additional rigid or strictly solid families of specializations (with respect to a given set of closures), the number of additional variables we need to add is globally bounded.

The collection of all these "Extra PS" (graded) test sequences (including the added variables) factor through a (canonical) collection of **maximal Extra** PS **limit groups** $ExtraPS_1,\dots,ExtraPS_\ell$. The analysis of graded formal limit groups presented in section 3 of [Se2] associates (canonically) with each Extra PS limit group $ExtraPS_i$ a graded formal Makanin–Razborov diagram, and each such graded formal resolution is in fact a one-level graded resolution, which is a graded formal closure of the graded resolution $PSHGHRes$, $GFCl(PSHGHRes)$. We denote such an *Extra PS* (graded) *resolution*

$$ExtraPSRes(u,v,r,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_1,w,p,a)$$

(the variables u are the variables added for the extra rigid and strictly solid specializations).

The Extra PS limit groups and their associated graded formal closures collect all the "generic" specializations (i.e., all the test sequences) of the PS limit groups $PSHGH$ for which there exist rigid or strictly solid families (with respect to the given set of closures) in addition to those specified by the specializations declared in the proof statement. For a general specialization of the PS limit groups $PSHGH$, i.e., a specialization which is not necessarily "generic", it may as well be that the additional rigid or strictly solid specializations, collected by the extra PS limit groups and their associated graded formal closures, do become non-rigid or non-strictly-solid or do coincide with the rigid or strictly solid families of the various specializations (g^1_j, h^1_0, w_0, p_0) declared in the proof statement. To collect all the specializations that factor through one of the extra PS graded formal closures associated with one of the extra PS limit groups $ExtraPS$, in which such a "collapse" of the additional rigid and strictly solid families occurs, we need to define the *Collapse Extra PS* (graded) *limit groups* and their associated (Collapse Extra PS) graded resolutions. We start by listing all possible collapsings of a specialization of an Extra PS resolution.

Definition 1.29: Let $ExtraPSRes$ be one of the Extra PS graded resolutions associated with one of the Extra PS graded limit groups $ExtraPS_i$. Note that by construction, $ExtraPSRes$ is a graded formal closure of the PS resolution $PSHGHRes$ with which we started the analysis.

We will say that a specialization that factors through the Extra PS graded resolution $ExtraPSRes$ is **collapsed** if the variables added for each of the additional rigid or strictly solid families of specializations (i.e., the ones that were not specified by the proof system) satisfy one of the following:

(1) A specialization of the variables added for one of the additional rigid specializations becomes flexible.

(2) A specialization of the variables added for one of the additional rigid specializations becomes equal to one of the rigid specializations specified by the proof system, i.e., with one of the specializations g^1_j in the specialization

$$(u, v, r, (h^2_1, g^1_1), \dots, (h^2_{d(ps)}, g^1_{d(ps)}), h^1_0, w_0, p_0).$$

(3) A specialization of the variables added for one of the additional strictly solid families of specializations (with respect to the given set of closures — ([Se3], 2.12)) factors through one of the given closures associated with

a flexible quotient associated with the corresponding solid limit group (see definitions 1.5 and 2.12 in [Se3]).

(4) A specialization of the variables added for one of the additional strictly solid families of specializations belongs to one of the strictly solid families of specializations specified by the proof system, i.e., with a family of one of the strictly solid families of specializations g_j^1 in the specialization

$$(u, v, r, (h_1^2, g_1^1), \ldots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0).$$

Note that, by definition, there are only finitely many ways that a specialization that factors through the Extra PS resolution, $ExtraPSRes$, can become a collapsed specialization. We will call each way a specialization can become collapsed a **collapse form**.

Having defined the finitely many possibilities for collapse forms, we collect all the specializations that are collapsed specializations in finitely many graded resolutions which we call *Collapse Extra PS* (graded) *resolutions*.

Definition 1.30: Let $ExtraPSRes$ be one of the extra PS graded resolutions associated with one of the extra PS graded limit groups $ExtraPS_i$. Note that, by construction, $ExtraPSRes$ is a graded formal closure of the PS resolution $PSHGHRes$ with which we started the analysis. With the Extra PS resolution we fix one of the (finitely many) collapse forms associated with it.

With each collapsed specialization that factors through the extra PS resolution

$$(u_0, v_0, r_0, (h_1^2(0), g_1^1(0)), \ldots, (h_{d(ps)}^2(0), g_{d(ps)}^1(0)), h_1(0), w_0, p_0, a))$$

which is collapsed in the particular collapse form we have fixed, we associate specializations of additional variables $\{c\}$, c_0, which demonstrate that the specializations of the additional variables added for one of the additional strictly solid families of specializations are either non-strictly solid with respect to the given set of closures, or belong to one of the strictly solid families of specializations specified by the proof system, according to our fixed collapse form. We pick the specialization c_0 to be the shortest among all possible ones. By our standard method presented in section 5 of [Se1], the collection of specializations

$$(c_0, u_0, v_0, r_0, (h_1^2(0), g_1^1(0)), \ldots, (h_{d(ps)}^2(0), g_{d(ps)}^1(0)), h_1(0), w_0, p_0, a)),$$

which are collapsed and demonstrate the collapse of the specializations of variables added for additional rigid and strictly solid families, factor through a (canonical) collection of finitely many maximal limit groups which we call *Collapse Extra PS* (graded) *limit groups*, denoted $CollapsExtraPS_i$.

Note that each collapsed specialization restricts, in particular, to a specialization of the completed resolution $Comp(PSHGHRes)$, so given one of the finitely many Collapse Extra PS limit groups, $CollapseExtraPS_i$, we are able to use the iterative procedure for the analysis of quotient (multi-graded) resolutions presented in this section (1.5–1.17) and based on section 1 of [Se4], to get a (canonical) collection of finitely many (graded) resolutions that contain the entire collection of (collapsed) specializations

$$(c_0, u_0, v_0, r_0, (h_1^2(0), g_1^1(0)), \dots, (h_{d(ps)}^2(0), g_{d(ps)}^1(0)), h_1(0), w_0, p_0, a))$$

that factor through the Extra PS resolution $ExtraPSRes$ and the collapsed extra PS limit group $CollapseExtraPS_i$. We call such a resolution a *Collapse Extra PS* (graded) *resolution*, and denote it $CollapseExtraPSRes_j$.

The Collapse Extra PS resolutions were obtained using the iterative procedure for the analysis of quotient resolutions presented earlier in this section. This iterative procedure was constructed in the first section of [Se4] in order to control the complexity of the obtained quotient resolutions. The Collapse Extra limit groups $CollapseExtraPS$ are not quotients of the completed (graded) resolution $Comp(PSHGH)$, since we have added the variables $\{u\}$ to demonstrate extra rigid and strictly solid families of solutions to obtain the Extra PS resolution $ExtraPSRes$, and additional variables $\{c\}$ to demonstrate their collapse according to the fixed collapse form.

The variables $\{u\}$ were added to demonstrate extra rigid and strictly solid solutions, hence the extra PS resolutions, $ExtraPSRes$, are in fact graded formal closures of the graded resolution $PSHGHRes$ we have started with. Therefore, apart from adding various roots to abelian vertex groups in the graded abelian JSJ decompositions associated with the various levels of the completion $Comp(PSHGHRes)$, the Extra PS resolution $ExtraPSRes$ may differ from the completion $Comp(PSHGHRes)$ only in the graded abelian JSJ decomposition associated with the terminal (bottom) level, in case the terminating limit group of the graded resolution $ExtraPSRes$ is solid.

The variables $\{c\}$ were added to demonstrate that the specializations of the variables added for extra strictly solid families of solutions actually factor through one of the given closures associated with flexible quotients of the corresponding solid limit group, or they belong to the same strictly solid family as one of the strictly solid families specified by the proof statement. By the minimal graded rank assumption, each of the limit groups $WPHG$ and $WPHGH$ do not admit a map onto a free group $F_k * F$ in which the factor F is non-trivial and the subgroup $AP =< a, p >$ is mapped onto F_k. Clearly, there are natural maps from the completion, $Comp(PSHGHRes)$, to each of the limit groups that appear along the graded resolution $CollapseExtraPSRes$. By the way the iterative procedure for the analysis of quotient resolutions that was used to construct the Collapse Extra PS resolutions $collapseExtraPSRes$ is defined, it follows that:

(i) If a graded limit group associated with one of the levels of the graded resolution $CollapseExtraPSRes$ admits a graded free decomposition, i.e., a non-trivial free decomposition in which the subgroup $AP =< a, p >$ is contained in one of the factors, then necessarily the entire image of the completion $Comp(PSHGHRes)$ is contained in the factor that contains the subgroup AP. Hence, if we modify the other factor to be the identity, the specializations of the completion $Comp(PSHGHRes)$ that factor through $CollapseExtraPSRes$ will not be changed. Therefore, we may assume that no limit group along the resolution $CollapseExtraPSRes$ admits a (non-trivial) graded free decomposition.

(ii) Let Q be a QH vertex group that appears in one of the graded abelian JSJ decompositions associated with the various levels along the graded resolution $CollapseExtraPSRes$ except, perhaps, the bottom level (in case the terminating limit group of the graded resolution $CollapseExtraPSRes$ is solid). Then there exists a subgroup of finite index in (a conjugate of) the QH subgroup Q which is in the image of the completion $Comp(PSHGHRes)$.

(iii) Suppose that all the QH vertex groups in the completion $Comp(PSHGH)$ are surviving QH subgroups and let A be an abelian vertex group in one of the graded abelian JSJ decompositions associated with the various levels along the graded resolution $CollapseExtraPSRes$ except, perhaps, the bottom level. Then there exist conjugates of subgroups in the image of the completion $Comp(PSHGHRes)$ which generate a subgroup of finite index in A.

Since otherwise, either the group $PSHGH$ admits a map onto a free group $F_k * F$ with a non-trivial F, so that $AP =< a, p >$ is mapped onto F, in contradiction to our assumptions, or the specializations of the variables c can be modified so that a QH vertex group will be eliminated or the rank of an abelian vertex group will be decreased, or a QH vertex group can be pushed to the bottom level of the graded resolution $CollapseExtraPSRes$ (see section 1 of [Se4] for a detailed discussion of the iterative procedure that constructs the graded resolutions $CollapseExtraPSRes$).

Given these properties of the Collapse Extra resolutions

$$CollapseExtraPSRes,$$

we need to slightly modify the way we measure their complexity in order to be able to control it in terms of the complexity of the completion

$$Comp(PSHGHRes)$$

with which we have started.

Definition 1.31: Let

$$CollapseExtraPSRes(c,u,v,r,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_1,w,p,a)$$

be one of the (well-structured, graded) Collapse Extra PS resolutions constructed by the iterative procedure for the analysis of quotient resolutions from the Extra PS resolution $ExtraPSRes$.

Recall that we have set the complexity of a (completed) minimal rank resolution (definition 1.14 in [Se4]) to be the tuple

$$Cmplx(Res(t,a)) = ((genus(S_1),|\chi(S_1)|),\dots,(genus(S_m),|\chi(S_m)|),Abrk(Res(t,a)))$$

where the ordered couples $(genus(S_j),|\chi(S_j)|)$ are associated with the QH subgroups that appear along the various levels of the resolution, and are ordered in a decreasing lexicographical order. The abelian rank, $Abrk(Res(t,a))$, is the sum

$$Abrk(Res(t,a)) = \Sigma(rk(A_j) - rk(eA_j))$$

over the abelian vertex groups A_j that appear along the various levels of the resolution, where $eA_j < A_j$ is the subgroup generated by the edge groups connected to an abelian vertex group.

Let $SubCollapseExtraPSRes$ be the subresolution induced by the image of the completion of the PS resolution $PSHGHRes$, in the completion of the Collapse Extra PS resolution, $CollapseExtraPSRes$ (see section 3 of [Se4] for the construction of the induced resolution).

We set the complexity of a Collapse Extra PS resolution,

$$CollapseExtraPSRes,$$

to be the tuple

$$Cmplx(CollapseExtraPSRes) = \\ ((genus(S_1), |\chi(S_1)|), \ldots, (genus(S_m), |\chi(S_m)|), Abrk(Res(t,a)))$$

where the ordered couples $(genus(S_j), |\chi(S_j)|)$ are associated with those QH subgroups that appear along the various levels of the Collapse Extra PS resolution, and a conjugate of them intersects the induced resolution, $SubCollapseExtraPSRes$, in a subgroup of finite index. These couples are ordered in a decreasing lexicographical order. The abelian rank, $Abrk(Res(t,a))$, is the sum

$$Abrk(Res(t,a)) = \Sigma(rk(A_j) - rk(eA_j))$$

over those abelian vertex groups A_j that appear along the various levels of the resolution, and a conjugate of them intersects $SubCollapseExtraPSRes$ in a subgroup of finite index. $eA_j < A_j$ is the subgroup generated by the edge groups connected to the abelian vertex group.

Note that, by construction, the QH and abelian vertex groups that contribute to the complexity of $CollapseExtraPSRes$ include all the QH and abelian vertex groups that appear in all levels of $CollapseExtraPSRes$ above the terminal level, and a subset of those that appear in the terminal level.

Once we have defined the complexity of the Collapse Extra PS resolutions properly, the iterative procedure for the construction of quotient resolutions (presented in section 1 of [Se4]) enables us to control the complexity of these resolutions by the complexity of the completion $Comp(PSHGHRes)$ with which we started.

THEOREM 1.32: *Let $CollapseExtraPSRes$ be one of the graded resolutions obtained from the Extra PS resolution $ExtraPSRes$, which is obtained from the PS resolution $PSHGHRes$. Then*

$$Cmplx(CollapseExtraPSRes) \leq Cmplx(Comp(PSHGHRes)),$$

and in case of equality the two graded resolutions are compatible, i.e., the resolution $CollapseExtraPSRes$ is a graded formal closure of the graded resolution $PSHGHRes$.

Proof: The resolutions $CollapseExtraPSRes$ were constructed by the iterative procedure for the analysis of quotient resolutions, hence the theorem follows using the same arguments used to prove Theorem 1.17, which is identical to the argument used to prove theorem 1.15 of [Se4]. ∎

In case the complexity of a Collapse Extra PS resolution,

$$CollapseExtraPSRes,$$

is equal to the complexity of the resolution $PSHGHRes$, i.e., in case $CollapseExtraPSRes$ is a graded formal closure of the graded resolution $PSHGHRes$, we call the resolution $CollapseExtraPSRes$ a **Generic** Collapse Extra PS resolution, and denote it $GenericCollapseExtraPSRes$.

To the list of non-Generic Collapse Extra PS resolutions we add resolutions obtained from the Extra PS resolutions. Given an Extra PS resolution, $ExtraPSRes$, we add roots to pegs of abelian groups that appear along the various levels of $ExtraPSRes$. The roots are of orders that divide the least common multiple of the finite index subgroups associated with the various resolutions (graded formal closures) that were constructed from the corresponding PS resolution, $PSHGHRes$. We analyze the resolutions obtained by adding the roots in the same way we analyze Collapse PS limit groups. We add those of the obtained resolutions that are not of maximal complexity to the list of (non-Generic) Collapse Extra PS resolutions.

The PS limit groups $PSHGH$ and their associated resolutions, together with the non-rigid and non-solid PS limit groups and their associated graded resolutions, the Left and the Root PS resolutions, the Extra PS resolutions and the Collapse and the Generic Collapse Extra PS resolutions, enable us to present the main principle which is the key in our general approach to the entire quantifier elimination process. Conceptually, we show that if a valid PS statement factors through one of the resolutions $PSHGHRes$, then either there exists a valid PS statement that factors through one of the Collapse Extra PS resolutions associated with $PSHGHRes$ which is not of maximal complexity (i.e., it is not a Generic Collapse Extra PS resolution), and these have strictly smaller complexity than the PS resolution, $PSHGHRes$, with which we started this branch, or the fiber that contains the valid PS statement in the bundle associated with the PS resolution $PSHGHRes$ contains a test sequence of valid PS statements (i.e., a "generic" point in that fiber is a valid PS statement).

Theorem 1.33: *Let $PSHGHRes$ be one of the PS graded resolutions, and suppose that there exists a valid PS statement*

$$(r_0,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

that factors through the completion of the resolution $PSHGHRes$ with respect to our fixed proof system. Then (at least) one of the following holds:

(1) *There exists a test sequence of specializations*

$$(v_n, r_n, (h_1^2(n), g_1^1(n)), \dots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_0, a))$$

that factor through the completion of the PS resolution $PSHGHRes$ (note that the specialization p_0 of the defining parameters p is fixed along the entire test sequence), for which the (restricted) specializations

$$((h_1^2(n), g_1^1(n)), \dots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_0, a))$$

are valid PS statements (with respect to our fixed proof system).

(2) *There exists a specialization*

$$(c_0, u_0, v_0, r_0, (h_1^2, g_1^1), \dots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a))$$

that factors through one of the non-Generic Collapse Extra PS resolutions $CollapseExtraPSRes$ associated with the resolution $PSHGHRes$, i.e., a collapse extra PS resolution for which

$$Cmplx(CollapseExtraPSRes) < Cmplx(Comp(PSHGHRes)).$$

Proof: Suppose that there exists a valid PS statement

$$(r_0, (h_1^2, g_1^1), \dots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

that factors through the completion of the resolution $PSHGHRes$ with respect to our fixed proof system, and for no specialization (c_0, u_0, v_0) the (combined) specialization

$$(c_0, u_0, v_0, r_0, (h_1^2, g_1^1), \dots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a))$$

factors through one of the non-Generic Collapse Extra PS resolutions

$$CollapseExtraPSRes$$

associated with the resolution $PSHGHRes$, i.e., those for which

$$Cmplx(CollapseExtraPSRes) < Cmplx(Comp(PSHGHRes)).$$

If the collection of the ungraded resolutions covered by the Non-Rigid PS resolutions, $NRgdPSRes$, the ungraded resolutions covered by the Non-Solid

PS resolutions, $NSldPSRes$, and the ungraded resolutions covered by the Left PS resolutions, $LeftPSRes$, and by the Root PS resolutions, $RootPSRes$, associated with the graded PS resolution $PSHGHRes$ form a covering closure of all the ungraded resolutions covered by the graded resolution $PSHGHRes$, then no valid PS statement of the form

$$(r_0,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

factors through the completion of the resolution $PSHGHRes$.

Furthermore, if the collection of the ungraded resolutions covered by the Non-Rigid PS resolutions, $NRgdPSRes$, the Non-Solid PS resolutions, $NSldPSRes$, and the Left PS resolutions, $LeftPSRes$, and the Root PS resolutions, $RootPSRes$, described above, together with the set of ungraded resolutions covered by the extra PS resolutions, $ExtraPSRes$, from which we take out the collection of specializations that factor through the set of Generic Collapse Extra PS resolutions, form a covering closure of all the ungraded resolutions covered by the graded resolution $PSHGHRes$,

$$PSHGHRes(r,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_1,w,p_0,a),$$

then for every valid PS statement of the form

$$(r_0,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$$

that factors through the completion of the resolution $PSHGHRes$, there must exist a specialization (c_0,u_0,v_0) for which the combined specialization

$$(c_0,u_0,v_0,r_0,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a))$$

factors through one of the non-Generic Collapse Extra PS resolutions, $CollapseExtraPSRes$.

Since we have assumed that there exists a valid PS statement that factors through the completion of the resolution $PSHGHRes$, and for which there is no specialization (c_0,u_0,v_0) for which the corresponding specialization is a factors through one of the non-Generic Collapsed Extra PS resolutions $CollapseExtraPSRes$, the collection of ungraded resolutions covered by the Non-Rigid PS resolutions, $NRgdPSRes$, the Non-Solid PS resolutions, $NSldPSRes$, the Left PS resolutions, $LeftPSRes$, the Root PS resolutions, $RootPSRes$, and the set of ungraded resolutions covered by the Extra PS resolutions, $ExtraPSRes$, from which we take out those specializations that factor through ungraded resolutions that are covered by the Generic Collapse Extra PS resolutions, described above, does not form a covering closure of the ungraded resolutions covered by the PS resolution $PSHGHRes$.

Therefore, there must exist a test sequence

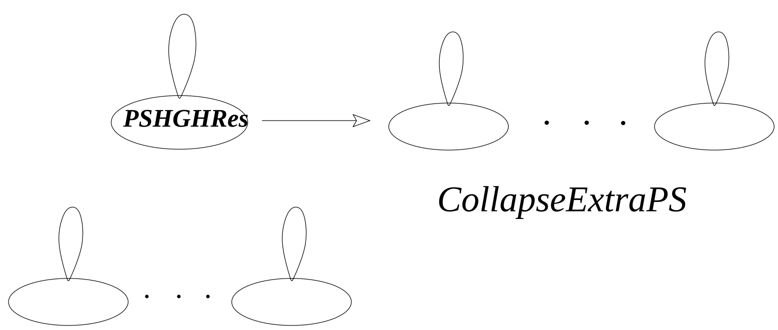
$$(v_n, r_n, (h_1^2(n), g_1^1(n)), \ldots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_0, a))$$

of specializations that factor through the completion of the PS resolution $PSHGHRes$, so that for every index n the corresponding specialization does not factor through any of the Non-Rigid, Non-Solid or the Left or the Root PS resolutions, $NRgdPSRes$, $NSldPSRes$ and $LeftPSRes$ and $RootPSRes$, and if it factors through an Extra PS resolution, $ExtraPSRes$, then it can be extended to a specialization that factors through a Generic Collapse Extra PS resolution associated with it. Hence, for large enough n, the corresponding (restricted) specialization

$$((h_1^2(n), g_1^1(n)), \ldots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_0, a))$$

is a valid PS statement, so we have found a test sequence of valid PS statements and the theorem follows. ∎

Theorem 1.33 is the key point in our analysis of the set $T_2(p)$. Conceptually, Theorem 1.33 reduces the analysis of the set $T_2(p)$ to the set of specializations of the defining parameters $P =< p >$ for which there exists a test sequence of valid PS statements that factor through the various resolutions $PSHGHRes$.



Proposition 1.34: *Let $PSHGHRes$ be one of the (non-trivial) graded PS resolutions. Let $TSPS(p)$ be the set of specializations p_0 of the defining parameters $P =< p >$, for which there exists a test sequence of specializations*

$$(v_n, r_n, (h_1^2(n), g_1^1(n)), \ldots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_0, a))$$

that factor through the completion of the PS resolution $PSHGHRes$, $Comp(PSHGHRes)$, and restricts to a sequence of valid PS statements

$$((h_1^2(n), g_1^1(n)), \ldots, (h_{d(ps)}^2(n), g_{d(ps)}^1(n)), h_1(n), w_n, p_0, a)).$$

Then $TSPS(p)$ is in the Boolean algebra of AE sets.

Proof: As we have already pointed out in the proof of Theorem 1.33, a specialization p_0 of the defining parameters $P =< p >$ satisfies $p_0 \in TSPS(p)$, if and only if the collection of ungraded resolutions covered by the Non-Rigid PS resolutions $NRgdPSRes$, the Non-Solid PS resolutions, the Left and the Root PS resolutions, together with the set of ungraded resolutions covered by the Extra PS resolutions, $ExtraPSRes$, from which we subtract the ungraded resolutions covered by the Generic Collapse Extra PS resolutions, associated with the ungraded resolution $PSHGHRes$, does not form a covering closure of all the ungraded resolutions covered by the graded resolution $PSHGHRes$. Therefore, to prove the proposition, we need to find a predicate in the Boolean algebra of AE predicates that defines the set of specializations p_0 for which there exist ungraded resolutions associated with them that satisfy this "non-covering" property.

By construction, there are finitely many PS graded resolutions $PSHGHRes$, $NRgdPSRes$, $NSldPSRes$, $LeftPSRes$, $RootPSRes$, $ExtraPSRes$, and $GenericCollapseExtraPSRes$, and each of these graded resolutions terminates in either a rigid or a solid graded limit group (with respect to defining parameters $P =< p >$). To these resolutions we add a collection of graded resolutions that indicate that certain pegs along one of these resolutions have a (non-trivial) root of an order that divides the least common multiple of the indices of the (finite index) subgroups that are associated with the closures associated with the various resolutions. These graded resolutions are constructed in the same way $RootPS$ was constructed (Definition 1.27).

By the global bounds on the number of rigid solutions of a rigid limit group ([Se3], 2.5), and on the number of strictly solid families with respect to a given set of closures of a solid limit group ([Se3], 2.13), for any specialization p_0 of the defining parameters $P =< p >$, there are finitely many combinations for the collections of ungraded resolutions covered by a PS resolution $PSHGHRes$, and the collections of ungraded resolutions that are covered by the other (auxiliary) graded resolutions associated with a PS resolution, $PSHGHRes$. These finitely many possibilities for the collections of ungraded resolutions form a stratification of the set of specializations of the defining parameters, obtained from the bases of all the graded resolutions that have been constructed, simultaneously. A specialization p_0 is in the set $TSPS(p)$ if and only if the ungraded resolutions covered by these resolutions and associated with it are divided according to a subset of these combinations and not according to the complement of this subset of combinations (i.e., $p_0 \in TSPS(p)$ if and only if it belongs to certain strata in the combined stratification, and not to the complement of these strata, but it depends only on the stratum, not on the particular specialization). The set of specializations p_0 associated with a given combination of rigid and strictly solid families of specializations (with respect to the given set of closures) of (finitely many) rigid and solid limit groups (a stratum in the simultaneous stratification) can be defined using a (finite) disjunction of conjunctions of an EA and an AE predicate (see section 3 of [Se3]). Hence, a finite union of such strata, the set $TSPS(p)$, is in the Boolean algebra of AE sets. ∎

At this stage we have all the tools needed for showing that the set $T_2(p)$ is in the Boolean algebra of AE sets. By construction, if $p_0 \in T_2(p)$ then there must exist a valid PS statement of the form

$$(r, (h_1^2, g_1^1), \ldots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

that factors through one of the PS resolutions $PSHGHRes$ constructed with respect to all proof systems of depth 2.

By Proposition 1.34, the sets $TSPS(p)$ associated with the various PS resolutions $PSHGHRes$, i.e., the sets of specializations p_0 of the defining parameters $P =< p >$ for which there exists a test sequence that factors through any of the PS resolutions $PSHGHRes$, and restricts to valid PS statements, are in the Boolean algebra of AE sets. By Theorem 1.33, if there exists a valid PS statement that factors through a PS resolution $PSHGHRes$, then either there exists a test sequence that factors through that PS resolution, and restricts to valid PS statements, or there must exist a specialization (c_0, u_0, v_0), so that the (combined) specialization

$$(c_0, u_0, v_0, r_0, (h_1^2, g_1^1), \ldots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

factors through one of the Extra collapse PS resolutions, $CollapseExtraPSRes$, associated with the PS resolution $PSHGHRes$, and for which

$$Cmplx(CollapseExtraPSRes) < Cmplx(Comp(PSHGHRes)).$$

We continue with each of the (non-Generic) Collapse Extra PS resolutions, $CollapseExtraPSRes$, i.e., those that satisfy

$$Cmplx(CollapseExtraPSRes) < Cmplx(Comp(PSHGHRes))$$

in parallel. Exactly as we did with each of the PS resolutions $PSHGHRes$, we associate (canonically) with such a resolution, $CollapseExtraPSRes$, its Non-Rigid and Non-Solid PS resolutions, Left PS resolutions, Root PS resolutions, Extra PS resolutions, and Collapse Extra PS resolutions. By Proposition 1.34 applied to the various resolutions $CollapseExtraPSRes$, the sets of specializations p_0 of the defining parameters $P =< p >$ for which there exists a test sequence that factors through any of the resolutions $CollapseExtraPSRes$, and restricts to valid PS statements, are in the Boolean algebra of AE sets. By Proposition 1.33 applied to the various resolutions $CollapseExtraPSRes$, if there exists a valid PS statement that factors through a resolution $CollapseExtraPSRes$, then either there exists a test sequence that factors through that extra collapse PS resolution, and restricts to valid PS statements, or there must exist a specialization (c_1, u_1, v_1), so that the (combined) specialization

$$(c_1, u_1, v_1, c_0, u_0, v_0, r_0, (h_1^2, g_1^1), \dots, (h_{d(ps)}^2, g_{d(ps)}^1), h_0^1, w_0, p_0, a)$$

factors through one of the Collapse Extra PS resolutions associated with the resolution $CollapseExtraPSRes$ so that their complexity is strictly smaller than the complexity of $CollapseExtraPSRes$.

Continuing iteratively with the associated Collapse Extra resolutions of strictly smaller complexity, we obtain a terminating iterative procedure by the proof of Theorem 1.18 which is identical to the proof of theorem 1.18 of [Se4]. The iterative procedure we have constructed has to terminate with either a rigid limit group with respect to the defining parameters $P =< p >$, or a solid limit group in which the subgroup $WP =< w, p >$ is contained in the distinguished vertex group in the associated graded abelian JSJ decomposition.

By iteratively applying Proposition 1.33 to the various resolutions that appear along the iterative procedure, if there exists a valid PS statement that factors through any of these resolutions, then either there exists a test sequence that factors through one of these resolutions, and restricts to valid PS statements, or there must exist a valid PS statement that factors through one of the terminating rigid or solid limit groups.

By iteratively applying Proposition 1.34 to the various resolutions constructed along the iterative procedure, the sets of specializations p_0 of the defining parameters $P =< p >$ for which there exists a test sequence that factor through any of these resolutions, and restricts to valid PS statements, are in the Boolean algebra of AE sets. By Proposition 1.24, the set of specializations p_0 of the defining parameters $P =< p >$ for which there exists a valid PS statement that factors through one of the terminating rigid or solid PS limit groups is in the Boolean algebra of AE sets. The entire set $T_2(p)$ is precisely the set of specializations p_0 of the defining parameters $P =< p >$ for which there exists a valid PS statement, hence the set $T_2(p)$ is in the Boolean algebra of AE sets, and the proof of Theorem 1.22 in the minimal (graded) rank case is completed. ∎

At this stage we are finally ready to show that the entire set $EAE(p)$ is in the Boolean algebra of AE sets. The tree of stratified sets has a finite depth, which (by definition) bounds the depth of all possible proof systems associated with the tree of stratified sets. For each integer d, we set $T_d(p)$ to be the set of specializations p_0 of the defining parameters $P =< p >$ for which there exists a valid PS statement for some proof system of depth d. Clearly, $EAE(p) = T_1(p) \cup \cdots \cup T_{d_0}(p)$ where d_0 is the depth of the tree of stratified sets. Hence, to show that the set $EAE(p)$ is in the Boolean algebra of AE sets, it is enough to show that each of the sets $T_d(p)$ is in the Boolean algebra of AE sets.

By the structure of the tree of stratified sets, and the global bounds on the possible numbers of distinct rigid and strictly solid families associated with each stratum in this tree, given a fixed depth d there exist finitely many possible proof systems of depth d. Given a fixed proof system of depth d, which we denote PS, we collect all its associated valid PS statements in a canonical collection of finitely many PS limit groups which we denote $PS(HG)^{d-1}H$. With each PS limit group $PS(HG)^{d-1}H$ we associate its canonical (strict Makanin–Razborov) resolutions $PS(HG)^{d-1}HRes$. With each of these PS resolutions we associate (canonically) the set of Non-Rigid and Non-Solid PS resolutions, the Left PS resolutions, the Root PS resolutions, the Extra PS resolutions, and the Collapse Extra PS resolutions. In exactly the same way we handled the set $T_2(p)$, we continue iteratively with those Collapse Extra PS resolutions that have strictly smaller complexity to the next step. The iterative procedure we construct terminates after finitely many steps since the complexity of the obtained resolutions strictly decreases in each step (Theorem 1.18). Theorem 1.33 applied iteratively to the resolutions along the iterative procedure implies that if there exists a valid PS statement that factors along the resolution $PS(HG)^{d-1}HRes$, then either there exists a test sequence that factors through one of the resolutions constructed along the iterative procedure, and restricts to valid PS statements, or there exists a valid PS statement that factors through one of the terminal rigid or solid PS limit groups. Theorems 1.34 and 1.24 applied to the various resolutions that appear along the iterative procedure, and the terminal rigid or solid PS limit groups finally imply that each of the sets $T_d(p)$ is in the Boolean algebra of AE sets, hence the entire set $EAE(p)$ is in the Boolean algebra of AE sets, and the proof of Theorem 1.4 in the minimal (graded) rank case is completed. ∎

The proof of Theorem 1.4 shows that a set $EAE(p)$ defined using a conjunction of a system of equalities and a system of inequalities,

$$EAE(p) = \exists w \ \forall y \ \exists x \ \Sigma(x, y, w, p, a) = 1 \wedge \Psi(x, y, w, p, a) \neq 1,$$

is in the Boolean algebra of AE sets. The generalization of that proof to a set $EAE(p)$ defined using a (finite) disjunction of conjunctions of a system of equalities and a system of inequalities,

$$\begin{aligned} EAE(p) = \exists w \ \forall y \ \exists x \ (\Sigma_1(x, y, w, p, a) = 1 \wedge \Psi_1(x, y, w, p, a) \neq 1) \vee \cdots \\ \cdots \vee (\Sigma_r(x, y, w, p, a) = 1 \wedge \Psi_r(x, y, w, p, a) \neq 1), \end{aligned}$$

is rather straightforward. Indeed, the only change required is in the construction of the tree of stratified sets. In proving Theorem 1.4, i.e., when the predicate defining the set $EAE(p)$ is the conjunction of a system of equalities and a system of inequalities, we have constructed the tree of stratified sets iteratively, where in each step we have first collected all the formal solutions defined over closures of the resolutions of the remaining y's from the previous step, and then applied the collections of formal solutions to the set of the remaining y's from the previous step, and analyzed the set of y's for which (at least) one of the inequalities from our given system is in fact an equality when we substitute the families of formal solutions we have collected, using an iterative procedure for the analysis of quotient resolutions.

When the set $EAE(p)$ is defined using a (finite) disjunction of conjunctions of a system of equalities and a system of inequalities, we work in parallel with each of the conjunctions in each step of the iterative procedure that constructs the tree of stratified sets. In each step of this iterative procedure, we do the following for the indices j, $1 \leq j \leq r$, in parallel. We first collect all the formal solutions of the system of equalities $\Sigma_j(x, y, w, p, a) = 1$ defined over closures of the resolutions of the remaining y's from the previous step, and then apply the collections of these formal solutions to the set of the remaining y's from the previous step, and analyze the set of y's for which (at least) one of the inequalities from the system $\Psi_j(x, y, w, p, a)$ is in fact an equality when we substitute the families of formal solutions we have collected, using the iterative procedure for the analysis of quotient resolutions.

The termination of this modified procedure for the construction of the tree of stratified sets follows using the same argument used in proving Theorem 1.4 (Theorem 1.18). Given the tree of stratified sets, the analysis of the set $EAE(p)$ is identical to the analysis described in proving Theorem 1.4. This finally shows that the set $EAE(p)$ is in the Boolean algebra of AE sets, also when $EAE(p)$ is defined using a (finite) disjunction of conjunctions of a system of equalities and a system of inequalities, and hence concludes the proof of Theorem 1.3 in the minimal (graded) rank case. ∎

2. The tree of stratified sets (general case)

To obtain quantifier elimination of elementary predicates over a free group, our goal is to show that the Boolean algebra of AE sets is invariant under projections. For presentation purposes, we presented our approach to the analysis of the projection of the Boolean algebra of AE sets assuming the (graded) limit groups that appear in our procedure are of minimal (graded) rank in the first section of this paper. In this section we combine the approach presented in the previous section, together with the concepts and techniques that appear in section 4 of [Se4], to generalize the construction of the tree of stratified sets, that is presented in the previous section assuming the limit groups in question are of minimal (graded) rank. The tree of stratified sets constructed in this section is combined with a generalization of the "sieve" method for identifying witnesses presented in the next paper, to finally prove the invariance of the Boolean algebra of AE sets under projection in the general case.

By Lemmas 1.1 and 1.2, the analysis of the projection of the Boolean algebra of AE sets reduces to the analysis of the projection of AE sets. Hence, in order to prove the invariance of the Boolean algebra of AE sets under projection it is enough to prove Theorem 1.3, i.e., to show that the projection of an AE set is in the Boolean algebra of AE sets. As we did in the previous section, we will start by proving Theorem 1.4, i.e., we will show that the projection of an AE set is in the Boolean algebra of AE sets, in the case the AE set has the form

$$AE(w,p) = \forall y\ \exists x\ \Sigma(x,y,w,p,a) = 1 \wedge \Psi(x,y,w,p,a) \neq 1.$$

The generalization to an arbitrary AE set, defined using a finite disjunction of conjunctions of equalities and inequalities, is fairly straightforward, and is presented afterwards.

Throughout this section we will use the notions presented in the previous section and in section 4 of [Se4]. Hence, we assume that the reader is already familiar with both. Since we are proving Theorems 1.3 and 1.4, we will use the notation used in the statement of these theorems. Recall that in stating Theorem 1.4 we defined the set $AE(w,p)$ to be

$$AE(w,p) = \forall y \ \exists x \ \Sigma(x,y,w,p,a) = 1 \wedge \Psi(x,y,w,p,a) \neq 1$$

and its projection $EAE(p)$ to be

$$EAE(p) = \exists w \ \forall y \ \exists x \ \Sigma(x,y,w,p,a) = 1 \wedge \Psi(x,y,w,p,a) \neq 1,$$

and our goal is to show that the set $EAE(p)$ is in the Boolean algebra of AE sets. We start in the same way we started in the minimal (graded) rank case.

Let F_y be the free group $F_y = < y_1, \dots, y_\ell >$, and let

$$\psi_1(x,y,w,p,a) = 1, \dots, \psi_q(x,y,w,p,a) = 1$$

be the defining equations of the system $\Psi(x,y,w,p,a) = 1$. By ([Se2], 1.2) for every $p_0 \in EAE(p)$, there exists some (witness) w_0, and a formal solution $x = x_{w_0,p_0}(y,a)$, so that the words corresponding to the defining equations of the system $\Sigma(x_{w_0,p_0}(y,a), y, w_0, p_0, a) = 1$ are trivial in the free group $F_k * F_y = < a, y >$, and the sentence

$$\exists y \ \psi_1(x_{w_0,p_0}(y,a), y, w_0, p_0, a) \neq 1 \wedge \cdots \wedge \psi_q(x_{w_0,p_0}(y,a), y, w_0, p_0, a) \neq 1$$

is a true sentence in F_k.

By the construction of graded formal limit groups presented in section 3 of [Se2], viewing $WP = < w,p >$ as the parameter subgroup, one can associate with the free group F_y and the system of equations $\Sigma(x,y,w,p,a) = 1$ a (canonical) finite collection of graded formal limit groups

$$GFL_1(x,y,w,p,a), \dots, GFL_r(x,y,w,p,a),$$

so that every formal solution $x = x_{w_0,p_0}(y,a)$ of the system $\Sigma(x,y,w,p,a) = 1$ factors through at least one of the resolutions of the graded formal Makanin–Razborov diagrams of the graded formal limit groups

$$GFL_1(x,y,w,p,a), \dots, GFL_r(x,y,w,p,a).$$

Viewing $WP = < w,p >$ as the parameter subgroup, each graded formal resolution in the graded formal Makanin–Razborov diagrams of the graded formal limit groups $GFL_1(x,y,w,p,a), \dots, GFL_r(x,y,w,p,a)$ terminates in either a rigid formal limit group of the form $WPRgd(h_R,w,p,a) * F_y$, where

$WPRgd(h_R,w,p,a)$ is a rigid (not formal!) graded limit group (with respect to WP), or in a solid formal limit group of the form $WPSld(h_S,w,p,a)*F_y$, where $WPSld(h_S,w,p,a)$ is a solid (not formal!) graded limit group. Note that by ([Se3], 2.5), for each specialization (w_0,p_0) there exists a global bound on the number of rigid solutions of the form (h_R,w_0,p_0,a) of any of the rigid graded limit groups $WPRgd(h_R,w,p,a)$, and by ([Se3], 2.9), for each specialization (w_0,p_0) there exists a global bound on the number of strictly solid families of solutions of the form (h_S,w_0,p_0,a) of any of the solid graded limit groups $WPSld(h_S,w,p,a)$.

Let $WPRgd(h_R,w,p,a)$ be one of the terminating rigid graded limit groups in the taut formal graded Makanin–Razborov diagram with respect to the defining parameters $WP = < w,p >$. The modular groups associated with each (ungraded) resolution that terminates in a rigid specialization of the rigid graded limit group $WPRgd(h_R,w,p,a)$ that lies outside the singular locus associated with $WPRgd(h_R,w,p,a)$ (see section 11 in [Se1] for the definition of the singular locus) are compatible with the graded formal resolution that terminates in the rigid graded formal limit group $WPRgd(h_R,w,p,a)*F_y$, i.e., the "tower" of modular groups in such formal graded resolutions is independent of the particular rigid specialization of $WPRgd(h_R,w,p,a)$. Therefore, we separate the various strata in the singular locus of the graded limit groups $WPRgd(h_R,w,p,a)$, and use the "tower" of modular groups that lie "above" each of the rigid formal graded limit groups $WPRgd(h_R,w,p,a)*F_y$ to associate a (usually infinite) system of equations (in the variables (y,h_R,w,p), and variables corresponding to the free variables that are dropped along the graded formal resolution, and coefficients in F_k) corresponding to each of the equations in the system $\Psi(x,y,w,p,a)=1$. By Guba's theorem [Gu], this infinite system of equations is equivalent to a finite system of equations. Since the variables corresponding to free variables dropped along the graded formal resolution can get an arbitrary value, the finite system of equations obtained using Guba's theorem is equivalent to a finite collection of systems of equations in the variables (y,h_R,w,p) and coefficients in F_k. We denote each of the obtained systems $\lambda_R(y,h_R,w,p,a)=1$.

Similarly, with each (stratum of the) terminating solid formal graded limit group $WPSld(h_S,w,p,a)*F_y$ we associate a finite collection of systems of equations that we denote $\lambda_S(y,h_S,w,p,a)=1$. Note that since we used the modular groups associated with the graded formal resolution to construct the systems $\lambda_S(y,h_S,w,p,a)$, if the system of equations holds for a particular value of the variables (y,h_S,w,p,a), then it continues to hold if we change the value of the variables h_S within its strictly solid family.

From now on we work with each of the graded formal resolutions

$$WPGFRes_i(x,y,w,p,a),$$

and its terminating rigid formal graded limit group, $WPRgd_i(h_R,w,p,a) * F_y$, or solid graded formal limit group, $WPSld_i(h_S,w,p,a) * F_y$, and the (finite collection of) systems of equations associated with them in parallel, so we may restrict our attention to one of these graded formal resolutions, omit its index and denote it $WPGFRes(x,y,w,p,a)$. Note that each of these graded formal resolutions is with respect to the parameter subgroup $WP = < w,p >$.

Suppose that the graded formal resolution $WPGFRes(x,y,w,p,a)$ terminates in the rigid graded formal limit group $WPRgd(h_R,w,p,a) * F_y$. Let

$$\lambda WPRGL_1(y,h_R,w,p,a),\ldots,\lambda WPRGL_d(y,h_R,w,p,a)$$

be the canonical collection of maximal graded limit groups (with respect to the parameter subgroup $WP = < w,p >$), that correspond to the set of specializations (y,h_R,w,p,a), for which (h_R,w,p,a) is a rigid specialization of $WPRgd(h_R,w,p,a)$ that lies outside the singular locus, (y,h_R,w,p,a) factors through the graded formal resolution $WPGFRes(x,y,w,p,a)$, and

$$\lambda_R(y,h_R,w,p,a) = 1.$$

Similarly, if the graded formal resolution $WPGFRes(x,y,w,p,a)$ terminates in the solid formal graded limit group $WPSld(h_S,w,p,a) * F_y$, we associate with the solid graded limit group $WPSld(h_S,w,p,a)$ and the system of equations $\lambda_S(y,h_S,w,p,a)$ the canonical collection of maximal graded limit groups

$$\lambda WPSGL_1(y,h_S,w,p,a),\ldots,\lambda WPSGL_d(y,h_S,w,p,a)$$

that correspond to the set of specializations (y,h_S,w,p,a), for which (h_S,w,p,a) is a strictly solid specialization of $WPSld(h_S,w,p,a)$ that lies outside the singular locus, (y,h_S,w,p,a) factors through the graded formal resolution $WPGFRes(x,y,w,p,a)$, and $\lambda_S(y,h_S,w,p,a) = 1$ (again, note that this set is invariant under changing the values of the variables h_S within their strictly solid family).

At this point we need to collect the "remaining" set of specializations of the variables y for each value of our parameters (w,p). Suppose that the terminating graded limit group of the formal graded resolution, $WPGFRes(x,y,w,p,a)$, is the rigid formal graded limit group, $Rgd(h_R, w, p, a) * F_y$, or the solid graded limit group, $Sld(h_S, w, p, a) * F_y$. With each of the graded limit groups

$$\lambda WPRGL_1(y, h_R, w, p, a), \ldots, \lambda WPRGL_d(y, h_R, w, p, a)$$

or $\lambda WPRGL_1(y, h_S, w, p, a), \ldots, \lambda WPRGL_d(y, h_S, w, p, a)$ (depending on whether the graded formal resolution $WPGFRes(x, y, w, p, a)$ terminates in a rigid or solid limit group with respect to $WP = < w, p >$), we associate its taut graded Makanin–Razborov diagram with respect to the parameter subgroup $< h_R, w, p >$, or $< h_S, w, p >$, in correspondence. (Note that before this step we considered the parameter subgroup to be $WP = < w, p >$, and from this point we enlarged it to be $< h_R, w, p >$ or $< h_S, w, p >$. We are able to do that, because of the global bound on the number of rigid and strictly solid families (theorems 2.5 and 2.9 in [Se3]), and since the set of the remaining y's depends only on the strictly solid family of a given specialization, (h_S, w, p, a), and not on the specific specialization in the family.) Each graded resolution $\lambda WPGRes(y, h_R, w, p, a)$ (or $\lambda WPGRes(y, h_S, p, a)$) in one of the taut graded diagrams of the graded limit groups $\lambda WPRGL_1(y, h_R, w, p, a), \ldots, \lambda WPRGL_d(y, h_R, w, p, a)$ (or

$$\lambda WPRGL_1(y, h_S, w, p, a), \ldots, \lambda WPRGL_d(y, h_S, w, p, a))$$

terminates in either a rigid graded limit group (with respect to $< h_R, w, p >$ or $< h_S, w, p >$), which we denote

$$\lambda WPRgd(g_R, h_R, w, p, a) \quad (\text{or } \lambda WPRgd(g_R, h_S, w, p, a)),$$

or a solid graded limit group which we denote

$$\lambda WPSld(g_S, h_R, w, p, a) \quad (\text{or } \lambda WPSld(g_S, h_S, w, p, a)).$$

With each graded resolution,

$$\lambda WPGRes(y, h_R, w, p, a) \quad \text{or } \lambda WPGRes(y, h_S, w, p, a),$$

we associate the finite set of taut resolutions and their terminal rigid or solid limit groups, associated with the various strata in the singular locus associated with the graded resolution. We add the resolutions associated with the singular loci to the list of resolutions containing the entire set of the remaining y's.

To construct the graded formal resolutions $GFRes(x, y, w, p, a)$, we have collected all the formal solutions $x_{(w,p)}(y, a)$ for which all the words corresponding to the equations in $\Sigma(x_{(w,p)}(y, a), y, w, p, a) = 1$ represent the trivial words in $F_{a,y} =< a, y >$. By theorem 1.18 of [Se2], if $p_0 \in EAE(p)$ then there must exist some "witness" w_0, and a formal solution $x_{(w_0,p_0)}(y,a)$, so that the maximal limit groups corresponding to each of the equations in the system $\Psi(x_{(w_0,p_0)}(y,a), y, w, p, a) = 1$ are all proper quotients of the free group $< a, y >= F_k * F_y$. Hence, for every $p_0 \in EAE(p)$ there must exist some witness w_0, and a rigid specialization (hr_0, w_0, p_0, a) of one of the rigid limit groups $WPRgd(h_R, w, p, a)$, or a strictly solid specialization (hs_0, w_0, p_0, a) of one of the solid limit groups $WPSld(h_S, w, p, a)$, so that every ungraded resolution $\lambda WPGRes(y, hr_0, w_0, p_0, a)$ (or $\lambda WPGRes(y, hs_0, w_0, p_0, a)$) does not correspond to the entire set of y's but rather to a resolution of a limit group which is a proper quotient of the free group $< a, y >= F_k * F_y$.

Therefore, the outcome of the initial step of our "trial and error" procedure is a decrease in the complexity (definitions 1.14 and 3.2 in [Se4]) of the ungraded resolutions of y's associated with each $p_0 \in AE(p)$, at least for one (rigid or strictly solid) specialization (hr_0, w_0, p_0) or (hs_0, w_0, p_0). Each of the next steps of the procedure is meant to sequentially decrease either the complexity of the "remaining" ungraded resolutions or the Zariski closures of certain sets of specializations associated with the data-structures we construct, as we did in the iterative procedure for validation of an AE sentence ([Se4], section 4). As in the minimal (graded) rank case, once the iterative procedure terminates, we present a second iterative procedure, that uses the outcome of the first iterative procedure (the *tree of stratified sets*) to sequentially approximate the set $EAE(p)$ by sets which are all in the Boolean algebra of AE sets. Finally, we show that the approximations we construct in the second iterative procedure become identical with the set $EAE(p)$ after finitely many steps. Since the approximations are all in the Boolean algebra of AE sets, this will imply that an EAE set is indeed in the Boolean algebra of AE sets, which finally proves Theorem 1.4.

For the continuation of the iterative procedure we will denote (for brevity) each of the limit groups $WPRgd(h_R, w, p, a)$ or $WPSld(h_S, w, p, a)$ by $WPH(h, w, p, a)$, and each of the limit groups

$$\lambda WPRgd(g_R, h_R, w, p, a), \lambda WPRgd(g_R, h_S, w, p, a), \lambda WPSld(g_S, h_R, w, p, a), \\ \lambda WPSld(g_S, h_S, w, p, a)$$

by $WPHG(g, h, w, p, a)$. We will also denote the graded resolution

$$\lambda WPGRes(y, h_R, w, p, a) \quad \text{or} \quad \lambda WPGRes(y, h_S, w, p, a)$$

by $\lambda WPHRes(y, h, w, p, a)$. Our treatment of these limit groups will be conducted in parallel, so we don't keep the indices associated with each of these (finite collections of) limit groups. Also, the rest of our "trial and error" procedure does not depend in an essential way on the type (rigid or solid) of the terminating graded limit groups in the preliminary two steps, hence we do not keep notation for the type of each of these terminating limit groups.

We continue to the next step of our iterative procedure for the analysis of the set $EAE(p)$ only with graded resolutions $\lambda WPHRes(y,h,w,p,a)$, for which each ungraded resolution, associated with a specialization of the corresponding terminating rigid or solid limit group $WPHG$, does not correspond to the entire limit group $F_k * F_y$, but rather to a resolution of a proper quotient of it. For each tuple (g_0,h_0,w_0,p_0,a), which is either a rigid or a strictly solid specialization of such a terminating limit group $WPHG$, there is an ungraded (well-separated) resolution associated. The associated ungraded resolution depends only on the strictly solid family of the specialization in case the corresponding terminating limit group is solid. Also, the ungraded resolution may be degenerate, so we separate the finitely many possible types of ungraded resolutions associated with a rigid or solid specialization (g_0,h_0,w_0,p_0,a) of $WPHG$ according to the strata of the corresponding singular locus (see section 11 of [Se1]), and treat those strata in parallel.

With the ungraded well-separated resolution, associated with a rigid or strictly solid specialization of the terminal rigid or solid limit group $WPHG$, we associate its completion. Given a tuple (g_0,h_0,w_0,p_0,a), which is a rigid or strictly solid specialization of $WPHG$, we collect all the formal solutions $\{x_{(g_0,h_0,w_0,p_0)}(s,z,y,a)\}$ for which the words corresponding to the equations in the system

$$\Sigma(x_{(g_0,h_0,w_0,p_0)}(s,z,y,a),y,w_0,p_0,a)=1$$

are the trivial words in some closure of the completion of the ungraded resolution associated with the given specialization. Using the construction of graded formal limit groups presented in section 3 of [Se2], and viewing the subgroup $<g,h,w,p>$ as parameters from the entire collection of formal solutions for all possible specializations (g_0,h_0,w_0,p_0,a), which are rigid or strictly solid specializations of $WPHG$, we can construct a graded formal Makanin–Razborov diagram so that any formal solution defined over a closure of an ungraded resolution associated with a rigid or strictly solid specialization of $WPHG$ factors through one of the graded formal Makanin–Razborov resolutions in this diagram.

Let $GFL_1(x,z,y,g_1,h_1,w,p,a),\ldots,GFL_r(x,z,y,g_1,h_1,w,p,a)$ be the maximal graded formal limit groups constructed from the collection of formal solutions associated with the graded limit group $WPHG$. By section 3 of [Se2], with each of the graded formal limit groups there is an associated graded formal Makanin–Razborov diagram with respect to the parameter subgroup $< g_1, h_1, w, p >$. By theorems 3.7 and 3.8 of [Se2], each of the graded formal resolutions in the graded formal Makanin–Razborov diagram associated with each of the graded formal limit groups

$$GFL_1(x,z,y,g_1,h_1,w,p,a), \ldots, GFL_r(x,z,y,g_1,h_1,w,p,a)$$

(with respect to $< g_1, h_1, w, p >$) terminates in either a group of the form

$$WPHGRgd(h_2^R, g_1, h_1, w, p, a) *_{Term(g_1,h_1,w,p,a)} GFCl(s,z,y,g_1,h_1,w,p,a),$$

where $WPHGRgd(h_2^R, g_1, h_1, w, p, a)$ is a graded (not formal!) limit group which is rigid with respect to the parameter subgroup $< g_1, h_1, w, p >$ and $GFCl(s,z,y,g_1,h_1,w,p,a)$ is a graded formal closure in the (well-separated) graded formal resolution

$$WPHGFRes(x,z,y,g_1,h_1,w,p,a)$$

associated with the graded limit group $WPHG$ (graded formal closure is presented in definition 3.4 in [Se2]). We will denote a terminating rigid graded limit group $WPHGRgd(h_2^R, g_1, h_1, w, p, a)$ by $WPHGH^R$.

Alternatively, the terminating graded formal limit group of a graded formal resolution in one of the graded formal Makanin–Razborov diagrams associated with the graded formal limit groups

$$GFL_1(x,z,y,g_1,h_1,w,p,a), \ldots, GFL_r(x,z,y,g_1,h_1,w,p,a)$$

is of the form

$$WPHGSld(h_2^S, g_1, h_1, w, p, a) *_{Term(g_1,h_1,w,p,a)} GFCl(s,z,y,g_1,h_1,w,p,a),$$

where $WPHGSld(h_2^S, g_1, h_1, w, p, a)$ is a graded (not formal!) limit group which is solid with respect to the parameter subgroup $< g_1, h_1, w, p >$, and $GFCl(s,z,y,g_1,h_1,w,p,a)$ is a graded formal closure in the graded formal resolution $WPHGFRes(x,z,y,g_1,h_1,w,p,a)$ associated with the graded limit group $WPHG$. We will denote the terminating solid graded limit group $WPHGSld(h_2^S, g_1, h_1, w, p, a)$ by $WHGH^S$.

With each terminal graded formal limit group of a resolution in the formal Makanin–Razborov diagrams of the graded formal limit groups

$$GFL(x,z,y,g_1,h_1,w,p,a)$$

we associate an **anvil**, which is set to be the corresponding terminal formal limit group. We denote the anvil $Anv(WPHGH)(s,z,y,h_2,g_1,h_1,w,p,a)$. With the anvil we associate a family of formal solutions $\{x_\alpha(s,z,y,h_2,g_1,h_1,w,p,a)\}$, obtained using the graded modular groups associated with the corresponding graded formal resolution $WPHGFRes$. Hence, by the construction of the graded formal resolution (presented in section 3 of [Se2]), for each specialization $(s_0,z_0,y_0,h2_0,g1_0,h1_0,w_0,p_0,a)$ that factors through the anvil, $Anv(WPHGH)$,

$$\Sigma(x_\alpha(s_0,z_0,y_0,h2_0,g1_0,h1_0,w_0,p_0,a),y_0,w_0,p_0,a)=1.$$

At this point we collected the remaining set of y's for every possible value of the tuple w,p,h_1 using the limit groups $\lambda WPHL$, and their resolutions $\lambda WPHRes$. We further collected all the possible formal solutions defined over (closures of) the resolutions $\lambda WPHRes$ using the graded formal resolutions $WPHGFRes$, that we also called anvils, and denoted $Anv(WPHGH)$.

To proceed to the next step of the procedure, we need to collect the set of specializations of the variables y that are left after applying the families of formal solutions encoded by the graded formal resolutions $WPHGFRes$, for each possible specialization of the tuple w,p,h_1, and we need to do it uniformly in the variables w,p,h_1. Hence, we do not need to consider all the specializations that factor through the anvils, but only those that satisfy the (usually infinite) systems of equations which correspond to the families of formal solutions that are encoded by the formal graded resolutions, $WPHGFRes$, which are defined over the anvils, $Anv(WPHGH)$.

In the iterative procedure for validation of an AE sentence, presented in section 4 in [Se4], we proceeded to the next step of the iterative procedure not with all the specializations of the anvil that satisfy one of the systems of equations associated with the graded formal resolutions, but rather with those specializations that are in *shortest form* (definition 4.1 in [Se4]). For technical reasons, in order to apply a similar procedure in the graded case (i.e., uniformly in the specializations of w,p,h_1), we do not use (graded) shortest form specializations, but we rather use (multi-graded) resolutions, modular groups, and limit groups that we associate with each level of the given anvils. These *auxiliary resolutions* and *auxiliary limit groups* enable us to apply the main principles of the iterative procedure presented in section 4 in [Se4] uniformly (in the specializations of w,p,h_1).

Definition 2.1: Let an anvil, $Anv(WPHGH)(s,z,y,h_2,g_1,h_1,w,p,a)$, be the terminal formal limit group in the graded formal resolution, $WPHGFRes$.

With the anvil we associate a collection of (multi-graded) **auxiliary resolutions** and (multi-graded) **auxiliary limit groups**.

Recall that the anvil has the structure of a completed resolution, and it is constructed from a graded formal closure of a graded resolution, $\lambda WPHRes(y,h_1,w,p,a)$. Suppose that this resolution, hence the anvil, contains ℓ levels. Let $Rlim(y,h_1,w,p,a)$ be the subgroup generated by $< y,h_1,w,p,a >$ in the limit group associated with the anvil, and let $Rlim(z_i,h_1,w,p,a>$ be its image in the (graded) limit group associated with the i-th level in the anvil, where $1 \leq i \leq \ell$.

With the anvil, we associate a taut multi-graded Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level 2, with respect to the non-QH, non-abelian vertex groups and edge groups in the (given) graded abelian decomposition associated with the top level of the anvil, i.e., the graded abelian decomposition associated with the subgroup $Rlim(y,h_1,w,p,a)$. Similarly, with each level i in the anvil, $1 \leq i \leq \ell-1$, we associate a multi-graded taut Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level $i+1$, with respect to the non-QH, non-abelian vertex groups and edge groups in the (given) graded abelian decomposition associated with the i-th level in the anvil, i.e., the graded abelian decomposition associated with the subgroup $Rlim(z_i,h_1,w,p,a)$.

We call each of the resolutions in these multi-graded diagrams a (multi-graded) **auxiliary resolution**, and its terminating solid or rigid limit group a (multi-graded) **auxiliary limit group**, which we denote

$$Aux(WPHGH)(s,z,y,h_2,g_1,h_1,w,p,a).$$

Naturally, with each auxiliary resolution we associate its modular groups, which we call **auxiliary modular groups**. In the sequel, we call the auxiliary resolutions associated with the tower containing all the levels up to level 2 (all the levels except the top level), the **highest level**.

At this stage we continue in a similar way to what we did in the initial step. Having constructed the anvils, $Anv(WPHGH)$, the families of formal solutions defined over them, and the auxiliary limit groups and resolutions, we are ready to collect all the specializations of the variables y (for any given value of the parameter subgroup $< w,p,h_1 >$) for which the (families of) formal solutions defined in the first two iterates of our procedure didn't provide a proof. The collection of the remaining y's needs to be done uniformly in the values of the parameters w,p,h_1.

In collecting the set of the remaining y's, we apply both the families of formal solutions that are encoded by the (graded) formal modular groups associated with the formal resolution that is associated with the anvil, and the auxiliary modular groups that are associated with the multi-graded auxiliary resolution that is associated with the anvil, $Anv(WPHGH)$.

Suppose that the auxiliary resolution that is associated with the anvil, $Anv(WPHGH)$, is highest level (i.e., it is associated with the limit group that appears in the second level of the anvil). Let

$$(s_0, z_0, y_0, h_2(0), g_2(0), h_1(0), w_0, p_0, a)$$

be a specialization of the anvil, $Anv(WPHGH)$, so that the restriction of this specialization to the limit group that is associated with the second level of the anvil factors through the auxiliary limit group, $Aux(WPHGH)$, that is associated with the anvil. Let φ_β be an element in the auxiliary modular group that is associated with the auxiliary resolution. φ_β acts on specializations of the auxiliary limit group, $Aux(WPHGH)$. Hence, given the specialization $(s_0, z_0, y_0, h_2(0), g_2(0), h_1(0), w_0, p_0, a)$, φ_β acts on the restriction of this specialization to the subgroup associated with the second level of the anvil. By the structure of the anvil, which is associated with a completion of a resolution, and the structure of the auxiliary resolution (that is multi-graded with respect to the non-abelian, non-QH vertex groups, and edge groups in the given graded abelian decomposition of the subgroup $< y, h_1, w, p, a >$), the specialization of the limit group associated with the second level of the anvil obtained by the action of φ_β, can be completed to a specialization of the ambient anvil, $Anv(WPHGH)$, without changing the specialization $(y_0, h_1(0), w_0, p_0, a)$ of the subgroup $< y, h_1, w, p, a >$, that is associated with the top level of the anvil.

We denote the obtained specialization of the anvil,

$$\varphi_\beta(s_0, z_0, y_0, h_2(0), g_2(0), h_1(0), w_0, p_0, a).$$

Note that by the above observation, to collect the remaining sets of y's, it is enough to collect all the specializations of the associated auxiliary limit group, and when we impose a formal solution on such a specialization, we are able to impose it not only on the specialization, but also on its images under the ("extended") actions of all the elements φ_β from the associated auxiliary modular group. Also, we have considered highest level auxiliary resolutions, but precisely the same argument applies to an arbitrary auxiliary resolution.

Given an anvil, $Anv(WPHGH)$, and an auxiliary resolution, we look at the entire set of rigid or strictly solid specializations of the associated auxiliary limit group (Definition 2.1), and their extensions to specializations of the anvil, $(s,z,y,h_2,g_1,h_1,w,p,a)$, for which for the entire family of formal solutions $\{x_\alpha(s,z,y,h_2,g_1,h_1,w,p,a)\}$ associated with the anvil (i.e., associated with the graded formal resolution, $WPHGFRes$, that is associated with the anvil), and for each element φ_β in the auxiliary modular group associated with the given auxiliary resolution

$$\psi_j(x_\alpha(\varphi_\beta(s,z,y,h_2,g_1,h_1,w,p,a)),y,w,p,a)=1$$

for at least one of the equations ψ_j in the system (of inequalities) $\Psi(x,y,w,p,a) \neq 1$ used to define the set $EAE(p)$. By the standard argument presented in section 5 of [Se1], the entire collection of such (extended) specializations, $(s,z,y,h_2,g_1,h_1,w,p,a)$, is contained in a finite set of maximal graded limit groups (that are all quotients of the anvil, $Anv(WPHGH)$, and of the corresponding auxiliary limit groups)

$$QRlim_1(s,z,y,h_2,g_1,h_1,w,p,a),\dots,QRlim_u(s,z,y,h_2,g_1,h_1,w,p,a),$$

that we call **quotient limit groups**. Note that with each such limit group there is an associated anvil, $Anv(WPHGH)$, and an associated auxiliary resolution (hence, an auxiliary limit group as well).

In constructing the system of equations associated with a given anvil, $Anv(WPHGH)$, and its associated auxiliary resolution, we applied the family of formal solutions associated with the graded formal resolution that is associated with the given anvil, and the auxiliary modular groups associated with the auxiliary resolution that is associated with the anvil. Hence, if a specialization of the anvil, $(s,z,y,h_2,g_1,h_1,w,p,a)$, factors through a quotient limit group, $QRlim_i(s,z,y,h_2,g_1,h_1,w,p,a)$, where $1 \leq i \leq u$, and restricts to a rigid or a strictly solid specialization of the associated auxiliary limit group, then the same holds for all the specializations of the form $\varphi_\beta(s,z,y,h_2,g_1,h_1,w,p,a)$, where φ_β is an element of the multi-graded auxiliary modular groups associated with the anvil. Hence, in analyzing the quotient limit groups, $QRlim_i$, $1 \leq i \leq u$, we consider the non-abelian and non-QH vertex groups and edge groups in the multi-graded abelian JSJ decomposition of the auxiliary limit group, as determined only up to (appropriate) conjugacy, and the abelian and QH vertex groups as “formal”, i.e., we are allowed to act on these with their associated modular groups. Adapting this point of view, which is essential along the entire iterative procedure presented in this section, replaces the role of restricting to shortest form specializations in the ungraded case (definition 4.1 in [Se4]), and enables us to exclude the variables that belong to lower levels of the anvil from taking part in the analysis of the (top part of the) quotient limit groups $QRlim_i$, i.e., it allows us to get (certain) "separation of variables" (of different levels) in the analysis of quotient limit groups.

In section 1 we have started the construction of the tree of stratified sets with a graded (minimal rank) resolution, and showed that the complexities of the various (quotient) graded resolutions associated with it, with which we need to continue to the next step of the iterative procedure, are strictly smaller than the complexity of the original resolution. As in the case of general AE sentences that are analyzed in section 4 of [Se4], in the construction of the tree of stratified sets in the general case we are not able to get a reduction in the complexity of the obtained (quotient) graded resolutions in each step of our iterative procedure. To "force" the "size" of the set of the remaining specializations y_0 associated with each specialization of the parameters $< w, p, h_1 >$ to actually decrease, we need to associate with each graded resolution information about certain (multi-graded) resolutions and abelian decompositions associated with it, together with Zariski closures of some subgroups associated with the graded resolution. To carry all the information attached to a graded resolution, we associate a (graded) *data-structure*, and (canonical) resolutions which we call graded *developing resolutions*, with each of the quotient resolutions. We construct the *data-structure* and graded *developing resolutions* iteratively, in a similar way to their construction in the iterative procedure for validation of an AE sentence presented in section 4 of [Se4]. As in the iterative procedure for validation of a sentence, we divide the construction of the graded developing resolution and the associated anvil into several cases, depending on the structure of the graded resolution $\lambda WPHRes(y, h_1, w, p, a)$ associated with the anvil, $Anv(WPHGH)$, with which we started the first step, and the structure of the multi-graded quotient resolutions constructed along the first step. We describe the first step of our iterative procedure, then the general step of the iterative procedure, and finally prove the termination of the iterative procedure for the analysis of an EAE set.

I: The first step. We start the analysis of the remaining set of y's by analyzing those quotient limit groups, $QRlim_i(s, z, y, h_2, g_1, h_1, w, p, a)$, that are associated with the anvils we constructed, and with auxiliary resolutions associated with the tower containing all the levels in those anvils except the top level (i.e., the highest level auxiliary resolutions). Since we analyze these quotient limit groups in parallel, we will omit their index and denote them $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$.

As parts (1) and (2) of the first step of the procedure indicate, we will analyze only multi-graded resolutions of these quotient limit groups that are not of maximal complexity, i.e., resolutions that do not contain a single level with abelian decomposition that has the same structure as the abelian decomposition associated with the top level of the associated anvil, $Anv(WPHGH)$. To analyze (specializations that factor through) multi-graded resolutions of maximal complexity, we will need to use the quotient limit groups associated with auxiliary resolutions that are not of highest level (this is done in part (3) of the first step of the procedure).

(1) Let $Q(y,h_1,w,p,a)$ be the graded limit group generated by $< y,h_1,w,p,a >$ in the limit group associated with the graded resolution $\lambda WPHRes(y,g_1,h_1,w,p,a)$, associated with the anvil,

$$Anv(WPHGH)(s,z,y,h_2,g_1,h_1,w,p,a).$$

$Q(y,h_1,w,p,a)$ is a quotient of the limit group $\lambda WPRGL$ or $\lambda WPSGL$, for which the graded resolution $\lambda WPHRes(y,h_1,w,p,a)$ is one of the resolutions in its graded taut Makanin–Razborov diagram. Let $Q^1(y,h_1,w,p,a)$ be the limit group generated by $< y,h_1,w,p,a >$ in the quotient limit group $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$. If $Q^1(y,h_1,w,p,a)$ is a proper quotient of the subgroup $Q(y,h_1,w,p,a)$, we continue this branch of the iterative procedure by replacing the quotient limit group $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$ with the quotient graded resolutions obtained by starting the initial step of the procedure with the limit group $Q^1(y,h_1,w,p,a)$ instead of the limit group associated with the graded resolution $\lambda WPHGRes(y,g_1,h_1,w,p,a)$ with which we started the initial step.

Since the graded resolution $\lambda WPHGRes(y,g_1,h_1,w,p,a)$ is not of maximal possible complexity, i.e., each ungraded resolution does not correspond to the entire free group F_y with which we started the iterative procedure, and the maximal complexity graded resolutions were collected separately in the initial step of the procedure, in analyzing graded resolutions of $Q^1(y,h_1,w,p,a)$ in this branch of the procedure we need to consider only those graded resolutions which are not of maximal possible complexity, i.e., only those graded quotient resolutions for which each ungraded resolution does not correspond to the entire free group F_y.

(2) At this stage we may assume that $Q^1(y,h_1,w,p,a)$ is isomorphic to $Q(y,h_1,w,p,a)$. We set the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$ to be the non-abelian, non-QH vertex groups in the graded abelian decomposition associated with the top level of the anvil,

$$Anv(WPHGH)(s,z,y,h_2,g_1,h_1,w,p,a).$$

Note that the subgroup generated by $< h_1,w,p,a >$ in the anvil is, by definition, a subgroup of $Base^1_{2,1}$. With the quotient limit group $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$ we associate its multi-graded Makanin–Razborov diagram with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$.

As we remarked earlier, in constructing the multi-graded diagram, we regard the QH and abelian vertex groups in the multi-graded abelian decomposition associated with the auxiliary limit group (which is associated with the anvil), that are all contained in the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$, as "formal", i.e., the only relations they satisfy are those coming from the abelian decomposition associated with the auxiliary limit group. We denote the obtained multi-graded resolutions in the obtained diagram, that we call quotient resolutions, by

$$MGQRes_1(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a),\dots,$$
$$MGQRes_q(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a).$$

If for some QH vertex group Q in the abelian decomposition associated with the top level of the graded resolution $\lambda WPHRes(y,h_1,w,p,a)$, the sequence of abelian decompositions that Q inherits from a multi-graded resolution, $MGQRes_j(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$, is not compatible with the specified collection of s.c.c. on Q that are mapped into the trivial element according to the (taut) graded resolution $\lambda WPHRes(y,h_1,w,p,a)$, we omit the multi-graded resolution $MGQRes_j$ from our list of multi-graded resolutions of the quotient limit group

$$QRlim(s,z,y,h_2,g_1,h_1,w,p,a).$$

We continue with each of the remaining multi-graded quotient resolutions in parallel, hence we omit the index of the specific resolution with which we continue.

In order to bound the complexity of the multi-graded resolutions, $MGQRes_j$, in terms of the complexity of the abelian decomposition associated with the top level of graded resolution, $\lambda WPHRes(y,h_1,w,p,a)$, with which we started the first step, we need to slightly modify the definition of the *complexity* of a resolution to be suitable for the multi-graded case (cf. definitions 1.14 and 3.2 in [Se4]).

Definition 2.2: Let $MGRes(v,P,R_1,\dots,R_d,a)$ be a well-separated complete multi-graded resolution. Let $Q_1,\dots,Q_m$ be the QH subgroups that appear in the abelian decompositions associated with the various levels of $MGRes$. With each QH vertex group Q_j we naturally associate its corresponding (punctured) surface S_j. With each (punctured) surface S_j we may associate an ordered couple $(genus(S_j),|\chi(S_j)|)$. We will assume that the QH subgroups $Q_1,\dots,Q_m$ are ordered according to the lexicographical (decreasing) order of the ordered couples associated with their corresponding surfaces. Let $rk(MGRes)$ be the sum of the ranks of the free factors that are being dropped along the various levels of $MGRes$, and let $factor(MGRes)$ be the number of factors it is terminating with (i.e., the number of factors in the free decomposition associated with its terminal multi-graded subgroup). Let $Abrk(MGRes)$ be the abelian rank of the resolution (see definition 1.14 of [Se4]).

We set the complexity of the multi-graded resolution $MGRes$, denoted $Cmplx(MGRes)$, to be

$$Cmplx(MGRes)=(rk(MGRes)+factor(MGRes),$$
$$(genus(S_1),|\chi(S_1)|),\dots,(genus(S_m),|\chi(S_m)|),\ Abrk(MGRes).$$

On the set of multi-graded resolutions we can define a partial order. Let $MGRes_1$ and $MGRes_2$ be two multi-graded resolutions. We say that $Cmplx(MGRes_1)=Cmplx(MGRes_2)$ if the tuples defining the two complexities are identical. We say that $Cmplx(MGRes_1)<Cmplx(MGRes_2)$ if:

(1) $rk(MGRes_1)+factor(MGRes_1)$ is smaller than

$$rk(MGRes_2)+factor(MGRes_2),$$

(2) the above numbers are equal and the tuple

$$((genus(S^1_1),|\chi(S^1_1)|),\dots,(genus(S^1_{m_1}),|\chi(S^1_{m_1}|))$$

is smaller in the lexicographical order than the tuple

$$((genus(S^2_1),|\chi(S^2_1)|),\dots,(genus(S^2_{m_2}),|\chi(S^2_{m_2}|)),$$

(3) the above numbers and tuples are equal and

$$Abrk(MGRes_1)<Abrk(MGRes_2).$$

Once we have modified the complexity to be suitable for multi-graded resolutions, we can bound the complexity of the quotient resolutions $MGQRes$ in terms of the complexity of the abelian decomposition associated with the top level of the completion of the resolution $\lambda WPHRes$ with which we started the first step.

Proposition 2.3: *The complexity of each of the quotient multi-graded resolutions $MGQRes$ that was not removed from our list (i.e., that is compatible with the given taut structure) is bounded by the complexity of the abelian decomposition associated with the top level of the completion of the resolution $\lambda WPHRes$ with which we started the first step. In case of equality, the multi-graded resolution $MGQRes$ has only one level and its structure is identical to the structure of the abelian decomposition associated with the top level of the completion of $\lambda WPHRes$.*

Proof: Identical to the proof of proposition 4.2 in [Se4]. ∎

By Proposition 2.3, the complexity of each of the multi-graded abelian decompositions associated with the various levels of the multi-graded resolution $MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$ is bounded by the complexity of the graded abelian decomposition associated with the top level of the graded resolution $\lambda WPHRes(y,h_1,w,p,a)$, that is associated with the anvil with which we started. In this part, we also assume that the multi-graded quotient resolution $MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$ is not of maximal possible complexity, i.e., it does not have a single level with a (multi-graded) abelian decomposition identical with the abelian decomposition associated with the top level of $\lambda WPHRes(y,h_1,w,p,a)$. The case of maximal complexity will be treated in the next part of the first step of the procedure. We treat such a multi-graded quotient resolution as we treated a quotient resolution in part (2) of the first step of the iterative procedure for validation of an AE sentence. To handle multi-graded resolutions that are not of maximal complexity, we need the following two basic observations.

Proposition 2.4: *Let $MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$ be one of the multi-graded quotient resolutions constructed above. Recall that the limit group $Q^1(y,h_1,w,p,a)$ is set to be the subgroup generated by $<y,h_1,w,p,a>$ in (the completion of) $MGQRes$. By the construction of a resolution, the limit group $Q^1(y,h_1,w,p,a)$ is mapped into the limit group associated with each of the* *levels of the multi-graded quotient resolution*

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a).$$

Let $Q^1_{term}(y,h_1,w,p,a)$ *be the image of* $Q^1(y,h_1,w,p,a)$ *in the terminal (rigid or solid) limit group of* $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$.

Then the multi-graded resolution $MGQRes$ *can be replaced by two finite collections of (well-separated) multi-graded resolutions, that are all compatible with the top level of the resolution* $\lambda WPHRes$ *associated with the anvil,* $Anv(WPHGH)$, *and are all obtained from* $MGQRes$ *by adding at most a single (terminal) level. Furthermore, all the resolutions in these collections are not of maximal complexity.*

We denote each of the resolutions in these collections, $MGQ'Res$.

- (i) *In the first (possibly empty) collection of multi-graded resolutions, the image of the subgroup* $Q^1(y,h_1,w,p,a)$ *in the terminal limit group of* $MGQ'Res$, $Q^1_{term}(y,h_1,w,p,a)$, *is a proper quotient of* $Q^1(y,h_1,w,p,a)$.
- (ii) *In the second (possibly empty) finite collection of multi-graded resolutions, the terminal limit group of* $MGQ'Res$ *is either a rigid or a solid limit group with respect to the parameter subgroup* $< w,p,h_1 >$, *i.e., the terminal limit group is rigid or solid with respect to the parameter subgroup* $< w,p,h_1 >$, *and not only with respect to the multi-grading with respect to the subgroups* $Base^1_{2,1},\dots,Base^1_{2,v_1}$, *that was used in the construction of the resolution,* $MGQRes$.

Proof: The argument is similar to the argument used to prove proposition 4.3 of [Se4]. By part (1) we may assume that $Q^1(y,h_1,w,p,a)$ is isomorphic to $Rlim(y,h_1,w,p,a)$. Let $MGQTerm(s,z,y,h_1,w,p,a)$ be the terminal rigid or solid (multi-graded) limit group of the multi-graded resolution $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$, and suppose that the image of the subgroup $Q^1(y,h_1,w,p,a)$ in the limit group $MGQTerm$, $Q^1_{term}(y,h_1,w,p,a)$, is isomorphic to $Q^1(y,h_1,w,p,a)$.

With the terminal limit group $MGQTerm(s,z,y,h_1,w,p,a)$, which is assumed to be (multi-graded) rigid or solid with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$, we associate the resolutions that appear in its graded taut Makanin–Razborov diagram with respect to the parameter subgroup $< h_1,w,p >$. We treat these resolutions in parallel, so let $GTRes(s,z,y,h_1,w,p,a)$ be one of these resolutions. By construction, the resolution $GTRes$ terminates in a rigid or solid limit group with respect to the parameter subgroup $< h_1,w,p >$.

Let $Q^1_{top}(y,h_1,w,p,a)$ be the image of $Q^1(y,h_1,w,p,a)$ in the limit group that is associated with the top level of the graded resolution $GTRes$. If Q^1_{top} is a proper quotient of Q^1, part (i) of the proposition holds. If the graded resolution $GTRes$ has a single level, i.e., if the limit group associated with its top level is rigid or solid with respect to $<h_1,w,p>$, part (ii) of the proposition holds. Hence, we may assume that Q^1_{top} is isomorphic to Q^1, and $GTRes$ has more than a single level.

Let Λ_{top} be the (essential) graded abelian decomposition associated with the top level of the graded resolution $GTRes(s,z,y,h_1,w,p,a)$ (graded with respect to the subgroup $<h_1,w,p>$; see definition 1.8 in [Se3] for an essential decomposition). $Q^1_{top}(y,h_1,w,p,a)$ inherits an (essential, graded) abelian decomposition from Λ_{top}. Since $Q^1_{top}(y,h_1,w,p,a)$ is isomorphic to $Rlim(y,h_1,w,p,a)$, the non-QH, non-abelian vertex groups and edge groups in the (essential) graded abelian JSJ decomposition of $Rlim(y,h_1,w,p,a)$ with respect to the parameter subgroup $<h_1,w,p,a>$ have to be elliptic in the graded abelian decomposition inherited by $Q^1_{top}(y,h_1,w,p,a)$ from Λ_{top}.

The auxiliary limit group $Aux(WPHGH)$, being a subgroup of the anvil, $Anv(WPHGH)$, is naturally mapped into the limit group associated with the top level of $GTRes$. Let Δ_{Aux} be the (essential) abelian decomposition associated with the auxiliary limit group (that is associated with the anvil, $Anv(WPHGH)$), $Aux(WPHGH)$. By construction, Δ_{Aux} is multi-graded with respect to the non-abelian, non-QH vertex groups, and edge groups in the abelian JSJ decomposition of $Rlim(y,h_1,w,p,a)$. Since the non-abelian, non-QH vertex groups and edge groups in the (essential) graded abelian JSJ decomposition of $Rlim(y,h_1,w,p,a)$ are elliptic in Λ_{top}, if a non-abelian, non-QH vertex group or an edge group in Δ_{Aux} is not elliptic in Λ_{top}, lemma 1.9 in [Se3] implies that the restriction of the specializations that factor through $GTRes$ to the auxiliary limit group, $Aux(WPHGH)$, are neither rigid nor strictly solid, so we may remove the graded resolution $GTRes$ from our list of graded resolutions.

Each of the subgroups $Base^1_{2,1},\ldots,Base^1_{2,v_1}$ is a factor in the (multi-graded) free decomposition of the auxiliary limit group, $Aux(WPHGH)$, with respect to the non-QH, non-abelian vertex groups and edge groups of $Rlim(y,h_1,w,p,a)$. Hence, each subgroup $Base^1_{2,j}$ inherits an abelian decomposition from Δ_{Aux}, which is a subgraph (of groups) of Δ_{Aux}. We set Θ_{term} to be the abelian decomposition obtained from the multi-graded abelian decomposition associated with the terminal level of $MGQRes$ (which is multi-graded with respect to the subgroups $Base^1_{2,1},\ldots,Base^1_{2,v_1}$), by replacing each of the vertex groups stabilized by one of the subgroups $Base^1_{2,j}$, with a (possibly degenerate) graph of groups obtained from the abelian decomposition associated with $Base^1_{2,j}$ in Δ_{Aux}. Θ_{term} is the multi-graded abelian decomposition of the terminal limit group of $MGQRes$, $MGQTerm$, with respect to the non-abelian, non-QH vertex groups and edge groups in the (essential) graded abelian JSJ decomposition of $Q^1_{term}(y,h_1,w,p,a)$ (which is assumed to be isomorphic to $Rlim(y,h_1,w,p,a)$).

If any of the non-abelian, non-QH vertex groups or any of the edge groups in the multi-graded abelian decomposition Θ_{term} is not elliptic in the graded abelian decomposition Λ_{top}, then the specializations that factor through the graded resolution $GTRes$ are neither rigid nor strictly solid specializations of the terminal limit group, $MGQTerm$, of the multi-graded resolution, $MGQRes$. Hence, in this case we may omit the resolution $GTRes$ from our list of graded resolutions.

We continue iteratively along the levels of the graded resolution $GTRes$, and conclude that if the resolution $GTRes$ was not removed from our list of graded resolutions, then as long as the image of $Q^1(y,h_1,w,p,a)$ in the limit group associated with some level j is isomorphic to $Q^1(y,h_1,w,p,a)$, then the images of all the non-QH, non-abelian vertex groups and the edge groups in the abelian decomposition Θ_{term}, in the graded abelian decomposition associated with level j in $GTRes$, are elliptic.

Therefore, either we get to the terminal level of $GTRes$, which is rigid or solid with respect to $< h_1,w,p >$ (possibility (ii) in the proposition), or we get to some level j for which the image of $Q^1(y,h_1,w,p,a)$ in the limit group associated with that level is a proper quotient of $Q^1(y,h_1,w,p,a)$ (possibility (i) in the proposition). Furthermore, since the non-abelian, non-QH vertex groups and edge groups in Θ_{term} are mapped to elliptic subgroups in all the levels until level j of $GTRes$, the modular groups associated with Θ_{term} are sufficient to map the terminal limit group of $MGQRes$, $MGQTerm$, onto the limit group associated with level j of $GTRes$, hence $MGQRes$ needs to be extended in at most a single level.

If the image of $Q^1(y,h_1,w,p,a)$ in the limit group associated with level j in $GTRes$ is a proper quotient of $Q^1(y,h_1,w,p,a)$, then it is enough to use the modular groups associated with $MGQTerm$ as a multi-graded rigid or solid limit group with respect to $Base^1_{2,1},\ldots,Base^1_{2,v_1}$, to map $MGQTerm$ onto a limit group in which the image of $Q^1(y,h_1,w,p,a)$ is a proper quotient of it. This allows one to continue viewing the QH and abelian vertex groups in the multi-graded abelian decomposition of the auxiliary limit group, $Aux(WPHGH)$, as "formal", i.e., we can still act on them with their associated modular groups in the terminal limit groups as well, and the obtained (extended) specializations still factor through the terminal limit group of $GTRes$, a point of view that is adapted throughout the whole iterative procedure.

If the limit group associated with level j in $GTRes$ is rigid or solid with respect to $< h_1, w, p >$, then we use this terminal limit group to express the rigid or strictly solid families associated with it. Still, in this case, the terminal level of the obtained resolution $MGQ'Res$ does not really take part in the next steps of the procedure (besides expressing the rigid or strictly solid families). Hence, for the purposes of the next steps in the procedure, we are allowed to continue viewing the QH and abelian vertex groups in the multi-graded abelian decomposition associated with the auxiliary resolution associated with the anvil, $Anv(WPHGH)$, as "formal". ∎

By Proposition 2.4, we can either drop the resolution $MGQRes$ from our list of multi-graded resolutions, or we can replace the resolution $MGQRes$ by finitely many resolutions, which for brevity we still denote $MGQRes$, and for each resolution we may assume that either the image of the subgroup $Q^1(y, h_1, w, p, a)$ in the terminal graded limit group of $MGQRes$, $Q^1_{term}(y, h_1, w, p, a)$, is a proper quotient of $Q^1(y, h_1, w, p, a)$, or the terminal graded limit group of $MGQRes$ is rigid or solid with respect to the parameter subgroup $< w, p, h_1 >$. We continue with the resolutions given by Proposition 2.4 in parallel, and continue to denote each of them, $MGQRes$. To continue handling the various multi-graded resolutions $MGQRes$ of the quotient limit group $QRlim(s, z, y, h_2, g_1, h_1, w, p, a)$, we also need the following proposition, which is similar to proposition 4.4 of [Se4].

Proposition 2.5: *Let $MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a)$ be one of the multi-graded quotient resolutions in our list of multi-graded quotient resolutions of*

$$QRlim(s, z, y, h_2, g_1, h_1, w, p, a).$$

Let $Q^1(y, h_1, w, p, a)$ and $Q^1(s, z, y, h_2, g_1, h_1, w, p, a)$ be the subgroups generated by $< y, h_1, w, p, a >$ and $< s, z, y, h_2, g_1, h_1, w, p, a >$ in correspondence, in the limit group associated with the multi-graded quotient resolution $MGQRes$. Let $Q^1_2(y, h_1, w, p, a)$ and $Q^1_2(s, z, y, h_2, g_1, h_1, w, p, a)$ be the images in the limit *group associated with the second level of*

$$MGQRes, GQlim_2(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a),$$

of the subgroups $Q^1(y,h_1,w,p,a)$ *and* $Q^1(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$*, in correspondence.*

Then $Q^1_2(y,h_1,w,p,a)$ *is a quotient of* $Q^1(y,h_1,w,p,a)$*, and*

$$Q^1_2(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

is a proper quotient of $Q^1(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$*.*

Proof: The proposition is simply a basic property of a multi-graded resolution. ∎

Suppose that the image of $Q^1(y,h_1,w,p,a)$ in the limit group associated with the second level of $MGQRes$, $Q^1_2(y,h_1,w,p,a)$, is a proper quotient of $Q^1(y,h_1,w,p,a)$. In this case we do the following.

With the subgroup $Q^1_2(y,h_1,w,p,a)$ we associate the graded resolutions that appear in its graded taut Makanin–Razborov diagram with respect to the subgroup $< h_1,w,p >$

$$GQRes_1(y,h_1,w,p,a),\dots,GQRes_t(y,h_1,w,p,a).$$

We continue with each of the graded resolutions $GQRes_j(y,h_1,w,p,a)$ in parallel.

If the subgroup generated by $< y,h_1,w,p,a >$ in the limit group associated with the resolution $GQRes_j(y,h_1,w,p,a)$ is a proper quotient of $Q^1_2(y,h_1,w,p,a)$, we replace the graded resolution $GQRes_j(y,h_1,w,p,a)$ by starting part (2) of the the initial step with the graded resolution obtained from $MGQRes$ by replacing its second limit group $Q^1_2(s,z,y,h_2,g_1,h_1,w,p,a)$ with the maximal limit groups obtained from all the specializations that factor through both $Q^1_2(s,z,y,h_2,g_1,h_1,w,p,a)$ and the subgroup generated by $< y,h_1,w,p,a >$ in the limit group associated with $GQRes_j(y,h_1,w,p,a)$. If the subgroup generated by $< s,z,y,h_2,g_1,h_1,w,p,a >$ in the obtained (one level) resolution, $QRlim'(s,z,y,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $QRlim(s,z,y,h_2,g_1,w,p,a)$, we replace the obtained resolution by starting the first step of our iterative procedure with the limit group $QRlim'(s,z,y,h_2,g_1,h_1,w,p,a)$ instead of the limit group

$$QRlim(s,z,y,h_2,g_1,h_1,w,p,a),$$

and since the resolution $MGQRes$ is not of maximal complexity, in analyzing the limit group $QRlim'$ we need to consider only those resolutions in its multi-graded (taut) Makanin–Razborov diagram that are not of maximal possible complexity. Hence, for the rest of this part we may assume that the subgroup generated by $< y, h_1, w, p, a >$ in the limit group associated with $GQRes_j(y, h_1, w, p, a)$ is isomorphic to the subgroup generated by $< y, h_1, w, p, a >$ in $Q_2^1(s, z, y, h_2, g_1, h_1, w, p, a)$.

Suppose that a graded quotient resolution $GQRes_j(y, h_1, w, p, a)$ is of maximal possible complexity, i.e., the limit group associated with it is of the form $< h_1, w, p, a > *F_y$. Since the limit group $Q_2^1(y, h_1, w, p, a)$ is a proper quotient of $Q^1(y, h_1, w, p, a)$, and since the limit group $Q_2^1(y, h_1, w, p, a)$ is naturally mapped onto the limit group $< h_1, w, p, a > *F_y$, associated with the graded resolution $GQRes_j(y, h_1, w, p, a)$, the Hopf property for limit groups implies that the subgroup generated by $< h_1, w, p, a >$ in the limit group associated with $GQRes_j(y, h_1, w, p, a)$ is a proper quotient of the limit group generated by $< h_1, w, p, a >$ in $Q^1(y, h_1, w, p, a)$. Hence, we can replace the resolution $GQRes_j(y, h_1, w, p, a)$ by starting the initial step of the procedure with the subgroup $< h_1, w, p, a > *F_y$, where $< h_1, w, p, a >$ is the subgroup generated by these elements in the limit group associated with $GQRes_j(y, h_1, w, p, a)$. Since resolutions of maximal possible complexity of the limit group $Q^1(y, h_1, w, p, a)$ with respect to the parameter subgroup $< h_1, w, p >$, i.e., those corresponding to the entire free group F_y, were already analyzed in the initial step of the procedure, we can omit a graded resolution $GQRes_j(y, h_1, w, p, a)$ of maximal complexity from our list of graded quotient resolutions $\{GQRes_j(y, h_1, w, p, a)\}$. Hence, for the rest of this part we may assume that the graded resolution $GQRes_j(y, h_1, w, p, a)$ with which we continue is not of maximal possible complexity.

Let $CRes_j(y, h_1, w, p, a)$ be the graded resolution obtained from the resolution induced by the subgroup $< y, h_1, w, p, a >$ from the completion of the graded resolution $MGQRes$ (see section 3 of [Se4] for the construction of the induced resolution), followed by the graded resolution $GQRes_j(y, h_1, w, p, a)$.

We now treat each of the graded resolutions $GQRes_j(y, h_1, w, p, a)$, and their associated resolution $CRes_j(y, h_1, w, p, a)$, in a similar way to our treatment of multi-graded quotient resolutions in part (4) of the general step of the iterative procedure for validation of an AE sentence, presented in section 4 of [Se4]. Let $WPHGHG(g_2, h_1, w, p, a)$ be the terminal rigid or solid limit group of the graded resolution $GQRes_j(y, h_1, w, p, a)$ (which is also the terminal rigid or solid limit group of its associated resolution $CRes_j(y, h_1, w, p, a)$). We start by collecting all the formal solutions defined over ungraded resolutions covered by the graded resolution $CRes_j(y, h_1, w, p, a)$. This collection of formal solutions factors through a canonical collection of graded formal limit groups. With each graded formal limit group we associate its graded formal Makanin–Razborov diagram as we did in section 3 of [Se2]. We continue with each of the graded formal resolutions that appear in these diagrams in parallel.

Let $GFRes(x, f, y, g_2, h_1, w, p, a)$ be a graded formal resolution in one of these diagrams, and let $WPHGHGH(h_3, g_2, h_1, w, p, a)$ be its terminating rigid or solid (not formal!) limit group. With the graded formal resolution $GFRes(x, f, y, h_3, g_2, h_1, w, p, a)$ we associate the resolution

$$GRes(f, y, h_3, g_2, h_1, w, p, a),$$

which is the graded (not formal!) resolution associated with the terminal formal limit group of the graded formal resolution $GFRes(x, f, y, h_3, g_2, h_1, w, p, a)$ (i.e., the graded formal closure associated with the formal resolution $GFRes$, amalgamated with the terminal rigid or solid limit group $WPHGHGH$ (see section 3 of [Se2] for the structure and construction of a (graded) formal resolution, and a graded formal closure)).

We set the developing resolution to be the graded resolution $GRes(f, y, h_3, g_2, h_1, w, p, a)$, which is the resolution associated with the terminal graded formal limit group in the graded formal resolution, $GFRes(x, f, y, h_3, g_2, h_1, w, p, a)$. We further set the anvils associated with the developing resolution to be the (canonical) finite set of maximal limit quotients of the group obtained as the amalgamated product of the completion of the developing resolution and the completion of the top level of the multi-graded resolution $MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a)$, amalgamated along the top part of the developing resolution, which was set to be the resolution induced by the subgroup $< y, h_1, w, p, a >$ from the top level of $MGQRes$, enlarged by replacing the subgroup associated with the bottom level in the induced resolution with $Q^1_2(y, h_1, w, p, a)$. We denote each of the (finitely many) anvils $Anv(MGQRes)(t, y, h_1, w, p, a)$. With the anvil and the developing resolution we also associate the terminal rigid or solid limit groups, $WPHGHG(g_2,h_1,w,p,a)$ and $WPHGHGH(h_3,g_2,h_1,w,p,a)$. Note that the completion of the developing resolution is canonically mapped into the anvil, hence the formal solutions encoded by the graded formal resolution, $GFRes$, that are defined over the developing resolution, can be naturally defined over the anvil.

Suppose that $Q^1_2(y,h_1,w,p,a)$ is isomorphic to $Q^1(y,h_1,w,p,a)$. In this case we continue to the next level of the multi-graded quotient resolution $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$. If for some level j of the multi-graded resolution, the image of $Q^1(y,h_1,w,p,a)$ in the limit group associated with this level, $Q^1_j(y,h_1,w,p,a)$, is a proper quotient of $Q^1(y,h_1,w,p,a)$, then from the highest such level j, we can continue as in case $Q^1_2(y,h_1,w,p,a)$ is a proper quotient of $Q^1(y,h_1,w,p,a)$, and associate with the multi-graded resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

a finite collection of developing resolutions, anvils, and a family of formal solutions defined over each of the developing resolutions and its associated anvil.

Finally, suppose that for every level j, the image of $Q^1(y,h_1,w,p,a)$ in the limit group associated with the j-th level of the multi-graded resolution $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$, $Q^1_j(y,h_1,w,p,a)$, is isomorphic to $Q^1(y,h_1,w,p,a)$. In this case, by Proposition 2.4, the terminal limit group of the multi-graded resolution $MGQRes$, $Q^1_{term}(s,z,y,h_1,w,p,a)$, is rigid or solid with respect to the parameter subgroup $<h_1,w,p>$.

We denote the terminal limit group of the multi-graded resolution $MGQRes$, $WPHGHG(g_2,h_1,w,p,a)$. Note that this terminal limit group is rigid or solid with respect to the parameter subgroup $<h_1,w,p>$. We set the graded resolution $CRes(y,g_2,h_1,w,p,a)$ to be the resolution obtained from the resolution induced from the multi-graded resolution $MGQRes$ by the subgroup generated by $<y,h_1,w,p,a>$, by enlarging its terminal limit group to be $WPHGHG$. We collect all the formal solutions defined over ungraded resolutions covered by the graded resolution $CRes(y,g_2,h_1,w,p,a)$. This collection of formal solutions factors through a canonical collection of graded formal limit groups. With each graded formal limit group we associate its graded formal Makanin–Razborov diagram as we did in section 3 of [Se2]. We continue with each of the graded formal resolutions that appear in these diagrams in parallel.

Let $GFRes(x,f,y,g_2,h_1,w,p,a)$ be a graded formal resolution in one of these diagrams, and let $WPHGHGH(h_3,g_2,h_1,w,p,a)$ be its terminating rigid or solid (not formal!) limit group. With the graded formal resolution $GFRes(x,f,y,h_3,g_2,h_1,w,p,a)$ we associate the resolution

$$GRes(f,y,h_3,g_2,h_1,w,p,a),$$

which is the graded (not formal!) resolution associated with the terminal formal limit group of the graded formal resolution

$$GFRes(x,f,y,h_3,g_2,h_1,w,p,a)$$

(i.e., its graded formal closure amalgamated with its terminal rigid or solid limit group). Note that the terminal rigid or solid limit groups of those resolutions is $WPHGHGH(h_3,g_2,h_1,w,p,a)$.

We set the developing resolution to be the resolution

$$GRes(f,y,h_3,g_2,h_1,w,p,a).$$

To set the finite collection of anvils, we first look at the amalgamation of (the completion of) $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ and the (completion of the) resolution, $GRes(f,y,h_3,g_2,h_1,w,p,a)$, amalgamated along the (completion of the) induced resolution, $CRes(y,g_2,h_1,w,p,a)$. With the obtained group we naturally associate a finite collection of maximal limit groups, and set each of them to be an anvil. With the anvil we further associate the terminal rigid or solid limit groups (with respect to $<h_1,w,p>$),

$$WPHGHG(g_2,h_1,w,p,a) \quad \text{and} \quad WPHGHGH(h_3,g_2,h_1,w,p,a).$$

With the developing resolution we associate the family of formal solutions $x_\alpha(f,y,h_3,g_2,h_1,w,p,a)$ encoded by the associated graded formal resolution, $GFRes$, and defined over the developing resolution. Since the developing resolution is canonically mapped into the anvil, the family of formal solutions associated with the developing resolution is naturally defined over the anvil as well.

(3) By part (1) we may assume that $Q^1(y,h_1,w,p,a)$ is isomorphic to $Q(y,h_1,w,p,a)$. Part (2) treats all the cases in which the multi-graded quotient resolution $MGQRes(w,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ is not of maximal possible complexity. Hence, the multi-graded resolutions that are left

in presenting the first step of our procedure for analyzing the set $EAE(p)$ are those of maximal possible complexity, i.e., those multi-graded resolutions $MGQRes$ that have a single level, and an abelian decomposition that has the same (taut) structure as the abelian decomposition associated with the top level of the anvil, $Anv(WPHGH)$.

Conceptually, we treat this case in a similar way to what we did in the minimal rank case, i.e., we continue to lower levels of the anvil and analyze it in a similar way to what we did with the top level. In parts (1) and (2), we have analyzed multi-graded resolutions of quotient limit groups, $QRlim(s, z, y, h_2, g_1, h_1, w, p, a)$, that were associated with the anvil, $Anv(WPHGH)$, and with an auxiliary resolution of highest level, i.e., an auxiliary resolution associated with the tower containing all the levels in the anvil up to level 2 (all levels except the top level).

An auxiliary resolution of highest level (Definition 2.1) is a multi-graded resolution of the subgroup of the anvil, $Anv(WPHGH)$, associated with all its levels except the top one, with respect to the subgroups which are the non-abelian, non-QH vertex groups in the graded abelian decomposition of $Rlim(y, h_1, w, p, a)$. Hence, such a resolution can be extended to a graded resolution of the entire anvil with respect to the subgroup associated with the top level. In the same way, an auxiliary resolution associated with all the levels up to level 3 can be extended to a graded resolution of the entire anvil with respect to the subgroup associated with the top two levels of the anvil.

Let (s, z, y, h_1, w, p, a) be a specialization of the anvil, $Anv(WPHGH)$. Suppose that for every auxiliary resolution of highest level, for which the restriction of the specialization (s, z, y, h_1, w, p, a) factors, the corresponding specialization of the associated auxiliary limit group extends to a corresponding specialization that factors through one of the associated quotient limit groups. Then for every auxiliary resolution that is associated with all the levels up to level 3, through which the restriction of the given specialization factors, the corresponding specialization of the auxiliary limit group extends to a corresponding specialization that factors through at least one of the associated quotient limit groups.

Furthermore, if in addition, for every quotient limit group (associated with auxiliary resolutions of highest level) through which such a specialization factors, it factors through a quotient resolution of maximal complexity of that quotient limit group, then the same holds for the top level of all the quotient resolutions associated with auxiliary resolutions that are associated with all the levels up to level 3. Therefore, to analyze such specializations we can replace the maximal complexity quotient resolutions associated with auxiliary resolutions of highest level, with (top) maximal complexity resolutions of quotient limit groups, that are associated with auxiliary resolutions which are associated with all the levels up to level 3.

Hence, to analyze maximal complexity multi-graded resolutions, we first replace the quotient limit groups associated with auxiliary resolutions of highest level, by those quotient limit groups associated with the anvil, $Anv(WPHGH)$, and with auxiliary resolutions that are associated with towers containing all the levels up to level 3, i.e., all the levels except the top two. We continue with those quotient limit groups in parallel, hence we will omit their index, and (still) denote the quotient limit group with which we continue, $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$.

We start with the multi-graded taut Makanin–Razborov diagram of the quotient limit group, $QRlim$, with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$, where the subgroups $Base^1_{2,j}$, $1 \leq j \leq v_1$, are the non-QH, non-abelian vertex groups and edge groups, in the graded abelian decomposition associated with the top level of the anvil, $Anv(WPHGH)$. We still denote these multi-graded resolutions $MGQRes$.

Since in this part we need to analyze specializations that factor through and are taut with respect to maximal complexity multi-graded resolutions of quotient limit groups associated with auxiliary resolutions of highest level, as we have already explained, we can continue only with those multi-graded resolutions in the taut Makanin–Razborov diagram of $QRlim$ that are of maximal complexity, i.e., that contain a single level with an abelian decomposition which has the same (taut) structure as the abelian decomposition associated with the top level of the anvil, $Anv(WPHGH)$.

If part (1) applies to such a multi-graded resolution $MGQRes$, i.e., if the limit group generated by $< y,h_1,w,p,a >$ in its completion is a proper quotient of the subgroup $Q(y,h_1,w,p,a)$ with which we started this branch of the procedure, we replace this resolution $MGQRes$ by starting the initial step of the procedure with the given proper quotient of $Q(y,h_1,w,p,a)$.

In case the abelian decomposition and the taut structure associated with $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ and the top level of the anvil, $Anv(WPHGH)$, are identical, we use the modular groups associated with the abelian decomposition associated with $MGQRes$ to map the subgroup $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$ into the subgroup of the anvil, $Anv(WPHGH)$, $QRlim_2(s,z,y,h_2,g_1,h_1,w,p,a)$, associated with its second level. We now set the subgroups $Base^1_{3,1},\dots,Base^1_{3,t_1}$ to be the subgroups of the anvil, $Anv(WPHGH)$, corresponding to the non-abelian, non-QH vertex groups in the abelian decomposition associated with the second level of the anvil.

At this point we analyze the quotient limit group

$$QRlim_2(s,z,y,h_2,g_1,h_1,w,p,a)$$

with respect to the subgroups $Base^1_{3,1},\dots,Base^1_{3,t_1}$, exactly as we analyzed the quotient limit group $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$ with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$ in parts (1) and (2), i.e., we associate with

$$QRlim_2(s,z,y,h_2,g_1,h_1,w,p,a)$$

all its multi-graded quotient resolutions with respect to the subgroups

$$Base^1_{3,1},\dots,Base^1_{3,t_1}$$

that are its subgroups, and analyze each of the obtained multi-graded quotient resolutions according to parts (1) to (the first part of) (3). If the multi-graded abelian decomposition associated with a multi-graded quotient resolution of $QRlim_2(s,z,y,h_2,g_1,h_1,w,p,a)$ with respect to the subgroups $Base^1_{3,1},\dots,Base^1_{3,t_1}$ is of maximal possible complexity, and its associated taut structure is identical to the one associated with the second level of the anvil, $Anv(WPHGH)$, i.e., if part (3) applies to an obtained quotient multi-graded resolution, we continue in a similar way to our approach in analyzing multi-graded resolutions for which their top level is of maximal complexity.

To analyze multi-graded resolutions that are of maximal complexity in their top two levels, we replace the quotient limit groups, and analyze quotient limit groups associated with the anvil, $Anv(WPHGH)$, and with auxiliary resolutions that are associated with towers containing all the levels up to level 4, i.e., all the levels apart from the top three.

Given such a quotient limit group, we start with its multi-graded taut Makanin–Razborov diagram with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$. We continue only with resolutions in this diagram that are of maximal complexity, and their abelian decomposition has the same taut structure as the one associated with the top level of the anvil. With such maximal complexity resolutions we continue to the second level. We look at the multi-graded taut Makanin–Razborov diagram of the subgroup of the quotient limit group associated with the second level of the maximal complexity resolutions, $QRlim_2(s,z,y,h_2,g_1,h_1,w,p,a)$, with respect to the subgroups $Base^1_{3,1},\dots,Base^1_{3,t_1}$ (where the subgroups $Base^1_{3,j}$ are the non-QH, non-abelian vertex groups in the abelian decomposition associated with the second level of the anvil, $Anv(WPHGH)$). Again, we continue only with resolutions that are of maximal complexity, and the taut structure associated with their abelian decomposition is identical to the taut structure associated with the second level of the anvil, $Anv(WPHGH)$.

Let $QRlim_3(s,z,y,h_2,g_2,h_1,w,p,a)$ be the limit group associated with the third level in such a multi-graded resolution (still denoted) $MGQRes$, that is assumed to be maximal complexity in its top two levels. We set the subgroups $Base^1_{4,1},\dots,Base^1_{4,r_1}$ to be the subgroups of the anvil, $Anv(WPHGH)$, corresponding to the non-abelian, non-QH vertex groups in the abelian decomposition associated with the third level of the anvil. At this point we analyze the quotient limit group

$$QRlim_3(s,z,y,h_2,g_1,h_1,w,p,a)$$

with respect to the subgroups $Base^1_{4,1},\dots,Base^1_{4,r_1}$ exactly as we analyzed the quotient limit group $QRlim(s,z,y,h_2,g_1,h_1,w,p,a)$ with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$, i.e., we associate with

$$QRlim_3(s,z,y,h_2,g_1,h_1,w,p,a)$$

all its multi-graded quotient resolutions with respect to the subgroups

$$Base^1_{4,1},\dots,Base^1_{4,r_1}$$

that are its subgroups, and analyze each of the obtained multi-graded quotient resolutions according to parts (1) to (the first part of) (3).

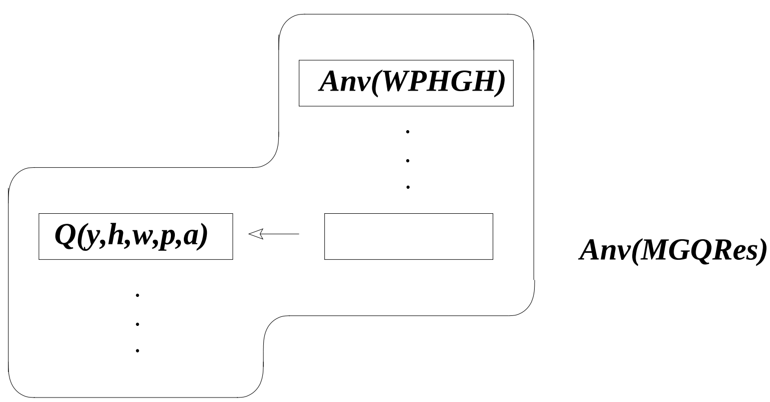
If the multi-graded abelian decomposition associated with a multi-graded quotient resolution of $QRlim_3(s,z,y,h_2,g_1,h_1,w,p,a)$ with respect to the subgroups $Base^1_{4,1},\dots,Base^1_{4,r_1}$ is of maximal possible complexity, and its associated taut structure is identical to the one associated with the third level of the anvil, $Anv(WPHGH)$, we continue to the next levels of the anvil in precisely the same way. At each level i, we consider the quotient limit groups associated with auxiliary resolutions that are associated with the tower containing all the levels up to level $i+1$. Then we analyze the taut Makanin–Razborov diagrams of the limit groups associated with the various levels (from level 1 to level $i-1$), and continue only with those resolutions that are of maximal complexity in all these levels, and the taut structures associated with their abelian decompositions are identical to those associated with the corresponding levels of the anvil, $Anv(WPHGH)$. Finally, we analyze the resolutions in the taut Makanin–Razborov diagram associated with the i-th level according to parts (1), (2), or (the first part of) (3), and continue iteratively.

Let $MGQRes$ be a multi-graded resolution obtained by the above iterative procedure. If there exists a level for which one of the parts (1)–(2) applies, we set a graded resolution, that is essentially the resolution induced by the image of the subgroup $< y, h_1, w, p, a >$, and an anvil, with the limit group associated with this level according to the part (1)–(2) that applies to it.

To set the developing resolutions associated with the resolution $MGQRes$, we first construct a resolution composed from the resolution induced by the subgroup $< y, h_1, w, p, a >$ from the parts of the resolution $MGQRes$ above the level for which parts (1) or (2) apply (i.e., the parts that are of maximal complexity), followed by the graded resolution constructed at that level according to part (1) or (2) (note that the obtained resolution is graded with respect to the parameter subgroup $< h_1, w, p >$). We denote the terminal rigid or solid limit group of the obtained graded resolution $WPHGHG(g_2, h_1, w, p, a)$. Then we collect all the formal solutions defined over the obtained (graded) resolution using the graded formal Makanin–Razborov diagram. We set each of the graded (not formal!) resolutions associated with the terminal limit groups in this graded formal Makanin–Razborov diagram to be a developing resolution. With each developing resolution we associate a family of formal solutions encoded by the graded formal modular groups associated with the graded formal resolution to which it belongs. We further associate with a developing resolution its terminal rigid or solid limit group, which we denote $WPHGHGH(h_3, g_2, h_1, w, p, a)$.

With the developing resolution we associate a finite collection of anvils, that are set to be the maximal limit groups corresponding to the group obtained as the amalgamation of the completion of the multi graded resolution $MGQRes$, and the developing resolution, amalgamated along the subgroup generated by the resolution induced by the subgroup $< y, h_1, w, p, a >$ from $MGQRes$, and the image of the subgroup $< y, h_1, w, p, a >$ in the limit group associated with the level in which part (1) or (2) applies (i.e., precisely as we constructed the anvil in part (2)). With the developing resolution and its associated anvil, we naturally associate a family of formal solutions, $x_\alpha(f, y, h_3, g_2, h_2, g_1, h_1, w, p, a)$, parameterized by the graded formal modular groups associated with the graded formal resolution associated with the developing resolution and the anvil. With the anvil we further associate the terminal rigid or solid limit groups, $WPHGHG$ and $WPHGHGH$, that were used in the construction of the developing resolution.

If all the abelian decompositions associated with the multi-graded resolution used for the construction of the developing resolution are of maximal complexity, i.e., if none of the parts (1)–(2) applies to any of these multi-graded resolutions, we reach a terminal state of our branching process. In this case, we do not include the obtained multi-graded resolution in our list of multi-graded resolutions and their associated anvils and developing resolutions with which we continue to the next step of the procedure. We just associate with the obtained multi-graded resolution its terminal rigid or solid limit group $WPHGHG(g_2, h_2, g_1, h_1, w, p, a)$.



Starting with the anvils, $Anv(WPHGH)$, their associated developing resolutions, and the auxiliary resolutions and quotient limit groups associated with them, we have constructed a finite collection of multi-graded resolutions, $MGQRes$, developing resolutions and anvils, $Anv(MGQRes)(t,y, w, p, a)$. With each couple of a developing resolution and an anvil, we have associated a family of formal solutions defined over them, and parameterized by the modular groups associated with the graded formal resolution associated with the developing resolution.

As in the initial step of the iterative procedure, to complete the *data-structure* with which we continue to the next step, we still need to associate with each anvil, $Anv(MGQRes)$, a finite collection of auxiliary resolutions and auxiliary modular groups (see Definition 2.1). Once the auxiliary resolutions are defined, we are able to get the new quotient limit groups which collect all the specializations that remain after applying the families of formal solutions we have collected and the auxiliary modular groups, precisely as we did before starting the first step of the procedure.

Definition 2.6: Recall that the developing resolution has the structure of a completed resolution, and the subgroup associated with each level of the developing resolution is naturally mapped into the subgroup associated with the corresponding level in the anvil, $Anv(MGQRes)$.

With the anvil, we associate a taut multi-graded Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level 2, with respect to the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian decomposition associated with the limit group that appears in the top level of the anvil, $Anv(MGQRes)$. Similarly, with each level i in the anvil, we associate a multi-graded taut Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level $i+1$, with respect to the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian decomposition associated with the limit group that appears in the i-th level in the anvil (cf. Definition 2.1).

We call each of the resolutions in these multi-graded diagrams a (multi-graded) **auxiliary resolution**, and its terminating solid or rigid limit group a (multi-graded) **auxiliary limit group**, which we denote $Aux(MGQRes)$. Naturally, with each auxiliary resolution we associate its modular groups, which we call **auxiliary modular groups**. In the sequel, we call the auxiliary resolutions associated with the tower containing all the levels up to level 2 (all the levels except the top level), **highest level**.

QH and abelian vertex groups in the abelian decomposition associated with the limit group, $Aux(WPHGH)$, that is associated with the anvil, $Anv(WPHGH)$, with which we started the first step, are considered "formal" along the analysis of a quotient resolution, i.e., it is possible to act on them with their modular group and still get a specialization that factors through the corresponding quotient limit group. When we construct the auxiliary resolution and modular groups associated with a quotient resolution, the QH and abelian vertex groups associated with both the previous auxiliary limit group and the newly constructed one are considered "formal" in the same way.

The collection of multi-graded resolutions, $MGQRes$, the developing resolutions and the anvils, $Anv(MGQRes)$, associated with them, the families of formal solutions defined over them, and their collections of auxiliary resolutions, limit groups, and modular groups, together with the data-structure constructed before starting the first step of the procedure, form the *data-structure* obtained as a result of the first step.

At this stage we continue in a similar way to what we did before starting the first step of the procedure. Given an anvil, $Anv(MGQRes)$, and an auxiliary resolution, we look at the entire set of rigid or strictly solid specializations of the associated auxiliary limit group (Definition 2.6), and their extensions to specializations of the anvil, for which for the entire family of formal solutions, $x_\alpha(t,y,w,p,a)$, associated with the anvil (i.e., associated with the graded formal resolution that is associated with the developing resolution associated with the anvil), and for each element φ_β in the auxiliary modular group associated with the given auxiliary resolution, $\psi_j(x_\alpha(\varphi_\beta(t,y,w,p,a)),y,w,p,a)=1$ for at least one of the equations ψ_j in the system (of inequalities) $\Psi(x,y,w,p,a)\neq 1$ used to define the set $EAE(p)$. By the standard argument presented in section 5 of [Se1], the entire collection of such (extended) specializations, (t,y,w,p,a), is contained in a finite set of maximal graded limit groups (that are all quotients of the anvil, $Anv(MGQRes)$),

$$Q^2Rlim_1(t,y,h_1,w,p,a),\ldots,Q^2Rlim_{u_2}(t,y,h_1,w,p,a),$$

which we call (second) **quotient limit groups**. Note that with each such limit group there is an associated anvil, $Anv(MGQRes)$, and an associated auxiliary resolution. The quotient limit groups we constructed, that collect (uniformly) all the remaining y's for every specialization of the parameter subgroup $<w,p,h_1>$, and their associated data-structure, are the input for the next step of the iterative procedure.

II. The general step. In the initial and first steps of the iterative procedure for analyzing the set $EAE(p)$ we have finally obtained a data structure with finitely many developing resolutions and anvils, and their associated graded formal resolutions, and auxiliary resolutions and limit groups. With a developing resolution and its associated anvil we have associated a family of formal solutions encoded by a graded formal resolution and defined over the developing resolution, which is mapped into the anvil. After presenting the initial and first steps, we finally present the general step of the procedure for analyzing the set $EAE(p)$, and then prove it terminates after finitely many steps.

We define the general step of the procedure inductively. For brevity, we denote the multi-graded resolutions that were obtained in the previous steps of the procedure, $MGQ^m Res(t, y, h_1, w, p, a)$, where m is the index of the step in which they were constructed. With each such multi-graded quotient resolution there is an associated developing resolution, graded formal resolution, (multi-graded) auxiliary resolution and limit group, and an anvil, which we denote $Anv(MGQ^m Res)(t, y, h_1, w, p, a)$. We start the general step of our iterative procedure for the analysis of the set $EAE(p)$ with the (finite) collection of multi-graded quotient resolutions constructed in the previous step, and their associated developing resolutions, graded formal resolutions, anvils, and (multi-graded) auxiliary resolutions and limit groups.

The ultimate goal of the general step of the iterative procedure is to obtain either a strict reduction in the complexity of certain decompositions and resolutions, or a strict reduction in the Zariski closures of certain limit groups associated with the anvils constructed in the previous steps of the procedure. The strict reduction in complexity and Zariski closures will finally guarantee the termination of the iterative procedure after finitely many steps.

Since we treat the anvils in parallel, we present the general (n-th) step of the procedure with one of the anvils, $Anv(MGQ^{n-1} Res)(t, y, h_1, w, p, a)$. With each anvil we have associated a family of formal solutions $x_\alpha(t, y, w, p, a)$ (that are defined over the developing resolution that is mapped into the anvil, and parameterized by the graded formal resolution associated with the developing resolution). Starting with the anvil, $Anv(MGQ^{n-1} Res)(t, y, h_1, w, p, a)$, we impose the family of formal solutions $x_\alpha(t, y, w, p, a)$ associated with the corresponding closure of the developing resolution. Fixing an auxiliary resolution associated with the anvil (see Definitions 2.1 and 2.6), we also use the auxiliary modular groups associated with it.

Given the anvil, $Anv(MGQ^{n-1} Res)$, and an auxiliary resolution associated with it, we look at the set of multi-graded rigid or strictly solid specializations of the auxiliary limit group, that can be extended to specializations

$$(t_0, y_0, h_1(0), w_0, p_0, a)$$

which factor through and are taut with respect to the resolutions associated with the anvil, $Anv(MGQ^{n-1}Res)(t,y,h_1,w,p,a)$, for which for the entire family of formal solutions, $x_\alpha(t,y,h_1,w,p,a)$, associated with the anvil (i.e., associated with the graded formal resolution that is associated with the developing resolution associated with the anvil), and for each element φ_β in the auxiliary modular group associated with the given auxiliary resolution, $\psi_j(x_\alpha(\varphi_\beta(t,y,h_1,w,p,a)),y,w,p,a) = 1$ for at least one of the equations ψ_j in the system (of inequalities) $\Psi(x,y,w,p,a) \neq 1$ used to define the set $EAE(p)$. By the standard argument presented in section 5 of [Se1], the entire collection of such (extended) specializations, (t,y,h_1,w,p,a), is contained in a finite set of maximal graded limit groups (that are all quotients of the anvil, $Anv(MGQ^{n-1}Res)$):

$$Q^nRlim_1(t,y,h_1,w,p,a),\dots,Q^nRlim_{u_n}(t,y,h_1,w,p,a),$$

that we call (n-th) **quotient limit groups**. Note that with each such limit group there is an associated anvil, $Anv(MGQ^{n-1}Res)$, a developing resolution, and an associated auxiliary resolution.

The quotient limit groups we constructed, that collect (uniformly) all the remaining y's for every specialization of the parameter subgroup $< h_1,w,p >$, and their associated data-structure, are the input for the next (n-th) step of the iterative procedure. Since our analysis of these (n-th) quotient limit groups is conducted in parallel, we will omit the indices from these (n-th) quotient limit groups and denote them $Q^nRlim(t,y,h_1,w,p,a)$.

We construct the *data-structure* and *developing resolutions* associated with the anvil and the n-th quotient limit group $Q^nRlim(w,y,a)$ iteratively, in a similar way to our analysis of quotient resolutions in the general step of the procedure for validation of a sentence, presented in section 4 of [Se4]. The analysis we carry out in the general step depends on the structure of the (previously constructed) data-structure, the developing resolutions, auxiliary resolutions, and the multi-graded resolutions constructed in the previous steps of the procedure (note that the construction depends on all the previous steps and not only the last steps). As in the initial and first steps of the procedure, our aim is to obtain a strict decrease in either the Zariski closure or the complexity of the resolution associated with some level of the data structure we construct.

We start the analysis of the remaining set of y's by analyzing those n-th quotient limit groups, $Q^nRlim(t,y,h_1,w,p,a)$, that are associated with the anvils we constructed, and with auxiliary resolutions associated with the tower containing all the levels in those anvils except the top level (i.e., the highest level auxiliary resolutions).

As parts (1)–(4) of the general step of the procedure indicate, we will analyze only multi-graded resolutions of these quotient limit groups that are not of maximal complexity, i.e., resolutions which do not contain a single level with abelian decomposition that has the same structure as the abelian decomposition associated with the top level of the associated anvil, $Anv(MGQ^{n-1}Res)$. To analyze (specializations that factor through) multi-graded resolutions of maximal complexity, we will need to use the quotient limit groups associated with auxiliary resolutions that are not of highest level (this is done in part (5) of the general step of the procedure).

(1) Let $Q^{n-1}(y,h_1,w,p,a)$ be the restricted limit group generated by $< y,h_1,w,p,a >$ in the $n-1$ quotient limit group,

$$Q^{n-1}Rlim(t,y,h_1,w,p,a).$$

Let $Q^n(y,h_1,w,p,a)$ be the limit group generated by $< y,h_1,w,p,a >$ in the n-th quotient limit group $Q^nRlim(t,y,h_1,w,p,a)$. If $Q^n(y,h_1,w,p,a)$ is a proper quotient of the subgroup $Q^{n-1}(y,h_1,w,p,a)$, we continue this branch of the iterative procedure, by starting the initial step of the procedure with the graded limit group $Q^n(y,h_1,w,p,a)$ instead of the graded limit group $Q^{n-1}(y,h_1,w,p,a)$.

Note that in continuing this branch of the procedure, we need to analyze only those resolutions in the graded taut Makanin–Razborov diagram of $Q^n(y,h_1,w,p,a)$ that are not of maximal complexity, i.e., that the ungraded resolutions associated with them do not correspond to the entire free group F_y.

(2) At this stage we may assume that $Q^n(y,h_1,w,p,a)$ is isomorphic to $Q^{n-1}(y,h_1,w,p,a)$. Along the process used to construct the anvil, $Anv(MGQ^{n-1}Res)(t,y,h_1,w,p,a)$, we enlarge the parameter subgroups each time the complexity of the abelian decomposition associated with the top level of the corresponding multi-graded quotient resolution is being reduced. At step m, $1 \leq m \leq n-1$, we set the parameter subgroups to be $Base^{s(m)}_{2,1},\ldots,Base^{s(m)}_{2,v_{s(m)}}$, and the corresponding multi-graded quotient resolution to be

$$MGQ^mRes(t_m,y,Base^{s(m)}_{2,1},\ldots,Base^{s(m)}_{2,v_{s(m)}},a),$$

where $s(m)$ is the number of places in which the parameter subgroups were enlarged along the process (up to step m), and the subindex 2 indicates that the parameters are associated with the second level of the multi-graded resolution.

For each index s, $1 \leq s \leq s(n-1)$, we set $f(s)$ to be the minimal index m, $1 \leq m \leq n-1$, for which $s = s(m)$, and $\ell(s)$ to be the maximal index m for which $s = s(m)$. For each couple of indices m_1, m_2, $1 \leq m_1 \leq m_2 \leq n$, let $Q^{m_2}(t_{m_1}, y, a) = Q^{m_2}(t_{m_1}, y, h_1, w, p, a)$ be the subgroup generated by $< t_{m_1}, y, h_1, w, p, a >$ in the limit group $Q^{m_2}Rlim(t_{m_2}, y, a) = Q^{m_2}Rlim(t_{m_2}, y, h_1, w, p, a)$.

In this part of the general step we assume that the multi-graded quotient resolution $MGQ^{n-1}Res(t, y, Base_{2,1}^{s(n-1)}, \ldots, Base_{2,v_{s(n-1)}}^{s(n-1)}, a)$ is of maximal complexity.

Suppose that for some index s, $1 \leq s \leq s(n-1)-1$, $Q^n(t_{\ell(s)}, y, a)$ is a proper quotient of $Q^{\ell(s)}(t_{\ell(s)}, y, a)$, and let s be the minimal index for which this happens. Then we omit the n-th limit group $Q^nRlim(t, y, h_1, w, p, a)$ from our list of n-th quotient limit groups, and replace it by going back to the $\ell(s)$-th step of the iterative procedure, and start it with the limit group $Q^n(t_{\ell(s)}, y, h_1, w, p, a)$ instead of the $\ell(s)$-th limit group $Q^{\ell(s)}Rlim(t_{\ell(s)}, y, h_1, w, p, a)$ used in the $\ell(s)$-th step of the process that leads to the construction of the anvil,

$$Anv(MGQ^{n-1}Res)(t, y, h_1, w, p, a).$$

Since, by definition of the index $\ell(s)$, the parameter subgroups were enlarged at step $\ell(s)+1$, in analyzing the quotient limit group

$$Q^nRlim(t_{\ell(s)}, y, h_1, w, p, a),$$

we need to take into account only those multi-graded quotient resolutions which are not of maximal complexity, i.e., only those multi-graded quotient resolutions which do not have a single level with a multi-graded abelian decomposition with the same structure as the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQ^{\ell(s)-1}Res(t_{\ell(s)-1}, y, Base_{2,1}^{s(\ell(s)-1)}, \ldots, Base_{2,v_{s(\ell(s)-1)}}^{s(\ell(s)-1)}, a)$$

used in the process of the construction of the anvil,

$$Anv(MGQ^{n-1}Res)(t, y, h_1, w, p, a).$$

Suppose that for $s(n-1)-1$ (hence, for every index s, $1 \leq s \leq s(n-1)-1$), $Q^n(t_{\ell(s)}, y, h_1, w, p, a)$ is isomorphic to

$Q^{\ell(s)}(t_{\ell(s)},y,h_1,w,p,a)$. We set $s(n)=s(n-1)$. Let

$$MGQ^nRes_1(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a),\dots,$$
$$MGQ^nRes_q(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

be the multi-graded resolutions in the multi-graded taut Makanin–Razborov diagram of $Q^nRlim(t,y,h_1,w,p,a)$ with respect to the parameter subgroups $Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)}$. We will treat the multi-graded quotient resolutions $MGQ^nRes_j(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$ in parallel, hence we omit their index.

If there exists a QH vertex group Q in the abelian decomposition associated with the top level of the $\ell(s(n)-1)$ quotient resolution

$$MGQ^{\ell(s(n)-1)}Res(t_{\ell(s(n)-1)},y,Base_{2,1}^{s(n)-1},\dots,Base_{2,v_{s(n)-1}}^{s(n)-1},a)$$

for which the sequence of abelian decompositions inherited by Q from the various levels of the multi-graded resolution

$$MGQ^nRes(t_n,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

is not compatible with the specified collection of s.c.c. on the QH vertex group Q that is mapped to the trivial element in the second level of the $\ell(s(n)-1)$ quotient resolution

$$MGQ^{\ell(s(n)-1)}Res(t_{\ell(s(n)-1)},y,Base_{2,1}^{s(n)-1},\dots,Base_{2,v_{s(n)-1}}^{s(n)-1},a),$$

we omit the multi-graded resolution

$$MGQ^nRes(t_n,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

from our list of multi-graded resolutions.

By Proposition 2.3, the complexities of the abelian decompositions associated with the various levels of each of the n-th multi-graded quotient resolution $MGQ^nRes(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$ are bounded by the complexity of the multi-graded quotient abelian decomposition associated with the top level of $MGQ^{n-1}Res(t,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a)$, and if the complexity of the abelian decomposition associated with some level of $MGQ^nRes(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$ is equal to the complexity of the abelian decomposition associated with the top level of

$$MGQ^{n-1}Res(t,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a),$$

then the n-th multi-graded quotient resolution

$$MGQ^nRes(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

has only one level above the terminating solid or rigid limit group, and the structure of the abelian decomposition associated with this level is identical with the structure of the abelian decomposition associated with the top level of $MGQ^{n-1}Res(t,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a)$. In this part of the n-th step of the procedure, we will also assume that the complexities of the abelian decompositions associated with the various levels of the n-th multi-graded quotient resolution

$$MGQ^nRes(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

are strictly smaller than the complexity of the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQ^{n-1}Res(t,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a).$$

We treat the n-th multi-graded quotient resolution

$$MGQ^nRes(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

according to part (4) of step $n-1$ of the iterative procedure.

(3) At this stage we may assume that $Q^n(y,h_1,w,p,a)$ is isomorphic to $Q^{n-1}(y,h_1,w,p,a)$. At this part we assume that the $n-1$ multi-graded quotient resolution $MGQ^{n-1}Res(t,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a)$ is not of maximal possible complexity.

Suppose that for some index s, $1 \leq s \leq s(n-1)$, $Q^n(t_{\ell(s)},y,a)$ is a proper quotient of $Q^{\ell(s)}(t_{\ell(s)},y,a)$, and suppose that s is the minimal index for which this happens. Then we omit the n-th limit group $Q^nRlim(t,y,h_1,w,p,a)$ from our list of n-th quotient limit groups, and replace it by going back to the $\ell(s)$-th step of the iterative procedure, and start it with the limit group $Q^n(t_{\ell(s)},y,a)$, the subgroup generated by $< t_{\ell(s)},y,a >$ in the n-th quotient limit group $Q^nRlim(t,y,h_1,w,p,a)$, instead of the $\ell(s)$-th limit group $Q^{\ell(s)}Rlim(t_{\ell(s)},y,h_1,w,p,a)$ used in the $\ell(s)$-th step of the process that leads to the construction of the anvil, $Anv(MGQ^{n-1}Res)(t,y,h_1,w,p,a)$. Since, by definition of the index $\ell(s)$, in case $\ell(s) < n-1$ the parameter subgroups were enlarged at step $\ell(s)+1$, and in case $\ell(s) = n-1$ the multi-graded quotient resolution

$MGQ^{n-1}Res(t,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a)$ is not of maximal possible complexity, in analyzing the quotient limit group

$$Q^nRlim(t_{\ell(s)},y,h_1,w,p,a)$$

we need to take into account only those multi-graded quotient resolutions that are not of maximal complexity, i.e., only those multi-graded quotient resolutions which do not have a single level with a multi-graded abelian decomposition with the same structure as the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQ^{\ell(s)-1}Res(t_{\ell(s)-1},y,Base_{2,1}^{s(\ell(s)-1)},\dots,Base_{2,v_{s(\ell(s)-1)}}^{s(\ell(s)-1)},a)$$

used in the process of the construction of the anvil,

$$Anv(MGQ^{n-1}Res)(t,y,h_1,w,p,a).$$

(4) In this part we may assume that the multi-graded quotient resolution $MGQ^{n-1}Res(t,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a)$ is not of maximal complexity, and that $Q^n(t_{n-1},y,h_1,w,p,a)$ is isomorphic to $Q^{n-1}(t_{n-1},y,h_1,w,p,a)$. We set $s(n)=s(n-1)+1$, and the parameter subgroups $Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)}$ to be the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of the anvil, $Anv(MGQ^{n-1}Res)$. Let

$$MGQ^nRes_1(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a),\dots,$$
$$MGQ^nRes_q(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

be the resolutions in the taut multi-graded Makanin–Razborov diagram of $Q^nRlim(t,y,h_1,w,p,a)$ with respect to the parameter subgroups

$$Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)}.$$

We analyze the n-th multi-graded quotient resolutions

$$MGQ^nRes_j(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

in parallel, hence we will omit their index.

If there exists a QH vertex group Q in the abelian decomposition associated with the top level of the $n-1=\ell(s(n)-1)$ quotient resolution

$$MGQ^{\ell(s(n)-1)}Res(t_{\ell(s(n)-1)},y,Base_{2,1}^{s(n)-1},\dots,Base_{2,v_{s(n)-1}}^{s(n)-1},a)$$

for which the sequence of abelian decompositions inherited by Q from the various levels of the multi-graded resolution

$$MGQ^n Res(t_n, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

is not compatible with the specified collection of s.c.c. on the QH vertex group Q that are mapped to the trivial element in the second level of the $n-1 = \ell(s(n)-1)$ quotient resolution

$$MGQ^{\ell(s(n)-1)} Res(t_{\ell(s(n)-1)}, y, Base_{2,1}^{s(n)-1}, \ldots, Base_{2,v_{s(n)-1}}^{s(n)-1}, a),$$

we omit the multi-graded resolution

$$MGQ^n Res(t_n, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

from our list of multi-graded resolutions.

In this part we will also assume that the n-th multi-graded quotient resolution

$$MGQ^n Res(t, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

is not of maximal possible complexity, i.e., it does not have a single level with a (multi-graded) abelian decomposition identical with the abelian decomposition associated with the top level of the multi-graded quotient resolution $MGQ^{n-1} Res(t, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$. The case of maximal complexity will be treated in the next part of the general step. To treat an n-th multi-graded quotient resolution which is not of maximal possible complexity we need the following observation, which is similar to Proposition 2.4.

LEMMA 2.7: *Let $MGQ^n Res(t, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$ be an n-th multi-graded quotient resolution that is not of maximal complexity. By construction, the limit group $Q^n(t_{n-1}, y, h_1, w, p, a)$ is mapped into the limit group associated with each of the levels of the multi-graded quotient resolution $MGQ^n Res$. Let $Q^n_{term}(t_{n-1}, y, h_1, w, p, a)$ be the image of $Q^n(t_{n-1}, y, h_1, w, p, a)$ in the terminal (rigid or solid) limit group of $MGQ^n Res$.*

Then the multi-graded resolution $MGQ^n Res$ can be replaced by two finite collections of multi-graded resolutions, that are all compatible with the top level of the resolution $MGQ^{\ell(s(n)-1)} Res$ associated with the anvil, $Anv(MGQ^{n-1} Res)$, and are all obtained from $MGQ^n Res$ by adding at most a single (terminal) level. Furthermore, all the resolutions in these collections are not of maximal complexity.

We denote each of the resolutions in these collections, $MGQ^{n'}Res$.

(i) *In the first (possibly empty) collection of multi-graded resolutions, the image of the subgroup $Q^n(t_{n-1},y,h_1,w,p,a)$ in the terminal limit group of $MGQ^{n'}Res$, $Q^n_{term}(t_{n-1},y,h_1,w,p,a)$, is a proper quotient of $Q^n(t_{n-1},y,h_1,w,p,a)$.*

(ii) *In the second (possibly empty) finite collection of multi-graded resolutions, the terminal limit group of $MGQ^{n'}Res$ is either a rigid or a solid limit group with respect to the parameter subgroup $< h_1,w,p >$, i.e., the terminal limit group is rigid or solid with respect to the parameter subgroup $< h_1,w,p >$, and not only with respect to the multi-grading with respect to the subgroups $Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{v_{s(n)}}$, that was used in the construction of the resolution, MGQ^nRes.*

Proof: The argument is a modification of the argument used to prove Proposition 2.4. We may assume that the image of $Q^n(t_{n-1},y,h_1,w,p,a)$ in the terminal limit group of MGQ^nRes, $Q^n_{term}(t_{n-1},y,h_1,w,p,a)$, is isomorphic to $Q^n(t_{n-1},y,h_1,w,p,a)$.

With the terminal limit group $MGQ^nTerm(t_n,y,h_1,w,p,a)$, which is assumed to be (multi-graded) rigid or solid with respect to the subgroups $Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{v_{s(n)}}$, we associate the resolutions that appear in its graded taut Makanin–Razborov diagram with respect to the parameter subgroup $< h_1,w,p,>$. We treat these resolutions in parallel, so let

$$GTRes(t_n,y,h_1,w,p,a)$$

be one of these resolutions. By construction, the resolution $GTRes$ terminates in a rigid or solid limit group with respect to the parameter subgroup $< h_1,w,p >$.

Let $Q^n_{top}(t_{n-1},y,h_1,w,p,a)$ be the image of $Q^n(t_{n-1},y,h_1,w,p,a)$ in the limit group that is associated with the top level of the graded resolution $GTRes$. If $Q^n_{top}(t_{n-1},y,h_1,w,p,a)$ is a proper quotient of $Q^n(t_{n-1},y,h_1,w,p,a)$, part (i) of the proposition holds. If the graded resolution $GTRes$ has a single level, i.e., if the limit group associated with its top level is rigid or solid with respect to $< h_1,w,p >$, part (ii) of the proposition holds. Hence, we may assume that Q^n_{top} is isomorphic to Q^n, and $GTRes$ has more than a single level.

Let Λ_{top} be the (essential) graded abelian decomposition associated with the top level of the graded resolution $GTRes(t_n,y,h_1,w,p,a)$ (graded with respect to the subgroup $< h_1,w,p >$; see definition 1.8 in [Se3] for an essential decomposition). $Q^n_{top}(t_{n-1},h_1,w,p,a)$ inherits an (essential, graded) abelian decomposition from Λ_{top}. Since $Q^n_{top}(t_{n-1},h_1,w,p,a)$ is isomorphic to

$Q^{n-1}Rlim(t_{n-1},h_1,w,p,a)$, the non-$QH$, non-abelian vertex groups and edge groups in the (essential) graded abelian JSJ decomposition of $Q^{n-1}Rlim(t_{n-1},h_1,w,p,a)$ with respect to the parameter subgroup $<h_1,w,p,a>$ have to be elliptic in the graded abelian decomposition inherited by $Q^n_{top}(t_{n-1},h_1,w,p,a)$ from Λ_{top}.

The auxiliary limit group $Aux(MGQ^{n-1}Res)$, being a subgroup of the anvil, $Anv(MGQ^{n-1}Res)$, is naturally mapped into the limit group associated with the top level of $GTRes$. Let Δ_{Aux} be the (essential) abelian decomposition associated with the auxiliary limit group (that is associated with the anvil, $Anv(MGQ^{n-1}Res)$), $Aux(MGQ^{n-1}Res)$. By construction, Δ_{Aux} is multi-graded with respect to the non-abelian, non-QH vertex groups, and edge groups in the abelian JSJ decomposition of $Q^{n-1}Rlim(t_{n-1},h_1,w,p,a)$. Since the non-abelian, non-QH vertex groups and edge groups in the (essential) graded abelian JSJ decomposition of $Q^{n-1}Rlim(t_{n-1},h_1,w,p,a)$ are elliptic in Λ_{top}, if a non-abelian, non-QH vertex group or an edge group in Δ_{Aux} is not elliptic in Λ_{top}, lemma 1.9 in [Se3] implies that the restriction of the specializations that factor through $GTRes$ to the auxiliary limit group, $Aux(MGQ^{n-1}Res)$, are neither rigid nor strictly solid, so we may remove the graded resolution $GTRes$ from our list of graded resolutions.

Each of the subgroups $Base^{s(n)}_{2,1},\ldots,Base^{s(n)}_{v_{s(n)}}$ is a factor in the (multi-graded) free decomposition of the auxiliary limit group, $Aux(MGQ^{n-1}Res)$, with respect to the non-QH, non-abelian vertex groups and edge groups of $Q^{n-1}Rlim(t_{n-1},h_1,w,p,a)$. Hence, each subgroup $Base^{s(n)}_{2,j}$ inherits an abelian decomposition from Δ_{Aux}, which is a subgraph (of groups) of Δ_{Aux}. We set Θ_{term} to be the abelian decomposition obtained from the multi-graded abelian decomposition associated with the terminal level of MGQ^nRes (which is multi-graded with respect to the subgroups $Base^{s(n)}_{2,1},\ldots,Base^{s(n)}_{v_{s(n)}}$), by replacing each of the vertex groups stabilized by one of the subgroups $Base^{s(n)}_{2,j}$ with a (possibly degenerate) graph of groups obtained from the abelian decomposition associated with $Base^{s(n)}_{2,j}$ in Δ_{Aux}. Θ_{term} is the multi-graded abelian decomposition of the terminal limit group of MGQ^nRes, MGQ^nTerm, with respect to the non-abelian, non-QH vertex groups and edge groups in the (essential) graded abelian JSJ decomposition of $Q^n_{term}(t_{n-1},h_1,w,p,a)$ (which is assumed to be isomorphic to $Q^{n-1}Rlim(t_{n-1},h_1,w,p,a)$).

If any of the non-abelian, non-QH vertex groups or any of the edge groups in the multi-graded abelian decomposition Θ_{term} is not elliptic in the graded abelian decomposition Λ_{top}, then the specializations that factor through the graded resolution $GTRes$ are neither rigid nor strictly solid specializations of the terminal limit group, MGQ^nTerm, of the multi-graded resolution, $MGQRes$. Hence, in this case we may omit the resolution $GTRes$ from our list of graded resolutions.

We continue iteratively along the levels of the graded resolution $GTRes$, and conclude that if the resolution $GTRes$ was not removed from our list of graded resolutions, then as long as the image of $Q^{n-1}Rlim(t_{n-1},y,h_1,w,p,a)$ in the limit group associated with some level j is isomorphic to $Q^{n-1}Rlim(t_{n-1},y,h_1,w,p,a)$, then the images of all the non-QH, non-abelian vertex groups and the edge groups in the abelian decomposition Θ_{term}, in the graded abelian decomposition associated with level j in $GTRes$, are elliptic.

Therefore, either we get to the terminal level of $GTRes$, which is rigid or solid with respect to $< h_1,w,p >$ (possibility (ii) in the proposition), or we get to some level j, for which the image of $Q^{n-1}Rlim(t_{n-1},y,h_1,w,p,a)$ in the limit group associated with that level is a proper quotient of $Q^{n-1}Rlim(t_{n-1},y,h_1,w,p,a)$ (possibility (i) in the proposition). Furthermore, since the non-abelian, non-QH vertex groups and edge groups in Θ_{term} are mapped to elliptic subgroups in all the levels until level j of $GTRes$, the modular groups associated with Θ_{term} are sufficient to map the terminal limit group of $MGQRes$, $MGQTerm$, onto the limit group associated with level j of $GTRes$, hence $MGQRes$ needs to be extended in at most a single level.

If the image of $Q^{n-1}Rlim(t_{n-1},y,h_1,w,p,a)$ in the limit group associated with level j in $GTRes$ is a proper quotient of $Q^{n-1}Rlim(t_{n-1},y,h_1,w,p,a)$, then it is enough to use the modular groups associated with $MGQTerm$ as a multi-graded rigid or solid limit group with respect to $Base^{s(n)}_{2,1},\ldots,Base^{s(n)}_{2,v_{s(n)}}$, to map $MGQTerm$ onto a limit group in which the image of $Q^n(t_{n-1},y,h_1,w,p,a)$ is a proper quotient of it. This allows one to continue viewing the QH and abelian vertex groups in the multi-graded abelian decomposition of the auxiliary limit group, $Aux(MGQ^{n-1}Res)$, as “formal”, i.e., we can still act on them with their associated modular groups in the terminal limit groups as well, a point of view that is adapted throughout the whole iterative procedure.

If the limit group associated with level j in $GTRes$ is rigid or solid with respect to $< h_1,w,p >$, then we use this terminal limit group to express the rigid or strictly solid families associated with it. Still, in this case, the terminal level of the obtained resolution $MGQ'Res$ does really take part in the next steps of the procedure (besides expressing the rigid or strictly solid families). Hence, for the purposes of the next steps in the procedure, we are allowed to continue viewing the QH and abelian vertex groups in the multi-graded abelian decomposition associated with the auxiliary resolutions that are associated with the anvil, $Anv(MGQ^{n-1}Res)$, and were constructed in previous steps of the procedure, as "formal". ∎

By Lemma 2.7 under the assumptions of part (4), we can either drop the n-th multi-graded quotient resolution MGQ^nRes from our list of n-th multi-graded quotient resolutions, or we may replace it by finitely many multi-graded resolutions, which for brevity we still denote MGQ^nRes, so that for each resolution we may assume that either $Q^n_{term}(t_{\ell(s(n)-1)}, y, h_1, w, p, a)$ is a proper quotient of $Q^n(t_{\ell(s(n)-1)}, y, h_1, w, p, a)$, or the terminal limit group of the multi-graded resolution MGQ^nRes is rigid or solid with respect to the parameter subgroup $< h_1, w, p >$. We continue with the resolutions from the collection given in Lemma 2.7 in parallel, and continue to denote them MGQ^nRes.

At this point we need the following lemma that is similar to Proposition 2.5.

Lemma 2.8: *Let MGQ^nRes be an n-th multi-graded quotient resolution which is not of maximal possible complexity. By construction, the limit group $Q^n(t, y, h_1, w, p, a)$ is mapped onto the limit group associated with each of the levels of the multi-graded quotient resolution MGQ^nRes. Let $Q^n_2(t_{n-1}, y, a)$, $Q^n_2(t_n, y, a)$ be the images of the subgroups $Q^n(t_{n-1}, y, h_1, w, p, a)$, $Q^n(t_n, y, h_1, w, p, a)$ in correspondence, in the limit group associated with the second level of the multi-graded quotient resolution MGQ^nRes. Then $Q^n_2(t_{n-1}, y, h_1, w, p, a)$ is a quotient of the subgroup $Q^n(t_{n-1}, y, h_1, w, p, a)$, and $Q^n_2(t_n, y, h_1, w, p, a)$ is a proper quotient of the subgroup $Q^n(t_n, y, h_1, w, p, a)$.*

Proof: The claim of the lemma is a basic property of a multi-graded resolution. ∎

Suppose that $Q^n_2(y, h_1, w, p, a)$ is a proper quotient of $Q^n(y, h_1, w, p, a)$. In this case we modify the procedure used in part (2) of the first step of the procedure. We decrease the parameter subgroup to be $< h_1, w, p >$, and associate with $Q^n_2(y, h_1, w, p, a)$ its taut graded Makanin–Razborov diagram with respect to the parameter subgroup $< h_1, w, p >$

$$GQRes_1(y, h_1, w, p, a), \ldots, GQRes_t(y, h_1, w, p, a).$$

We continue with each of the graded resolutions $GQRes_j(y, h_1, w, p, a)$ in parallel.

If the subgroup generated by $< y, h_1, w, p, a >$ in the limit group associated with the resolution $GQRes_j(y, h_1, w, p, a)$ is a proper quotient of $Q_2^n(y, h_1, w, p, a)$, we replace the graded resolution $GQRes_j(y, h_1, w, p, a)$ by starting part (4) of the the general step with the multi-graded resolution obtained from MGQ^nRes, by replacing its second limit group $Q_2^n(t, y, h_1, w, p, a)$ with the maximal limit groups obtained from all specializations that factor through both $Q_2^n(t, y, h_1, w, p, a)$ and the subgroup generated by $< y, h_1, w, p, a >$ in the limit group associated with $GQRes_j(y, h_1, w, p, a)$. If the subgroup generated by $< t, y, w, p, a >$ in the obtained (one level) resolution, $QRlim'(t, y, h_1, w, p, a)$, is a proper quotient of $QRlim(t, y, h_1, w, p, a)$, we replace the obtained resolution by starting the n-th step of our iterative procedure with the limit group $QRlim'(t, y, h_1, w, p, a)$ instead of the limit group $QRlim(t, y, h_1, w, p, a)$, and since the resolution MGQ^nRes is not of maximal complexity, in analyzing the limit group $QRlim'(t, y, h_1, w, p, a)$ we need to consider only those resolutions in its multi-graded Makanin–Razborov diagram that are not of maximal possible complexity. Hence, for the rest of this part we may assume that the subgroup generated by $< y, h_1, w, p, a >$ in the limit group associated with $GQRes_j(y, h_1, w, p, a)$ is isomorphic to the subgroup generated by $< y, h_1, w, p, a >$ in $Q_2^n(t, y, h_1, w, p, a)$.

Suppose that a graded quotient resolution $GQRes_j(y, h_1, w, p, a)$ is of maximal possible complexity, i.e., the limit group associated with it is of the form $< h_1, w, p, a > *F_y$. Since the limit group $Q_2^n(y, h_1, w, p, a)$ is a proper quotient of $Q^n(y, h_1, w, p, a)$, and since the limit group

$$Q_2^n(y, h_1, w, p, a)$$

is naturally mapped onto the limit group $< h_1, w, p, a > *F_y$ associated with the graded resolution $GQRes_j(y, h_1, w, p, a)$, the Hopf property for limit groups implies that the subgroup generated by $< h_1, w, p, a >$ in the limit group associated with $GQRes_j(y, h_1, w, p, a)$ is a proper quotient of the limit group generated by $< h_1, w, p, a >$ in $Q^n(y, h_1, w, p, a)$. Hence, we can replace the resolution $GQRes_j(y, h_1, w, p, a)$ by starting the initial step of the procedure with the subgroup $< h_1, w, p, a > *F_y$, where $< h_1, w, p, a >$ is the subgroup generated by these elements in the limit group associated with $GQRes_j(y, h_1, w, p, a)$. Since resolutions of maximal possible complexity of the limit group $Q^n(y, h_1, w, p, a)$ with respect to the parameter subgroup $< h_1, w, p >$, i.e., those corresponding to the entire free group F_y, were already analyzed in the initial step of the procedure, we can omit a graded resolution $GQRes_j(y, h_1, w, p, a)$ of maximal complexity from our list of graded quotient resolutions $\{GQRes_j(y, h_1, w, p, a)\}$. Hence, for the rest of this part we may assume that the graded resolution $GQRes_j(y, h_1, w, p, a)$ with which we continue is not of maximal possible complexity.

Let $CRes_j(y, h_1, w, p, a)$ be the graded resolution obtained from the resolution induced by the subgroup $< y, h_1, w, p, a >$ from the top level of the completion of the multi-graded resolution MGQ^nRes, followed by the graded resolution $GQRes_j(y, h_1, w, p, a)$ (see section 3 of [Se4] for the construction of the induced resolution). If the subgroup generated by $< y, h_1, w, p, a >$ in the limit group corresponding to the graded resolution $CRes_j(y, h_1, w, p, a)$ is a proper quotient of $Q^n(y, h_1, w, p, a)$, we replace the graded resolution $GQRes_j(y, h_1, w, p, a)$ by starting the initial step of the procedure with the subgroup generated by $< y, h_1, w, p, a >$ in the limit group corresponding to the resolution $CRes_j(y, h_1, w, p, a)$ and treat only those graded resolutions of this limit group (with respect to the parameter subgroup $< h_1, w, p >$) that are not of maximal possible complexity, i.e., those graded resolutions which do not cover ungraded ones that correspond to the entire free group F_y. Hence, we may assume that for the rest of this part, the subgroup generated by $< y, h_1, w, p, a >$ in the limit group corresponding to $CRes_j(y, h_1, w, p, a)$ is isomorphic to $Q^n(y, h_1, w, p, a)$. In particular, we may assume that each of the graded resolutions $GQRes_j(y, h_1, w, p, a)$ in question is not of maximal possible complexity.

We now treat each of the graded resolutions $GQRes_j(y, h_1, w, p, a)$, and their associated resolutions $CRes_j(y, h_1, w, p, a)$, in a similar way to our treatment of multi-graded quotient resolutions in part (4) of the general step of the iterative procedure for validation of a sentence, presented in section 4 of [Se4]. Let $WP(HG)^{n+1}(g_{n+1}, h_1, w, p, a)$ be the terminal rigid or solid limit group of the graded resolution $GQRes_j(y, h_1, w, p, a)$ (which is also the terminal rigid or solid limit group of its associated resolution $CRes_j(y, h_1, w, p, a)$). We start by collecting all the formal solutions defined over ungraded resolutions covered by the graded resolution $CRes_j(y, h_1, w, p, a)$. According to section 3 of [Se2], this collection of formal solutions factors through a canonical collection of graded formal limit groups. With each graded formal limit group we associate its graded formal Makanin–Razborov diagram (as we did in section 3 of [Se2]). We continue with each of the graded formal resolutions that appear in these diagrams in parallel.

Let $GFRes(x,f,y,g_{n+1},h_1,w,p,a)$ be a graded formal resolution in one of these diagrams, and let $WP(HG)^{n+1}H(h_{n+2},g_{n+1},h_1,w,p,a)$ be its terminating rigid or solid (not formal!) limit group. With the graded formal resolution $GFRes(x,f,y,h_{n+2},g_{n+1},h_1,w,p,a)$ we associate the resolution $GRes(f,y,h_{n+2},g_{n+1},h_1,w,p,a)$, which is the graded (not formal!) resolution associated with the terminal formal limit group of the graded formal resolution $GFRes(x,f,y,h_{n+2},g_{n+1},h_1,w,p,a)$. Note that the terminal rigid or solid limit group of the graded resolution $GRes$ is the (rigid or solid) limit group $WP(HG)^{n+1}H(h_{n+2},g_{n+1},h_1,w,p,a)$ as well.

We set the developing resolution to be the resolution

$$GRes(f,y,h_{n+2},g_{n+1},h_1,w,p,a).$$

We further set the anvils associated with the developing resolution to be the (canonical) finite set of maximal limit quotients of the group obtained as the amalgamated product of the completion of the developing resolution and the completion of the top level of the multi-graded resolution

$$MGQ^nRes(t,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

amalgamated along the top part of the developing resolution, which was set to be the subgroup generated by the resolution induced by the subgroup $<y,h_1,w,p,a>$ from the top level of MGQ^nRes, and the subgroup $Q_2^n(y,h_1,w,p,a)$ (which is the image of $Q^n(y,h_1,w,p,a)$ in the limit group associated with the second level of MGQ^nRes). We denote each of the (finitely many) anvils $Anv(MGQ^nRes)(t,y,a)$. Note that the completion of the developing resolution is canonically mapped into the anvil, hence the formal solutions encoded by the graded formal resolution, $GFRes$, that are defined over the developing resolution, can be naturally defined over the anvil. With the anvil and its developing resolution we further associate the terminal rigid or solid limit groups (graded with respect to the parameter subgroup $<h_1,w,p>$), $WP(HG)^{n+1}$ and $WP(HG)^{n+1}H$.

Suppose that $Q_2^n(y,h_1,w,p,a)$ is isomorphic to $Q^n(y,h_1,w,p,a)$, and $Q_2^n(t_{n-1},y,h_1,w,p,a)$ is a proper quotient of $Q^n(t_{n-1},y,h_1,w,p,a)$. We set s, $1 \leq s \leq s(n)-1$, to be the minimal index for which

$Q_2^n(t_{\ell(s)},y,h_1,w,p,a)$ is a proper quotient of $Q^n(t_{\ell(s)},y,h_1,w,p,a)$, which is assumed to be isomorphic to $Q^{\ell(s)}(t_{\ell(s)},y,h_1,w,p,a)$. Let

$$MGQRes_1(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a),\dots,$$
$$MGQRes_d(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

be the resolutions in the taut multi-graded diagram of

$$Q_2^n(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

with respect to the parameter subgroups $Base^s_{2,1},\dots,Base^s_{2,v_s}$.

Given a multi-graded resolution

$$MGQRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a),$$

we set the multi-graded quotient resolution

$$CRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

to be the multi-graded resolution obtained from the resolution induced by the subgroup $< t_{\ell(s)},y,w,p,a >$ from the top level of the completion of the multi-graded quotient resolution

$$MGQ^nRes(t,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a),$$

followed by the multi-graded resolution

$$MGQRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a).$$

If the subgroup generated by $< t_{\ell(s)},y,w,p,a >$ in the limit group associated with the resolution $CRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$ is a proper quotient of the $\ell(s)$ quotient limit group

$$Q^{\ell(s)}Rlim(t_{\ell(s)},y,h_1,w,p,a),$$

we set q, $1\leq q\leq s$, to be the minimal index for which the subgroup generated by $< t_{\ell(q)},y,w,p,a >$ in the limit group associated with the resolution $CRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$ is a proper quotient of the $\ell(q)$-th quotient limit group $Q^{\ell(q)}Rlim(t_{\ell(q)},y,h_1,w,p,a)$. We now replace the multi-graded quotient resolution

$$MGQRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

of the limit group associated with the second level of the multi-graded resolution $MGQ^nRes(t,y,Base_{2,1}^{s(n)},\ldots,Base_{2,v_{s(n)}}^{s(n)},a)$, by starting the $\ell(q)$-th step of our process with the limit group generated by $< t_{\ell(q)},y,w,p,a >$ in (the closure of) $CRes_j(t_{\ell(s)},y,Base_{2,1}^{s},\ldots,Base_{2,v_s}^{s},a)$, instead of the limit group $Q^{\ell(q)}Rlim(t_{\ell(q)},y,h_1,w,p,a)$, which is assumed to be isomorphic to $Q^nRlim(t_{\ell(q)},y,h_1,w,p,a)$, and was used in the $\ell(q)$-th step of the procedure.

If there exists a QH vertex group Q in the abelian decomposition associated with the top level of the $\ell(s-1)$ quotient resolution

$$MGQ^{\ell(s-1)}Res(t_{\ell(s-1)},y,Base_{2,1}^{s-1},\ldots,Base_{2,v_{s-1}}^{s-1},a)$$

for which the sequence of abelian decompositions inherited by Q from the various levels of the multi-graded resolution

$$CRes_j(t_{\ell(s)},y,Base_{2,1}^{s},\ldots,Base_{2,v_s}^{s},a)$$

is not compatible with the specified collection of s.c.c. on the QH vertex group Q that are mapped to the trivial element in the second level of the $\ell(s-1)$ quotient resolution,

$$MGQ^{\ell(s-1)}Res(t_{\ell(s-1)},y,Base_{2,1}^{s-1},\ldots,Base_{2,v_{s-1}}^{s-1},a),$$

we omit the multi-graded resolution

$$MGQRes_j(t_{\ell(s)},y,Base_{2,1}^{s},\ldots,Base_{2,v_s}^{s},a)$$

from our list of multi-graded resolutions.

Suppose that a multi-graded quotient resolution

$$MGQRes_j(t_{\ell(s)},y,Base_{2,1}^{s},\ldots,Base_{2,v_s}^{s},a)$$

is of maximal possible complexity, i.e., that it has a single level with an abelian decomposition of the same structure as the abelian decomposition associated with the top level of the multi-graded quotient resolution $MGQ^{\ell(s)-1}Res(t_{\ell(s)-1},y,Base_{2,1}^{s(\ell(s)-1)},\ldots,Base_{2,v_{s(\ell(s)-1)}}^{s(\ell(s)-1)},a)$. Each of the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of the anvil,

$$Anv(MGQ^{\ell(s)-1}Res)(t_{\ell(s)},y,h_1,w,p,a),$$

is naturally mapped into the $\ell(s)$ quotient limit group

$$Q^{\ell(s)}(t_{\ell(s)},y,h_1,w,p,a),$$

which is assumed to be isomorphic to $Q^n(t_{\ell(s)}, y, h_1, w, p, a)$. Since

$$Q^n(t_{\ell(s)}, y, h_1, w, p, a)$$

is mapped onto $Q_2^n(t_{\ell(s)}, y, h_1, w, p, a)$, the image in $Q^n(t_{\ell(s)}, y, h_1, w, p, a)$ of each of the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of the anvil,

$$Anv(MGQ^{\ell(s)-1}Res)(t_{\ell(s)}, y, h_1, w, p, a),$$

is naturally mapped into $Q_2^n(t_{\ell(s)}, y, h_1, w, p, a)$. Since

$$Q_2^n(t_{\ell(s)}, y, h_1, w, p, a)$$

is assumed to be a proper quotient of $Q^n(t_{\ell(s)}, y, h_1, w, p, a)$, and the abelian decomposition associated with the multi-graded resolution

$$MGQRes_j(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$$

has a single level with an abelian decomposition of the same structure as the abelian decomposition associated with the top level of the multi-graded quotient resolution $MGQ^{\ell(s)-1}Res$, the map of the image in

$$Q^n(t_{\ell(s)}, y, h_1, w, p, a),$$

of at least one of the non-abelian, non-QH vertex groups in the abelian decomposition of the anvil, $Anv(MGQ^{\ell(s)-1}Res)(t_{\ell(s)}, y, h_1, w, p, a)$, into $Q_2^n(t_{\ell(s)}, y, h_1, w, p, a)$, is not a monomorphism.

Hence, in this case of $MGQRes_j(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$ being a resolution of maximal possible complexity, we do the following. We set the multi-graded quotient resolution

$$CRes_j(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$$

to be the multi-graded resolution obtained from the resolution induced by the subgroup $< t_{\ell(s)}, y, h_1, w, p, a >$ from the completion of the multi-graded quotient resolution $MGQ^nRes(t, y, Base^{s(n)}_{2,1}, \ldots, Base^{s(n)}_{2,v_{s(n)}}, a)$, followed by the multi-graded resolution

$$MGQRes_j(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a).$$

We set q, $1 \leq q \leq s$, to be the minimal index for which the subgroup generated by $< t_{\ell(q)}, y, w, p, a >$ in (the closure of)

$$CRes_j(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$$

is a proper quotient of $Q^n(t_{\ell(q)},y,h_1,w,p,a) = Q^{\ell(q)}(t_{\ell(q)},y,h_1,w,p,a)$ (by the above argument there must exist such an index q). We now replace the multi-graded quotient resolution

$$MGQRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

of the limit group associated with the second level of

$$MGQ^nRes(t,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a)$$

by starting the $\ell(q)$-th step of our process with the limit group generated by $< t_{\ell(q)},y,w,p,a >$ in (the closure of)

$$CRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a),$$

instead of the limit group

$$Q^{\ell(q)}Rlim(t_{\ell(q)},y,h_1,w,p,a) = Q^nRlim(t_{\ell(q)},y,h_1,w,p,a)$$

used in the $\ell(q)$-th step of the procedure. In analyzing the new $\ell(q)$-th quotient limit group, we need to consider only its multi-graded resolutions that are not of maximal complexity, as (multi-graded) resolutions of maximal possible complexity are analyzed in different branches of the iterative procedure.

By the above argument, we may consider only those multi-graded quotient resolutions, $MGQRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$, which are not of maximal possible complexity. In this case we analyze each of the multi-graded quotient resolutions,

$$MGQRes_j(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a),$$

as we did in step $\ell(s)$ of our iterative procedure. First, we associate with the multi-graded resolution $MGQRes(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$ a multi-graded resolution $CRes(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$, obtained from the resolution induced by the subgroup $< t_{\ell(s)},y,a >$ from the top level of the completion of the multi-graded resolution

$$MGQ^nRes(t,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a)$$

followed by the multi-graded resolution

$$MGQRes(t_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a).$$

If the multi-graded resolution $CRes(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$ is not compatible with the collections of s.c.c. associated the various QH vertex groups in the multi-graded abelian decomposition associated with the top level of the multi-graded resolution

$$MGQ^{\ell(s-1)}Res(t_{\ell(s-1)}, y, Base^{s-1}_{2,1}, \ldots, Base^{s-1}_{2,v_{s-1}}, a),$$

we omit the multi-graded resolution

$$MGQRes(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$$

from our list of multi-graded resolutions of $Q^n_{term}(t_{\ell(s)}, y, a)$. Otherwise, we continue as in step $\ell(s)$ of the iterative procedure, and associate with the multi-graded resolution $MGQRes(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$ a canonical collection of graded resolutions that are induced by the (image of the) subgroup $< y, h_1, w, p, a >$, and their associated anvils.

As we did in part (4) of the general step of the iterative procedure for validation of an AE sentence in [Se4], given an anvil associated with the graded resolution $MGQRes(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$, we set a resolution $IRes(u, y, h_1, w, p, a)$ obtained from the resolution induced by the subgroup $< y, h_1, w, p, a >$ from the top level of the n-th multi-graded quotient resolution

$$MGQ^nRes(t, y, Base^{s(n)}_{2,1}, \ldots, Base^{s(n)}_{2,v_{s(n)}}, a)$$

followed by the resolution induced by the (image of the) subgroup $< y, h_1, w, p, a >$, that is associated with the anvil which is associated with the multi-graded resolution $MGQRes(t_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$.

We collect all the formal solutions defined over (ungraded resolutions covered by the graded resolution) $IRes(u, y, h_1, w, p, a)$ in a taut graded formal Makanin–Razborov diagram. We set each graded (not formal!) resolution associated with a terminal graded formal limit group in this graded formal Makanin–Razborov diagram (which is a graded closure of the resolution $IRes(u, y, h_1, w, p, a)$) to be a developing resolution (see section 3 in [Se2] for the construction of a graded formal diagram). With each developing resolution we associate a (graded) family of formal solutions parameterized by the graded formal modular groups associated with the graded formal resolution in the graded formal Makanin–Razborov diagram that is associated with the developing resolution.

With each developing resolution we also associate a finite collection of anvils that we denote $Anv(MGQ^nRes)$. The anvils are set to be the finite collection of maximal limit quotients of the group generated by the developing resolution, the corresponding closure of the top level of the multi-graded resolution

$$MGQ^nRes(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

and the anvil associated with $MGQRes(t_{\ell(s)}, y, Base_{2,1}^s, \dots, Base_{2,v_s}^s, a)$, where the top part of the developing resolution is identified with its image in the (closure of the) top part of MGQ^nRes, the tail of the developing resolution (i.e., the developing resolution except its top part) is identified with its image in the anvil associated with the resolution $MGQRes$, and the corresponding images of the subgroup $Q_2^n(t_{\ell(s)}, y, a)$ in the second level of the completion of MGQ^nRes and in the anvil associated with the multi-graded resolution $MGQRes(t_{\ell(s)}, y, Base_{2,1}^s, \dots, Base_{2,v_s}^s, a)$ are identified as well. Since the developing resolution is mapped into the anvil, $Anv(MGQ^nRes)(t, y, a)$, the family of formal solutions associated with and defined over the developing resolution is naturally defined over the anvil as well.

We still need to consider the case in which both $Q_2^n(y, h_1, w, p, a)$ is isomorphic to $Q^n(y, h_1, w, p, a)$, and $Q_2^n(t_{n-1}, y, h_1, w, p, a)$ is isomorphic to $Q^n(t_{n-1}, y, h_1, w, p, a)$. In this case we continue to the next level of the multi-graded quotient resolution

$$MGQ^nRes(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a).$$

If for some level j of the multi-graded resolution, the image of $Q^n(y, h_1, w, p, a)$ in the limit group associated with this level, $Q_j^n(y, h_1, w, p, a)$, is a proper quotient of $Q^n(y, h_1, w, p, a)$, or the image of the limit group $Q^n(t_{n-1}, y, h_1, w, p, a)$ in the limit group associated with this level, $Q_j^n(t_{n-1}, y, h_1, w, p, a)$, is a proper quotient of

$$Q^n(t_{n-1}, y, h_1, w, p, a),$$

then from the highest such level j we can continue as in case $Q_2^n(y, h_1, w, p, a)$ is a proper quotient of $Q^n(y, h_1, w, p, a)$, or $Q_2^n(t_{n-1}, y, h_1, w, p, a)$ is a proper quotient of $Q^n(t_{n-1}, y, h_1, w, p, a)$, and associate with the multi-graded resolution

$$MGQ^nRes(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

a finite collection of developing resolutions, anvils, and families of formal solutions defined over each of the developing resolutions and its associated anvil.

Finally, suppose that for every level j, the image of

$$Q^n(t_{n-1}, y, h_1, w, p, a)$$

in the limit group associated with the j-th level of the multi-graded resolution $MGQ^nRes(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$, $Q_j^n(t_{n-1}, y, h_1, w, p, a)$, is isomorphic to $Q^n(t_{n-1}, y, h_1, w, p, a)$. In this case, by Lemma 2.7, the terminal limit group of the multi-graded resolution

$$MGQ^nRes(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a),$$

$Q^n_{term}(t, y, h_1, w, p, a)$, is rigid or solid with respect to the parameter subgroup $< h_1, w, p >$.

We denote the terminal limit group of the multi-graded resolution MGQ^nRes, $WP(HG)^{n+1}(g_{n+1}, h_1, w, p, a)$. Note that this terminal limit group is rigid or solid with respect to the parameter subgroup $< h_1, w, p >$. We set the graded resolution $CRes(y, g_{n+1}, h_1, w, p, a)$ to be the resolution obtained from the resolution induced by the subgroup generated by $< y, h_1, w, p, a >$ from the multi-graded resolution MGQ^nRes, by enlarging its terminal group to be $WP(HG)^{n+1}$. We collect all the formal solutions defined over ungraded resolutions covered by the graded resolution $CRes(y, g_{n+1}, h_1, w, p, a)$. This collection of formal solutions factors through a canonical collection of graded formal limit groups. With each graded formal limit group we associate its graded formal Makanin–Razborov diagram as we did in section 3 of [Se2]. We continue with each of the graded formal resolutions that appear in these diagrams in parallel.

Let $GFRes(x, f, y, g_{n+1}, h_1, w, p, a)$ be a graded formal resolution in one of these diagrams, and let $WP(HG)^{n+1}H(h_{n+2}, g_{n+1}, h_1, w, p, a)$ be its terminating rigid or solid (not formal!) limit group. With the graded formal resolution $GFRes$ we associate the resolution

$$GRes(f, y, h_{n+2}, g_{n+1}, h_1, w, p, a),$$

which is the graded (not formal!) resolution associated with the terminal formal limit group of the graded formal resolution $GFRes$ (i.e., its graded formal closure amalgamated with its terminal rigid or solid limit group). Note that the terminal rigid or solid limit groups of those resolutions is

$WP(HG)^{n+1}H(h_{n+2}, g_{n+1}, h_1, w, p, a)$.

We set the developing resolution to be the resolution

$$GRes(f, y, h_{n+2}, g_{n+1}, h_1, w, p, a).$$

To set the finite collection of anvils, we first look at the amalgamation of (the completion of) MGQ^nRes and the (completion of the) resolution, $GRes$, amalgamated along the (completion of the) induced resolution, $CRes(y, g_{n+1}, h_1, w, p, a)$. With the obtained group we naturally associate a finite collection of maximal limit groups and set each of them to be an anvil. With the developing resolution we associate the family of formal solutions $x_\alpha(f, y, h_{n+2}, g_{n+1}, h_1, w, p, a)$ encoded by the associated graded formal resolution, $GFRes$, and defined over the developing resolution. Since the developing resolution is canonically mapped into the anvil, the family of formal solutions associated with the developing resolution is naturally defined over the anvil as well.

(5) By part (1) we may assume that $Q^n(y, h_1, w, p, a)$ is isomorphic to the limit group $Q^{n-1}(y, h_1, w, p, a)$ associated with the anvil,

$$Anv(MGQ^{n-1}Res)(t, y, h_1, w, p, a),$$

and by parts (2)–(3) we may assume that $Q^n(t_{s(n)-1}, y, h_1, w, p, a)$ is isomorphic to the subgroup $Q^{n-1}(t_{s(n)-1}, y, h_1, w, p, a)$ associated with the anvil. Parts (2) and (4) treat all the cases in which the n-th multi-graded quotient resolution $MGQ^nRes(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$ is not of maximal complexity. Hence, the only case left in presenting the general step of our procedure for validation of a sentence is the case of an n-th multi-graded quotient resolution

$$MGQ^nRes(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

of maximal possible complexity, i.e., a multi-graded quotient resolution

$$MGQ^nRes(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

consists of a single level, and the abelian decomposition associated with this level is identical to the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQ^{n-1}Res(t, y, Base_{2,1}^{s(n-1)}, \dots, Base_{2,v_{s(n-1)}}^{s(n-1)}, a)$$

with which we started the n-th step of the procedure.

Conceptually, we treat this case in a similar way to what we did in the minimal rank case, and in the general step of the procedure for validation of a sentence (in section 4 of [Se4]), i.e., we continue to lower levels of the anvil and analyze it in a similar way to what we did with the top level. In parts (1)–(4), we have analyzed multi-graded resolutions of quotient limit groups, $Q^nRlim(t, y, h_1, w, p, a)$, that were associated with the anvil, $Anv(MGQ^{n-1}Res)$, and with an auxiliary resolution of highest level, i.e., an auxiliary resolution associated with the tower containing all the levels in the anvil up to level 2 (all levels except the top level).

As we did in the first step of the iterative procedure, to analyze maximal complexity multi-graded resolutions, we first replace these quotient limit groups by those quotient limit groups associated with the anvil, $Anv(MGQ^{n-1}Res)$, and with auxiliary resolutions that are associated with towers containing all the levels up to level 3, i.e., all the levels apart from the top two. We continue with those quotient limit groups in parallel, hence we will omit their index, and (still) denote the quotient limit group with which we continue, $Q^nRlim(t, y, h_1, w, p, a)$.

We start with the multi-graded taut Makanin–Razborov diagram of the quotient limit group, Q^nRlim, with respect to the subgroups

$$Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)},$$

where those subgroups are the non-QH, non-abelian vertex groups and edge groups in the abelian decomposition associated with the top level of the anvil, $Anv(MGQ^{n-1}Res)$. We still denote these multi-graded resolutions MGQ^nRes. Note that, as we explained in part (3) of the first step of the procedure, since the auxiliary modular groups associated with auxiliary resolutions of highest level are "bigger" than auxiliary modular groups associated with auxiliary resolutions associated with towers of lower level, all the specializations that factor through the developing resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$, and can be extended to specializations that factor only through maximal complexity multi-graded resolutions of quotient limit groups associated with auxiliary resolutions of highest level, can be extended to specializations that factor through maximal complexity multi-graded resolutions of quotient limit groups associated with auxiliary resolutions associated with towers containing all the levels up to level 3. Since in this part we need to analyze specializations that factor only through maximal complexity multi-graded resolutions of quotient limit groups associated with auxiliary resolutions of highest level, we can certainly continue only with those multi-graded resolutions in the taut Makanin–Razborov diagram of Q^nRlim that are of maximal complexity, i.e., that contain a single level with an abelian decomposition that have the same (taut) structure as the abelian decomposition associated with the top level of the anvil, $Anv(MGQ^{n-1}Res)$.

If part (1) applies to such a multi-graded resolution MGQ^nRes, i.e., if the limit group generated by $< y, h_1, w, p, a >$ in its completion is a proper quotient of the subgroup $Q^{n-1}(y, h_1, w, p, a)$ with which we started this branch of the procedure, we replace this resolution MGQ^nRes by starting the initial step of the procedure with the given proper quotient of $Q(y, h_1, w, p, a)$.

In case the abelian decomposition and the taut structure associated with MGQ^nRes and the top level of the anvil $Anv(MGQ^{n-1}Res)$, are identical, we use the modular groups associated with the abelian decomposition associated with MGQ^nRes to map the subgroup Q^nRlim into the subgroup of the anvil $Anv(MGQ^{n-1}Res)$, Q^nRlim_2, associated with its second level. We now set the subgroups $Base_{3,1}^{s_2(n)}, \ldots, Base_{3,t_{s_2(n)}}^{s_2(n)}$ to be the subgroups of the anvil, $Anv(MGQ^{n-1}Res)$, corresponding to the non-abelian, non-QH vertex groups in the multi-graded abelian decomposition associated with the second level of the anvil, $Anv(MGQ^{n-1}Res)$.

At this point, we analyze the quotient limit group Q^nRlim_2 with respect to the subgroups $Base_{3,1}^{s_2(n)}, \ldots, Base_{3,t_{s_2(n)}}^{s_2(n)}$ exactly as we analyzed the quotient limit group Q^nRlim with respect to the subgroups

$$Base_{2,1}^1, \ldots, Base_{2,v_1}^1$$

in steps (1)–(4), i.e., we associate with Q^nRlim_2 all its multi-graded quotient resolutions with respect to the subgroups

$$Base_{3,1}^{s_2(n)}, \ldots, Base_{3,t_{s_2(n)}}^{s_2(n)}$$

that are its subgroups, and analyze each of the obtained multi-graded quotient resolutions according to parts (1) to (the first part of) (5). If the multi-graded abelian decomposition associated with a multi-graded quotient resolution of $QRlim_2$ with respect to the subgroups

$$Base_{3,1}^{s_2(n)}, \ldots, Base_{3,t_{s_2(n)}}^{s_2(n)}$$

is of maximal possible complexity, and its associated taut structure is identical to the one associated with the second level of the anvil,

$$Anv(WPHGH),$$

i.e., if part (5) applies to an obtained quotient multi-graded resolution, we continue in a similar way to our approach in analyzing multi-graded resolutions the top level of which is of maximal complexity (see also part (3) of the first step of the procedure).

We continue to the next levels of the anvil in precisely the same way. At each level i, we consider the quotient limit groups associated with auxiliary resolutions that are associated with the tower containing all levels up to level $i+1$ in the anvil, $Anv(MGQ^{n-1}Res)$. Then we analyze the taut Makanin–Razborov diagrams of the limit groups associated with the various levels (from level 1 to level $i-1$), and continue only with those resolutions that are of maximal complexity in all these levels, and the taut structures associated with their abelian decompositions are identical to those associated with the corresponding levels of the anvil, $Anv(MGQ^{n-1}Res)$. Finally, we analyze the resolutions in the taut Makanin–Razborov diagram associated with the i-th level according to parts (1)–(4), or (the first part of) (5), and continue iteratively.

Let MGQ^nRes be a multi-graded resolution obtained by the above iterative procedure. If there exists a level for which one of the parts (1)–(4) applies, we set a developing resolution, and an anvil with the limit group associated with this level according to the part (1)–(4) that applies to it.

To set the developing resolutions associated with the resolution MGQ^nRes, we first construct a resolution composed from the resolution induced by the subgroup $< y, h_1, w, p, a >$ from the parts of the resolution MGQ^nRes above the level for which parts (1)–(4) apply (i.e., the parts that are of maximal complexity), followed by the graded resolution constructed at that level according to part (1)–(4) that applies (which is also composed from graded resolutions induced by the subgroup $< y, h_1, w, p, a >$). We denote the terminal rigid or solid limit group of the obtained graded resolution (where the grading is with respect to the parameter subgroup $< h_1, w, p >$), $WP(HG)^{n+1}(g_{n+1}, h_1, w, p, a)$. Then we collect all the formal solutions defined over the obtained (graded) resolution using the graded formal Makanin–Razborov diagram. We set each of the graded (not formal!) resolutions associated with the terminal limit groups in this graded formal Makanin–Razborov diagram to be a developing resolution. With each developing resolution we associate a family of formal solutions encoded by the graded formal modular groups associated with the graded formal resolution to which it belongs.

With the developing resolution we associate a finite collection of anvils, which are set to be the maximal limit quotients of the group generated by the completion of the multi-graded resolution MGQ^nRes, the anvil constructed at the level in which one of the parts (1)–(4) applies, where the top part of this anvil is identified with the bottom part of the completion of the multi-graded resolution MGQ^nRes, and the corresponding parts of the developing resolution are identified with their images in the given anvil, and the completion of MGQ^nRes (precisely as we did in part (4)).

With the developing resolution and its associated anvil, we naturally associate a family of formal solutions, $x_\alpha(f, y, h_{n+2}, g_{n+1}, , h_1, w, p, a)$, parameterized by the graded formal modular groups associate with the graded formal resolution associate with the developing resolution and the anvil. With the anvil we also associated a rigid or solid limit group of the form $WP(HG)^{n+1}H(h_{n+2}, g_{n+1}, h_1, w, p, a)$, which is the terminal rigid or solid limit group of the developing resolution (with respect to the parameter subgroup $< h_1, w, p >$), and the rigid or solid limit group $WP(HG)^{n+1}$, as we did according to the part (1)–(4) that applies to the multi-graded resolution associated with the corresponding level.

If all the abelian decompositions associated with the multi-graded resolutions used for the construction of the developing resolution are of maximal complexity, i.e., if none of the parts (1)–(4) applies to any of these multi-graded resolutions, we examine the structure of the developing resolution. The developing resolution is built from a sequence of induced resolutions. Each of the induced resolutions is a resolution induced by the (image of the) subgroup $< y, h_1, w, p, a >$, and with each level of the induced resolution there is associated an (induced) abelian decomposition (see section 3 of [Se4] for the construction of the induced resolution).

PROPOSITION 2.9: *Suppose that all the abelian decompositions associated with the multi-graded resolutions used for the construction of the developing resolution are of maximal possible complexity. Let $< v, y, h_1, w, p, a >$ be the subgroup generated by the closure of the developing resolution in the anvil $Anv(MGQ^{n-1}Res)(t, y, h_1, w, p, a)$. From each of the multi-graded resolutions used to construct the developing resolution (in step n of the procedure), there is a resolution induced by the (image of the) subgroup $< v, y, h_1, w, p, a >$.*

Then either the structure of the resolution composed from the resolutions *induced by the subgroup* $< v, y, h_1, w, p, a >$ *from the various multi-graded resolutions used to construct the developing resolution is identical to the structure of the developing resolution, or there exists some level* j *so that the structure of the abelian decompositions associated with the resolutions induced by the subgroup* $< v, y, h_1, w, p, a >$ *above level* j *are identical to the structure of the abelian decompositions associated with the developing resolution, and in level* j*, the number of factors in the free decomposition associated with the abelian decomposition associated with the resolution induced by* $< v, y, h_1, w, p, a >$ *is strictly smaller than the number of factors in the corresponding free decomposition associated with the abelian decomposition associated with level* j *in the developing resolution, and in case of equality in the number of factors, the complexity of the abelian decomposition associated with the resolution induced by* $< v, y, h_1, w, p, a >$ *is strictly smaller than the complexity of the abelian decomposition associated with level* j *in the developing resolution.*

Proof: Identical to the proof of proposition 4.8 of [Se4]. ∎

If the structure of the resolution composed from the resolutions induced by the subgroup $< v, y, h_1, w, p, a >$ from the various multi-graded resolutions used to construct the developing resolution is identical to the structure of the developing resolution, we have reached a terminal point of our branching procedure. With the multi-graded resolution

$$MGQ^n Res(t, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

we associate the terminal rigid or solid limit group of the resolution composed from the resolutions induced by the subgroup $< v, y, h_1, w, p, a >$, which we denote $WP(HG)^{n+1}$.

If the structure of the resolution composed from the resolutions induced by the subgroup $< v, y, h_1, w, p, a >$ from the various multi-graded resolutions used to construct the developing resolution is not identical to the structure of the developing resolution, Proposition 2.9 implies that there exists some level j for which the structure of the abelian decompositions associated with the resolutions induced by the subgroup $< v, y, h_1, w, p, a >$ above level j are identical to the structure of the abelian decompositions associated with the developing resolution, and in level j, the number of factors in the free decomposition associated with the abelian decomposition associated with the resolution induced by $< v, y, h_1, w, p, a >$ is strictly smaller than the number of factors in the corresponding free decomposition associated with the developing resolution, and in case of equality in the number of factors, the complexity of the abelian decomposition associated with the resolution induced by $< v, y, h_1, w, p, a >$ is strictly smaller than the complexity of the abelian decomposition associated with the developing resolution. In this case we do the following.

Let $CRes(u, y, h_1, w, p, a)$ be the graded resolution that is composed from the resolutions induced by the subgroup $< v, y, h_1, w, p, a >$. With the resolution $CRes(u, y, h_1, w, p, a)$ we associate its terminal rigid or solid limit group (with respect to the parameter subgroup $< h_1, w, p >$), which we denote $WP(HG)^{n+1}(g_{n+1}, h_{n+1}, g_n, h_1, w, p, a)$. We collect all the formal solutions defined over ungraded resolutions covered by the graded resolution $CRes(u, y, h_1, w, p, a)$. This collection of formal solutions factors through a canonical collection of graded formal limit groups. With each graded formal limit group we associate its graded formal Makanin–Razborov diagram as we did in section 3 of [Se2]. We continue with each of the graded formal resolutions that appear in these diagrams in parallel.

Let $GFRes(x, u, y, g_{n+1}, h_{n+1}, g_n, h_1, w, p, a)$ be a graded formal resolution in one of these diagrams, and let

$$WP(HG)^{n+1}H(h_{n+2}, g_{n+1}, h_{n+1}, g_n, h_1, w, p, a)$$

be its terminating rigid or solid (not formal!) limit group. With the graded formal resolution $GFRes(x, u, y, h_{n+2}, g_{n+1}, h_{n+1}, h_1, w, p, a)$ we associate the resolution $GRes(\hat{u}, y, h_{n+2}, g_{n+1}, h_{n+1}, h_1, w, p, a)$, which is the graded (not formal!) resolution associated with the terminal formal limit group of the graded formal resolution $GFRes(x, u, y, h_{n+2}, g_{n+1}, h_{n+1}, h_1, w, p, a)$. With a graded formal resolution we naturally associate a family of formal solutions parameterized by elements of the graded formal modular groups associated with the graded formal resolution.

We continue with each of the graded formal resolutions separately. We set the developing resolution to be the graded resolution

$$GRes(\hat{u}, y, h_{n+2}, g_{n+1}, h_{n+1}, h_1, w, p, a).$$

We set the finite collection of anvils, which we denote

$$Anv(MGQ^n Res)(t, y, h_1, w, p, a),$$

to be the (finite) collection of maximal (quotient) limit groups associated with the amalgamation of the (previous) anvil

$$Anv(MGQ^{n-1}Res)(t, y, h_1, w, p, a),$$

with the graded resolution $GRes$, amalgamated along the developing resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$ (that is naturally mapped into both). With the anvil, $Anv(MGQ^nRes)(t,y,h_1,w,p,a)$, we naturally associate the (graded) family of formal solutions, which are defined over the developing resolution, and parameterized by the graded formal modular groups associated with the graded formal resolution $GFRes$. We further associate with the anvil the terminal rigid or solid limit groups, $WP(HG)^{n+1}$ and $WP(HG)^{n+1}H$.

Starting with the anvils, $Anv(MGQ^{n-1}Res)$, their associated developing resolutions, and the auxiliary resolutions and quotient limit groups associated with them, we have constructed a finite collection of multi-graded resolutions, MGQ^nRes, developing resolutions and anvils, $Anv(MGQ^nRes)(t,y,h_1,w,p,a)$. With each couple of a developing resolution and an anvil we have associated a family of formal solutions defined over them, and parameterized by the modular groups associated with the graded formal resolution associated with the developing resolution.

As in the initial and first steps of the iterative procedure, to complete the *data-structure* with which we continue to the next step, we still need to associate with each anvil, $Anv(MGQ^nRes)$, a finite collection of auxiliary resolutions and auxiliary modular groups (see Definitions 2.1 and 2.6). We construct the associated auxiliary resolutions, and limit groups, precisely as we did in the first step, i.e., by applying the construction presented in Definition 2.6.

The collection of multi-graded resolutions, MGQ^nRes, the developing resolutions and the anvils, $Anv(MGQ^nRes)$, associated with them, the families of formal solutions defined over them, and their collections of auxiliary resolutions, limit groups, and modular groups, together with the data-structure constructed before starting the n-th step of the procedure, form the *data-structure* obtained as a result of the n-th step.

At this stage, we continue in a similar way to what we did before starting the first step of the procedure. Given an anvil, $Anv(MGQ^nRes)$, and an auxiliary resolution, we look at the entire set of multi-graded rigid or strictly solid specializations of the associated auxiliary limit group (Definition 2.6), and their extensions to specializations of the anvil, for which for the entire family of formal solutions, $x_\alpha(t,y,w,p,a)$, associated with the anvil (i.e., associated with the graded formal resolution that is associated with the developing resolution associated with the anvil), and for each element φ_β in the auxiliary modular group associated with the given auxiliary resolution, $\psi_j(x_\alpha(\varphi_\beta(t,y,w,p,a)),y,w,p,a) = 1$

for at least one of the equations ψ_j in the system (of inequalities) $\Psi(x,y,w,p,a) \neq 1$ used to define the set $EAE(p)$. By the standard argument presented in section 5 of [Se1], the entire collection of such (extended) specializations, (t,y,w,p,a), is contained in a finite set of maximal graded limit groups (that are all quotients of the anvil, $Anv(MGQ^nRes)$)

$$Q^{n+1}Rlim_1(t,y,h_1,w,p,a),\ldots,Q^{n+1}Rlim_{u_{n+1}}(t,y,h_1,w,p,a),$$

which we call $(n+1)$ **quotient limit groups**. Note that with each such limit group there is an associated anvil, $Anv(MGQ^nRes)$, and an associated auxiliary resolution. The quotient limit groups we constructed, which collect (uniformly) all the remaining y's for every specialization of the parameter subgroup $< h_1, w, p >$, and their associated data-structure, are the input for the next $(n+1)$ step of the iterative procedure.

III. Termination of the iterative procedure. Defining the initial, first and general steps of our iterative procedure for analyzing the set $EAE(p)$, we are required to prove its termination. To prove termination of our iterative procedure in the minimal rank case (section 1), we used the strict decrease in the complexity of the resolutions associated with successive steps of the procedure, a strict decrease that forces termination. Unlike our procedure in the minimal rank case, in the general procedure we do not obtain a strict decrease in the complexity of the resolutions associated with successive steps of the procedure. To obtain termination in the general case, we need to look at limit groups (or alternatively Zariski closures) and complexities of various resolutions and decompositions associated with the developing resolutions and anvils constructed along the steps of the procedure. Our ultimate goal in proving the termination of the procedure is to show that after finitely many steps of it, the iterative procedure is applied not to specializations of the limit group $QRlim(y,h_1,w,p,a)$ but rather to specializations of a proper quotient of it. The entire argument is similar to the one used to prove the termination of the procedure for validation of a sentence ([Se4], 4.12), hence we refer the interested reader to the argument presented in [Se4].

Theorem 2.10: *The iterative procedure for the analysis of the set $EAE(p)$ terminates after finitely many steps.*

Proof: Identical to the proof of theorem 4.12 of [Se4]. ∎

As the minimal rank case, the outcome of the "trial and error" procedure presented in this section gives us a finite diagram, constructed along the various steps of the iterative procedure, a diagram which is a directed tree in which on every vertex we place a rigid or solid limit group (with respect to the parameter subgroup $< h_1, w, p >$) of the form $WP(HG)^n$ or $WP(HG)^nH$, which is the basis of a bundle of the set of the remaining y's or the set of formal solutions defined over the bundle of the remaining y's analyzed along the iterative procedure, which we call the **tree of stratified sets**. This tree encodes all the (finitely many) possible sequences of forms of (families of) formal solutions that are needed in order to validate that a certain specialization p_0 of the defining parameters p is indeed in the set $EAE(p)$. This tree and the stratification associated with its various rigid and solid limit groups is the basis for our analysis of the structure of the set $EAE(p)$.

Given the tree of stratified sets, to analyze the set $EAE(p)$ we still need a sieve procedure, similar to the one presented in the minimal rank case. However, obtaining a terminating sieve procedure in the general case is much more involved than in the minimal rank case. This sieve procedure is the goal of the next paper in the sequence, that finally proves quantifier elimination for general predicates over a free group.

3. The sieve method in a few special cases

As in the procedure for the analysis of an EAE set in the minimal (graded) rank case, presented in the first section of this paper, the outcome of the "trial and error" procedure presented in the previous section gives us a *tree of stratified sets* which encodes all the (finitely many) possible sequences of families of formal solutions that are needed in order to validate that a certain specialization p_0 of the defining parameters p is indeed in the set $EAE(p)$. This stratification is the basis for our analysis of the structure of the set $EAE(p)$ in the general case as well.

Let $EAE(p)$ be the set defined by the predicate

$$EAE(p) = \exists w \ \forall y \ \exists x \ \Sigma(x, y, w, p, a) = 1 \wedge \Psi(x, y, w, p, a) \neq 1.$$

Recall (Definition 1.19) that a specialization w_0 of the variables w is said to be a *witness* for a specialization p_0 of the defining parameters p if the following sentence:

$$\forall y \quad \exists x \quad \Sigma(x, y, w_0, p_0, a) = 1 \wedge \Psi(x, y, w_0, p_0, a) \neq 1$$

is a true sentence. Clearly, if there exists a witness for a specialization p_0 then $p_0 \in EAE(p)$, and every $p_0 \in EAE(p)$ has a witness.

In order to show that a specialization p_0 of the defining parameters p is in the set $EAE(p)$, we need to find a witness w_0 for the specialization p_0. As in the minimal rank case, the construction of the tree of stratified sets guarantees that a witness w_0 for a specialization p_0 proves that $p_0 \in EAE(p)$ using a certain "proof system" which is built from a finite sequence of (families of) formal solutions that corresponds to boundedly many paths along the tree of stratified sets (Definition 1.20).

Given $p_0 \in EAE(p)$, we are not able to say much about a possible witness for p_0 using the information we have collected so far. However, as in the minimal rank case, with each "proof system", i.e., with each collection of paths (a subtree) of (families of) formal solutions that goes along the tree of stratified sets, one can associate a certain Diophantine set of possible witnesses. As demonstrated in the first section, the bound on the form and number of all possible "proof systems", associated with all possible witnesses suggested by the tree of stratified sets, forces every possible witness for p_0 to belong to one of the finitely many Diophantine sets associated with the (finite) collection of all proof systems. As in the minimal rank case, in the sequel we will construct a finite set of Diophantine sets associated with each proof system (subtree of the tree of stratified sets), and show that if a specialization $p_0 \in EAE(p)$, and it can be shown that $p_0 \in EAE(p)$ using a witness w_0 and a specific proof system, then every "generic" specialization of w which belongs to some Diophantine set associated with the specific proof system is a witness for p_0 using the same proof system as w_0. This will reduce the analysis of the set $EAE(p)$ to the analysis of the Diophantine sets associated with each "proof system", and eventually will enable us to show that the set $EAE(p)$ is in the Boolean algebra of AE sets, which finally concludes the proof of Theorem 1.4.

As we indicated in the minimal rank case (Definition 1.20), if $p_0 \in EAE(p)$ is a specialization of the defining parameters p, and w_0 is a witness for p_0, then the construction of the tree of stratified sets implies that one can associate a *proof system* with the couple (w_0, p_0), which corresponds to a (finite) collection of paths in the tree of stratified sets. Note that there may be several proof systems associated with a given couple (w_0, p_0), but the construction of the tree of stratified sets guarantees that the number of proof systems associated with the couple (w_0, p_0) is globally bounded. As in the first section we will say that a given proof system associated with the couple (w_0, p_0) is of *depth* d, if all the paths associated with the proof system terminate after d steps (levels) of the tree of stratified sets.

As we did in the minimal rank case, we will start by demonstrating our approach for the analysis of the set $EAE(p)$, by analyzing those specializations of the defining parameters p that have witnesses with proof systems of depth 1, i.e., those that have witnesses with proof systems that terminate after the initial step of the construction of the tree of stratified sets. We will continue by analyzing the specializations of the defining parameters p for which there are witnesses with proof systems of depth at most 2, i.e., those that have witnesses with proof systems that terminate after the first step of the construction of the tree of stratified sets, and then present the analysis of the entire set $EAE(p)$.

LEMMA 3.1: *Let $T_1(p) \subset EAE(p)$ be the subset of all specializations $p_0 \in EAE(p)$ of the defining parameters p that have a witness with a proof system which terminates after the initial step of the construction of the tree of stratified sets. Then $T_1(p)$ is an EA set.*

Proof: The initial step of the iterative procedure for the construction of the tree of stratified sets in the general case is identical to the first step of the iterative procedure for the construction of the tree of stratified sets in the minimal rank case. Hence, the proof of Lemma 3.1 is identical with the proof of Lemma 1.21. ∎

Lemma 3.1 proves that the set of specializations p_0 of the defining parameters p that have a witness with a proof system which terminates after the initial step of the iterative procedure for the construction of the tree of stratified sets is an EA set. Before analyzing the entire set $EAE(p)$, we analyze the set of specializations p_0 which have witnesses with a proof system that terminates after the first step of the iterative procedure for the construction of the tree of stratified sets. The analysis of specializations $p_0 \in EAE(p)$ having witnesses with such proof systems is much more complicated than the analysis of witnesses with proof systems that terminate after the initial step, and it is based (though it is somewhat different) on the analysis of the set of specializations p_0 having proof systems of depth 2, $T_2(p)$, in the minimal rank case (Theorem 1.22).

THEOREM 3.2: *Let $T_2(p) \subset EAE(p)$ be the subset of all specializations $p_0 \in EAE(p)$ of the defining parameters p that have witnesses with a proof system which terminates after the first step of the procedure for the construction of the tree of stratified sets. Then $T_2(p)$ is in the Boolean algebra of AE sets.*

Proof: To start the proof of Theorem 3.2 recall that a *valid PS* statement, presented in Definition 1.23, is a statement that satisfies a list of properties required by the corresponding proof system (see Definition 1.23).

As in the minimal rank case, to each valid PS statement we add specializations that extend the specializations of the limit groups $WPHGH$ in the valid PS statement, by adding specializations of primitive roots of edge groups and pegged abelian vertex groups in the graded abelian JSJ decompositions of the limit groups $WPHG$ that occur along the given proof system (i.e., we add specializations of primitive roots of a fixed set of elements in the valid PS statement). We further add specializations that demonstrate how all the multiples of these primitive roots, multiples up to the least common multiple of the indices of the finite index subgroups associated with the closure domains associated with the various groups $WPHGH$, can be extended to specializations that factor through the finite set of (graded formal) closures specified by the valid PS statement (in fact these closures are specified by the proof system, not just by the proof statement), i.e., the closures associated with the various limit groups $WPHGH$. This is equivalent to demonstrating that the given set of closures (associated with the specializations of the groups $WPHGH$) is a covering closure for the ungraded resolutions associated with the specializations (specified by the proof statement) of the groups $WPHG$.

For brevity, in the sequel we still call such extended specializations valid PS statements and denote them $(r,(h_1^2,g_1^1),\dots,(h_{\nu(ps)}^2,g_{\nu(ps)}^1),h_0^1,w_0,p_0,a)$. By the standard arguments presented in section 5 of [Se1], the entire collection of (extended) valid PS statements factor through a (canonical) collection of maximal limit groups $PSHGH_1,\dots,PSHGH_m$, which we call PS (proof system) **limit groups**.

By construction, for each $p_0 \in T_2(p)$ there exists some witness w_0 and a corresponding proof system, so that a specialization of the form

$$(r,(h_1^2,g_1^1),\dots,(h_{\nu(ps)}^2,g_{\nu(ps)}^1),h_0^1,w_0,p_0,a)$$

that is associated with the specialization p_0, the witness w_0 and that proof system is a valid PS statement (i.e., it satisfies conditions (i)–(iv) of Definition 1.23), and factors through a limit group $PSHGH_j$. As in the minimal rank case, our main goal will be to show that these valid PS statements are “generic” in some Diophantine set associated in the sequel with each of the PS limit groups $PSHGH$. The “sieve” procedure for the analysis of the valid PS statements that factor through a given PS limit group combines the procedure presented in the first section for the minimal ranks case with tools used in the construction of the tree of stratified sets in the general case, and with the notion of a *core resolution* presented in the next section. As in the first section (Proposition

1.24), we start by analyzing those PS limit groups which are rigid or solid with respect to the parameter subgroup $P =< p >$.

PROPOSITION 3.3: *Suppose that a PS limit group $PSHGH$ is rigid or solid with respect to the parameter subgroup $P =< p >$, and if it is solid suppose that the subgroup $WP =< w, p >$ is a subgroup of the distinguished vertex group in the graded JSJ decomposition of $PSHGH$ (i.e., the vertex stabilized by the subgroup $AP =< p, a >$). The set of specializations p_0 that have a witness w_0, and a rigid or strictly solid specialization $(r, (h_1^2, g_1^1), \ldots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_0^1, w_0, p_0, a)$ of $PSHGH$, which is a valid PS statement that we denote $PS(p)$, is in the Boolean algebra of AE sets.*

Proof: Identical to the proof of Proposition 1.24. ∎

Proposition 3.3 proves Theorem 3.2 in case the PS limit groups $PSHGH$ are rigid or solid with no flexible quotients, and the subgroup $WP =< w, p >$ is a subgroup of the distinguished vertex group in the graded JSJ decomposition of $PSHGH$. In this special case, the number of possible witnesses w_0 associated with each specialization p_0 of the defining parameter p is finite and globally bounded. In the general case, the number of possible witnesses associated with each specialization p_0 of the defining parameters p is infinite. Our goal in the analysis of the set $T_2(p)$ in the general case is to construct finitely many Diophantine sets associated with each of the PS limit groups $PSHGH$, so that a specialization p_0 of the defining parameters p admits a valid proof statement that factors the PS limit group $PSHGH$, if and only if a generic point in a fiber associated with p_0 in one of the Diophantine sets associated with the PS limit group $PSHGH$ is a valid proof statement. To achieve this goal, i.e., to construct the finitely many Diophantine sets associated with each of the PS limit groups $PSHGH$, we present a "trial and error" procedure, based on the one used to construct the tree of stratified sets (in the general case). As in the minimal rank case, the output of the iterative "trial and error" procedure, i.e., the finitely many Diophantine sets associated with each PS limit group $PSHGH$, are later used to derive a predicate in the Boolean algebra of AE predicates that describes the set $T_2(p)$.

Let $P =< p >$ be the group of defining parameters. With each of the limit groups $PSHGH_i$ we associate its canonical taut graded Makanin–Razborov diagram (with respect to the parameter subgroup P), which contains finitely many graded resolutions that we denote $PSHGHRes_j$, and each graded resolution $PSHGHRes_j$ is defined over the rigid or solid limit group $PB_j(b, p, a)$. We will treat the limit groups $PSHGH_i$, and their graded resolutions $PSHGHRes_j$ and their terminal rigid or solid limit groups $PB_j(b,p,a)$, in parallel, hence we will omit the indices of the limit group and its graded resolution. In the sequel, we will treat each stratum in the singular locus of the graded resolutions $PSHGHRes$ separately, and do it in parallel.

Precisely as we did in the minimal rank case, we start our analysis of the set of valid PS statements by associating with (the completion of) the graded PS resolution $PSHGHRes$ a (canonical) finite collection of *Non-Rigid* and *Non-Solid PS limit groups* (see Definition 1.25). We denote the *non-rigid PS limit groups* associated with the PS resolution $PSHGHRes$,

$$NRgdPS_1, \dots, NRgdPS_q,$$

and the *non-solid PS limit groups* associated with $PSHGHRes$,

$$NSldPS_1, \dots, NSldPS_r.$$

Recall (Definition 1.25) that the graded formal closures associated with the collection of non-rigid and non-solid PS limit groups determine those "generic" specializations that factor through and are taut with respect to the various PS resolutions $PSHGHRes$, but fail to be valid PS statements with respect to the (fixed) proof system because certain specializations that are required to be rigid or strictly solid actually factor through flexible quotients of the corresponding rigid or solid limit groups. As in the minimal rank case, "generic" specializations that factor through the PS resolutions $PSHGHRes$ can fail to be valid PS statements in other ways as well.

In particular, we need to collect all the test sequences that factor through the PS resolutions $PSHGHRes$, and for which for at least one of the tuples $(h^2_j(n), g^1_j(n), h_1(n), w_n, p_n, a)$ there exists some specialization $g^2_j(n)$ so that the (combined) specialization $(g^2_j(n), h^2_j(n), g^1_j(n), h_1(n), w_n, p_n, a)$ factors through (at least) one of the limit groups $WP(HG)^2$. Recall (Definition 1.26) that the collection of all these test sequences factors through a (canonical) collection of *Left PS limit groups* $LeftPS_1, \dots, LeftPS_m$. Each Left PS limit group $LeftPS_i$ is in fact a graded formal closure of the graded resolution $PSHGHRes$, $GFCl(PSHGHRes)$. Clearly, no specialization

$$(r, (h^2_1, g^1_1), \dots, (h^2_{\nu(ps)}, g^1_{\nu(ps)}), h^1_0, w_0, p_0, a)$$

that factors through the resolution $PSHGHRes$, and which is a valid PS statement with respect to our fixed proof system, factors through one of the $LeftPS$ limit groups $LeftPS_1, \dots, LeftPS_m$ and their associated resolutions.

To a valid PS statement we have added additional variables, so that their specializations are supposed to be primitive roots of the specializations of pegs of abelian groups that appear in the graded formal closures associated with the groups $WPHGH$, in order to demonstrate that the given sets of closures (specified by the proof system) form a covering closure (for the specializations given by the proof statement). This demonstration remains valid if the orders of the specializations of the variables that are supposed to be primitive roots are prime to the indices of the finite index subgroups associated with the (finitely many) closures. The demonstration may fail to be valid if the orders of these specializations are not prime to the order of the finite index subgroups. To check if this failure occurs for a generic specialization of a PS resolution, $PSHGHRes$, we construct *Root PS* limit groups and resolutions, precisely as we did in the minimal rank case (Definition 1.27). We denote the *Root PS* limit groups, $RootPS$, and the *Root PS* resolutions, $RootPSRes$.

No specialization $(r,(h_1^2,g_1^1),\dots,(h_{d(ps)}^2,g_{d(ps)}^1),h_0^1,w_0,p_0,a)$ that factors through the resolution $PSHGH$ (a virtual proof), and which is a valid PS statement with respect to our fixed proof system, factors through one of the $RootPS$ limit groups $RootPS_1,\dots,RootPS_m$ and their associated Root PS resolutions.

"Generic" specializations that factor through the PS resolutions $PSHGHRes$ can fail to be valid PS statements also if there exist additional rigid or strictly solid specializations of the limit groups $WPHG(g_1,h_1,w,p,a)$ that are not specified by the given proof statements. As in the minimal rank case, the "generic" specializations for which there exists a "surplus" in rigid or families of strictly solid specializations are collected in *Extra PS* (graded) *limit groups* and graded resolutions (Definition 1.28). We denote the *Extra PS limit groups* associated with the graded PS resolution $PSHGHRes$, $ExtraPS_1,\dots,ExtraPS_\ell$.

The extra PS limit groups collect all the "generic" specializations (i.e., all the test sequences) of the PS limit groups $PSHGH$ for which there exist rigid or strictly solid families in addition to those specified by the generic specializations. For a general specialization of the PS limit groups $PSHGH$, i.e., a specialization which is not necessarily "generic", it may as well be that the additional rigid or strictly solid specializations, collected by the extra PS limit groups, do become flexible or do coincide with the rigid or strictly solid families of the various specializations (g_j^1,h_0^1,w_0,p_0).

In the minimal rank case, we collected all the specializations that factor through one of the Extra PS resolution, in which such a "collapse" of the additional rigid and strictly solid families occurs, in a canonical collection of *Collapse Extra PS* (graded) *limit groups* and their associated (Collapse Extra PS) graded resolutions (Definition 1.29). In the general case, the analysis of the specializations that factor through one of the extra PS resolutions, in which such a "collapse" of the additional rigid and strictly solid families occurs, and the construction of the associated *Collapse Extra PS* (graded) *limit groups* and their *Collapse Extra PS* (*graded*) *resolutions*, is more involved, and so is the iterative procedure for the analysis of the set $T_2(p)$. Hence, mostly for presentation purposes, in this section we present the construction of the Collapse Extra PS limit groups and their associated graded resolutions, and the iterative procedure for the analysis of the set $T_2(p)$, in (the special) case all the limit groups $WPHG$ are rigid. The approach we use in the rigid case can be somewhat generalized; however, the general case of general rigid and solid $WPHG$ seems to be considerably more involved and is treated in the next paper in the sequel.

For the rest of this section we assume that all the limit groups $WPHG$ associated with one of the paths associated with our fixed proof systems are rigid limit groups. In this case the analysis of the set $T_2(p)$ is conceptually similar to the procedure presented in the minimal rank case, combined with the iterative procedure for the construction of the tree of stratified sets (in the general case) presented in the second section.

Let $ExtraPSRes$ be one of the extra PS graded resolutions associated with one of the extra PS graded limit groups $ExtraPS_i$. Note that by construction, $ExtraPSRes$ is a graded closure of the PS resolution $PSHGHRes$ with which we started the analysis. In case the groups $WPHG$ are all rigid, we will say that a specialization that factors through and is taut with respect to the extra PS graded resolution $ExtraPSRes$ is **collapsed** if the variables added for each of the additional rigid specializations (i.e., the ones that were not specified by the proof statement) satisfy one of the following:

(1) A specialization of the variables added for one of the additional rigid specializations becomes flexible.

(2) A specialization of the variables added for one of the additional rigid specializations becomes equal to one of the rigid specializations specified by the proof statement, i.e., one of the specializations g_j^1 in the specialization

$$(u, v, r, (h_1^2, g_1^1), \ldots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_0^1, w_0, p_0).$$

After defining the finitely many possibilities for collapse forms, we collect all the test sequences of specializations that are collapsed specializations in finitely many closures of the resolution $PSHGHRes$ we have started with, which we call *Generic Collapse Extra PS* (graded) *resolutions*, and the remaining collapsed specializations in finitely many graded limit groups, which we call *Collapse Extra PS* (graded) *limit groups*.

We start by collecting all the test sequences that factor through an Extra PS resolution, and for which each of the extra rigid specializations (i.e., those that are not specified by the (virtual) proof statement) satisfies the conditions of one of the possible collapse forms. These test sequences factor through finitely many *Generic Collapse Extra PS* limit groups and resolutions. The definition of the Generic Collapse Extra PS resolutions (Definition 3.4 below) is general and does not depend on the graded limit groups $WPHG$ being rigid or solid.

Definition 3.4: Let $ExtraPSRes$ be one of the Extra PS graded resolutions associated with one of the Extra PS graded limit groups $ExtraPS_i$. Note that by construction, $ExtraPSRes$ is a graded closure of the PS resolution $PSHGHRes$ with which we started the analysis. With the Extra PS resolution we associate all its possible collapse forms.

Let $ExtraPSRes(u,v,r,(h_1^2,g_1^1),\dots,(h_{\nu(ps)}^2,g_{\nu(ps)}^1),h_1,w,p,a)$ be an Extra PS resolution. We look at the entire collection of test sequences that factor through the Extra PS resolution.

From the collection of all test sequences, we look at those graded test sequences for which each specialization of the sequence satisfies the conditions of one of the (finitely many) possible collapse forms associated with the extra PS resolution. The collection of all these (graded collapse) test sequences factors through a (canonical) collection of *maximal Generic Collapse Extra PS limit groups*:

$$GenericCollapseExtraPS_1,\dots,GenericCollapseExtraPS_t.$$

The analysis of graded formal limit groups presented in section 3 of [Se2] associates (canonically) with each Generic Collapse Extra PS limit group $GCollapseExtraPS$ a graded formal Makanin–Razborov diagram, and each such graded formal resolution is in fact a one-level graded resolution, which is a graded formal closure of the graded resolution $PSHGHRes$,

$$GFCl(PSHGHRes).$$

We call each such graded formal closure a *Generic Collapse Extra PS* (graded) *resolution*.

Definition 3.5: Let $ExtraPSRes$ be the Extra PS (graded) resolutions associated with the PS resolutions $PSHGHRes$ that appear in the taut Makanin–Razborov diagram of a PS limit group $PSHGH$. With each of the Extra PS resolutions we associate all its possible collapse forms.

The resolutions $ExtraPSRes$ are graded closures of the PS resolution $PSHGHRes$ with which we started the analysis, so with the resolutions $ExtraPSRes$ we can associate their canonical collection of graded auxiliary resolutions and limit groups (Definition 2.1). Note that if a rigid or strictly solid specialization of the graded auxiliary limit group associated with an Extra PS resolution, $ExtraPSRes$, extends to a specialization of $ExtraPSRes$ that satisfies the conditions of one of the collapse forms associated with it, the extended specialization has to satisfy a (fixed) system of equations associated with the specific collapse form.

Also, note that since the parameter subgroup $< h_1, w, p >$ does not change with an action of the auxiliary modular group, if a strictly solid specialization of the auxiliary limit group extends to a specialization of $ExtraPSRes$ which is collapsed, then the corresponding extensions of all the specializations of the auxiliary limit group that are in the same strictly solid family are all collapsed. This last observation allows us to apply the iterative procedure for the construction of the tree of stratified sets, to get a sieve procedure in case all the graded limit groups $WPHG$ are rigid.

We go over all the (finitely many) graded auxiliary resolutions and all the (finitely many) collapse forms associated with the Extra PS resolutions, $ExtraPSRes$. Given a graded auxiliary limit group and a collapse form, we look at all the rigid or strictly solid specializations of the given graded auxiliary limit group that extends to specializations of $ExtraPSRes$ which satisfy the system of equations associated with the specified collapse form associated with the associated resolution $ExtraPSRes$.

By our standard method presented in section 5 of [Se1], the collection of (extended) specializations of $ExtraPSRes$ that restricts to rigid or solid specializations of the given auxiliary limit group, so that the extended specializations satisfy the system of equations associated with a specified collapse form, factor through a canonical (finite) collection of maximal limit groups, which we call *Collapse Extra PS limit groups* and denote

$$CollapseExtraPS_1, \ldots, CollpaseExtraPS_d.$$

Note that by construction, each Collapse Extra PS limit group is a proper quotient of the limit group associated with the extra PS resolution, $ExtraPSRes$, with which we started.

The PS limit groups $PSHGH$ and their associated PS resolutions, together with the non-rigid, Left, and Root PS limit groups and their associated graded resolutions, the Extra PS resolutions, the Generic Collapse Extra PS resolutions, and the Collapse Extra PS limit groups, enable us to generalize the main principle that was used in the procedure for quantifier elimination under the minimal (graded) rank assumption, to the case in which the graded limit group $WPHG$ is all rigid. As in the minimal rank case, we show that if a valid PS statement factors through one of the resolutions $PSHGHRes$, then either there exists a valid PS statement that factors through one of the (non-Generic) Collapse extra PS limit groups and their associated graded auxiliary resolutions, or there exists a "generic" valid PS statement, i.e., a test sequence of valid PS statements that factor through the PS resolution $PSHGHRes$. This principle does not depend on the graded limit groups $WPHG$ being rigid.

THEOREM 3.6: *Let $PSHGHRes$ be the PS resolutions that appear in the taut Makanin–Razborov diagram of a PS limit group $PSHGH$. Suppose that there exists a valid PS statement $(r,(h_1^2,g_1^1),\dots,(h_{\nu(ps)}^2,g_{\nu(ps)}^1),h_0^1,w_0,p_0,a)$ which factors through the PS limit group $PSHGH$. Then one of the following holds:*

(1) *There exists a test sequence of specializations*

$$(v_n,r_{,n}\,,(h_1^2(n),g_1^1(n)),\dots,(h_{d(ps)}^2(n),g_{\nu(ps)}^1(n)),h_1(n),w_n,p_0,a))$$

that factor through the completion of one of the PS resolutions $PSHGHRes$ (note that the specialization p_0 of the defining parameters p are fixed along the entire test sequence), which restricts to a sequence of valid PS statements (with respect to our fixed proof system)

$$((h_1^2(n),g_1^1(n)),\dots,(h_{d(ps)}^2(n),g_{\nu(ps)}^1(n)),h_1(n),w_n,p_0,a)).$$

(2) *The valid PS statement factors through one of the (non-Generic) Collapse Extra PS limit groups associated with the multi-graded auxiliary limit groups that are associated with the Extra PS resolutions $ExtraPSRes$.*

Proof: Similar to the proof of Theorem 1.33. ∎

As in the minimal rank case, Theorem 3.6 reduces the analysis of the set $T_2(p)$ to the set of specializations of the defining parameters $P=<p>$ for which there exists a test sequence of valid PS statements that factor through the various resolutions $PSHGHRes$.

PROPOSITION 3.7: *Let $PSHGHRes$ be one of the (non-trivial) PS graded resolutions. Let $TSPS(p)$ be the set of specializations p_0 of the defining parameters $P=<p>$, for which there exists a test sequence of specializations*

$$(v_n, r_n, (h_1^2(n), g_1^1(n)), \ldots, (h_{d(ps)}^2(n), g_{\nu(ps)}^1(n)), h_1(n), w_n, p_0, a))$$

that factor through the completion of the PS resolution $PSHGHRes$, $Comp(PSHGHRes)$, and restricts to valid PS statements

$$((h_1^2(n), g_1^1(n)), \ldots, (h_{d(ps)}^2(n), g_{\nu(ps)}^1(n)), h_1(n), w_n, p_0, a)).$$

Then $TSPS(p)$ is in the Boolean algebra of AE sets.

Proof: Identical to the proof of Proposition 1.34. ∎

As in the minimal (graded) rank case, at this stage we have all the tools needed for showing that the set $T_2(p)$ is in the Boolean algebra of AE sets (in case all the groups $WPHG$ are rigid). By construction, if $p_0 \in T_2(p)$ then there must exist a valid PS statement of the form $(r_0, (h_1^2, g_1^1), \ldots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_0^1, w_0, p_0, a)$ that factors through one of the PS resolutions $PSHGHRes$ constructed with respect to a proof system of depth 2.

By Proposition 3.7, the sets $TSPS(p)$ associated with the various PS resolutions $PSHGHRes$, i.e., the sets of specializations p_0 of the defining parameters $P=<p>$, for which there exists a test sequence that factors through a PS resolution, $PSHGHRes$, and restricts to valid PS statements, is in the Boolean algebra of AE sets. By Theorem 3.6, if there exists a valid PS statement that factors through a PS limit group, $PSHGH$, then either there exists a test sequence that factors through (at least) one of the PS resolutions associated with it, and restricts to valid PS statements, or there must exist a valid PS statement that extends to a specialization of one of the PS resolutions, $PSHGHRes$, and this extended specialization restricts to a specialization that factors through one of the graded auxiliary limit groups associated with one of the extra PS resolutions $ExtraPSRes$, that is associated with the PS resolution, $PSHGHRes$, and the extended specialization factors through one of its associated (non-Generic) Collapse Extra PS limit groups.

We continue with each of the (non-Generic) Collapse Extra PS limit groups associated with the various PS resolutions $PSHGHRes$. Given a (non-Generic) Collapse Extra PS limit group, we apply the procedure for the analysis of quotient resolutions, presented in the general step of the procedure for the construction of the tree of stratified sets (in the previous section). The output of this general step is a finite collection of anvils and their associated developing resolutions.

Exactly as we did with each of the PS resolutions $PSHGHRes$, we associate (canonically) with each developing resolution its Non-Rigid, Left, and Root PS resolutions, Extra PS resolutions, and Generic Collapse Extra PS resolutions. With each graded auxiliary limit group associated with an Extra PS resolution, we further associate its Collapse Extra PS limit groups. By Proposition 3.7, the sets of specializations p_0 of the defining parameters $P =< p >$, for which there exists a test sequence that factors through any of the developing resolutions, and restricts to valid PS statements, are in the Boolean algebra of AE sets. By Theorem 3.6 applied to the (finite) collection of the (non-Generic) collapse extra PS limit groups, if there exists a valid PS statement that can be extended to a specialization which factors through a Collapse Extra PS limit group, then either there exists a test sequence that factors through one of the developing resolutions associated with it, and restricts to valid PS statements, or there must exist a valid PS statement that extends to a specialization that factors through one of the associated (non-Generic) Collapse Extra PS limit groups.

Continuing iteratively with a procedure similar to the one used to construct the tree of stratified sets (in the previous section), we obtain a terminating iterative procedure by the proof of Theorem 2.10, which is identical to the proof of theorem 4.12 of [Se4]. The iterative procedure we have constructed has to terminate with either a rigid limit group with respect to the defining parameters $P =< p >$, or a solid limit group in which the subgroup $WP =< w,p >$ is contained in the distinguished vertex group in the associated graded abelian JSJ decomposition.

By iteratively applying Proposition 3.7 to the various developing resolutions constructed along the iterative procedure, the sets of specializations p_0 of the defining parameters $P =< p >$, for which there exists a test sequence that factors through any of the developing resolutions constructed along the terminating procedure, and restricts to valid PS statements, are in the Boolean algebra of AE sets. By Proposition 3.3, the set of specializations p_0 of the defining parameters $P =< p >$, for which there exists a valid PS statement that can be extended to a rigid or strictly solid specialization of the terminating rigid or solid PS limit groups, is in the Boolean algebra of AE sets. The entire set $T_2(p)$ is precisely the set of specializations p_0 of the defining parameters $P =< p >$ for which there exists a valid PS statement that factors through one of the PS limit groups, $PSHGH$, we started with. Hence, the set $T_2(p)$ is a finite union of sets that are in the Boolean algebra of AE sets, so the set $T_2(p)$ is in the Boolean algebra of AE sets, and the proof of Theorem 3.2, in case all the graded limit groups $WPHG$ are rigid, is completed. ∎

4. Core resolutions

In the first section we have shown that an EAE set is in the Boolean algebra of AE sets, assuming the limit groups associated with it are of minimal rank. The procedure used to analyze the structure of an EAE set in the minimal rank case, presented in the first section, is combined from two iterative procedures. The first iterative procedure constructs the tree of stratified sets, from which a finite collection of proof systems is obtained, and the second is a sieve procedure for finding those specializations of the defining parameters for which there exists a valid proof statement which is in the form of one of the proof systems derived from the tree of stratified sets. In the second section, we combined concepts and techniques used in the minimal rank case, with the iterative procedure for validation of an AE sentence presented in section 4 of [Se4], to generalize the iterative procedure for the construction of the tree of stratified sets to general EAE sets.

In the third section we considered a special case of EAE sets, the case in which all the groups $WPHG$ associated with an EAE set along the tree of stratified sets are rigid, for which the techniques used in the procedure for validation of an AE sentence can be used in order to get a sieve procedure that is similar in concept to the sieve procedure used in the minimal rank case. However, the techniques used in the procedure for validation of a sentence presented in [Se4] do not seem to be sufficient in order to construct a sieve procedure for a general EAE set. In order to construct a general sieve procedure, we will need several additional tools and notions. In this section we present some of the needed tools, and in the next paper in the series we use these tools to construct the sieve procedure for a general EAE set.

To construct the iterative procedure for validation of an AE sentence, presented in section 4 of [Se4], we needed to present geometric and induced resolutions (see section 3 of [Se4] for these notions). We start this section with the definition of the main object introduced in this section, a **core resolution** of a resolution $Res(t,v,a)$ and its complexity.

Definition 4.1: Let $Res(t,v,a)$ be a a well-separated resolution and let $Comp(Res)(u,t,v,a)$ be its completion. Let $CRes(r,v,a)$ be a geometric sub-resolution of the completed resolution $Res(u,t,v,a)$. We say that the resolution

$CRes(r,v,a)$ is a core resolution of the subgroup $< v,a >$ in the completion $Comp(Res)(u,t,v,a)$, if the resolution $CRes(r,v,a)$ is a firm subresolution of the subgroup $< r,v,a >$ ([Se4], definition 3.9), i.e., it has the following properties:

(i) The rank of the resolution $CRes(r,v,a)$, $rk(CRes(r,v,a))$, is equal to the rank of the subgroup $< r,v,a >$ with respect to the completed resolution $Comp(Res)(u,t,v,a)$.

(ii) There exists a firm test sequence for the subgroup $< r,v,a >$.

(iii) Let $A_1,\ldots,A_m$ be all the non-cyclic pegged abelian groups that appear along the completed resolution $Comp(Res)(t,v,a)$, let $peg_1,\ldots,peg_m$ be the pegs of the abelian groups $A_1,\ldots,A_m$, and let $\{peg_i,q^i_1,\ldots,q^i_{j_i}\}_{i=1}^m$ be an arbitrary basis for the collection of the subgroups $A_1,\ldots,A_m$. Then for any set of integers $\{(s^i_j,n^i_j)\}$, where $n^i_j \geq 2$ and $0 \leq s^i_j < n^i_j$, there exists a firm test sequence $\{t_n,v_n,a\}$ of the subgroup $< t,v,a >$ so that for every index n, the specialization of each of the pegs peg_i is an element h_i, and the specialization of each of the basis elements q^i_j is $h_i^{r^i_j}$, where $r^i_j = u^i_j \cdot n^i_j + s^i_j$ for some positive integer u^i_j.

We denote a core resolution, $Core(< v,a >, Res(t,v,a))(r,v,a)$. In exactly the same way we define a *graded core resolution* and a *multi-graded core resolution*. Since a core resolution is, in particular, a geometric subresolution, we set the complexity of a core resolution to be its complexity as a geometric subresolution (definition 3.2 in [Se4]).

A (graded, multi-graded) core resolution is generally not unique, but given a well-separated resolution $Res(t,v,a)$ and a subgroup $< v,a >$ of its corresponding limit group, there always exists a core resolution of the subgroup $< v,a >$, simply the completion $Comp(Res)(u,t,v,a)$ itself. However, in general, picking the completion of the ambient resolution as a core resolution for a given subgroup is not "economical", i.e., its complexity is going to be much larger than the complexity of other core resolutions that can be associated with the same subgroup. Hence, we continue by modifying the procedure for the construction of the induced resolution, presented in section 3 of [Se4], to get a procedure for the construction of a core resolution.

The construction of a core resolution is composed from two iterative procedures. The procedure used for the first part is essentially identical to the iterative procedure used to construct the induced resolution. In the second part we use an iterative procedure that either reduces the rank of the resolution constructed in the first part, or alternatively, shows that the resolution constructed in the first part is a firm subresolution, hence it is a core resolution. For presentation purposes, we present the second step of the construction in case none of the abelian decompositions associated with the various levels of the resolution $Res(t,v,a)$ with which we started contains QH vertex groups, and then generalize the construction to an arbitrary (well-separated) resolution.

Let $Comp(Res)(t,v,a)$ be a completed well-separated resolution and let $< v,a >< Rlim(t,v,a)$ be a subgroup of the limit group $Rlim(t,v,a)$. With each level of the completed resolution $Comp(Res)(t,v,a)$, there is associated a (possibly trivial) free decomposition and abelian decompositions of the factors. We denote the decompositions associated with the various levels of the completed resolution $Comp(Res)(t,v,a)$ by $\Lambda_1,\dots,\Lambda_q$, and the canonical epimorphisms between consecutive levels we denote by $\eta_1,\dots,\eta_{q-1}$. Furthermore, each vertex group in these decompositions which is neither quadratically hanging nor abelian is embedded into the next level by the canonical epimorphisms between consecutive levels.

For the first part of the construction of a core resolution of the subgroup $< v,a >$ in the completed resolution $Comp(Res)(t,v,a)$ we use a finite iterative procedure, which is essentially identical to the first step in the construction of the induced resolution, presented in section 3 of [Se4]. We start by describing its first step.

(i) Using standard Bass–Serre theory, the subgroup $< v,a >< Rlim(t,v,a)$ inherits a decomposition Δ_1 with abelian and trivial edge groups from the decomposition Λ_1. Note that if $< v,a >$ intersects a conjugate of a QH vertex group in the decomposition Λ_1 in a subgroup of finite index, then the intersection appears as a QH vertex group in the inherited decomposition Δ_1. If $< v,a >$ intersects a conjugate of a QH vertex group in the decomposition Λ_1 in a non-trivial subgroup of infinite index, then by ([Se4], 1.4) the intersection gives rise to a free factorization (and a possible free factor) in the decomposition Δ_1 of the group $< v,a >$.

(ii) Suppose that the free decomposition inherited by the subgroup $< v,a >$ from the decomposition Δ_1 is

$$< v,a >= F_1 * < v_1,a > * < v_2 > * \cdots * < v_b >,$$

where F_1 is a free group which is the free product of free factors contributed by subgroups of infinite index in QH vertex groups in Λ_1, and cyclic subgroups generated by Bass–Serre generators associated with loops with trivial stabilizer in Δ_1; $< v_1,a >$ is the connected component that contains the vertex stabilized by $F_k =< a_1,\dots,a_k >$ itself.

We continue with each of the factors $< v_1, a >, < v_2 >, \dots, < v_b >$ separately. We will denote each of these factors by V_i. Each factor V_i inherits an abelian splitting Δ_1^i from the decomposition Λ_1. Each edge e in Δ_1^i that connects two non-QH, non-abelian vertex groups is composed from a couple of edges e_1 and e_2 that are adjacent and are both in the orbit of the same edge e' in the Bass–Serre tree corresponding to the graph of groups Λ_1 of $Comp(Res)(t,v,a)$. Furthermore, e' connects a non-abelian vertex group to an abelian vertex group in the decomposition Λ_1.

Let A be the abelian vertex group that stabilizes the common vertex v of e_1 and e_2 in the Bass–Serre tree corresponding to the decomposition Λ_1. There exists a (unique) element $a \in A$ that is mapped to the identity element by η_1, and conjugates the vertex adjacent to v in e_2 to the vertex adjacent to v in e_1. We modify the factor V_i by adding to its generators the element a. We act in the same way on the factor V_i, in case an abelian vertex group in Δ_1^i is connected to two non-QH, non-abelian vertex groups, which are necessarily in the same orbit of a non-QH, non-abelian vertex group in the graph of groups Λ_1. Repeating this operation for all the edges connecting two non-QH, non-abelian vertex groups in the decomposition Δ_1^i of the factor V_i, and for all couples of edges connecting an abelian vertex group to two non-QH non-abelian vertex groups in Δ_1^i, we get a subgroup $\hat{V}_i < Comp(Res)(t,v,a)$, so that $V_i < \hat{V}_i$, $\eta_1(V_i) = \eta_1(\hat{V}_i)$, and in the decomposition $\hat{\Delta}_1^i$ inherited by $\hat{V}_i$ from the decomposition Λ_1, which is a "folding" of the decomposition Δ_1^i of V_i, a non-QH, non-abelian vertex group is connected only to QH vertex groups and to abelian vertex groups and not to any other non-QH, non-abelian vertex groups. Furthermore, an abelian vertex group in $\hat{\Delta}_1^i$ is connected to at most one non-QH, non-abelian vertex group as it is in the decomposition Λ_1.

(iii) As we did with a general well-separated resolution (definition 2.2 in [Se4]), we associate a decomposition $\hat{\Theta}_1^i$ with the decomposition $\hat{\Delta}_1^i$, by cutting each of the punctured surfaces that correspond to QH vertex groups in $\hat{\Delta}_1^i$ along the collection of disjoint, non-homotopic (non-boundary parallel) s.c.c. which are the pre-images of the s.c.c. along which each of the QH vertex groups in the ambient resolution $Comp(Res)(t,v,a)$ is cut. As we did in completing a general well-structured resolution ([Se2], definition 1.12), we modify the factor $\hat{V}_i$ by adding Bass–Serre elements so that each connected punctured subsurface in the decomposition $\hat{\Theta}_1^i$ that is connected to a non-QH vertex group in $\hat{\Theta}_1^i$ will be connected to a unique non-QH

vertex group, and get a subgroup $\tilde{V}_i$ with corresponding graphs of groups $\tilde{\Delta}_1^i$ and $\tilde{\Theta}_1^i$.

As in the construction of the completed resolution of a well-separated resolution, every connected component in the decomposition $\tilde{\Theta}_1^i$ that contains a non-QH vertex group contains a unique non-QH, non-abelian vertex group, and (possibly) few abelian vertex groups all connected (only) to the (unique) non-QH, non-abelian vertex group in their connected component. All the QH vertex groups in a connected component in $\tilde{\Theta}_1^i$ are also connected only to the (unique) non-QH, non-abelian vertex group in that connected component.

Since $Comp(Res)(t,v,a)$ is a completed and well-separated resolution, all the (conjugating) Bass–Serre elements we have added to the factor $\hat{V}_i$ are naturally mapped to elements of the completed resolution $Rlim(t,v,a)$, and the subgroup $\tilde{V}_i$ obtained after adding the Bass–Serre generators is naturally mapped into the completed limit group $Rlim(t,v,a)$ (note that it is not necessarily embedded). Since (the image in $Rlim(t,v,a)$ of) the conjugating (Bass–Serre) elements that we added are mapped to the identity element by the map η_1, the addition of these elements does not change the image of the map η_1, i.e., $\eta_1(\hat{V}_i) = \eta_1(\tilde{V}_i)$.

(iv) With each factor $\tilde{V}_i$ and each connected component in the decomposition $\tilde{\Theta}_1^i$ that contains a non-QH vertex group, we associate a subgroup J_s^i and continue to the second level of the completed resolution $Comp(Res)(t,v,a)$ with each of the subgroups J_s^i separately.

For each index i, and each connected component (indexed by s) in $\tilde{\Theta}_1^i$, we set J_s^i to be the image under the canonical epimorphism η_1 of the fundamental group of the corresponding connected component. Note that the Bass–Serre elements, and the elements from abelian vertex groups that were added to the various factors V_i, do not change the images of the factors V_i under the epimorphism η_1.

(v) As we did in part (i), for each of the subgroups J_s^i we use standard Bass–Serre theory to get a decomposition $\Delta_2^{(i,s)}$, inherited by the subgroup J_s^i from the decomposition Λ_2 associated with the second level of the completed resolution $Comp(Res)(t,v,a)$. From the decomposition $\Delta_2^{(i,s)}$, the subgroup J_s^i inherits a free decomposition and an abelian decomposition of each of the factors. We add elements for each edge connecting two non-QH, non-abelian vertex groups in the abelian decompositions of the different factors as we did in part (ii), and conjugating elements corresponding to boundary components of QH vertex groups in one of the abelian decompositions of the different factors using the corresponding decomposition $\Theta_2^{(i,s)}$ associated with $\Delta_2^{(i,s)}$ as we did in part (iii). With each factor of J_s^i we associate finitely many subgroups $M_{(i,s,j)}$ corresponding to the different connected components in the decomposition $\Theta_2^{(i,s)}$ exactly in the same way we associated the subgroups J_s^i with the factors V_i in part (iv). We continue to the third level of the completed resolution $Comp(Res)(t,v,a)$ with the subgroups $M_{(i,s,j)}$ associated with each of the connected components of the various graphs $\Theta_2^{(i,s)}$ separately, analyze the decompositions inherited by each of the subgroups $M_{(i,s,j)}$ from the the abelian decomposition Λ_3 associated with the third level of the completed resolution $Comp(Res)(t,v,a)$, add additional elements according to parts (ii) and (iii), and subsequently continue to the next levels of the completed resolution $Comp(Res)(t,v,a)$.

The first step in the iterative procedure used in the first part of the construction of the core resolution constructs a resolution $Res(u,v,a)$ of the subgroup $<v,a>\,<Rlim(t,v,a)$ by going through the levels of the completed resolution $Comp(Res)(t,v,a)$ and applying steps (i)–(v) above.

To start the second step in the iterative procedure used for the first part of the construction of the core resolution, we set $IRes_1(u,v,a)$ to be the subgroup of the completed resolution $Comp(Res)(t,v,a)$ generated by the different factors $\tilde{V}_i$ and their images in the lower levels of the completed resolution $Comp(Res)(t,v,a)$ obtained by steps (i)–(v) above. Defining the subgroup $IRes_1(u,v,a) < Rlim(t,v,a)$, we start going through the levels of the completed resolution $Comp(Res)(t,v,a)$ starting with the subgroup $IRes_1(u,v,a)$, instead of the subgroup $<v,a>$ with which we started in the first step.

$IRes_1(u,v,a)$, being a subgroup of $Rlim(t,v,a)$, inherits a decomposition from the abelian decomposition Λ_1 associated with the first level of the completed resolution $Comp(Res)(t,v,a)$. Let this decomposition be $I\Delta_1$. Since the subgroups $<v,a>$ and $IRes_1(u,v,a)$ differ only in the stabilizers of the unique non-abelian, non-QH vertex group in each connected component of the various decompositions $\tilde{\Theta}_1^i$ associated with the decompositions $\tilde{\Delta}_1^i$ of the factors $\tilde{V}_i$, if the graph of groups $I\Delta_1$ is not combinatorially similar to the graph of groups $\tilde{\Delta}_1$ then one of the following occurs:

(1) In the free decomposition $I\Delta_1$ inherited by the subgroup $IRes_1(u,v,a)$ from the graph of groups Λ_1, either the number of factors is dropping, or the rank of the free factor corresponding to Bass–Serre generators of loops

with trivial edge stabilizers and free factors contributed by infinite index subgroups of QH vertex groups in Θ_1 is dropping. In this case, each of the factors V_i of $< v, a >$, and the subgroups $\tilde{V}_i$ associated with it, are subgroups of a factor in the free decomposition of $IRes_1(u, v, a)$ inherited from Λ_1.

(2) The number of factors and the rank of the free group corresponding to Bass–Serre generators of loops with trivial edge stabilizers and free factors contributed by infinite index subgroups of QH vertex groups in the free decomposition inherited by $IRes_1(u, v, a)$ from Λ_1 remain identical to their values in the free decomposition inherited by the subgroup $< v, a >$ from the decomposition $\tilde{\Delta}_1$. In this case, each of the factors V_i of $< v, a >$ is a subgroup of a unique factor in the free decomposition of $I\Delta_1$ inherited by $IRes_1(u, v, a)$ from Λ_1.

The combinatorics of the graph of groups $I\Delta_1$ is strictly smaller than the combinatorics of the graphs of groups $\tilde{\Delta}_1^i$ in correspondence, i.e., for at least one of the factors in the decomposition $I\Delta_1$, the combinatorics of its corresponding graph of groups is strictly smaller than the combinatorics of the corresponding graph of groups in $\tilde{\Delta}_1^i$, i.e., either the number of edges and vertices is smaller, or the genus or the (absolute value of the) Euler characteristic of some of the QH vertex groups is smaller.

Applying steps (i)–(v) to the subgroup $IRes_1(u, v, a)$, we set $IRes_2(u, v, a)$ to be the subgroup of the completed limit group $Rlim(t, v, a)$, generated by the different factors of the subgroup $IRes_1(u, v, a)$ and their images in the lower levels of the completed resolution $Comp(Res)(t, v, a)$ obtained in the second step of the iterative procedure, i.e., obtained by performing steps (i)–(v) above starting with the subgroup $IRes_1(u, v, a) < Rlim(t, v, a)$. Defining the subgroup $IRes_2(u, v, a) < Rlim(t, v, a)$, we say that the procedure for the construction of the induced resolution terminates if $IRes_2(u, v, a) = IRes_1(u, v, a)$. Otherwise, we perform the third step of the iterative procedure for constructing the induced resolution by going through the levels of the completed resolution $Comp(Res)(t, v, a)$, starting with the subgroup $IRes_2(u, v, a)$ instead of the subgroups $< v, a >$ and $IRes_1(u, v, a)$ with which we started in the first two steps.

In the general step, we say that the iterative procedure used in the first part of the construction of the core resolution terminates if $IRes_n(u, v, a) = IRes_{n-1}(u, v, a)$. Otherwise, we continue to the next step by applying steps (i)–(v) starting with the subgroup $IRes_n(u, v, a) < Rlim(t, v, a)$.

LEMMA 4.2:

(i) *The various ranks of the resolutions constructed along the iterative procedure used in the first part of the construction of the core resolution is a non-increasing sequence.*

(ii) *The iterative procedure used in the first part of the construction of the core resolution terminates after finitely many steps.*

Proof: Part (ii) is identical to lemma 3.3 in [Se4]. To prove part (i), note that by construction, at each step j of the iterative procedure, the terminal free group of the resolution constructed in the j-th step of the procedure, $IRes_j(u,v,a)$, is an image of the terminal free group of the resolution constructed in step $j-1$ of the procedure, $IRes_{j-1}(u,v,a)$. ∎

A core resolution is a geometric subresolution, hence the output of the first part of the procedure needs to be a completed resolution.

LEMMA 4.3: *Let $IRes_f(u,v,a)$ be the terminal resolution obtained by the above procedure. Then $IRes_f(u,v,a)$ is a geometric subresolution of the completed resolution, $Comp(Res)(t,v,a)$.*

Proof: Identical to lemma 3.4 of [Se4]. ∎

The first part of the construction of the core resolution is essentially identical to the construction of the induced resolution. The resolution obtained by the iterative procedure used in this part, $IRes_f(u,v,a)$, is a geometric subresolution of the completed well-separated resolution $Comp(Res)(t,v,a)$ with which we started. Hence, if $IRes_f(u,v,a)$ is a resolution of minimal rank, it is a core resolution. However, in general it is not a firm subresolution, hence it cannot serve as a core resolution. To get a firm subresolution, we usually need to modify the resolution $IRes_f(u,v,a)$ by sequentially reducing its rank. We do that by a sequentially "fill-in" of various abelian and QH vertex groups associated with the resolution $Comp(Res)(t,v,a)$, and related to the resolution $IRes_f(u,v,a)$, in a way that does not increase the ranks of the obtained resolutions, and so that if the entire sequence of "fill-in" operations does not manage to eventually reduce the rank of the obtained resolution, then it is guaranteed that the resolution $IRes_f(u,v,a)$ with which we started the second step is a firm subresolution, so it can be taken as a core resolution.

Before presenting the second part of the construction of a core resolution, we present two preliminary iterative procedures. The first preliminary procedure fills in *inefficient* QH vertex groups, and the second preliminary procedure adds pegs to the abelian vertex groups in the various abelian decompositions associated with the various levels of a given induced resolution. As we will see in the sequel, the iterative procedure we use for the second part of the construction of a core resolution takes care of these two operations, so the preliminary procedures are not really necessary for the construction of the core but they may be of importance for other purposes.

Since the construction of the core resolution involves (only) the completion of a well-separated resolution (and not the resolution itself), to save notation in the sequel, we will assume that the resolution $Res(t,v,a)$ we started with is a completed resolution, i.e., we will assume that $Comp(Res)(t,v,a)$ is identical to $Res(t,v,a)$.

Definition 4.4: Let $IRes_f(u,v,a)$ be the terminal resolution obtained by the iterative procedure used in the first part of the construction of the core resolution. Let Q be a QH vertex group in the abelian decomposition associated with one of the levels of the resolution $IRes_f(u,v,a)$. By construction, Q is a finite index subgroup of a QH vertex group $\hat{Q}$ in the completed resolution $Res(t,v,a)$ with which we started. With the QH vertex group Q there is an associated surface S, and with the QH vertex group $\hat{Q}$ there is an associated surface $\hat{S}$ that is finitely covered by S. The resolutions $IRes_f(u,v,a)$ and $Res(t,v,a)$ are well-separated, so with S and $\hat{S}$ there is an associated collection of non-homotopic disjoint s.c.c. that are mapped to the identity in the next levels of the corresponding resolutions.

We say that the QH vertex group Q (or the corresponding surface S) of $IRes_f(u,v,a)$ is **inefficient** if it is not of minimal rank and one of the following conditions hold:

- (i) The rank of the free group dropped from the QH vertex group Q is strictly bigger than the rank of the QH vertex group dropped from the QH vertex group $\hat{Q}$.
- (ii) The number of connected components in the surface S after cutting it along its associated collection of s.c.c. is strictly bigger than the number of connected components of the surface $\hat{S}$ after cutting it along its associated collection of s.c.c.

We start the second part of the construction of the core resolution by a sequential "fill" of inefficient surfaces (QH vertex groups). By the way inefficient surfaces are defined, each time we fill an inefficient surface, the rank of the associated (geometric) resolution strictly decreases.

We start the second part of the construction of the core resolution with the resolution obtained by the first part of the construction, $IRes_f(u,v,a)$. If the resolution $IRes_f(u,v,a)$ contains no inefficient surfaces (QH vertex groups), we do not change the resolution $IRes_f(u,v,a)$ and continue with it to the next iterative procedure used in the second part of the construction of the core resolution. Suppose Q is an inefficient QH vertex group in one of the levels of the resolution $IRes_f(u,v,a)$ and let $\hat{Q}$ be the QH vertex group that contains Q in the resolution $Res(t,v,a)$ with which we started the procedure. With the (completed) resolution $IRes_f(u,v,a)$ we naturally associate the subgroup $< u,v,a >$ that is generated by the limit groups associated with the various levels of $IRes_f(u,v,a)$. By the construction of the resolution $IRes_f(u,v,a)$, the subgroup $< u,v,a >$ is naturally a subgroup of the limit group $Rlim(t,v,a)$ associated with the (completed) resolution $Res(t,v,a)$.

Let $< q,u,v,a >$ be the subgroup generated by the subgroups $< u,v,a >$ and $\hat{Q}$ in the limit group $Rlim(t,v,a)$. At this point, we apply the procedure used in the first part of the construction to the limit group $< q,u,v,a >$ and the resolution $Res(t,v,a)$ we started with (i.e., we construct the resolution induced by the subgroup $< q,u,v,a >$ from the resolution $Res(t,v,a)$). Let the obtained (geometric) resolution be $FiRes(s,v,a)$.

PROPOSITION 4.5: *The rank of the resolution $FiRes(s,v,a)$, obtained by filling the QH subgroup Q, is strictly smaller than the rank of the resolution $IRes_f(u,v,a)$ with which we started the second step.*

Proof: The terminal free group of the resolution $IRes_f(u,v,a)$ is naturally mapped into the terminal free group of the resolution $FiRes(s,v,a)$. Since the QH vertex group $\hat{Q}$ is obtained from the QH vertex group Q by adding finitely many elements which are roots of elements in Q, the terminal free group of the resolution $FiRes(s,v,a)$ is generated by the image of the terminal free group of the resolution $IRes_f(u,v,a)$ in addition to finitely many roots of elements in that image. Hence, the rank of the resolution $FiRes(s,v,a)$ is bounded by the rank of the resolution $IRes_f(u,v,a)$.

The QH vertex group Q is inefficient. If the rank of the free factor dropped from it in the resolution $IRes_f(u,v,a)$ is strictly bigger than the rank of the free factor dropped from the QH vertex group $\hat{Q}$ in the resolution $Res(t,v,a)$, then by the above structure of the terminal free group of $FiRes(s,v,a)$, the rank of that terminal free group is strictly smaller than the rank of the terminal free group of $IRes_f(u,v,a)$.

Let S be the surface associated with the QH vertex group Q and let $\hat{S}$ be the surface associated with the QH vertex group $\hat{Q}$. S finitely covers $\hat{S}$, and with each of them there is an associated collection of s.c.c. that are mapped to the identity in the next levels of the resolutions $IRes_f(u,v,a)$ and $Res(t,v,a)$ in correspondence.

Suppose that the ranks of the free factors dropped from Q and $\hat{Q}$ are identical. In that case, the number of connected components of the surface S after we cut it along its associated collection of s.c.c. is strictly bigger than the number of components of $\hat{S}$ after we cut it along its associated collection of s.c.c. Let $C_1,\ldots,C_\ell$ be the subgroups associated with the various components of the surface S. Let $q_1,\ldots,q_d \in \hat{Q}$ be elements that conjugate distinct couples of components of the surface S in the QH subgroup $\hat{Q}$. For each i, let C_i and $C_{j(i)}$ be the subgroups of $< u,v,a >$ associated with the two components conjugated by the element $q_i \in \hat{Q}$. Since the groups C_i intersect $q_i C_{j(i)} q_i^{-1}$ non-trivially, the rank of the image of the subgroup $< C_i, q_i C_{j(i)} q_i^{-1} >$ in the terminal free group of the resolution $FiRes(s,v,a)$ is strictly smaller than the sum of the ranks of the images of C_i and $C_{j(i)}$ in the terminal free group of the resolution $IRes_f(u,v,a)$ with which we started the second step.

Let ν be the map from the subgroup $H =< C_1,\ldots,C_\ell,\hat{Q} >$ into the terminal free group of the resolution $FiRes(s,v,a)$, and suppose that the ranks of the terminal free groups of the resolutions $IRes_f(u,v,a)$ and $FiRes(s,v,a)$ are identical. Then the rank of the free group $\nu(H)$ is equal to the rank of the image of the subgroup $< C_1,\ldots,C_\ell >$ in the terminal free group of the resolution $IRes_f(u,v,a)$, which is the sum of the ranks of the images of the various subgroups C_i in that terminal free group. Since, in addition, the rank of the subgroup $\nu(< C_i, q_i C_{j(i)} q_i^{-1} >)$ is strictly smaller than the sum of the ranks of the images of C_i and $C_{j(i)}$ in the terminal free group of $IRes_f(u,v,a)$, $\nu(H)$ admits a free decomposition $\nu(H) = \nu(q_1) * \cdots * \nu(q_d) * \nu(H_1)$, and since for some integer n, $q_1^n \in Q$, $\nu(q_1)^n \in \nu(<< q_2,\ldots,q_d,H_1 >, q_1^{-1} < q_2,\ldots,q_d,H_1 > q_1 >)$. But clearly, if $G =< a > *B$ and the element a is of infinite order, then $< B, a^{-1}Ba >= B * a^{-1}Ba$, hence for every $n > 1$, $a^n \notin < B, a^{-1}Ba >$, a contradiction. ∎

If the resolution $FiRes(s,v,a)$ contains no inefficient QH vertex groups, we replace the resolution $IRes_f(u,v,a)$ by the resolution $FiRes(u,v,a)$ and proceed to the next iterative procedure used in the second step of the construction of the core resolution. If the resolution $FiRes(s,v,a)$ contains an inefficient QH vertex group Q, we “fill” this inefficient surface precisely as we filled an inefficient surface in the resolution $IRes_f(u,v,a)$, obtained by the first part of the construction. We continue iteratively by filling inefficient QH vertex groups in the previously obtained resolutions. Since, by Proposition 4.5, each filling of an inefficient QH vertex group strictly decreases the rank of the obtained resolution, this (filling) iterative procedure terminates after finitely many steps, which enable us to replace the resolution $IRes_f(u,v,a)$ by the obtained resolution, and continue to the next iterative procedure used in the second step of the construction of the core resolution.

Let $IRes_e(u,v,a)$ be the resolution obtained by iteratively filling inefficient QH vertex groups. The resolution $IRes_e(u,v,a)$ is a geometric subresolution of $Res(t,v,a)$ by construction, but in general it may not be a firm subresolution. The second preliminary procedure adds pegs to abelian vertex groups in the abelian decompositions associated with the various levels of the resolution $IRes_e(u,v,a)$.

Suppose that the resolution $IRes_e(u,v,a)$, obtained by filling inefficient QH vertex groups in the resolution $IRes_f(u,v,a)$, constructed in the first part of the construction of the core resolution, is not of minimal rank. In the second preliminary procedure we add pegs of abelian subgroups intersected non-trivially (i.e., in a subgroup that is not contained in the cyclic subgroup generated by the peg) by the subgroup associated with the resolution $IRes_e(u,v,a)$ iteratively.

(1) We start by adding the pegs of abelian subgroups associated with the top level of the resolution $Res(t,v,a)$ that are intersected non-trivially by the subgroup associated with the resolution $IRes_e(u,v,a)$.

(2) We construct the resolution induced from the resolution $Res(t,v,a)$ by the subgroup generated by the subgroup associated by the resolution $IRes_e(u,v,a)$ and the additional pegs, and denote the obtained resolution $IRes_1(u,v,a)$. Since the pegs we added are either roots of elements that belong to the subgroup associated with $IRes_e(u,v,a)$, or they commute with (non-trivial) elements in the subgroup associated with $IRes_e(u,v,a)$, the rank of the obtained resolution $IRes_1(u,v,a)$ is bounded by the rank of the resolution $IRes_e(u,v,a)$

$$rk(IRes_1(u,v,a)) \leq rk(IRes_e(u,v,a)).$$

(3) If $rk(IRes_1(u,v,a)) < rk(IRes_e(u,v,a))$, we start the second step of the procedure with the resolution $IRes_1(u,v,a)$, so we may suppose that $rk(IRes_1(u,v,a)) = rk(IRes_e(u,v,a))$. In this case, if a peg that was added to the subgroup associated with the resolution $IRes_e(u,v,a)$ is not a root of an element of this subgroup, then the additional peg has to be mapped into a free generator in the terminal free subgroup of the obtained (induced) resolution $IRes_1(u,v,a)$, and the subgroup generated by

the collection of those additional pegs is mapped into a free factor (of the same rank) in the terminal free subgroup of $IRes_1(u, v, a)$.

Let Λ_1 be the abelian decomposition associated with the top level of the resolution $Res(t, v, a)$. In the free decomposition $I\Delta_1$, inherited by the subgroup $IRes_1(u, v, a)$ from Λ_1, either the number of factors is dropping, or the rank of the free factor corresponding to Bass–Serre generators of loops with trivial edge stabilizers and free factors contributed by infinite index subgroups of QH vertex groups is dropping, or the number of free factors and the rank of the free factor corresponding to Bass–Serre generators with trivial edge stabilizers and free factors contributed by infinite index subgroups of QH vertex groups are identical to the corresponding number and rank in the abelian decomposition inherited by the subgroup (associated with) $IRes_e(u, v, a)$ from Λ_1.

In case the number of free factors and the rank of the above free factor in $I\Delta_1$ are identical to the corresponding number of factors and rank in the abelian decomposition inherited by the subgroup $IRes_e(u, v, a)$ from the abelian decomposition Λ_1, the complexity of the graph of groups $I\Delta_1$, inherited by the subgroup $IRes_1(u, v, a)$ from Λ_1, is bounded by complexity of the graph of groups inherited by $IRes_e(u, v, a)$ from Λ_1 in addition to the number of abelian groups for which we have added a peg and this peg is not a root of an element in the subgroup associated with $IRes_e(u, v, a)$. Note that the number of those abelian groups is bounded by the rank of the resolution $IRes_e(u, v, a)$, $rk(IRes_e(u, v, a))$.

(4) We continue as in steps (1) and (2), by adding all the pegs of abelian subgroups associated with the top level of the resolution $Res(t, v, a)$ that are intersected non-trivially by the subgroup associated with the resolution $IRes_1(u, v, a)$, and constructing the resolution induced from the resolution $Res(t, v, a)$ by the subgroup generated by the subgroup associated by the resolution $IRes_1(u, v, a)$ and the additional pegs, and denote the obtained resolution $IRes_2(u, v, a)$, and continue iteratively. By construction, the ranks of all the obtained resolutions, which we denote $IRes_n(u, v, a)$, satisfy

$$rk(IRes_n(u, v, a)) \leq rk(IRes_{n-1}(u, v, a)) \leq rk(IRes_e(u, v, a)).$$

If at some step n, $rk(IRes_n(u, v, a)) < rk(IRes_{n-1}(u, v, a))$, we start the second step of the procedure with the resolution $IRes_n(u, v, a)$, hence we may assume that the ranks of the obtained resolutions $IRes_n(u, v, a)$ do not decrease along the iterative procedure.

The subgroup generated by the pegs that were added to the subgroups associated with the resolutions

$$IRes_e(u,v,a), IRes_1(u,v,a), \ldots, IRes_{n-1}(u,v,a),$$

that are not a root of an element in the subgroup associated with the preceding resolution, is mapped isomorphically onto a free factor (of the same rank) in the terminal free subgroup of the obtained (induced) resolution $IRes_n(u,v,a)$. In the free decomposition inherited by the subgroup $IRes_n(u,v,a)$ from Λ_1, the abelian decomposition associated with the top level of $Res(t,v,a)$, either the number of factors is dropping, or the rank of the free factor corresponding to Bass–Serre generators of loops with trivial edge stabilizers and free factors contributed by infinite index subgroups of QH vertex groups is dropping, or the number of free factors and the rank of the free factor corresponding to Bass–Serre generators with trivial edge stabilizers and free factors contributed by infinite index subgroups of QH vertex groups are identical to the corresponding number and rank in the abelian decomposition inherited by the subgroup $IRes_{n-1}(u,v,a)$ from Λ_1. In case the number of free factors and the rank of the above free factor in the abelian decomposition inherited by $IRes_n(u,v,a)$ from Λ_1 are identical to the corresponding number of factors and rank in the abelian decomposition inherited by the subgroup $IRes_{n-1}(u,v,a)$ from the abelian decomposition Λ_1, the complexity of the graph of groups inherited by the subgroup $IRes_n(u,v,a)$ from Λ_1 is bounded by complexity of the graph of groups inherited by $IRes_{n-1}(u,v,a)$ from Λ_1 in addition to the number of abelian groups for which we have added a peg and this peg is not a root of an element in the subgroup associated with $IRes_{n-1}(u,v,a)$. Since the total number of those abelian groups that are added to the entire sequence of resolutions $IRes_e(u,v,a), IRes_1(u,v,a), \ldots, IRes_n(u,v,a)$ is bounded by the rank of the resolution $IRes_e(u,v,a)$, the sequence terminates after a finite time, i.e., we obtain a resolution $IRes_n(u,v,a)$ for which if the subgroup associated with it intersects a (conjugate of an) abelian vertex group in the graded abelian decomposition associated with the top level of $Res(t,v,a)$, then this intersection contains the peg of the abelian subgroup.

(5) At this point we continue by adding pegs to abelian subgroups associated with the second level of the resolution $Res(t,v,a)$ that are intersected non-trivially by the subgroup associated with the resolution $IRes_n(u,v,a)$, and continue iteratively as we did in part (4). Note that after adding pegs of

abelian subgroups associated with the second level of $Res(t,v,a)$, we may need to add pegs of abelian subgroups associated with the top level of $Res(t,v,a)$ as well. Since eventually, either the rank of an obtained resolution decreases or the complexity of the abelian decompositions inherited by the obtained resolutions from the abelian decomposition associated with either the top level or the second level of the resolution $Res(t,v,a)$ decreases, the iterative procedure for adding pegs to abelian subgroups associated with the top two levels of the resolution $Res(t,v,a)$ terminates after finitely many steps. Continuing to the next levels of the resolution $Res(t,v,a)$, we finally obtain a resolution, which we denote $IRes_p(t,v,a)$, for which if a conjugate of an abelian vertex group associated with one of the levels of $Res(t,v,a)$ intersects non-trivially the subgroup associated with the obtained resolution $IRes_p(u,v,a)$, then this intersection contains the peg of the abelian subgroup.

The iterative procedure for the addition of pegs terminates after a finite number of steps with a resolution, $IRes_p(u,v,a)$. To the obtained resolution $IRes_p(u,v,a)$ we apply the procedure for filling inefficient QH vertex groups, and if an inefficient vertex group was indeed filled, we apply the procedure for adding pegs once again, until we obtain a resolution, which we still denote $IRes_p(u,v,a)$, that does not contain any inefficient QH vertex groups, and for which if a conjugate of an abelian vertex group associated with one of the levels of $Res(t,v,a)$ intersects non-trivially the subgroup associated with the obtained resolution $IRes_p(u,v,a)$, then this intersection contains the peg of the abelian subgroup. However, the obtained resolution $IRes_p(u,v,a)$, which is a geometric subresolution by construction, is not necessarily a firm subresolution. To modify the resolution $IRes_p(u,v,a)$ (or equivalently the induced resolution $IRes_f(u,v,a)$) in order to obtain a firm subresolution, we use an iterative procedure that is aimed at sequentially reducing the rank of the obtained resolutions. The iterative procedure we present either reduces the rank after finitely many steps, or guarantees (after finitely many steps) that the resolution $IRes_p(u,v,a)$ (or the resolution $IRes_f(u,v,a)$) is indeed a firm subresolution. Unlike the procedure for the addition of pegs, the procedure used in the second part of the second step of the construction of the core resolution starts from the bottom level of the resolution $Res(t,v,a)$ and iteratively climbs towards its top level.

(1) The terminal level (which we denote by ℓ) of the resolution $Res(t,v,a)$ is a free group $F_k * F$, hence we start the procedure from the level above the terminal one (level $\ell-1$). According to the construction of an induced res-

olution, the subgroup associated with the resolution $IRes_p(u,v,a)$ inherits a (possibly trivial) free decomposition from each of the levels of the resolution $IRes_p(u,v,a)$, and it is mapped to a subgroup associated with each of the levels. In particular, in accordance with the free decomposition inherited by the subgroup associated with the resolution $IRes_p(u,v,a)$ from the levels that lie above the $\ell-1$ level, the image of the subgroup associated with $IRes_p(u,v,a)$ in the subgroup associated with the $\ell-1$ level of the (completed) ambient resolution $Res(t,v,a)$, $G^{\ell-1}$, admits a free decomposition when mapped into the $\ell-1$ level, $G^{\ell-1}=H_1^{\ell-1}*\cdots*H_{s(\ell-1)}^{\ell-1}$. We treat the factors $H_i^{\ell-1}$ in parallel.

(2) We fix a system of generators of $H_i^{\ell-1}$, $H_i^{\ell-1}=<h_1^{\ell-1},\dots,h_{r(\ell-1,i)}^{\ell-1}>$. If no (non-trivial) subgroup of the factor $H_i^{\ell-1}$ fixes a vertex in the Bass–Serre tree associated with the $\ell-1$ level of the resolution $Res(t,v,a)$ with which we started, we leave the factor $H_i^{\ell-1}$ unchanged. Suppose that a (non-trivial) subgroup of $H_i^{\ell-1}$ fixes a vertex in the abelian decomposition associated with the $\ell-1$ level of the resolution $Res(t,v,a)$.

Let $T_{\ell-1}$ be the Bass–Serre tree corresponding to the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and let $\Lambda_{H_i^{\ell-1}}$ be the graph of groups inherited by $H_i^{\ell-1}$ from its action on the Bass–Serre tree $T_{\ell-1}$.

Definition 4.6: Suppose that the abelian decomposition $\Lambda_{H_i^{\ell-1}}$, inherited by $H_i^{\ell-1}$ from its action on the Bass–Serre tree $T_{\ell-1}$, contains a couple of QH vertex groups, Q_1,Q_2, that satisfy the following conditions:

(i) Q_1 is a finite index subgroup in a subgroup Q_1' that is conjugate to a QH vertex group in the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and Q_2 is a finite index subgroup in a subgroup Q_2' that is conjugate to a QH vertex group in the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and Q_1' is conjugate to Q_2' in the limit group associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$.

(ii) The QH vertex group Q_1' (hence also Q_2') is not of minimal rank, i.e., there exists a s.c.c. on S_1' (the surface associated with Q_1') that is mapped to the trivial element in the next level of the resolution $Res(t,v,a)$.

In this case we say that the abelian decomposition $\Lambda_{H_i^{\ell-1}}$ contains a *reducing QH couple.*

If the abelian decomposition $\Lambda_{H_i^{\ell-1}}$ contains a reducing QH couple, Q_1

and Q_2, we set the subgroup $\hat{H}_i$ to be the subgroup generated by $H_i^{\ell-1}$, the QH vertex groups Q_1' and Q_2' that contain Q_1 and Q_2 as subgroups of finite index, and an element in the limit group associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$ that conjugates Q_1' to Q_2'. We set H_i' to be the limit group associated with the resolution induced by the subgroup $\hat{H}_i$ from the ambient resolution $Res(t,v,a)$. Since the QH vertex groups Q_1' and Q_2' are not of minimal rank, and since Q_1 and Q_2 are not conjugate in the subgroup $H_i^{\ell-1}$ with which we started, and Q_1' and Q_2' are conjugate in H_i', it is not difficult to see that the elements we have added reduce the rank of the obtained group, i.e.,

$$rk(H_i') < rk(H_i^{\ell-1})$$

in case there exists a reducing QH couple in $\Lambda_{H_i^{\ell-1}}$. In this case we replace the factor $H_i^{\ell-1}$ by H_i', and repeat part (2) of the construction with the newly obtained subgroup $H_i^{\ell-1}$.

Suppose that the abelian decomposition $\Lambda_{H_i^{\ell-1}}$ does not contain a reducing QH couple. Let $\eta_{\ell-1}$ be the map from the limit group associated with the $\ell-1$ level of $Res(t,v,a)$ to the (free) limit group associated with the (terminal) ℓ level of $Res(t,v,a)$. Let $t_0 \in T_{\ell-1}$ be its base point, and let $\hat{T}_{\ell-1}$ be the finite subtree of $T_{\ell-1}$ spanned by the points $t_0, h_1^{\ell-1}(t_0), \ldots, h_{r(\ell-1,i)}^{\ell-1}(t_0)$. We further increase the finite subtree $\hat{T}_{\ell-1}$. To each vertex stabilized by a QH subgroup in the finite subtree $\hat{T}_{\ell-1}$ we add edges connected to the various boundary components of the QH subgroup (an edge for each orbit of edges in $\Lambda_{H_i^{\ell-1}}$, that is connected to the QH vertex group under the action of the subgroup $H_i^{\ell-1}$), and the vertices connected to these edges (note that these new vertices are stabilized by non-abelian, non-QH vertex groups (in $T_{\ell-1}$), and that since $H_i^{\ell-1}$ is f.g. we have added only finitely many edges and vertices). We denote the obtained finite subtree $T'_{\ell-1}$. To continue our treatment of the factor $H_i^{\ell-1}$ we need the notions of **floating** and **absorbed** surfaces.

Definition 4.7: Let Q be a QH vertex group in the finite tree $T'_{\ell-1}$ and let S be its associated surface. Since the ambient resolution $Res(t,v,a)$ is well-separated, the image of the QH vertex group Q in the next level of the resolution $Res(t,v,a)$ is non-abelian. We say that Q is a *floating* QH vertex group (S is a *floating* surface) with respect to the geometric subresolution $IRes_p(u,v,a)$, if Q does not intersect the subgroup $H_i^{\ell-1}$ in a subgroup of finite index, and (at least) one of the three conditions hold:

(i) The QH vertex group Q is not of minimal rank, i.e., there exists a s.c.c. on the surface $\hat{S}$ that is mapped to the trivial element in the limit group associated with the next level of the ambient resolution $Res(t,v,a)$.
(ii) None of the vertex groups that are adjacent to Q in $T'_{\ell-1}$ intersects non-trivially the subgroup $H_i^{\ell-1}$ (see Definition 4.7 for an absorbed surface).
(iii) The QH vertex group Q is of minimal rank, and for every vertex group V in $T'_{\ell-1}$ that is adjacent to Q in $T'_{\ell-1}$ and intersects $H_i^{\ell-1}$ non-trivially,

$$rk(\eta_{\ell-1}(< H_i^{\ell-1} \cap V, Q >)) > rk(\eta_{\ell-1}(H_i^{\ell-1} \cap V)).$$

We say that Q is an *absorbed* vertex group (S is an *absorbed* surface) if it is not floating.

In case there is no reducing QH couple, we set $\hat{H}_i$ to be the subgroup generated by the factor $H_i^{\ell-1}$ and one of the following if it exists:
(i) An abelian vertex group in the finite tree $T'_{\ell-1}$ that is intersected non-trivially by the factor $H_i^{\ell-1}$, but is not contained in it.
(ii) An absorbed QH vertex group in the finite tree $T'_{\ell-1}$ (with respect to the subgroup $H_i^{\ell-1}$) that is not contained in $H_i^{\ell-1}$ (see Definition 4.7 for an absorbed surface).
(iii) An (abelian) edge group E in the finite tree $T'_{\ell-1}$ that is not contained in $H_i^{\ell-1}$ and is adjacent to a non-abelian, non-QH vertex group V in $T'_{\ell-1}$ that is intersected non-trivially by $H_i^{\ell-1}$, for which

$$rk(\eta_{\ell-1}(< H_i^{\ell-1} \cap V, E >)) \leq rk(\eta_{\ell-1}(H_i^{\ell-1} \cap V)).$$

(iv) An element $v_0 \in V$, $v_0 \notin H_i^{\ell-1}$, where V is a non-abelian, non-QH vertex group in the finite tree $T'_{\ell-1}$ that is intersected non-trivially by the subgroup $H_i^{\ell-1}$, for which there exist two edge groups E_1, E_2 that are adjacent to V in the finite tree $T'_{\ell-1}$ and are not conjugate in the subgroup $H_i^{\ell-1}$, so that v_0 conjugates E_1 to E_2 in the limit group associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and

$$rk(\eta_{\ell-1}(< H_i^{\ell-1} \cap V, v_0 >)) \leq rk(\eta_{\ell-1}(H_i^{\ell-1} \cap V)).$$

We further set H'_i to be the subgroup associated with the resolution induced by the subgroup $\hat{H}_i$. Since $\hat{H}_i$ is generated by $H_i^{\ell-1}$, the absorbed QH vertex groups in $T'_{\ell-1}$ with respect to $H_i^{\ell-1}$, abelian vertex groups that are intersected non-trivially by $H_i^{\ell-1}$, and additional elements that do not increase the ranks of the corresponding vertex groups, the rank of

H_i' is bounded by the rank of $H_i^{\ell-1}$, $rk(H_i') \leq rk(H_i^{\ell-1})$. Furthermore, both groups, H_i' and $H_i^{\ell-1}$, inherit free decompositions from the abelian decompositions associated with their corresponding actions on the Bass–Serre tree $T_{\ell-1}$, and both the number of factors as well as the rank of the additional free group in the abelian decomposition associated with H_i' (the Kurosh rank) are bounded by the number of factors and the corresponding rank in the abelian decomposition associated with $H_i^{\ell-1}$.

If H_i' is identical to $H_i^{\ell-1}$, we have concluded our treatment of the factor $H_i^{\ell-1}$. Otherwise, we replace the factor $H_i^{\ell-1}$ by H_i' and repeat the construction of a subgroup $\hat{H}_i$ associated with the newly obtained subgroup $H_i^{\ell-1}$. In constructing $\hat{H}_i$ we do not change the finite subtree $\hat{T}_{\ell-1}$, but the number of (orbits of) edges connected to QH vertex groups in $\hat{T}_{\ell-1}$ may increase, and with it the finite subtree $T'_{\ell-1}$. From $\hat{H}_i$ we further construct the subgroup H_i' that is associated with the resolution induced by $\hat{H}_i$ from the ambient resolution $Res(t,v,a)$, and repeat our treatment of the subgroup H_i'.

Since in each step we either reduce the rank of the obtained subgroup or we add a new edge group, an abelian vertex group or absorbed QH vertex group from the finite tree $T'_{\ell-1}$ to the subgroup associated with $H_i^{\ell-1}$, or we add an element that conjugates two edge groups in $T'_{\ell-1}$ that were not conjugate previously, we conclude our treatment of the factor $H_i^{\ell-1}$ after finitely many steps. If the ranks of at least one of the factors $H_i^{\ell-1}$ strictly decreased by the iterative procedure, we replace the resolution $IRes_p(u,v,a)$ with the resolution induced by the subgroup generated by the subgroup associated with $IRes_p(u,v,a)$ and the newly obtained factors $H_1^{\ell-1},\ldots,H_{s(\ell-1)}^{\ell-1}$ and denote the obtained resolution $IRes_{\ell-1}(u,v,a)$. In this case, $rk(IRes_{\ell-1}(u,v,a)) < rk(IRes_p(u,v,a))$, and we continue by starting the second step of the construction of the core resolution with the resolution $IRes_{\ell-1}(u,v,a)$ (instead of $IRes_f(u,v,a)$). If none of the ranks of the various factors $H_i^{\ell-1}$ decreases, we continue by analyzing the next $\ell-2$ level of the resolution $IRes_p(u,v,a)$.

(3) The iterative procedure presented in part (2) concludes our treatment of the various factors $H_i^{\ell-1}$ of the subgroup $G^{\ell-1} = H_1^{\ell-1} * \cdots * H_{s(\ell-1)}^{\ell-1}$, which is the image of the subgroup associated with the resolution $IRes_p(u,v,a)$ in the subgroup associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$. We continue by iteratively increasing the index m, and analyzing the various factors of the image of the subgroup $G^{\ell-m} =$

$H_1^{\ell-m} * \cdots * H_{s(\ell-m)}^{\ell-m}$, which is the image of the subgroup associated with the resolution $IRes_p(u,v,a)$ in the subgroup associated with the $\ell - m$ level of the ambient resolution $Res(t,v,a)$, according to the procedure presented in part (2).

Suppose that we analyzed all the images of the subgroup associated with the resolution $IRes_p(u,v,a)$ in the subgroups associated with all the levels below the $\ell - m$ level of the ambient resolution $Res(t,v,a)$, $G^{\ell-1}, \ldots, G^{\ell-m+1}$, and suppose that the rank of the obtained resolution did not decrease along these levels. In accordance with the free decomposition inherited by the subgroup associated with the resolution $IRes_p(u,v,a)$ from the levels that lie above the $\ell - m$ level, the image of the subgroup associated with $IRes_p(u,v,a)$, $G^{\ell-m}$, admits a free decomposition when mapped into the $\ell - m$ level, $G^{\ell-m} = H_1^{\ell-m} * \cdots * H_{s(\ell-m)}^{\ell-m}$. We treat the factors $H_i^{\ell-m}$ according to the procedure presented in part (2).

We start by fixing a system of generators of $H_i^{\ell-m}$,

$$H_i^{\ell-m} = < h_1^{\ell-m}, \ldots, h_{r(\ell-m,i)}^{\ell-m} > .$$

If no (non-trivial) subgroup of the factor $H_i^{\ell-m}$ fixes a vertex in the Bass–Serre tree associated with the $\ell - m$ level of the resolution $Res(t,v,a)$ with which we started, we leave the factor $H_i^{\ell-m}$ unchanged. Suppose that a (non-trivial) subgroup of $H_i^{\ell-m}$ fixes a vertex in the abelian decomposition associated with the $\ell - m$ level of the resolution $Res(t,v,a)$.

Let $T_{\ell-m}$ be the Bass–Serre tree corresponding to the abelian decomposition associated with the $\ell - m$ level of the ambient resolution $Res(t,v,a)$, and let $\Lambda_{H_i^{\ell-m}}$ be the graph of groups inherited by $H_i^{\ell-m}$ from its action on the Bass–Serre tree $T_{\ell-m}$. If the abelian decomposition $\Lambda_{H_i^{\ell-m}}$ contains a reducing QH couple, Q_1 and Q_2, we set the subgroup $\hat{H}_i$ to be the subgroup generated by $H_i^{\ell-m}$, the QH vertex groups Q_1' and Q_2' that contain Q_1 and Q_2 as subgroups of finite index, and an element in the limit groups associated with the $\ell - m$ level of the ambient resolution $Res(t,v,a)$ that conjugates Q_1' to Q_2'. We set H_i' to be the limit group associated with the resolution induced by the subgroup $\hat{H}_i$ from the ambient resolution $Res(t,v,a)$. Since the QH vertex groups Q_1' and Q_2' are not of minimal rank, and since Q_1 and Q_2 are not conjugate in the subgroup $H_i^{\ell-m}$ we started with, and Q_1' and Q_2' are conjugate in H_i',

$$rk(H_i') < rk(H_i^{\ell-m})$$

in case there exists a reducing QH couple in $\Lambda_{H_i^{\ell-m}}$. In this case we replace the factor $H_i^{\ell-m}$ by H_i', and start the second part of the construction of the core resolution, with the resolution induced from the ambient resolution $Res(t,v,a)$, by the subgroup generated by the newly obtained subgroup $H_i^{\ell-m}$, and the subgroup associated with the resolution $IRes_p(u,v,a)$. Note that the rank of the obtained induced resolution is strictly smaller than the rank of the resolution $IRes_p(u,v,a)$ with which we started the second step.

Suppose that the abelian decomposition $\Lambda_{H_i^{\ell-m}}$ does not contain a reducing QH couple. Let $\eta_{\ell-m}$ be the map from the limit group associated with the $\ell-m$ level of $Res(t,v,a)$ to the (free) limit group associated with the (terminal) ℓ level of $Res(t,v,a)$. Let V be a non-abelian, non-QH vertex group in $T_{\ell-m}$ and let $M<V$ be a subgroup. Note that V, hence M, are naturally embedded into the subgroup associated with level $\ell-m+1$ in $Res(t,v,a)$. We denote by $rk(\eta_{\ell-m}(M))$ the rank of resolution obtained by starting with the subgroup M and applying the iterative procedure for the construction of the core resolution (restricted to the bottom m levels). Note that we can apply the procedure restricted to the bottom m levels, by our induction hypothesis.

Let $t_0 \in T_{\ell-m}$ be its base point, and let $\hat{T}_{\ell-m}$ be the finite subtree of $T_{\ell-m}$ spanned by the points $t_0, h_1^{\ell-m}(t_0), \ldots, h_{r(\ell-m,i)}^{\ell-m}(t_0)$. We further increase the finite subtree $\hat{T}_{\ell-m}$. To each vertex stabilized by a QH subgroup in the finite subtree $\hat{T}_{\ell-m}$ we add edges connected to the various boundary components of the QH subgroup (an edge for each orbit of edges connected to the QH vertex group (in $\Lambda_{H_i^{\ell-m}}$) under the action of the subgroup $H_i^{\ell-m}$), and the vertices connected to these edges (note that these new vertices are stabilized by non-abelian, non-QH vertex groups in $T_{\ell-m}$). We denote the obtained finite subtree $T'_{\ell-m}$. We set $\hat{H}_i$ to be the subgroup generated by the factor $H_i^{\ell-m}$ and one of the following if it exists:

(i) An abelian vertex group in the finite tree $T'_{\ell-m}$ that is intersected non-trivially by the factor $H_i^{\ell-m}$ but is not contained in it.

(ii) An absorbed QH vertex group in the finite tree $T'_{\ell-m}$ (with respect to the subgroup $H_i^{\ell-m}$) that is not contained in $H_i^{\ell-m}$ (see Definition 4.7 for an absorbed surface).

(iii) An (abelian) edge group E in the finite tree $T'_{\ell-m}$ that is not contained in $H_i^{\ell-m}$ and is adjacent to a non-abelian, non-QH vertex group V in $T'_{\ell-m}$

that is intersected non-trivially by $H_i^{\ell-m}$, for which

$$rk(\eta_{\ell-m}(< H_i^{\ell-m} \cap V, E >)) \leq rk(\eta_{\ell-m}(H_i^{\ell-m} \cap V)).$$

(iv) An element $v_0 \in V$, $v_0 \notin H_i^{\ell-m}$, where V is a non-abelian, non-QH vertex group in the finite tree $T'_{\ell-m}$ that is intersected non-trivially by the subgroup $H_i^{\ell-m}$, for which there exist two edge groups E_1, E_2 that are adjacent to V in the finite tree $T'_{\ell-m}$, and are not conjugate in the subgroup $H_i^{\ell-m}$, so that v conjugates E_1 to E_2 in the limit group associated with the $\ell - m$ level of the ambient resolution $Res(t, v, a)$, and

$$rk(\eta_{\ell-m}(< H_i^{\ell-m} \cap V, v_0 >)) \leq rk(\eta_{\ell-m}(H_i^{\ell-m} \cap V)).$$

We further set H'_i to be the subgroup associated with the resolution induced by the subgroup $\hat{H}_i$. Since $\hat{H}_i$ is generated by $H_i^{\ell-m}$, the absorbed QH vertex groups in $T'_{\ell-m}$ with respect to $H_i^{\ell-m}$, abelian vertex groups that are intersected non-trivially by $H_i^{\ell-m}$, and additional elements that do not increase the ranks of the corresponding vertex groups, the rank of H'_i is bounded by the rank of $H_i^{\ell-m}$, $rk(H'_i) \leq rk(H_i^{\ell-m})$. Furthermore, both groups H'_i and $H_i^{\ell-m}$ inherit free decompositions from the abelian decompositions associated with their corresponding actions on the Bass–Serre tree $T_{\ell-m}$, and both the number of factors as well as the rank of the additional free group in the abelian decomposition associated with H'_i (the Kurosh rank) are bounded by the number of factors and the corresponding rank in the abelian decomposition associated with $H_i^{\ell-m}$.

If H'_i is identical to $H_i^{\ell-m}$ we have concluded our treatment of the factor $H_i^{\ell-m}$. Otherwise, we replace the factor $H_i^{\ell-m}$ by H'_i, and repeat the analysis of the resolution induced by the (newly obtained) subgroup $H_i^{\ell-m}$ from the ambient resolution $Res(t, v, a)$, according to parts (2) and (3), without changing the finite tree $T'_{\ell-m}$.

Since in each step we either reduce the rank of the obtained subgroup, or we add a new edge group, an abelian vertex group or absorbed QH vertex group from the finite tree $T'_{\ell-m}$ to the subgroup associated with $H_i^{\ell-m}$, or we add an element that conjugates two edge groups in $T'_{\ell-m}$ that were not conjugated previously, we conclude our treatment of the factor $H_i^{\ell-m}$ after finitely many steps. If the ranks of at least one of the factors $H_i^{\ell-m}$ strictly decreased by the iterative procedure, we replace the resolution $IRes_p(u, v, a)$ with the resolution induced by the subgroup generated by the subgroup associated with $IRes_p(u, v, a)$ and the newly obtained factors $H_1^{\ell-m},\dots,H_{s(\ell-m)}^{\ell-m}$, and denote the obtained resolution $IRes_{\ell-m}(u,v,a)$. In this case, $rk(IRes_{\ell-m}(u,v,a)) < rk(IRes_p(u,v,a))$, and we continue by starting the second step of the construction of the core resolution with the resolution $IRes_{\ell-m}(u,v,a)$ (instead of $IRes_f(u,v,a)$). If none of the ranks of the various factors $H_i^{\ell-1}$ decreases, we continue by analyzing the next $(\ell-(m+1))$ level of the resolution $IRes_p(u,v,a)$.

The iterative procedure used for the second part of the construction of the core resolution terminates after finitely many steps. Using it we obtain a geometric subresolution of the ambient resolution $Res(t,v,a)$, which we denote $IRes_s(u,v,a)$, that is set to be either the resolution $IRes_f(u,v,a)$ obtained by the first part of the construction, in case the procedure used for the second part of the construction of the core resolution has not reduced the rank of the resolution it has constructed, or it is the resolution constructed by the procedure used in the second part of the construction of the core resolution, in case this resolution is of strictly smaller rank than the resolution $IRes_f(u,v,a)$, constructed by the procedure used in the first part of the construction. The obtained resolution $IRes_s(u,v,a)$ is a geometric subresolution of $Res(t,v,a)$ by construction; in addition, it is guaranteed to be a firm subresolution by the following theorem, hence it may serve as a core resolution, $Core(<v,a>,Res(t,v,a))$.

THEOREM 4.8: *The resolution $IRes_s(u,v,a)$, obtained by the procedure for the construction of a core resolution, is a firm subresolution of the resolution $Res(t,v,a)$.*

Proof: From the two parts of the construction of the core resolution, we obtain a geometric subresolution of the ambient resolution, $Res(t,v,a)$, that we denote $IRes_{sec}(u,v,a)$. We set the resolution $IRes_s(u,v,a)$ to be the resolution $IRes_{sec}(u,v,a)$ in case $rk(IRes_{sec}) < rk(IRes_f)$, and $IRes_f(u,v,a)$ in case $rk(IRes_{sec}) = rk(IRes_f)$. Furthermore, if $rk(IRes_{sec}) = rk(IRes_f)$, then for any specialization of the ambient resolution $Res(t,v,a)$, the specialization of the terminal free group of the resolution $IRes_{sec}(u,v,a)$ is obtained from the specialization of the terminal free group of the resolution $IRes_f(u,v,a)$ by successively adding elements that do not increase the rank. Hence, if $rk(IRes_{sec}) = rk(IRes_f)$ and $IRes_{sec}(u,v,a)$ is a firm subresolution, so is the resolution $IRes_f(u,v,a)$ which is set to be the resolution $IRes_s(u,v,a)$ in this case. Therefore, to prove Theorem 4.8, it is enough to prove that the resolution obtained by the procedure for the construction of a core resolution, $IRes_{sec}(u,v,a)$, is a firm subresolution.

We first prove that $IRes_{sec}(u,v,a)$ is a firm subresolution We start by proving it in case the ambient resolution, $Res(t,v,a)$, is a one-level resolution, and the associated abelian decomposition contains no QH vertex groups. We continue by proving it for ambient resolutions, $Res(t,v,a)$, for which the abelian decompositions associated with their various levels contain no QH vertex groups, and then we generalize the proof to arbitrary ambient resolutions.

LEMMA 4.9: *Suppose that the ambient resolution, $Res(t,v,a)$, contains a single level, and the abelian decomposition associated with that level contains no QH vertex groups. Then the resolution $IRes_{sec}(u,v,a)$, obtained by the procedure for the construction of a core resolution, is a firm subresolution of the resolution $Res(t,v,a)$.*

Proof: Let H^1 be the subgroup associated with the resolution $IRes_{sec}(u,v,a)$, and let T be the Bass–Serre tree corresponding to the abelian decomposition associated with the single level of the ambient resolution $Res(t,v,a)$. In case the ambient resolution contains a single level, only parts (1) and (2) of the procedure used for the second part of the construction of a core resolution are applied along the iterative procedure.

Suppose that no (non-trivial) subgroup of H^1 fixes a vertex in the Bass–Serre tree T. Then there exists a test sequence of the ambient resolution $Res(t,v,a)$, so that the corresponding sequence of specializations of the subgroup H^1 are bi-Lipschitz equivalent to the action of H^1 on the Bass–Serre tree T. Hence, the rank of each specialization of H^1 (from the test sequence) is equivalent to the rank of H^1, which implies that $IRes_{sec}(u,v,a)$ is a firm subresolution of $Res(t,v,a)$.

Suppose that there are non-trivial subgroups of H^1 that fix vertices in the Bass–Serre tree T. In this case H^1 inherits a free decomposition from its action on the Bass–Serre tree T: $H^1 = B_1 * \cdots * B_m * F_s$, where each of the factors B_j inherits a (possibly trivial) abelian decomposition with non-trivial abelian edge groups from its action on the tree T. Furthermore, the canonical map from the limit group associated with the top level of the ambient resolution, $Res(t,v,a)$, to the limit group associated with its terminal (second) level, maps each of the factors B_j onto a free group D_j, which is a subgroup of the terminal free group of the ambient resolution, $Res(t,v,a)$. Since $IRes_{sec}(u,v,a)$ is a completed resolution, H^1 naturally contains a subgroup isomorphic to $M = D_1 * \cdots * D_m * F_s$, where each of the subgroups D_j is a subgroup of a vertex group in the Bass–Serre tree T.

Then there exists a test sequence of the ambient resolution $Res(t,v,a)$, so that the corresponding sequence of specializations of the subgroup $M < H^1$ are isomorphic to M. Hence, the rank of each of the specializations of H^1 is equivalent to the rank of the subgroup M, which is equivalent to the rank of $IRes_{sec}(u,v,a)$; this implies that $IRes_{sec}(u,v,a)$ is a firm subresolution of $Res(t,v,a)$. ∎

An argument similar to the one used to prove Lemma 4.9 allows us to prove Theorem 4.8 for ambient resolutions for which their associated abelian decompositions contain no QH vertex groups.

LEMMA 4.10: *Suppose that the abelian decompositions associated with the various levels of the ambient resolution, $Res(t,v,a)$, contain no QH vertex groups. Then the resolution $IRes_{sec}(u,v,a)$, obtained by the procedure for the construction of a core resolution, is a firm subresolution of the resolution $Res(t,v,a)$.*

Proof: Let $H = H^1$ be the subgroup associated with the obtained resolution $IRes_{sec}(u,v,a)$, and let T_1 be the Bass–Serre tree corresponding to the abelian decomposition associated with the top level of the ambient resolution, $Res(t,v,a)$. The subgroup H^1 inherits an abelian decomposition from its action on the Bass–Serre tree T_1, an abelian decomposition that gives rise to a free decomposition: $H^1 = B_1^1 * \cdots * B_{m_1}^1 * F_{s_1}$. Each of the factors B_j^1 is mapped into the limit group associated with the second level of the ambient resolution $Res(t,v,a)$, and we denote its image H_j^2. Setting T_2 to be the Bass–Serre tree corresponding to the abelian decomposition associated with the second level of the ambient resolution $Res(t,v,a)$, each of the subgroups H_j^2 inherits an abelian decomposition from its action on the Bass–Serre tree T_2, an abelian decomposition that gives rise to a free decomposition of the subgroup H_j^2. Each of the factors B_j^2 in the free decomposition of H_j^2 is naturally mapped into the limit group associated with the third level of the ambient resolution $Res(t,v,a)$, and we denote its image H_j^3. Continuing inductively, by going down through the levels of the ambient resolution $Res(t,v,a)$, we obtain the set of subgroups $H_j^{\ell-m}$ of the limit group associated with the $\ell - m$ level of the ambient resolution $Res(t,v,a)$, $m = 0,\dots,\ell-1$, $j = 1,\dots,q_{\ell-m}$, and the factors in the free decompositions the subgroups $H_j^{\ell-m}$ inherit from their action on the Bass–Serre tree $T_{\ell-m}$, which we denote $B_j^{\ell-m}$.

We prove that the obtained resolution $IRes_{sec}(u,v,a)$ is a firm subresolution, by going through the levels of the ambient resolution $Res(t,v,a)$ from bottom to top, and showing inductively that the resolutions induced by the subgroups

$H_j^{\ell-m}$ associated with the corresponding level are indeed firm subresolutions of the ambient resolution $Res(t,v,a)$.

The subgroups associated with the bottom level, H_j^{ℓ}, are subgroups of the terminal free group of the ambient resolution, $Res(t,v,a)$, and the resolutions induced by them from the ambient resolution, $Res(t,v,a)$, are trivial, so these are indeed firm subresolutions of $Res(t,v,a)$. According to Lemma 4.9, the resolutions induced by the various subgroups associated with the $\ell-1$ level, $H_j^{\ell-1}$, from $Res(t,v,a)$ are firm subresolutions of $Res(t,v,a)$.

Suppose that the resolutions induced by the various subgroups $H_j^{\ell-m+1}$ from the ambient resolution $Res(t,v,a)$ are firm subresolutions, and let $H_j^{\ell-m}$ be one of the factors in the free decomposition inherited by the image of the subgroup associated with the resolution $IRes_{sec}(u,v,a)$ in the limit group associated with the $\ell-m$ level of the ambient resolution $Res(t,v,a)$.

Suppose that no (non-trivial) subgroup of $H_j^{\ell-m}$ fixes a vertex in the Bass–Serre tree $T_{\ell-m}$. Then there exists a test sequence of the ambient resolution $Res(t,v,a)$, so that the corresponding sequence of specializations of the subgroup $H_j^{\ell-m}$ are bi-Lipschitz equivalent to the action of $H_j^{\ell-m}$ on the Bass–Serre tree $T_{\ell-m}$. Hence, the rank of each specialization of $H_j^{\ell-m}$ is equivalent to the rank of $H_j^{\ell-m}$, which implies that the part of $IRes_{sec}(u,v,a)$ corresponding to the factor $H_j^{\ell-m}$ is a firm subresolution of $Res(t,v,a)$.

Suppose that there are non-trivial subgroups of $H_j^{\ell-m}$ that fix vertices in the Bass–Serre tree $T_{\ell-m}$. In this case $H_j^{\ell-m}$ inherits a free decomposition from its action on the Bass–Serre tree $T_{\ell-m}$: $H_j^{\ell-m} = B_1^{\ell-m} * \cdots * B_r^{\ell-m} * F_s$, where each of the factors $B_k^{\ell-m}$ inherits a (possibly trivial) abelian decomposition with non-trivial abelian edge groups from its action on the tree $T_{\ell-m}$. Furthermore, the canonical map from the limit group associated with the $\ell-m$ level of the ambient resolution, $Res(t,v,a)$, to the limit group associated with its $\ell-m+1$ level maps each of the factors $B_k^{\ell-m}$ onto a subgroup $H_k^{\ell-m+1}$. Since by our inductive assumption, the resolution induced by each of the factors $H_k^{\ell-m+1}$ is a firm subresolution, each of the factors $H_k^{\ell-m+1}$ is mapped onto the terminal free group of the resolution induced by the various factors $H_k^{\ell-m+1}$, which we denote $D_k^{\ell-m+1}$. Since $IRes_{sec}(u,v,a)$ is a completed resolution, $H_j^{\ell-m}$ naturally contains a subgroup isomorphic to $M = D_1^{\ell-m+1} * \cdots * D_r^{\ell-m+1} * F_s$, where each of the subgroups $D_k^{\ell-m+1}$ is contained in a vertex stabilizer in the Bass–Serre $T_{\ell-m}$.

Then there exists a test sequence of the ambient resolution $Res(t,v,a)$, so that the corresponding sequence of specializations of the subgroup $M < H_j^{\ell-m}$

are isomorphic to M. Hence, the rank of each of the specializations of $H_j^{\ell-m}$ is equivalent to the rank of the subgroup M, which is equivalent to the rank of the resolution induced by the subgroup $H_j^{\ell-m}$ from the ambient resolution $Res(t,v,a)$, which implies that the resolution induced by the subgroup $H_j^{\ell-m}$ is a firm subresolution of $Res(t,v,a)$. By induction on the various levels of the ambient resolution $Res(t,v,a)$, the resolution $IRes_{sec}(u,v,a)$ is a firm subresolution of the ambient resolution $Res(t,v,a)$. ∎

In Lemmas 4.9 and 4.10 we proved Theorem 4.8 in case the abelian decompositions associated with the various levels of the ambient resolution $Res(t,v,a)$ contain no QH vertex groups. To prove Theorem 4.8 in the general case we use the combinatorial properties of (quadratic) test sequences (definition 1.5 in [Se2]), together with the arguments used to prove Lemmas 4.9 and 4.10. As in analyzing the resolution $IRes_{sec}(u,v,a)$ in case the ambient resolution, $Res(t,v,a)$, contains no QH vertex groups, we analyze the resolution $IRes_{sec}(u,v,a)$ inductively from bottom to top, starting with the factors associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$.

Let $H_i^{\ell-1}$ be a factor in the free decomposition inherited by the subgroup associated with the resolution $IRes_{sec}(u,v,a)$, from the abelian decompositions associated with all the levels of the ambient resolution $Res(t,v,a)$ that lie above the $\ell-1$ level. Let $T_{\ell-1}$ be the Bass–Serre tree associated with the $\ell-1$ level in the ambient resolution, $Res(t,v,a)$, and let $\Lambda_{H_i^{\ell-1}}$ be the abelian decomposition inherited by $H_i^{\ell-1}$ from its action on $T_{\ell-1}$. Let $H_i^{\ell-1} =< h_1^{\ell-1},\dots,h_{r(\ell-1,i)}^{\ell-1} >$ be a fixed generating set for $H_i^{\ell-1}$, let $t_0 \in T_{\ell-1}$ be its base point, and let $\hat{T}_{\ell-1}$ be the finite subtree of $T_{\ell-1}$ spanned by the points $t_0, h_1^{\ell-1}(t_0),\dots,h_{r(\ell-1,i)}^{\ell-1}(t_0)$. Recall that along the construction of the resolution $IRes_{sec}(u,v,a)$ we increased the finite subtree $\hat{T}_{\ell-1}$. To each vertex stabilized by a QH subgroup in the finite subtree $\hat{T}_{\ell-1}$, we added edges connected to the various boundary components of the QH subgroup that correspond to orbits of edges in the graph of groups $(\Lambda_{H_i^{\ell-1}})$, and the vertices connected to these edges (note that these new vertices are stabilized by non-abelian, non-QH vertex groups in $T_{\ell-1}$). We denoted the obtained finite subtree $T'_{\ell-1}$.

By lemma 1.4 of [Se4], from the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, the factor $H_i^{\ell-1}$ inherits an abelian decomposition that we (still) denote $\Lambda_{H_i^{\ell-1}}$, in which all the (non-trivial) edge groups are edge groups in the Bass–Serre tree $T_{\ell-1}$, and the vertex groups are either abelian or QH vertex groups in $T_{\ell-1}$, or they are contained in non-QH, non-abelian vertex groups in $T_{\ell-1}$.

If the finite subtree $T'_{\ell-1}$ contains no floating QH vertex groups, the resolution induced by the subgroup $H_i^{\ell-1}$ is firm by the argument used to prove Lemma 4.9. Suppose that the finite subtree $T'_{\ell-1}$ contains a floating QH vertex group. Let Q be a QH vertex group in the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, let Θ_Q be the abelian decomposition obtained from the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$ by collapsing all the edges except those that are connected to the QH vertex group Q, and suppose that the free decomposition inherited by the factor $H_i^{\ell-1}$ from Θ_Q is a non-trivial free decomposition of $H_i^{\ell-1}$ (cf. lemma 1.4 in [Se4]), which implies that some conjugates of the QH vertex group Q are floating QH vertex groups in the finite tree $T'_{\ell-1}$.

Recall that, given a test sequence of the resolution $Res(t,v,a)$, a quadratic test sequence is associated with the QH vertex Q. A quadratic test sequence is presented in definition 1.5 of [Se2]. According to this definition, with the surface S associated with the QH vertex group Q, we associate a collection of non-homotopic, non-boundary parallel s.c.c. $b_1,\dots,b_q$ so that $S \setminus \cup\{b_1,\dots,b_q\}$ is a disjoint union of three-punctured spheres and once-punctured Möbius bands, and another collection of non-homotopic, non-boundary parallel s.c.c. $d_1,\dots,d_t$ so that each of the curves d_i intersects non-trivially at least one of the curves b_j, and the collection of curves $b_1,\dots,b_q,d_1,\dots,d_t$ fills the surface S. Let $\varphi_1,\dots,\varphi_q$ be the automorphisms of Q that correspond to Dehn twists along the s.c.c. $b_1,\dots,b_q$, and $\psi_1,\dots,\psi_t$ be the automorphisms of Q that correspond to Dehn twists along the s.c.c. $d_1,\dots,d_t$. To construct a quadratic test sequence, we define the following sequences of automorphisms of the QH vertex group Q, $\{\nu_n,\tau_n\}$, iteratively. We set $\tau_1=id.$, and ν_1 to be

$$\nu_1=\psi_1^{\ell_1^1}\circ\psi_2^{\ell_2^1}\circ\cdots\circ\psi_t^{\ell_t^1}.$$

For every index $n>1$ we define τ_n to be

$$\tau_n=\varphi_1^{m_1^n}\circ\varphi_2^{m_2^n}\circ\cdots\circ\varphi_q^{m_q^n}\circ\nu_{n-1}$$

and

$$\nu_n=\psi_1^{\ell_1^n}\circ\psi_2^{\ell_2^n}\circ\cdots\circ\psi_t^{\ell_t^n}\circ\tau_n.$$

The automorphisms $\{\nu_n,\tau_n\}$ naturally extend to automorphisms of the limit group $Rlim_{\ell-1}(t,v,a)$ associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and we set the sequence of homomorphisms $\lambda_n\colon Rlim_{\ell-1}(t,v,a)\to F_k$ to be a sequence of homomorphisms of the form

$$\lambda_n=\mu_n\circ\varphi_1^{e_n}\circ\varphi_2^{e_n}\circ\cdots\circ\varphi_q^{e_n}\circ\nu_n$$

where μ_n is a homomorphism obtained from a composition of the automorphisms associated with the other parts of the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and the (canonical) epimorphism $\eta_{\ell-1}$ from the group associated with the $\ell-1$ level of $Res(t,v,a)$ to the free group associated with the (terminal) ℓ level.

The decomposition of the QH vertex group Q along the collection of s.c.c. $b_1,\dots,b_q$ can naturally be extended to a decomposition of the limit group $Rlim_{\ell-1}(t,v,a)$, associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$. Let G_Q be the Bass–Serre tree corresponding to this decomposition of $Rlim_{\ell-1}(t,v,a)$. Before analyzing the image under the homomorphism λ_n of the factor $H_i^{\ell-1}$, $\lambda_n(H_i^{\ell-1})$, in the terminal free group of the ambient resolution $Res(t,v,a)$, we analyze the action on the tree G_Q of the image of the factor $H_i^{\ell-1}$ under the automorphism ν_n of $Rlim_{\ell-1}(t,v,a)$.

Proposition 4.11: *Let Γ_n be the graph of groups inherited by the subgroup $\nu_n(H_i^{\ell-1})$ from its action on the Bass–Serre tree G_Q, and let Γ'_n be the free decomposition of $\nu_n(H_i^{\ell-1})$ obtained from the graph of groups Γ_n by collapsing all the edges with non-trivial stabilizers. The free decomposition Γ'_n of $\nu_n(H_i^{\ell-1})$ naturally transfers to a free decomposition (still denoted Γ'_n) of $H_i^{\ell-1}$. Then there exists a test sequence with corresponding sequence of automorphisms, $\{(\tau_n,\nu_n)\}$, for which there exists an index n_0, so that for every index $n>n_0$, the free decompositions inherited by $H_i^{\ell-1}$ from Γ'_n are identical to the free decomposition inherited by $H_i^{\ell-1}$ from the decomposition Θ_Q (according to lemma 1.4 in [Se4]).*

Furthermore, the test sequence can be chosen so that in the limit action of the group $Rlim_{\ell-1}(t,v,a)$ on a real tree Y, obtained from the sequence of actions of $Rlim(t,v,a)$ on the Bass–Serre tree G_Q via the automorphisms ν_n, the action of the subgroup $H_i^{\ell-1}$ on the real tree Y contains orbits of only discrete and IET components.

Proof: Let $\Delta_{H_i^{\ell-1}}$ be the free decomposition inherited by $H_i^{\ell-1}$ from the decomposition Θ_Q according to lemma 1.4 of [Se4]. By our assumptions, $\Delta_{H_i^{\ell-1}}$ is a non-trivial free decomposition of $H_i^{\ell-1}$.

Recall that the collection of s.c.c. $b_1,\dots,b_q$ and $d_1,\dots,d_t$ are chosen to fill the surface S associated with the QH vertex group Q. By the way a test sequence is defined, larger and larger (finite) sets of elements in the subgroup $H_i^{\ell-1}$, that are either hyperbolic with respect to the abelian decomposition Θ_Q or are conjugate to non-boundary parallel elements in the QH subgroup

Q, are guaranteed to be mapped by the automorphisms ν_n to elements that act hyperbolically on the Bass–Serre tree G_Q which corresponds to the abelian decomposition obtained from Θ_Q, by decomposing the QH subgroup Q along the disjoint, non-homotopic collection of s.c.c. $b_1, \ldots, b_q$. Furthermore, the axis of the image of such elements under the automorphisms ν_n when acting on G_Q is composed from high powers of elements that are conjugate to the s.c.c. $b_1, \ldots, b_q$ and $d_1, \ldots, d_t$ (see the construction of test sequences in definition 1.20 in [Se2]).

The subgroup $H_i^{\ell-1}$ inherits an abelian decomposition from the decomposition Θ_Q of the ambient limit group $Rlim_{\ell-1}(t, v, a)$. Let $M_1, \ldots, M_s$ be those vertex groups in this decomposition that can be conjugated into infinite index subgroups of the QH vertex Q, and are not free product of conjugates of boundary subgroups in Q.

By the work of P. Scott [Sc], every subgroup of a surface group is geometric, i.e., every subgroup of a given surface group is the fundamental group of a subsurface of a finite cover of the given surface. Since the collection of curves $b_1, \ldots, b_q, d_1, \ldots, d_t$ fills the given surface S (with fundamental group Q), the collection of lifts of these curves fills any given finite cover of S.

Since the subgroups $M_1, \ldots, M_s$ are of infinite index in conjugates of the QH vertex group Q, and they are not free products of boundary subgroups in Q, each is the fundamental group of a corresponding proper subsurface in some finite cover of the surface S. Let X_i be the cover of the surface S that is associated with the subgroup M_i, and let $S_i \subset X_i$ be the proper subsurface (not necessarily connected) with fundamental group M_i. Since S_i is a proper subsurface, and its fundamental group is not a free product of boundary subgroups in Q, S_i has boundary components $c_1^i, \ldots, c_u^i$ that are non-boundary parallel in X_i.

Since $c_1^i, \ldots, c_u^i$ are non-boundary parallel in X_i, they represent non-boundary parallel curves on the surface S, hence they intersect non-trivially some of the s.c.c. $b_1, \ldots, b_q, d_1, \ldots, d_t$ (that fill S). Therefore, by the construction of a test sequence, for some index n_1, and for every index i, $1 \leq i \leq s$, each of the curves $\nu_{n_1}(c_1^i), \ldots, \nu_{n_1}(c_u^i)$ intersects non-trivially all the s.c.c. $b_1, \ldots, b_q, d_1, \ldots, d_t$. Hence, again by the construction of a test sequence, there exists some index n_0, so that for every couple of indices i and j, the collection of curves $\nu_{n_0}(c_j^i)$ and $b_1, \ldots, b_q$ fills the surface S, so their lifts fill the cover X_i. This implies that for every index i, $1 \leq i \leq s$, $\nu_{n_0}(M_i)$ contains no non-trivial elliptic elements in acting on the tree G_Q, which is the Bass–Serre tree that corresponds to the abelian decomposition obtained by cutting the QH subgroup Q along the collection of s.c.c. $b_1, \ldots, b_t$. Hence, Γ'_{n_0} is precisely $\Delta_{H_i^{\ell-1}}$, i.e., the free decomposition inherited by $H_i^{\ell-1}$ from Θ_Q according to lemma 1.4 of [Se4].

Once the free decompositions inherited by the subgroups $M_1, \ldots, M_s$ from the decompositions Γ'_{n_0} are identical to those guaranteed by lemma 1.4 of [Se4], by choosing the powers that define the next automorphisms $\{(\tau_n, \mu_n)\}$ in our test sequence to be large enough, we guarantee that all the next free decompositions Γ'_n are identical to the free decomposition Γ_{n_0}. By the combinatorial properties of a quadratic test sequence (presented in detail in [Se2]), a limit action obtained from the test sequence we constructed contains orbits of only discrete and IET components, and the proposition follows. ∎

Suppose that the (non-trivial) free decomposition inherited by the subgroup $H_i^{\ell-1}$ from the abelian decomposition Θ_Q according to lemma 1.4 of [Se4] is $H_i^{\ell-1} = B_1 * \cdots * B_m * F_s$, where F_s is a free group. Since the surface Q is floating, we can modify the homomorphism μ of the ambient limit group $Rlim(t,v,a)$, so that for every index i, $1 \leq i \leq t$, $\mu(b_i)$ cannot be conjugated into the subgroup $H_i^{\ell-1} \cap V$, for every vertex group V that is adjacent to the QH vertex group Q in the finite subtree $T'_{\ell-1}$. Since by Proposition 4.11 there exists a sequence of automorphisms $\{(\tau_n, \mu_n)\}$, for which there exists some index n_0, so that for every index $n > n_0$ the free decomposition Γ'_n of $H_i^{\ell-1}$ is identical to the free decomposition $H_i^{\ell-1} = B_1 * \cdots * B_m * F_s$, then the sequence of automorphisms $\{(\tau_n, \mu_n)\}$ can be extended to a test sequence of the two bottom levels of the ambient resolution $Res(t,v,a)$, so that for every $n > n_0$,

$$\lambda_n(H_i^{\ell-1}) = \lambda_n(B_1) * \cdots * \lambda_n(B_m) * \lambda_n(F_s),$$

where λ_n maps F_s isomorphically onto $\lambda_n(F_s)$, which clearly implies that the given test sequence is firm with respect to the entire subgroup $H_i^{\ell-1}$. By iteratively adding the QH vertex groups in the abelian decomposition associated with the $\ell - 1$ level of the ambient resolution $Res(t,v,a)$, we get that the resolutions induced by the various subgroups $H_i^{\ell-1}$ are indeed firm subresolutions of $Res(t,v,a)$.

Analyzing the resolutions induced by the various subgroups $H_i^{\ell-1}$ from the ambient resolution $Res(t,v,a)$, to complete the proof of Theorem 4.8, i.e., to show that the entire resolution $IRes_{sec}(u,v,a)$ is a firm subresolution, we continue the analysis of the resolution $IRes_{sec}(u,v,a)$ by climbing through the levels of the ambient resolution $Res(t,v,a)$ as we did in proving Lemma 4.10, and for each level we use the same argument used in the analysis of the factors $H_i^{\ell-1}$. ∎

Given a well-separated completed resolution $Res(t,v,a)$, and a subgroup

$< v, a >$ of the limit group $Rlim(t, v, a)$ associated with this resolution, the procedure for the construction of a core resolution enables us to start with the resolution induced by the subgroup $< v, a >$ from the resolution $Res(t, v, a)$, and modify it sequentially by reducing the ranks of the obtained resolution to get a firm geometric subresolution of the resolution $Res(t, v, a)$, which can be taken to be a core resolution, $Core(< v, a >, Res(t, v, a))(r, v, a)$. Naturally, the procedure used for the construction of a core resolution generalizes to the construction of a (graded) core of a (well-separated) graded resolution, which we denote $GCore(< v, a >, GRes(t, v, p, a))(r, v, a)$, and to the construction of a (multi-graded) core of a multi-graded resolution, which we denote $MGCore(< v, a >, MGRes)(r, v, a)$. Note that in the graded and multi-graded cases, the second part of the construction of the core resolution either reduces the sum of the rank of the terminating free factor and the number of the terminating factors of the resolution $IRes_f(u, v, a)$ obtained in the first part (i.e., reduces the Kurosh rank of the free decomposition associated with its terminal limit group), or it proves that it is indeed a firm subresolution, hence may serve as a (graded, multi-graded) core resolution.

Also, note that, unlike ungraded resolution, in analyzing the set of specializations that factor and are taut with respect to a completed graded or multi-graded resolution, $MGRes$, it is possible that the restrictions of certain specializations to some levels of the multi-graded resolution, $MGRes$, cannot be obtained from the restrictions of the same specializations to the successive (lower) levels of the resolution, using the associated (multi-graded) modular automorphisms. This phenomenon arises since there are specializations of (graded, multi-graded) solid limit groups that do not factor through any flexible quotient of the solid limit group (after applying an element of the associated modular groups), but still they are not strictly solid specializations of the solid limit group, hence they are assumed to factor through (the limit group associated with the completion of) a resolution associated with at least one of the flexible quotients of the given solid limit group (see definitions 1.4 and 1.5 in [Se3]).

However, the Kurosh rank of the restriction of such specialization to the subgroup associated with the core of such resolution $MGRes$ is at most the Kurosh rank of the constructed core. This observation is needed in proving certain inequalities on the complexity of the core resolution (see Theorems 4.18 and 4.19).

In the procedure for the construction of the tree of stratified sets, we were able to bound the complexity of multi-graded resolutions by the complexity of multi-graded resolutions produced in previous steps of the iterative procedure. In the general sieve method, presented in the next paper in the sequel, we will need similar bounds on the cores of the constructed multi-graded resolutions. To state these bounds, we first need to present the *complexity* of a (graded, multi-graded) core resolution, which is a slight modification of our definition of the complexity of a resolution (cf. Definition 2.2).

Definition 4.12: Let $Res(t,v,a)$ be an (ungraded) well-separated completed resolution and let $Core(<v,a>,Res(t,v,a))$ be a core resolution. We set the complexity of the core $Core(<v,a>,Res(t,v,a))$,

$$Cmplx(Core(<v,a>,Res(t,v,a))),$$

to be the complexity of the core $Core(<v,a>,Res(t,v,a))$, viewed as an induced resolution ([Se4], definition 3.2). Unlike the ungraded case, in case the ambient resolution is multi-graded (or graded), we need to slightly modify the complexity of the core.

Let $MGRes(t,v,P,R_1,\dots,R_d,a)$ be a well-separated completed multi-graded resolution and let $MGCore(<v,a>,MGRes(t,v,P,R_1,\dots,R_d,a))$ be a (multi-graded) core resolution. Let $Q_1,\dots,Q_m$ be the QH subgroups that appear in the core $MGCore(<v,a>,MGRes(t,v,P,R_1,\dots,R_d,a))$. Each QH vertex group Q_j is a subgroup of finite index in a QH vertex group in the ambient (multi-graded) resolution, so with each QH we associate the (punctured) surface S_j that is associated with the QH vertex group of the ambient resolution that contains Q_j. With each (punctured) surface S_j we may associate an ordered couple $(genus(S_j),|\chi(S_j)|)$. We will assume that the QH subgroups $Q_1,\dots,Q_m$ are ordered according to the lexicographical (decreasing) order of the ordered couples associated with their corresponding surfaces. Let $rk(MGCore)$ be the rank of the free factor in the free decomposition associated with the terminal limit group of the core, $MGCore(<v,a>,MGRes(t,v,P,R_1,\dots,R_d,a))$, and let $factor(MGCore)$ be the number of factors it is terminating with (i.e., the number of factors in the free decomposition inherited by the terminal subgroup of the multi-graded core from the free decomposition of the terminal subgroup of the ambient multi-graded resolution). Note that $rk(MGCore)+factor(MGCore)$ is precisely the Kurosh rank of the free decomposition associated with the terminal limit group of the core $MGCore$. Let $Abrk(MGCore)$ be the abelian rank of the core resolution (see definition 1.14 of [Se4]).

We set the complexity of the multi-graded core

$$MGCore(<v,a>,MGRes(t,v,P,R_1,\dots,R_d,a)),$$

denoted $Cmplx(MGCore)$, to be

$$Cmplx(MGCore) = (rk(MGCore) + factor(MGCore),$$
$$(genus(S_1), |\chi(S_1)|), \ldots, (genus(S_m), |\chi(S_m)|), Abrk(MGCore), rk(MGCore)).$$

On the set of core resolutions we can define a partial order. Let $MGCore_1$ and $MGCore_2$ be two (multi-graded) core resolutions. We say that

$$Cmplx(MGCore_1) = Cmplx(MGCore_2)$$

if the tuples defining the two complexities are identical. We say that

$$Cmplx(MGCore_1) < Cmplx(MGCore_2)$$

if:

(1) $rk(MGCore_1) + factor(MGCore_1)$ is smaller than

$$rk(MGCore_2) + factor(MGCore_2),$$

(2) the above numbers are equal and the tuple

$$((genus(S^1_1), |\chi(S^1_1)|), \ldots, (genus(S^1_{m_1}), |\chi(S^1_{m_1}|))$$

is smaller in the lexicographical order than the tuple

$$((genus(S^2_1), |\chi(S^2_1)|), \ldots, (genus(S^2_{m_2}), |\chi(S^2_{m_2}|)),$$

(3) the above numbers and tuples are equal and

$$Abrk(MGCore_1) < Abrk(MGCore_2),$$

(4) the above numbers and tuples are equal and

$$rk(MGCore_1) < rk(MGCore_2).$$

To get bounds on the complexity of core resolutions of the multi-graded resolutions constructed along our quantifier elimination iterative procedure, we need to study some basic properties of the core resolution. These properties of the core resolution seem to be basic tools for analyzing Diophantine sets in general.

THEOREM 4.13: *Let $MGRes_1(v, R_1, \ldots, R_m, P, a)$ be a well-separated completed multi-graded resolution containing two levels and a unique quotient map between the two limit groups associated with the two levels, and let Λ_1 be* *the graph of groups with fundamental group $Rlim(v,P,a)$ associated with the top level of $MGRes_1$, where $R_1,\dots,R_m,P$ are the non-QH, non-abelian vertex groups in the graph of groups Λ_1. Let $MGRes_2(t,v,L_1,\dots,L_n,P,a)$ be a well-separated completed resolution of a limit group $Rlim(t,v,P,a)$, so that there is an embedding $\nu\colon Rlim(v,P,a)\to Rlim(t,v,P,a)$ that maps each of the subgroups $R_1,\dots,R_m,P$ into a conjugate of one of the subgroups L_j.*

Since the multi-graded resolution $MGRes_1(v,R_1,\dots,R_m,P,a)$ is well-separated, with each QH vertex group in its associated multi-graded abelian decomposition, Λ_1, there is an associated collection of s.c.c. that are mapped to the trivial element in the terminal level of the resolution

$$MGRes_1(v,R_1,\dots,R_m,P,a).$$

Each QH vertex group in Λ_1 naturally inherits a sequence of abelian decompositions from the multi-graded resolution $MGRes_2(t,v,L_1,\dots,L_n,P,a)$. Suppose that for every such QH vertex group Q, this sequence of multi-graded abelian decompositions is compatible with the collection of s.c.c. on Q that are mapped to the trivial element in the terminal level of the multi-graded resolution $MGRes_1(v,R_1,\dots,R_m,P,a)$.

Then the multi-graded resolution $MGRes_2(t,v,L_1,\dots,L_n,P,a)$ can be modified (without changing the Diophantine set associated with its completion) to a well-separated multi-graded resolution $MGRes_3(u,t,v,L_1,\dots,L_n,P,a)$, so that:

(i) *Every specialization of the limit group $Rlim(t,v,p,a)$ that can be extended to a specialization that factors through the (limit group associated with the) completion of the multi-graded resolution $MGRes_2$, $Comp(MGRes_2)$, can be extended to a specialization that factors through the completion of the multi-graded resolution $MGRes_3$, $Comp(MGRes_3)$, and vice versa, i.e., the Diophantine sets associated with the completions of $MGRes_2$ and $MGRes_3$ are identical.*

(ii) *Let (v,p,a) be a specialization of the limit group $Rlim(v,p,a)$ that factors through and is taut with respect to the resolution $MGRes_1$. Suppose that (v,p,a) can be extended to a specialization (t,v,p,a) of the limit group $Rlim(t,v,p,a)$ that factors and is taut with respect to the resolution $MGRes_2$. Then (t,v,p,a) can be extended to a specialization (u,t,v,p,a) that factors and is taut with respect to the resolution $MGRes_3$, i.e., the part of the modular block associated with the resolution, $MGRes_2$, that projects to the modular block associated with the resolution, $MGRes_1$, can be "lifted" to the modular block associated with $MGRes_3$.*

Furthermore, there exists a core for $MGRes_3$,

$$MGCore(< v, P, a >, MGRes_3),$$

produced by modifying the above procedure for the construction of a (multi-graded) core resolution, that has the following properties:

(iii) *The complexity of the multi-graded core is bounded by the complexity of the multi-graded resolution $MGRes_1$:*

$$Cmplx(MGCore(< v, P, a >, MGRes_3)) \leq Cmplx(MGRes_1).$$

(iv) *If these complexities are equal, then the structure of the core,*

$$MGCore(< v, P, a >, MGRes_3),$$

is identical to the structure of the multi-graded resolution $MGRes_1$, i.e., the core has one level, and the abelian decomposition associated with this one level has the same structure as Λ_1, the abelian decomposition associated with $MGRes_1$.

Proof: Since (the taut structure of) the multi-graded resolution $MGRes_2$ is "compatible" with the multi-graded resolution $MGRes_1$, the rank of the induced resolution, $IndRes(< v, P, a >, MGRes_2)$, is bounded by the rank of the multi-graded resolution $MGRes_1$, and

$$rk(IndRes(< v, P, a >, MGRes_2)) + factor(IndRes(< v, P, a >, MGRes_2)) \\ \leq rk(MGRes_1) + factor(MGRes_1).$$

Applying the iterative procedure for the construction of the core resolution to the subgroup $< v, P, a >$ and the multi-graded resolution $MGRes_2$, we obtain a multi-graded core, $MGCore(< v, P, a >, MGRes_2)$. By construction, the sum of the rank and the number of factors of this core is bounded by the sum of the rank and the number of factors of the induced resolution, $IndRes(< v, P, a >, MGRes_2)$, and if the two sums are equal, then the second part of the construction was not applied, i.e., the multi-graded core and the induced resolution are identical. Hence

$$rk(MGCore(< v, P, a >, MGRes_2)) + factor(MGCore(< v, P, a >, MGRes_2)) \\ \leq rk(MGRes_1) + factor(MGRes_1).$$

If

$$rk(MGCore(<v,P,a>,MGRes_2))+factor(MGCore(<v,P,a>,MGRes_2)) < rk(MGRes_1)+factor(MGRes_1),$$

then we set the core to be the output of the procedure,

$$MGCore(<v,P,a>,MGRes_2),$$

and set the multi-graded resolution $MGRes_3$ to be the multi-graded resolution $MGRes_2$, and the theorem follows. Hence, we may assume

$$rk(MGCore(<v,P,a>,MGRes_2))+factor(MGCore(<v,P,a>,MGRes_2)) = rk(MGRes_1)+factor(MGRes_1).$$

In this case, the multi-graded core constructed by our iterative procedure, $MGCore(<v,P,a>,MGRes_2)$, is identical to the induced resolution, $IndRes(<v,P,a>,MGRes_2)$.

Suppose that the multi-graded abelian decomposition Λ_1, associated with the multi-graded resolution $MGRes_1$, contains no surviving QH vertex groups (see Definition 1.8 for the notion of a surviving surface). In this case, we set $MGRes_3$ to be the resolution $MGRes_2$ and modify the core, $MGCore(<v,p,a>,MGRes_2)$, which is in fact the induced resolution, $IndRes(<v,p,a>,MGRes_2)$. We replace the part of the induced resolution which is the subresolution induced by the free factor that is dropped in the resolution $MGRes_1$, by the image of that factor in $MGRes_2$. The modified resolution is a firm geometric subresolution of $MGRes_2$ that contains the subgroup $<v,p,a>$, since the previously constructed core,

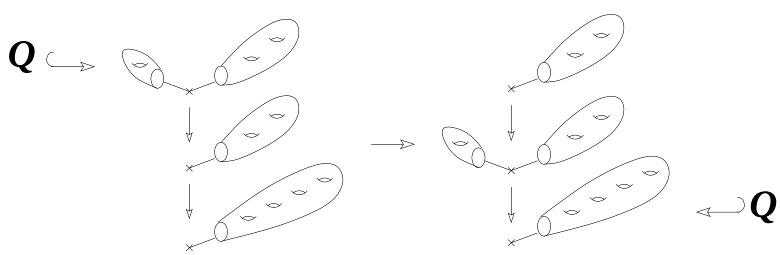
$$MGCore(<v,p,a>,MGRes_2),$$

is a firm geometric subresolution, and its Kurosh rank is identical to the Kurosh rank of the original resolution, $MGRes_1$. Hence, we set the modified resolution to be the core resolution, $MGCore(<v,p,a>,MGRes_2)$.

Suppose that $MGRes_1$ contains a QH vertex group. Since we assume that the core contains no surviving surface, its complexity is strictly bounded by the complexity of the resolution $MGRes_1$ by our analysis of taut homomorphisms of maximal rank, presented in section 2 of [Se4]. If $MGRes_1$ contains no QH vertex groups, the complexity of the core is bounded by the complexity of $MGRes_1$, and in case of equality their structures are identical.

Suppose that the multi-graded abelian decomposition Λ_1 of $MGRes_1$ contains a surviving QH vertex group. In this case, we modify the resolution $MGRes_2$ in a somewhat different way than we did in the minimal rank case (section 1 in [Se4] and in this paper), so that the corresponding Diophantine set will not change (as well as the projection of the associated modular block as indicated in the statement of the theorem).

Let Q be a surviving vertex group in Λ_1 and let $\hat{Q}$ be a QH vertex group in an abelian decomposition associated with one of the levels of $MGRes_2$, so that Q is mapped isomorphically onto $\hat{Q}$, and Q is not mapped isomorphically onto a QH vertex group in an abelian decomposition associated with a level that lies above the level to which $\hat{Q}$ belongs in the multi-graded resolution $MGRes_2$. We vary the multi-graded resolution by "pushing down" a QH vertex isomorphic to $\hat{Q}$, which is set to be the image of the QH vertex Q, and is mapped (in the new resolution) to (the image in the new resolution of) the image of the QH subgroup Q in $MGRes_1$, i.e., we push down a QH vertex group which is set to be the image of the QH subgroup Q in the new resolution. The image of this new subgroup in the next level is set to be the image of the QH subgroup Q in the multi-graded resolution $MGRes_1$. We leave the levels below the QH vertex group $\hat{Q}$ unchanged, and change the order of the QH vertex groups that lie above $\hat{Q}$ into which the QH vertex group Q is mapped. We repeat this "pushing down" operation for a maximal collection of surviving QH vertex groups Q_j in the abelian decomposition Λ_1, that are mapped isomorphically onto non-conjugate QH vertex groups $\hat{Q}_j$ in the various levels of $MGRes_2$. We denote the obtained resolution $MGRes_3(u, t, v, L_1, \ldots, L_n, P, a)$.



By construction, since we have only changed the order of the appearance of certain QH vertex groups in modifying the resolution $MGRes_2$, the Diophantine sets associated with the subgroup (t, v, p, a) and the (limit groups associated with the) completions of the resolutions $MGRes_2$ and $MGRes_3$ are identical. Furthermore, the part of the modular block associated with $MGRes_2$ that projects to the modular block associated with $MGRes_1$ can be "lifted" to the modular block associated with $MGRes_3$ (part (ii) in the claim of the theorem).

With the multi-graded resolution $MGRes_3$ we associate a core,

$$MGCore(<v,P,a>, MGRes_3),$$

by applying our algorithm for the construction of a core resolution. If

$$rk(MGCore(<v,P,a>, MGRes_3)) + factor(MGCore(<v,P,a>, MGRes_3)) < rk(MGRes_1) + factor(MGRes_1),$$

the theorem follows. Suppose

$$rk(MGCore(<v,P,a>, MGRes_3)) + factor(MGCore(<v,P,a>, MGRes_3)) = rk(MGRes_1) + factor(MGRes_1).$$

We modify the core, $MGCore(<v,p,a>, MGRes_3)$, which is in fact the induced resolution, $IndRes(<v,p,a>, MGRes_3)$, as we did in case there are no surviving surfaces. We replace the part of the induced resolution, which is the subresolution induced by the free factor that is dropped in the resolution $MGRes_1$, by the image of that factor in $MGRes_3$. The modified resolution is a firm geometric subresolution of $MGRes_3$ that contains the subgroup $<v,p,a>$, since the previously constructed core, $MGCore(<v,p,a>, MGRes_3)$, is a firm geometric subresolution, and its Kurosh rank is identical to the Kurosh rank of the original resolution, $MGRes_1$. Hence, we set the modified resolution to be the core resolution, $MGCore(<v,p,a>, MGRes_3)$.

If there are no two surviving QH vertex groups in Λ_1 that are mapped onto conjugate QH vertex groups in $MGRes_3$, or, more generally, if every two surviving QH vertex groups in Λ_1 that are mapped onto conjugate vertex groups in $MGRes_3$ belong to the same factor in the abelian decomposition Λ_1 of $MGRes_1$, the bound on the complexity of the obtained core (by the complexity of $MGRes_1$) follows by our analysis of taut homomorphisms of maximal rank ([Se4], section 2). Otherwise, every two surviving QH vertex groups Q_1 and Q_2 in Λ_1 that do not belong to the same factor in Λ_1, and are mapped isomorphically onto conjugate QH vertex groups in $MGRes_3$ must belong to different factors in the free decomposition associated with the terminal level of the core $MGCore(<v,P,a>, MGRes_3)$. Let Q_1 and Q_2 be such surfaces. In this case we replace the subgroup $<v,P,a>$ by the subgroup G_1 generated by $<v,P,a>$ and the element that conjugates $\hat{Q}_1$ to $\hat{Q}_2$ in the subgroup associated with the resolution $MGRes_3$. We repeat this operation for all such couples of surviving surfaces until we obtain a group G_s. Note that by adding such a conjugating element, we may increase the rank of the core of the corresponding subgroup by at most 1, but we necessarily reduce the number of terminating factors by at least 1.

At this point we look at the core, $MGCore(G_s, MGRes_3)$, obtained by our procedure for the construction of a core resolution. By construction

$$rk(MGCore(G_s, MGRes_3)) + factor(MGCore(G_s, MGRes_3)) \\ \leq rk(MGRes_1) + factor(MGRes_1),$$

and if the inequality is strict the theorem follows. If

$$rk(MGCore(G_s, MGRes_3)) + factor(MGCore(G_s, MGRes_3)) \\ = rk(MGRes_1) + factor(MGRes_1),$$

then the core, $MGCore(G_s, MGRes_3)$, is just the induced resolution,

$$IndRes(G_s, MGRes_3).$$

As in case there are no surviving surfaces, we replace the part of the induced resolution (which is the subresolution induced by the free factor that is the free product of the (free) factor that is dropped in the resolution $MGRes_1$ with the free factor generated by the new elements we have added to get the subgroup G_s) by the image of that factor in $MGRes_3$. The modified resolution is a firm geometric subresolution of $MGRes_3$ that contains the subgroup G_s, hence contains the subgroup $< v, p, a >$, since the previously constructed core, $MGCore(G_s, MGRes_3)$, is a firm geometric subresolution, and its Kurosh rank is identical to the Kurosh rank of the original resolution, $MGRes_1$. Hence, we set the modified resolution to be the core resolution,

$$MGCore(< v, p, a >, MGRes_3).$$

Furthermore, by our analysis of taut homomorphisms of maximal rank,

$$Cmplx(MGCore(G_s, MGRes_3)) < Cmplx(MGRes_1)$$

and the theorem follows. ∎

Given a one-level multi-graded resolution associated with a variety, Theorem 4.13 bounds the complexity of the core of a multi-graded resolution associated with a Diophantine set contained in the given variety, assuming the resolution associated with the Diophantine set is compatible with the resolution of the variety. In a similar way, given a Diophantine set, a multi-graded resolution associated with it, and a core of that multi-graded resolution, it is possible to use that core to bound the complexity of the core of a multi-graded resolution associated with a Diophantine set that is contained in the given Diophantine set, assuming the multi-graded resolution associated with the new (smaller) Diophantine set is compatible with the multi-graded resolution associated with the given Diophantine set.

THEOREM 4.14: *Let*

$$MGRes_1(v, R_1, \ldots, R_m, P, a) \quad \text{and} \quad MGRes_2(t, v, L_1, \ldots, L_n, P, a)$$

be multi-graded resolutions that satisfy the assumptions of Theorem 4.13 and let $< y, a >$ *be a subgroup of the limit group* $Rlim(v, P, a)$. *Let* $MGCore(< y, a >, MGRes_1)$ *be a given multi-graded core resolution of the subgroup* $< y, a >$ *in the multi-graded resolution* $MGRes_1$.

Then the multi-graded resolution $MGRes_2(t, v, L_1, \ldots, L_n, P, a)$ *can be modified to a well-separated multi-graded resolution*

$$MGRes_3(u, t, v, L_1, \ldots, L_n, P, a):$$

- (i) *The Diophantine sets associated with the completions of* $MGRes_2$ *and* $MGRes_3$ *are identical.*
- (ii) *Let* (y, a) *be a specialization of the limit group* $Rlim(y, a)$ *which can be extended to a specialization that factors and is taut with respect to the resolution* $MGRes_1$. *Suppose that* (y, a) *can be extended to a specialization that factors and is taut with respect to the resolution* $MGRes_2$. *Then* (y, a) *can be extended to a specialization that factors and is taut with respect to the resolution* $MGRes_3$, *i.e., the part of the modular block associated with the resolution,* $MGRes_2$, *that projects to the projection of the modular block associated with the resolution,* $MGRes_1$, *can be "lifted" to the modular block associated with* $MGRes_3$.

Furthermore, there exists a core for $MGRes_3$, $MGCore(< y, a >, MGRes_3)$, *produced by modifying the above procedure for the construction of a (multi-graded) core resolution, that has the following properties:*

- (iii) *The complexity of the multi-graded core is bounded by the complexity of the given core of the multi-graded resolution* $MGRes_1$*:*

$$Cmplx(MGCore(< y, a >, MGRes_3)) \leq Cmplx(MGCore(< y, a >, MGRes_1)).$$

(iv) *If these complexities are equal, then the structure of the core* $MGCore(<y,a>, MGRes_3)$ *is identical to the structure of the given core of the multi-graded resolution* $MGRes_1$,

$$MGCore(<y,a>, MGRes_1)$$

(i.e., the core has one level, and the abelian decomposition associated with this one level has the same structure as the abelian decomposition associated with the given core of $MGRes_1$, $MGCore(<y,a>, MGRes_1)$*).*

Proof: Let G_1 be the image of the subgroup corresponding to the core, $MGCore(<v,a>, MGRes_1)$, in the group $<t,v,P,a>$. The argument used to prove Theorem 4.13 applied to the core, $MGCore(G_1, MGRes_2)$, proves Theorem 4.14. ∎

Given a one-level multi-graded resolution associated with a Diophantine set, and a core associated with the given resolution, Theorem 4.14 bounds the complexity of the core of a multi-graded resolution associated with a Diophantine set contained in the given Diophantine set, assuming the resolution associated with the new (smaller) Diophantine set is compatible with the resolution associated with the given Diophantine set.

In analyzing Diophantine sets, we will need a bound not only on the complexity of the core associated with an entire resolution, but also on the complexity of a core associated with a resolution composed from some successive levels in a compatible ambient resolution.

THEOREM 4.15: *Let*

$$MGRes_1(v, R_1, \ldots, R_m, P, a) \quad \text{and} \quad MGRes_2(t, v, L_1, \ldots, L_n, P, a)$$

be well-separated completed multi-graded resolutions that satisfy the assumptions of Theorem 4.13, and let $<y,a> < Rlim(v,P,a)$. *Suppose that the multi-graded resolution* $MGRes_2(t, v, L_1, \ldots, L_n, P, a)$ *has* ℓ *levels, let* $1 \leq i_1 \leq i_2 \leq \ell$, *and let* $MGRes_3(t, v, L_1, \ldots, L_n, P, a)$ *be the multi-graded resolution composed from levels* $i_1, \ldots, i_2$ *of the multi-graded resolution*

$$MGRes_2(t, v, L_1, \ldots, L_n, P, a).$$

Then the multi-graded resolution $MGRes_3(t, v, L_1, \ldots, L_n, P, a)$ *can be modified to a well-separated multi-graded resolution*

$$MGRes_4(u, t, v, L_1, \ldots, L_n, P, a),$$

so that:

(i) $MGRes_3$ *and* $MGRes_4$ *satisfy properties (i) and (ii) of Theorem 4.14.*

(ii) *If* $MGRes_3$ *contains a single quotient map, then* $MGRes_4$ *is identical to* $MGRes_3$.

Furthermore, there exists a core, $MGCore(<y,a>, MGRes_4)$*, that has the following properties:*

(iii) *the complexity of the multi-graded core is bounded by the complexity of the corresponding core of the multi-graded resolution* $MGRes_1$*:*

$$Cmplx(MGCore(<y,a>, MGRes_4)) \\ \leq Cmplx(MGCore(<y,a>, MGRes_1)).$$

(iv) *If these complexities are equal, then the structure of the core,* $MGCore(<y,a>, MGRes_4)$*, is identical to the structure of the corresponding core of the multi-graded resolution* $MGRes_1$*,*

$$MGCore(<y,a>, MGRes_1).$$

(v) *If the complexities are equal, then the (ambient) multi-graded resolution*

$$MGRes_2(t,v,L_1,\dots,L_n,P,a)$$

can be modified to a well-separated multi-graded resolution

$$MGRes_5(u,t,v,L_1,\dots,L_n,P,a)$$

so that $MGRes_2$ *and* $MGRes_5$ *satisfy properties (i) and (ii) of Theorem 4.14, and for which there exists a core,* $MGCore(<y,a>, MGRes_5)$*, that has the same structure as the corresponding core of the multi-graded resolution* $MGRes_1$*,* $MGCore(<y,a>, MGRes_1)$*.*

Proof: Parts (i)–(iv) follow by the argument used to prove Theorem 4.14. Suppose that the complexities, $Cmplx(MGCore(<y,a>, MGRes_4))$ and $Cmplx(MGCore(<y,a>, MGRes_1))$, are equal. To prove part (v), suppose that using the construction employed in the proofs of Theorems 4.13 and 4.14, the multi-graded resolution $MGRes_2(t,v,L_1,\dots,L_n,P,a)$ can be modified to a well-separated multi-graded resolution $MGRes_5(u,t,v,L_1,\dots,L_n,P,a)$ that satisfies parts (i) and (ii) in Theorem 4.14, and

$$Cmplx(MGCore(<y,a>, MGRes_5)) \\ < Cmplx(MGCore(<y,a>, MGRes_1)).$$

If

$$rk(MGCore(<y,a>, MGRes_5)) + factor(MGCore(<y,a>, MGRes_5))$$
$$< rk(MGCore(<y,a>, MGRes_1)) + factor(MGCore(<y,a>, MGRes_1)),$$

then by construction, the multi-graded resolution $MGRes_5$ is identical to the multi-graded resolution $MGRes_2$, so

$$rk(MGCore(<y,a>, MGRes_3)) + factor(MGCore(<y,a>, MGRes_3))$$
$$\leq rk(MGCore(<y,a>, MGRes_2)) + factor(MGCore(<y,a>, MGRes_2))$$
$$< rk(MGCore(<y,a>, MGRes_1)) + factor(MGCore(<y,a>, MGRes_1))$$

and part (v) follows in this case. Hence, for the rest of the argument we may assume

$$rk(MGCore(<y,a>, MGRes_5)) + factor(MGCore(<y,a>, MGRes_5))$$
$$= rk(MGCore(<y,a>, MGRes_1)) + factor(MGCore(<y,a>, MGRes_1)).$$

Since $MGRes_3$ is composed from a "block" of consecutive levels of the multi-graded resolution $MGRes_2$, every surviving QH vertex group in the multi-graded abelian decomposition associated with the core

$$MGCore(<y,a>, MGRes_1)$$

with respect to the multi-graded resolution $MGRes_3$ is also a surviving QH vertex group with respect to the multi-graded resolution $MGRes_2$. Therefore

$$\begin{aligned} Cmplx(MGCore(<y,a>, MGRes_4)) & \\ &\leq Cmplx(MGCore(<y,a>, MGRes_5)) \\ &< Cmplx(MGCore(<y,a>, MGRes_1)) \end{aligned}$$

and part (v) of the theorem follows. ∎

If we restrict Theorem 4.15 to a single level of the multi-graded resolution $MGRes_2$, we are able to bound the complexity of the core of each of the multi-graded abelian decompositions associated with the various levels of a multi-graded resolution in terms of the complexity of the core of $MGRes_1$.

COROLLARY 4.16: *Let*

$$MGRes_1(v, R_1, \dots, R_m, P, a) \quad \text{and} \quad MGRes_2(t, v, L_1, \dots, L_n, P, a)$$

be well-separated completed multi-graded resolutions that satisfy the assumptions of Theorem 4.13 and let $<y,a>< Rlim(v,P,a)$. *Let* $\Theta_1,\dots,\Theta_\ell$ *be the multi-graded abelian decompositions associated with the various levels of the multi-graded resolution* $MGRes_2(t,v,L_1,\dots,L_n,P,a)$. *Then either:*

(i) *The multi-graded resolution* $MGRes_2(t,v,L_1,\dots,L_n,P,a)$ *can be modified to a well-separated multi-graded resolution*

$$MGRes_3(u,t,v,L_1,\dots,L_n,P,a),$$

so that $MGRes_2$ *and* $MGRes_3$ *satisfy properties (i) and (ii) in Theorem 4.14, and for which there exists a core,* $MGCore(<y,a>,MGRes_3)$, *that has the same structure as the core of the multi-graded resolution* $MGRes_1$, $MGCore(<y,a>,MGRes_1)$.

(ii) *For every abelian decomposition,* Θ_i, *associated with a level of the multi-graded resolution,* $MGRes_2(t,v,L_1,\dots,L_n,P,a)$, *the complexity of the core associated with that level,* $MGCore(<y,a>,\Theta_i)$, *satisfies*

$$Cmplx(MGCore(<y,a>,\Theta_i)) < Cmplx(MGCore(<y,a>,MGRes_1)).$$

As in the iterative procedure used for the construction of the tree of stratified sets, *resolutions of maximal complexity* play an essential role in the general sieve procedure.

Definition 4.17: Let

$$MGRes_1(v,R_1,\dots,R_m,P,a) \quad \text{and} \quad MGRes_2(t,v,L_1,\dots,L_n,P,a)$$

be well-separated completed multi-graded resolutions that satisfy the assumptions of Theorem 4.13 and let $<y,a>< Rlim(v,P,a)$. We say that the multi-graded resolution $MGRes_2(t,v,L_1,\dots,L_n,P,a)$ is a *resolution of maximal complexity*, if the multi-graded resolution $MGRes_2(t,v,L_1,\dots,L_n,P,a)$ can be modified to a well-separated multi-graded resolution

$$MGRes_3(u,t,v,L_1,\dots,L_n,P,a),$$

so that $MGRes_2$ and $MGRes_3$ satisfy properties (i) and (ii) in Theorem 4.14, and for which there exists a core, $MGCore(<y,a>,MGRes_3)$, that has the same structure as the given core of the multi-graded resolution $MGRes_1$, $MGCore(<y,a>,MGRes_1)$.

In case the core $MGCore(<y,a>,MGRes_3)$ has the same structure as the core $MGCore(<y,a>,MGRes_1)$, we call the part of the core that includes the edge groups and the abelian and QH vertex groups, the **formed part** of the (maximal complexity) core resolution.

Maximal core resolutions and their properties play an essential role in the sieve procedure presented in the next paper in this sequel. One of their main properties is the fact that only maximal core resolutions can "cover" maximal core resolutions.

THEOREM 4.18: *Let $MGRes_1(v,R_1,\dots,R_m,P,a)$ be a well-separated completed multi-graded resolution containing a single level with corresponding limit group $Rlim_1(v,P,a)$, so that the subgroups $R_1,\dots,R_m,P$ are the non-abelian, non-QH vertex groups in the multi-graded abelian decomposition associated with*

$$MGRes_1(v,R_1,\dots,R_m,P,a),$$

and let $<y,a>< Rlim_1(v,P,a)$. Let

$$Rlim_2(t,v,M_1,\dots,M_d,P,a) \quad \text{and} \quad Rlim_3(w,t,v,L_1,\dots,L_n,P,a)$$

be limit groups for which:

(1) *There exists an embedding μ: $Rlim_1(v,P,a) \to Rlim_2(t,v,P,a)$ that maps each of the subgroups $R_1,\dots,R_m$ into a conjugate of one of the subgroups M_j.*
(2) *There exists a homomorphism ν: $Rlim_2(t,v,P,a) \to Rlim_3(w,t,v,P,a)$ that embeds naturally the subgroup $<v,P,a>$, and maps each of the subgroups $M_1,\dots,M_d$ into a conjugate of one of the subgroups L_j.*

Let $MGRes_3(w,t,v,L_1,\dots,L_n,P,a)$ be a multi-graded resolution of the limit group $Rlim_3(w,t,v,L_1,\dots,L_n,P,a)$, so that the resolutions $MGRes_1(v,R_1,\dots,R_m,P,a)$ and $MGRes_3(w,t,v,L_1,\dots,L_n,P,a)$ satisfy the assumptions of Theorem 4.14. Then either:

(i) *The resolution $MGRes_3(w,t,v,L_1,\dots,L_n,P,a)$ can be modified to a resolution $MGRes_4(u,w,t,v,L_1,\dots,L_n,P,a)$, so that $MGRes_3$ and $MGRes_4$ satisfy properties (i) and (ii) of Theorem 4.14. Furthermore, there exists a core, $MGCore(<y,a>,MGRes_4)$, that has smaller complexity than the (given) core, $MGCore(<y,a>,MGRes_1)$.*
(ii) *The multi-graded resolution $MGRes_3$ is of maximal complexity. There exist (multi-graded) closures of $MGRes_3$:*

$$Cl_1(MGRes_3),\dots,Cl_c(MGRes_3),$$

that are all not of maximal complexity (i.e., satisfy part (i) of the theorem), and resolutions in the taut multi-graded Makanin–Razborov diagram of $Rlim_2(t,v,M_1,\ldots,M_d,P,a)$,

$$MGRes_1^2(t,v,M_1,\ldots,M_d,P,a),\ldots,MGRes_e^2(t,v,M_1,\ldots,M_d,P,a),$$

that are all compatible with the resolution $MGRes_1(v,R_1,\ldots,R_m,P,a)$ *(i.e., satisfy the assumptions of Theorem 4.14), and all are of maximal complexity, so that:*

(1) *The Diophantine set (of specializations of the subgroup* $< y,a >$*) associated with the completion of* $MGRes_3$ *is contained in the union of the Diophantine sets associated with the completions of the closures,* $Cl_1(MGRes_3),\ldots,Cl_c(MGRes_3)$*, and the Diophantine sets associated with the maximal complexity resolutions,* $MGRes_1^2,\ldots,MGRes_e^2$.

(2) *Let* (y_0,a) *be a specialization of the limit group* $Rlim(y,a)$ *that can be extended to a specialization that factors and is taut with respect to the resolution* $MGRes_1$*. Suppose that* (y_0,a) *can be extended to a specialization that factors and is taut with respect to the resolution* $MGRes_3$*. Then either* (y_0,a) *can be extended to a specialization that factors and is taut with respect to one of the given closures of the resolution* $MGRes_3$*,* $Cl_1(MGRes_3),\ldots,Cl_c(MGRes_3)$*, or* (y_0,a) *is contained in at least one of the Diophantine sets associated with the maximal complexity resolutions,* $MGRes_1^2,\ldots,MGRes_e^2$.

Proof: If $MGRes_3$ is not a resolution of maximal complexity, part (i) of the theorem holds, hence we may assume that $MGRes_3$ is a resolution of maximal complexity. $MGRes_3(w,t,v,L_1,\ldots,L_n,P,a)$ is a multi-graded resolution of the limit group $Rlim_3(w,t,v,P,a)$, and there is a homomorphism $\nu\colon Rlim_2(t,v,P,a)\to Rlim_3(w,t,v,P,a)$ that maps the subgroups $M_1,\ldots,M_d$ into conjugates of the subgroups $L_1,\ldots,L_n$. Hence, every specialization of the subgroup $Rlim_2(t,v,P,a)$ which can be extended to a specialization that factors through the multi-graded resolution $MGRes_3$, and for which the corresponding specialization of the subgroup $< v,P,a >$ factors and is taut with respect to the multi-graded resolution $MGRes_1$, factors and is taut with respect to at least one of the resolutions in the taut multi-graded Makanin–Razborov diagram of the limit group $Rlim_2(t,v,P,a)$ that are compatible with the resolution $MGRes_1(v,R_1,\ldots,R_m,P,a)$.

We look at the collection of all the test sequences that factor through the resolution $MGRes_3$, and for which the corresponding specializations of the subgroup $< v, P, a >$ factor through and are taut with respect to the resolution $MGRes_1$. The specializations of the subgroup $Rlim_2(t, v, P, a)$ corresponding to the specializations of each such test sequence factor and are taut with respect to at least one of the resolutions in the multi-graded taut Makanin–Razborov diagram of $Rlim_2(t, v, M_1, \ldots, M_d, P, a)$ that are compatible with the multi-graded resolution $MGRes_1$. Hence, by passing to appropriate subsequences, we can assume that the specializations of the subgroup $Rlim_2(t, v, P, a)$ corresponding to such a test sequence of $Rlim_3(w, t, v, P, a)$ factor and are taut with respect to a fixed resolution in the multi-graded taut Makanin–Razborov diagram of $Rlim_2(t, v, M_1, \ldots, M_d, P, a)$, and this resolution is compatible with $MGRes_1$.

With each test sequence of $MGRes_3$ for which the corresponding specializations factor and are taut with respect to $MGRes_1$, and for which the corresponding specializations of $Rlim_2$ factor and are taut with respect to a (fixed) multi-graded resolution $MGRes_i^2$, which is a resolution in the taut multi-graded Makanin–Razborov diagram of $Rlim_2(t, v, M_1, \ldots, M_d, P, a)$, we apply the techniques used for the construction of formal solutions (section 1 of [Se2]), and associate (canonically) a finite collection of closures of the multi-graded resolution $MGRes_3$, and with each such closure we further associate a map from the completion of $MGRes_i^2$ into this closure. We further apply the techniques used for the construction of graded and multi-graded formal limit groups, and associate with the entire collection of such test sequences of $MGRes_3$ a finite collection of closures of the multi-graded resolution $MGRes_3$ that by construction forms a covering closure, and with each such closure we further associate a map from the completion of one of the corresponding resolutions that appear in the taut multi-graded Makanin–Razborov diagram of $Rlim_2(t, v, M_1, \ldots, M_d, P, a)$, $MGRes_i^2$, into this closure.

If all the resolutions $MGRes_i^2$ that are mapped into the closures of $MGRes_3$ are of maximal complexity, part (ii) of the theorem follows. Hence, we may assume that at least one of the multi-graded resolutions $MGRes_i^2$ is not of maximal complexity, i.e.,

$$Cmplx(MGCore(< y, a >, MGRes_i^2)) < Cmplx(MGCore(< y, a >, MGRes_1)).$$

Suppose

$$rk(MGCore(<y,a>, MGRes_i^2)) + factor(MGCore(<y,a>, MGRes_i^2))$$
$$= rk(MGCore(<y,a>, MGRes_1)) + factor(MGCore(<y,a>, MGRes_1)).$$

Since $MGRes_3$ is of maximal complexity, every QH vertex group in the abelian decomposition associated with the core, $MGCore(<y,a>, MGRes_i^2)$, is a surviving surface with respect to the resolution $MGRes_3$, hence it must be a surviving surface with respect to the resolution $MGRes_i^2$. So the multi-graded resolution $MGRes_i^2$ has to be of maximal complexity as well, a contradiction. Therefore, we may assume

$$rk(MGCore(<y,a>, MGRes_i^2)) + factor(MGCore(<y,a>, MGRes_i^2))$$
$$< rk(MGCore(<y,a>, MGRes_1)) + factor(MGCore(<y,a>, MGRes_1)).$$

Let G_i be the image of the subgroup associated with the core of $MGRes_i^2$ in a corresponding closure of $MGRes_3$. At this point, we apply our iterative procedure for the construction of the core resolution, to construct the multi-graded core, $MGCore(G_i, Cl(MGRes_3))$, where $Cl(MGRes_3)$ is the corresponding (multi-graded) closure of $MGRes_3$. If

$$rk(MGCore(G_i, Cl(MGRes_3))) + factor(MGCore(G_i, Cl(MGRes_3)))$$
$$< rk(MGCore(<y,a>, MGRes_1)) + factor(MGCore(<y,a>, MGRes_1)),$$

then since $<y,a> < G_i$, we may associate with the core,

$$MGCore(G_i, Cl(MGRes_3)),$$

a corresponding core,

$$MGCore(<y,a>, cl(MGRes_3)),$$

for which

$$rk(MGCore(<y,a>, cl(MGRes_3)))$$
$$+ factor(MGCore(<y,a>, cl(MGRes_3)))$$
$$< rk(MGCore(<y,a>, MGRes_1)) + factor(MGCore(<y,a>, MGRes_1))$$

and part (i) of the theorem follows for that closure of $MGRes_3$.

Hence, for the rest of the argument we may assume that

$$rk(MGCore(G_i, Cl(MGRes_3))) + factor(MGCore(G_i, Cl(MGRes_3)))$$
$$\geq rk(MGCore(<y,a>, MGRes_1)) + factor(MGCore(<y,a>, MGRes_1)).$$

In this case, for every test sequence that factors through the given closure $Cl(MGRes_3)$ and is firm with respect to the subgroup G_i, the corresponding specializations of the limit group $Rlim_2(t,v,P,a)$ cannot factor through the multi-graded resolution $MGRes_i^2$. Hence, they must factor through another resolution in the taut multi-graded Makanin–Razborov diagram of $Rlim_2(t,v,P,a)$ that is compatible with the multi-graded resolution $MGRes_1$. Therefore, we may omit the resolution $MGRes_i^2$, and the corresponding map from its completion into the closure $Cl(MGRes_3)$, from our list of such closures and maps. Repeating these alterations for all the multi-graded resolutions $MGRes_i^2$ in the taut multi-graded Makanin–Razborov diagram of $Rlim_2(t,v,P,a)$ that appear in our list, we either construct a core, $MGCore(<y,a>, Cl(MGRes_3))$, which is not of maximal complexity, or we are left with resolutions $MGRes_i^2$ which are all of maximal complexity, so part (ii) of the theorem holds. ∎

Remark: Note that in analyzing the set of specializations that factor and are taut with respect to a completed multi-graded resolution, $MGRes$, it is possible that the restrictions of certain specializations to some levels of the multi-graded resolution, $MGRes$, cannot be obtained from the restrictions of the same specializations to the successive (lower) levels of the resolution, using the associated (multi-graded) modular automorphisms. This phenomenon arises since there are specializations of (graded, multi-graded) solid limit groups that do not factor through any flexible quotient of the solid limit group (after applying an element of the associated modular groups), but still they are not strictly solid specializations of the solid limit group, hence they are assumed to factor through a resolution associated with at least one of the flexible quotients of the given solid limit group (see definitions 1.4 and 1.5 in [Se3]).

However, the Kurosh rank of the restriction of such specialization to the subgroup associated with the core of such resolution $MGRes$ is at most the Kurosh rank of the core. This is needed in order to allow us to drop resolutions, $MGRes_i^2$, that are not of maximal complexity and their map into closures of the multi-graded resolution $MGRes_3$ (while proving Theorem 4.18), in case the Kurosh rank of the image of the subgroup associated with the core of $MGRes_i^2$ in $MGRes_3$ exceeds the Kurosh rank of the core in $MGRes_i^2$.

In addition to the natural "covering" property of maximal core resolutions, presented in Theorem 4.18, we need a correspondence between maximal core resolutions, and maximal core resolutions containing two parts, the top obtained by enlarging the parameter subgroups to include the formed part of the core resolution, and the bottom obtained using the original parameter subgroups.

THEOREM 4.19: *Let*

$$MGRes_1(v, R_1, \dots, R_m, P, a) \quad \text{and} \quad MGRes_2(t, v, L_1, \dots, L_n, P, a)$$

be multi-graded resolutions that satisfy the assumptions of Theorem 4.13, let $<y, a> < Rlim_1(v, P, a)$, *and suppose that* $MGRes_2(t, v, L_1, \dots, L_n, P, a)$ *is of maximal complexity.*

Then either there exists a multi-graded core, $MGCore(<y, a>, MGRes_2)$, *that has strictly smaller complexity than the core,* $MGCore(<y, a>, MGRes_1)$, *or there exist a collection of closures of the resolution* $MGRes_2$ *with strictly smaller complexity cores,* $Cl_1(MGRes_2), \dots, Cl_c(MGRes_2)$, *and a collection of multi-graded resolutions*

$$MGRes_1^2(t, v, L_1, \dots, L_n, P, a), \dots, MGRes_e^2(t, v, L_1, \dots, L_n, P, a)$$

of the limit group $Rlim_2(t, v, L_1, \dots, L_n, P, a)$ *that satisfy the following properties:*

(1) *Each resolution* $MGRes_j^2$ *is compatible with the resolution* $MGRes_1$ *(i.e., the two resolutions satisfy the assumptions of Theorem 4.13).*
(2) *Each resolution* $MGRes_j^2$ *is of maximal complexity.*
(3) *Each resolution* $MGRes_j^2$ *is composed from two parts: the top,* $TMGRes_j^2$, *being a resolution in the taut multi-graded Makanin–Razborov diagram of the limit group* $Rlim_2(t, v, P, a)$ *with respect to the parameter subgroups* $L_1, \dots, L_n, P$ *and the formed part of the core resolution,* $MGCore(<y, a>, MGRes_1)$, *and the bottom part being a one-level resolution of the terminal limit group of the top* $TMGRes_j^2$ *that has the same structure as the formed part of the core,* $MGCore(<y, a>, MGRes_1)$.
(4) *The Diophantine set (of specializations of the subgroup* $<y, a>$*) associated with the completion of* $MGRes_2$ *is contained in the union of the Diophantine sets associated with the completions of the closures,* $Cl_1(MGRes_2), \dots, Cl_c(MGRes_2)$, *and the Diophantine sets associated with the maximal complexity resolutions,* $MGRes_1^2, \dots, MGRes_e^2$.
(5) *Let* (y_0, a) *be a specialization of the limit group* $Rlim(y, a)$ *which can be extended to a specialization that factors and is taut with respect to the resolution* $MGRes_1$. *Suppose that* (y_0, a) *can be extended to a specialization which factors and is taut with respect to the resolution* $MGRes_2$. *Then either* (y_0, a) *can be extended to a specialization that factors and is taut with respect to one of the given closures of the resolution* $MGRes_2$, $Cl_1(MGRes_2), \dots, Cl_c(MGRes_2)$, *or it is contained in the Diophantine*

sets associated with at least one of the maximal complexity resolutions, $MGRes_1^2, \dots, MGRes_e^2$.

Proof: We start with the collection of multi-graded resolutions of the limit group $Rlim_2(t,v,L_1,\dots,L_n,P,a)$ that are composed from two parts: the top is a resolution in the multi-graded taut Makanin–Razborov diagram of $Rlim_2(t,v,P,a)$ with respect to the subgroups $L_1,\dots,L_n,P$ and the formed part of the core, $MGCore(<y,a>, MGRes_1)$, that we denote $TMGRes$. The bottom is a resolution in the taut multi-graded Makanin–Razborov diagram of the terminal rigid or solid (multi-graded) limit group of $TMGRes$ with respect to the subgroups $L_1,\dots,L_n,P$, that we denote $BMGRes$.

By Theorem 4.18, either there exists a core, $MGCore(<y,a>, MGRes_2)$, which has strictly smaller complexity than the core,

$$MGCore(<y,a>, MGRes_1),$$

or there exists a collection of closures of $MGRes_2$ with strictly smaller complexity cores, $Cl_1(MGRes_2),\dots,Cl_c(MGRes_2)$, and a collection of multi-graded resolutions,

$$MGRes_1^2(t,v,L_1,\dots,L_n,P,a),\dots,MGRes_e^2(t,v,L_1,\dots,L_n,P,a),$$

from the list described above, that are all of maximal complexity, compatible with the resolution $MGRes_1$ (i.e., the resolutions satisfy the assumptions of Theorem 4.13), and properties (4) and (5) hold for the union of the collections of closures of $MGRes_2$, and the maximal complexity resolutions $MGRes_j^2$.

Since each of the multi-graded resolutions, $MGRes_j^2$, is of maximal complexity, we can modify the bottom parts of each of these resolutions, $BMGRes_j^2$, as we did in analyzing maximal complexity resolutions along the proof of Theorem 4.13, to be a single-level (multi-graded) resolution that has the same structure as the formed part of the core, $MGCore(<y,a>, MGRes_1)$, and the theorem follows. ∎

Given a well-separated resolution $Res(t,v,a)$ and a subgroup $<v,a>$ of its corresponding limit group $Rlim(t,v,a)$, we presented an iterative procedure for the construction of a core resolution, $Core(<v,a>, Res(t,v,a))$, and used the algorithm to prove some basic properties of core resolutions, which play an essential role in the general sieve procedure presented in the next paper in the sequel.

The core resolution, $Core(< v,a >, Res(t,v,a))$, is embedded in the (completed) ambient resolution $Res(t,v,a)$. However, its terminal free group is not embedded "discretely" into the resolution $Res(t,v,a)$, i.e., the image of the terminal free group of a core resolution can intersect a QH vertex group in the ambient resolution, in a subgroup of finite index, or it can inherit a cyclic decomposition from an abelian vertex group in the ambient resolution. To guarantee the termination of the sieve procedure, we need the core resolutions we study to be "comparable" (metrically) with their embeddings in the ambient resolutions. Hence, we need to prevent "indiscrete" embeddings of the terminal free group of the core resolutions we study. To achieve that we present *penetrated core resolutions*.

Definition 4.20: Let $Res(t,v,a)$ be a well-separated (completed) resolution, and suppose we fix an order on the various QH and abelian vertex groups in each of the levels of the resolution $Res(t,v,a)$.

Let $< v,a >$ be a subgroup of $Rlim(t,v,a)$, the limit group associated with the resolution $Res(t,v,a)$. We say that a core resolution,

$$Core(< v,a >, Res(t,v,a)),$$

is a **penetrated core resolution**, if the abelian decompositions induced by the core from its embedding into the ambient resolution are identical to the abelian decompositions associated with it, i.e., the embedding of the core resolution into the ambient resolution does not contain a finite index subgroup of a QH vertex group which is not a QH vertex group in the core itself, and every abelian decomposition inherited by the core from its embedding into the ambient resolution is the natural image of (parts of) abelian decompositions associated with the core itself.

In short, we say that a core resolution is a *penetarted core resolution* if the core resolution is identical to the resolution induced by its image (subgroup) in the ambient resolution.

The algorithm for the construction of a core resolution presented in the beginning of this section actually constructs penetrated core resolutions (the resolution $IRes_{sec}(u,v,a)$ constructed by the procedure), since it is based on the iterative procedure for the construction of the induced resolution. However, its final output, and the modification of this algorithm used for the construction of core resolutions that satisfy the inequalities of Theorems 4.13–4.19, does not give penetrated core resolutions in general.

Along the sieve procedure, we use penetrated core resolutions only in case the (multi-graded) resolutions in question are of maximal complexity. Hence, for the purposes of the sieve procedure, we present a process that extends a core resolution to a penetrated core resolution, in case the resolution in question is of maximal complexity (Definition 4.17).

Let $MGRes_1(v, R_1, \ldots, R_m, P, a)$ be a well-separated completed multi-graded resolution containing a single level, and let Λ_1 be the graph of groups with fundamental group $Rlim(v, R_1, \ldots, R_m, P, a)$ associated with the single level of $MGRes_1$, where $R_1, \ldots, R_m, P$ are the non-QH, non-abelian vertex groups in the graph of groups Λ_1. Let $< y, a >< Rlim(v, R_1, \ldots, R_m, P, a)$. Let $MGRes_2(t, v, L_1, \ldots, L_n, P, a)$ be the completion of a resolution of a limit group $Rlim(t, v, L_1, \ldots, L_n, P, a)$, so that there is an embedding ν: $Rlim(v, P, a) \to Rlim(t, v, L_1, \ldots, L_n, P, a)$ that maps each of the subgroups $R_1, \ldots, R_m$ into a conjugate of one of the subgroups L_j. Suppose that the resolutions $MGRes_1$ and $MGRes_2$ satisfy the assumptions of Theorem 4.13, that the resolution $MGRes_2$ is of maximal complexity (Definition 4.17), and that the resolution $MGRes_2$ is composed from two parts, the top, $TMGRes_2$, being a multi-graded resolution in the taut (multi-graded) Makanin–Razborov diagram of the limit group $Rlim(t, v, L_1, \ldots, L_n, P, a)$ with respect to the parameter subgroups $L_1, \ldots, L_n, P$ and the formed part of the core, $MGCore(< y, a >, MGRes_1)$, and the bottom being a one-level resolution that has the same structure as the formed part of the core, $MGCore(< y, a >, MGCore_1)$.

PROPOSITION 4.21: *Let $PenMGCore(< y, a >, MGRes_2)$ be the resolution induced by the subgroup associated with the core, $MGCore(< y, a >, MGRes_1)$, from the multi-graded resolution $MGRes_2$. Then:*

(i) *$PenMGCore(< y, a >, MGRes_2)$ is a penetrated core resolution.*

(ii) *Every QH vertex group that appears in*

$$PenMGCore(< y, a >, MGRes_2),$$

and not in the core, $MGCore(< y, a >, MGRes_2)$, is of minimal rank.

(iii) *The images of the subgroup $< y, a >$ in each of the limit groups associated with the various levels of the resolution $MGRes_2$ inherit abelian decompositions from the abelian decompositions associated with the various levels of $MGRes_2$. In particular, these abelian decompositions of the images of the subgroup $< y, a >$ give rise to (possibly trivial) free decompositions.*

Hence, the subgroup $< y, a >$ inherits a (possibly trivial) free decomposition, $H * F_1$ (where F_1 is a free factor) from the abelian decomposition *associated with the first level of $MGRes_2$. The image of the subgroup $H_1 << y,a >$ in the limit group associated with the second level of $MGRes_2$ inherits a (possibly trivial) free decomposition, $H_2 * F_2$ (where F_2 is a free factor), from the abelian decomposition associated with the second level of $MGRes_2$. Iteratively, the image of H_{i-1} in the limit group associated with the i-th level of $MGRes_2$ inherits a (possibly trivial) free decomposition, $H_i * F_i$ (where F_i is a free factor), from the abelian decomposition associated with the i-th level of $MGRes_2$. Let $J_i(y,a)$ be the subgroup $H_i * F_i * \cdots * F_1$. Then for every level i, except the bottom level, the natural maps τ_i: $Rlim(y,a) \to J_i(y,a)$ and η_i: $J_i(y,a) \to J_{i+1}(y,a)$ are isomorphisms.*

(iv) *No QH vertex group that appears in $PenMGCore(< y,a >, MGRes_2)$, and not in the core, $MGCore(< y,a >, MGRes_2)$, belongs to the bottom two levels of the penetrated core, $PenMGCore(< y,a >, MGRes_2)$.*

(v) *The subgroup associated with the m-th level of the resolution, $PenMGCore(< y,a >, MGRes_2)$, inherits a free decomposition from the abelian decompositions associated with all the levels of the ambient resolution $Res(t,v,a)$ that lie above the m-th level. Let Pen_m be one of the factors in this free decomposition of the limit group associated with the m-th level of $PenMGCore(< y,a >, MGRes_2)$. Then either there exists a (multi-graded) core, $MGCore(< y,a >, MGRes_2)$, that is not of maximal complexity, or there exists a test sequence of $Res(t,v,a)$ which is firm with respect to the penetrated core, $PenMGCore(< y,a >, MGRes_2)$, for which the sequence of specializations of the subgroup Pen_m corresponding to the given test sequence converge into a (possibly trivial) action of this subgroup on a real tree Y. The action of the group Pen_m on the real tree Y is either trivial or it is faithful and geometric, and it is composed from finitely many orbits of discrete and IET components. Furthermore, the free decomposition induced by the subgroup Pen_m from this action is precisely the free decomposition it induces from the abelian decomposition associated with the m-th level of the resolution $MGRes_2$.*

Proof: $MGCore(< y,a >, MGRes_2)$ is a core resolution, hence it is a firm geometric subresolution of the multi-graded resolution $MGRes_2$. The resolution, $PenMGCore(< y,a >, MGRes_2)$, is a resolution induced by the subgroup associated with the core, $MGCore(< y,a >, MGRes_2)$. Hence, it is a firm geometric subresolution of the multi-graded resolution $MGRes_2$. Since it is an induced resolution, the abelian decompositions associated with its various levels are induced from the abelian decompositions associated with the various levels of the multi-graded resolution $MGRes_2$. Therefore, $PenMGCore(<y,a>, MGRes_2)$ is a penetrated core resolution, and we get part (i) of the proposition.

Let M be the limit group associated with the formed part of the core $MGCore(<y,a>, MGRes_2)$. Since the penetrated core,

$$PenMGCore(<y,a>, MGRes_2),$$

is a firm subresolution, for every level i of $PenMGCore(<y,a>, MGRes_2)$, except the bottom two levels, the map η_i from the limit group associated with the i-th level to the $i+1$-th level of $PenMGCore(<y,a>, MGRes_2)$ is an isomorphism that fixes the subgroup M elementwise, and we get part (iii) of the proposition. In particular, every QH vertex group in an abelian decomposition associated with one of the levels of $PenMGCore(<y,a>, MGRes_2)$, except the two bottom levels, is necessarily of minimal rank (otherwise, the maps η_i are not isomorphisms), and we get part (ii) of the proposition.

To prove part (iv), note that if there exists a QH vertex group Q that appears in $PenMGCore(<y,a>, MGRes_2)$, and not in the core,

$$MGCore(<y,a>, MGRes_2),$$

and Q belongs to one of the bottom two levels of the penetrated core, $PenMGCore(<y,a>, MGRes_2)$, then necessarily

$$\begin{aligned} &rk(PenMGCore(<y,a>, MGRes_2)) \\ &\qquad + factor(PenMGCore(<y,a>, MGRes_2)) \\ &\leq rk(MGCore(<y,a>, MGRes_2)) + factor(MGCore(<y,a>, MGRes_2)) - 1, \end{aligned}$$

which clearly implies that the core, $MGCore(<y,a>, MGRes_2)$, is not a firm subresolution of the multi-graded resolution $MGRes_2$, a contradiction to our assumptions.

Part (v) follows, since the penetrated core is a firm subresolution by part (i), and by the construction of test sequences presented in the proof of Proposition 4.11 that guarantee that the action of each of the subgroups Pen_m on their corresponding real trees contain only discrete and IET components, and that the abelian decompositions obtained from these actions are precisely the abelian decompositions which the subgroups Pen_m inherit from the abelian decompositions associated with the corresponding levels of the ambient resolution $MGRes_2$. ∎

Finally, under the assumptions of Proposition 4.21, we call the resolution, $PenMGCore(< y, a >, MGRes_2)$, the penetrated core resolution of the subgroup $< y, a >$ in the maximal complexity resolution $MGRes_2$.

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