DIOPHANTINE GEOMETRY OVER GROUPS I: MAKANIN-RAZBOROV DIAGRAMS

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This paper is the first in a sequence on the structure of sets of solutions to systems of equations in a free group, projections of such sets, and the structure of elementary sets defined over a free group. In the first paper we present the (canonical) Makanin-Razborov diagram that encodes the set of solutions of a system of equations. We continue by studying parametric families of sets of solutions, and associate with such a family a canonical graded Makanin-Razborov diagram, that encodes the collection of Makanin-Razborov diagrams associated with the individual members in the parametric family.

Sets of solutions to equations defined over a free group have been studied extensively, mostly since Alfred Tarski presented his fundamental questions on the elementary theory of free groups around 1945. Considerable progress in the study of such sets of solutions was made by G. S. Makanin, who constructed an algorithm that decides if a system of equations defined over a free group has a solution [Ma1], and showed that the universal and positive theories of a free group are decidable [Ma2]. A. A. Razborov was able to give a description of the entire set of solutions to a system of equations defined over a free group [Ra1], a description that was further developed by O. Kharlampovich and A. Myasnikov [Kh-My].

A set of solutions to equations defined over a free group is clearly a discrete set, and all the previous techniques and methods that studied these sets are combinatorial in nature. Naturally, the structure of sets of solutions defined over a free group is very different than the structure of sets of solutions (varieties) to systems of equations defined over the complexes, reals or a number field. Still, perhaps surprisingly, concepts from complex algebraic geometry and from Diophantine geometry can be borrowed to study varieties defined over a free group.

In this paper we start a sequence that borrows concepts and techniques from geometric group theory, low dimensional topology, and Diophantine geometry to study the structure of varieties defined over a free group. Our techniques and point of view on the study of these varieties is rather different than any of the pre-existing techniques in this field, though, as one can expect, some of our preliminary results overlap with previously known ones. As we will see in this paper and mostly in the following ones, the techniques and concepts we use enable the study of the structure of varieties defined over a free group and their projections, and in particular, give us the possibility to answer some questions that seem to be essential in any attempt to understand the structure of elementary sentences and predicates defined over

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a free group. At this stage, at least as it seems to us, none of the pre-existing (combinatorial) techniques can approach the difficulties involved in tackling these questions.

The sequence of papers on Diophantine geometry over groups is organized as follows. In the first paper we associate a canonical Makanin-Razborov diagram with a given system of equations defined over a free group. This Makanin-Razborov diagram encodes the entire set of solutions to the system. Later on we study systems of equations with parameters. With each such system we associate a (canonical) graded Makanin-Razborov diagram, that encodes the Makanin-Razborov diagrams of the systems of equations associated with each specialization of the defining parameters. In the second paper of the sequence, we generalize Merzlyakov theorem on the existence of a $formal\ solution$ associated with a positive sentence [Me]. We first construct a formal solution to a general AE sentence which is known to be true over some variety, and then develop tools that enable us to analyze the collection of all such formal solutions. In the third paper, we use techniques developed in the first two, to further study exceptional solutions of parametric systems of equations, which are one of the inherent pathologies in the theory that must be understood in studying the first order theory of a free group. Perhaps the main result of this paper is the existence of a global bound (independent of the parameters specialization) on the number of families of exceptional solutions. In the fourth paper we study AE sentences, and present a terminating iterative procedure for validation of such sentences. This terminating iterative procedure is the basis for our analysis of elementary sets defined over a free group presented in the last two papers. In the fifth paper we analyze the Boolean Algebra of AE sets and show that this Boolean algebra is invariant under projections. This implies that every elementary set defined over a free group is in the Boolean algebra of AE sets. The sixth paper uses the results of the fifth one to study elementary sets defined over a free group. In particular, we answer affirmatively some of A. Tarski's problems on the elementary theory of a free group, and obtain a classification of the f.g. groups that are elementary equivalent to a non-abelian f.g. free group.

We start the first paper of the sequence by studying *limit groups* that are obtained from (Gromov) limits of sequences of homomorphisms. In section 2 we study the (canonical) abelian JSJ decomposition of these groups, which is shown to be non-trivial. In the third section we study the (canonical) cyclic JSJ decomposition of a limit group, and show that this cyclic decomposition can not be trivial as well. In the fourth section we use the canonical cyclic JSJ decomposition to associate an *analysis lattice* with a limit group, which, in particular, implies that limit groups are finitely presented, and that a f.g. group is a limit group if and only if it is an ω-residually free group (f.g. ω-residually free groups were also shown to be f.p. by Kharlampovich and Myasnikov in [Kh-My]). The fifth section associates a canonical Makanin-Razborov diagram with a limit group. The properties of this diagram are similar to the ones obtained by Razborov [Ra2], and Kharlampovich and Myasnikov [Kh-My]. However, for the structure theory we develop, and for analyzing elementary predicates, it seems essential to have its canonical properties (which can not be guaranteed by combinatorial approaches), and to be able to have a "close relation" between the algebraic structure of a limit group and some of its (canonical) quotients with the structure of the diagram and the modular groups associated with it.

In the sixth section we present a family of ω-residually free groups that is more geometric in nature. The family of *towers* we present, plays an essential role when we classify f.g. groups that are elementary equivalent to a non-abelian free group in the last paper in the sequence. In section 7 we use our study of limit groups to study residually free groups.

At this point we begin our study of systems of equations with parameters. In section 9 we present graded limit groups, and in section 10 we associate a canonical graded Makanin-Razborov diagram with a graded limit group. We view the set of solutions to a system of equations with parameters as a bundle, where the groups of parameters lies in the base of this bundle. The graded Makanin-Razborov diagram is then a (flat) "connection" that allows one to compare different fibers associated with different specializations of the defining parameters. Section 11 studies the singular locus of that bundle, which is equivalent to studying the degeneracies in the graded Makanin-Razborov diagram. Section 12 generalizes our study of graded limit groups and diagrams to the study of multi-graded ones. Graded and multi-graded limit groups and diagrams are the basic objects needed in our analysis of sentences and elementary sets defined over a free group.

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§1. Limit Groups

Following [Ra1] with a finite system of equations Φ over a free group $F_k = < a_1, \ldots, a_k >$ it is natural to associate a f.p. group $G(\Phi)$. If the system Φ is defined by the coefficients $a_1, \ldots, a_k$, the unknowns $x_1, \ldots, x_n$ and the equations:

$$\begin{gathered} w_1(a_1, \ldots, a_k, x_1, \ldots, x_n) = 1 \\ \vdots \\ w_s(a_1, \ldots, a_k, x_1, \ldots, x_n) = 1 \end{gathered}$$

we set the associated f.p. group $G(\Phi)$ to be:

$$G(\Phi) \; = < \; a_1, \ldots, a_k, x_1, \ldots, x_n \,|\, w_1, \ldots, w_s \; >$$

Clearly, every solution of the system Φ corresponds to a homomorphism $h : G(\Phi) \to F_k$ for which $h(a_i) = a_i$, and every such homomorphism corresponds to a solution of the system Φ. Therefore, the study of sets of solutions to systems of equations in a free group is equivalent to the study of all homomorphisms from a fixed f.p. group G into a free group F_k for which a given prescribed set of elements in G is mapped to a fixed basis of the free group F_k.

Throughout this paper, this is the point of view we adapt in order to study sets of solutions to systems of equations in a free group. Since the techniques we use and the structure theory we obtain are valid for f.g. groups and not only for f.p. ones, we set G to be a f.g. group $G =< g_1, \ldots, g_m >$, $\gamma_1, \ldots, \gamma_k \in G$ to be a prescribed set of elements in G, and F_k to be a free group with a fixed basis $F_k =< a_1, \dots, a_k >$. With these notation our main goal is to get a structure theory for understanding the set of all homomorphisms:

$$Hom((G, \{\gamma_i\}), (F_k, \{a_i\})) \ = \ \{ \, h \, | \, h : G \to F_k \, , \, h(\gamma_i) = a_i \, \}$$

which for brevity we will denote $Hom(G, F_k)$. We set X to be the Cayley graph of F_k with respect to its given basis $a_1, \dots, a_k$. Given any homomorphism $h \in Hom(G, F_k)$, G admits a natural action λ_h on X given by $\lambda_h(g, x) = h(g)(x)$ for every $g \in G$ and $x \in X$.

For presentation purposes, we start our analysis with the entire set of homomorphisms $\{ \, h \, | \, h : G \to F_k \, \}$, without restricting the image of a finite subset of elements of the f.g. group G in the free group F_k. As we will see in the sequel, the results we obtain for the entire (unrestricted) set of homomorphisms can be slightly modified in order to study the set of restricted homomorphisms.

Let $\{h_j\} \subset Hom(G, F_k)$ be a set of (unrestricted) homomorphisms from G to F_k, and suppose that the homomorphisms $\{h_j\}$ belong to distinct conjugacy classes (i.e., for every j_1, j_2, $1 \leq j_1 < j_2$, and every $f \in F_k$, $f h_{j_1} f^{-1} \neq h_{j_2}$). For each index j we fix an element $f_j \in F_k$ having "minimal displacement" under the action λ_{h_j} and set μ_j to be:

$$\mu_j = \max_{1 \leq u \leq m} d_X(id., f_j h_j(g_u) {f_j}^{-1}) = \min_{f \in F_k} \max_{1 \leq u \leq m} d_X(id., f h_j(g_u) f^{-1})$$

Since the homomorphisms in the sequence $\{h_j\} \subset Hom(G, F_k)$ are distinct, the sequence of stretching factors $\{\mu_j\}$ does not contain a bounded subsequence. We set $\{(X_j, x_j)\}_{j=1}^{\infty}$ to be the pointed metric spaces obtained by rescaling the metric on the Cayley graph of F_k, $(X, id.)$, by μ_j. (X_j, x_j) is endowed with a left isometric action of our f.g. group G via the homomorphisms $\tau_{f_j} \circ h_j$ where τ_{f_j} is the inner automorphism of F_k defined by f_j. This sequence of actions of G on the metric spaces $\{(X_j, x_j)\}_{j=1}^{\infty}$ allows us to obtain an action of G on a real tree by passing to a Gromov-Hausdorff limit.

Proposition 1.1 ([Pa], 2.3). *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of δ_j-hyperbolic spaces with $\delta_\infty = \underline{\lim}\, \delta_j = 0$. Let H be a countable group isometrically acting on X_j. Suppose there exists a base point x_j in X_j such that for every finite subset P of H, the sets of geodesics between the images of x_j under P form a sequence of totally bounded metric spaces. Then there is a subsequence converging in the Gromov topology to a δ_∞-hyperbolic space X_∞ endowed with a left isometric action of H.*

Our spaces $\{(X_j, x_j)\}_{j=1}^{\infty}$ endowed with the left isometric action of G, satisfy the assumptions of the proposition and they are all trees, so they are 0-hyperbolic, hence, X_∞ is a real tree endowed with an isometric action of G. By construction the action of G on the real tree X_∞ is non-trivial. Let $\{j_n\}_{n=1}^{\infty}$ be the subsequence for which $\{(X_{j_n}, x_{j_n})\}_{n=1}^{\infty}$ converge to the limit real tree X_∞ and let (Y, y_0) denote this (pointed) limit real tree. For convenience for the rest of this section we denote the homomorphism $f_{j_n} h_{j_n} {f_{j_n}}^{-1} : G \to F_k$ by h_n.

With the limit tree we obtained by using the Gromov-Hausdorff topology we associate natural algebraic objects, the *kernel of the action* of G on this (limit) real tree and the quotient of G by this kernel which we call the *limit group*. The first goal in our approach to the structure of sets of solutions to equations in a free group is understanding some of the basic properties of limit groups, and in particular to show they are f.p. groups and that every maximal abelian subgroup of them is a f.g. free abelian group.

Definition 1.2. *The kernel of the action of the group G on the limit tree Y is defined to be:*

$$K_\infty = \{g \in G \mid \forall y \in Y \ g(y) = y\}$$

Having the kernel of the action we define the limit group to be: $L_\infty = G/K_\infty$ and set $\eta : G \to L_\infty$ to be the natural quotient map.

The following simple facts on the kernel of the action and the limit group are important observations and will serve us throughout the rest of the paper.

Lemma 1.3. *With the notation of definition 1.2:*

(i) *L_∞ is a f.g. group.*

(ii) *If Y is isometric to a real line then the limit group L_∞ is f.g. free abelian.*

(iii) *If $g \in G$ stabilizes a tripod in Y then for all but at most finitely many n's $g \in ker(h_n)$ (recall that a tripod is a finite tree with 3 endpoints). In particular, if $g \in G$ stabilizes a tripod then $g \in K_\infty$.*

(iv) *Let $g \in G$ be an element which does not belong to K_∞. Then for all but at most finitely many n's $g \notin ker(h_n)$.*

(v) *L_∞ is torsion-free.*

(vi) *Let $[y_1, y_2] \subset [y_3, y_4]$ be a pair of non-degenerate segments of Y and assume the stabilizer of $[y_3, y_4]$ in L_∞, $stab([y_3, y_4])$, is non-trivial. Then $stab([y_3, y_4])$ is an abelian subgroup of L_∞ and:*

$$stab([y_1, y_2]) = stab([y_3, y_4])$$

Hence, the action of L_∞ on the real tree Y is stable.

Proof: L_∞ is a quotient of the f.g. group G, so it is f.g. as well. To prove (ii) note that if Y is isometric to a real line, then L_∞ is a f.g. subgroup of $Isom(R)$. If $g \in G$ preserves the orientation of Y and $g' \in G$ reverses it, then for large enough n, $h_n(g')h_n(g)h_n(g')^{-1} = h_n(g^{-1})$, and since the image of the homomorphisms h_n is a subgroup of the free group F_k, necessarily $h_n(g) = 1$ for all but finitely many n's, hence, $g \in K_\infty$. In addition, since ${g'}^2$ preserves the orientation of Y, $h_n({g'}^2) = 1$ for all but finitely many n's, so $h_n(g') = 1$ for all but finitely many n's and $g' \in K_\infty$. Therefore, the ambient group G preserves the orientation of Y, and since L_∞ is a f.g. subgroup of $Isom(R)$, L_∞ is f.g. free abelian group.

To prove (iii) let $T(A, B, C)$ be a tripod in Y and let N be the three valence vertex in that tripod. Let $g \in G$ fix $T(A, B, C)$ and let $(A_n, B_n, C_n) \subset X_n$ be a sequence of triples of points converging into the triple (A, B, C). Let $\ell = \min\{d_Y(A, N), d_Y(B, N), d_Y(C, N)\}$. From the convergence of the metric spaces $\{(X_n, x_n)\}_{n=1}^{\infty}$ to the real tree (Y, y_0) we get for large enough n:

$$\max\{d_{X_n}(A_n, h_n(g)(A_n)), d_{X_n}(B_n, h_n(g)(B_n)), d_{X_n}(C_n, h_n(g)(C_n))\} < \frac{\ell}{3}$$

Let N_n be the center of a tripod with vertices A_n, B_n, C_n in X_n. By the bound on the displacement of A_n, B_n, C_n by $h_n(g)$, for large enough n $h_n(g)(N_n) = N_n$, so $h_n(g) = 1$ for all but finitely many n's, and in particular $g \in K_\infty$.

To prove (iv) suppose $h_{n_v}(g) = 1$ for an infinite subsequence of indices $\{n_v\}$. Since the actions of G on the metric spaces (X_n, x_n) via the homomorphisms h_n converge in the Gromov-Hausdorff topology into an action of G on the real tree Y, the subsequence of actions of G on the metric spaces (X_{n_v}, x_{n_v}) via the homomorphisms h_{n_v} converges into the same action of G on the real tree Y. Since we assume $h_{n_v} = 1$ for all v, g acts trivially on the limit tree Y, so $g \in K_\infty$ and we may conclude part (iv) of the lemma.

To prove L_∞ is torsion-free suppose $g \in G$ and $g^p \in K_\infty$. Note that if the limit tree Y is isometric to a real line then L_∞ is a f.g. free abelian group, and in particular torsion-free. Hence, we may assume Y is not isometric to a real line, which implies that g^p stabilizes a tripod in Y so by part (iii) of the lemma $h_n(g^p) = 1$ for all but finitely many n's, and necessarily $h_n(g) = 1$ for all but finitely many n's and $g \in K_\infty$ which concludes part (v). The proof of part (vi) is identical with the proof of proposition 4.2 of [Ri-Se1].

□

Lemma 1.3 gives us the basic properties of the action of the limit group L_∞ on the real tree Y. These properties are essential in applying Rips' classification of (stable) actions of groups on real trees which is going to be used extensively in the sequel. In addition to the properties of the action we will need the following basic algebraic facts on the limit group L_∞.

In a free group every solvable subgroup is infinite cyclic and every maximal cyclic subgroup is malnormal. These properties are naturally inherited by the limit group L_∞.

Lemma 1.4. *With the notation of definition 1.2:*

(i) *Let u_1, u_2, u_3 be non-trivial elements of L_∞, and suppose that $[u_1, u_2] = 1$ and $[u_1, u_3] = 1$. Then $[u_2, u_3] = 1$. It follows that every abelian subgroup in L_∞ is contained in a unique maximal abelian subgroup.*

(ii) *Every maximal abelian subgroup of L_∞ is malnormal.*

(iii) *Every solvable subgroup of the limit group L_∞ is abelian.*

Proof: Let $g_1, g_2, g_3 \in G$ be elements for which $\eta(g_1) = u_1$, $\eta(g_2) = u_2$, $\eta(g_3) = u_3$. Since g_1, g_2, g_3 project to non-trivial elements in L_∞, part (iv) of lemma 1.3 implies that for all but finitely many n's h_n maps g_1, g_2, g_3 to non-trivial elements in F_k. Since $\eta(g_1)$ commutes with both $\eta(g_2)$ and $\eta(g_3)$, part (iii) of lemma 1.3 implies that for all but finitely many n's $h_n(g_1)$ commutes with both $h_n(g_2)$ and $h_n(g_3)$. Hence, for all but finitely many n's the subgroup generated by $h_n(g_1), h_n(g_2), h_n(g_3)$ is infinite cyclic, and, in particular, $h_n(g_2)$ commutes with $h_n(g_3)$. By part (iv) of lemma 1.3 $u_2 = \eta(g_2)$ commutes with $u_3 = \eta(g_3)$ and we get part (i) of the lemma.

To prove part (ii) let A be a maximal abelian subgroup of L_∞. If A is not malnormal there must exist elements $g_1, g_2, g \in G$ so that $\eta(g_1), \eta(g_2) \in A$, $\eta(g_1) \neq 1$ and $\eta(g)\eta(g_1)\eta(g^{-1}) = \eta(g_2)$. Since by part (iii) of lemma 1.3 if $r \in K_\infty$ then for all but finitely many n's $h_n(r) = 1$, for all but finitely many n's $h_n(g)h_n(g_1)h_n(g^{-1}) = h_n(g_2)$ and $[h_n(g_1), h_n(g_2)] = 1$, so $< h_n(g_1), h_n(g_2) >$ is a cyclic subgroup of F_k. Since a maximal cyclic subgroup of F_k is malnormal, for all but finitely many n's

$h_n(g)$ commutes with both $h_n(g_1)$ and $h_n(g_2)$, so part (iv) of lemma 1.3 implies that $\eta(g)$ commutes with both $\eta(g_1)$ and $\eta(g_2)$ so $\eta(g) \in A$ and A is malnormal.

To prove part (iii) let S be a solvable subgroup of L_∞. By induction we may assume that its derived subgroup S' is abelian. By part (i) every (non-trivial) abelian subgroup of L_∞ is contained in a unique maximal abelian subgroup, hence, the derived subgroup S' is contained in a (unique) maximal abelian subgroup M of L_∞, and by part (ii), M is malnormal in L_∞. Since S' is normal in S, every element $u \in S$ normalizes S', so uMu^{-1} intersects M non-trivially. Since M is malnormal, $u \in M$, so u commutes with S'. Therefore, S is abelian.

□

Proposition 1.3 shows the action of L_∞ on the real tree Y is stable. The original analysis of stable actions of groups on real trees applies to f.p. groups ([Be-Fe1]), and the limit group L_∞ is only known to be f.g. at this point, by part (i) of lemma 1.3. Still, given the basic properties of the action of L_∞ on the real tree Y that we already know, we are able to apply a generalization of Rips' work to f.g. groups obtained in [Se3]. In [Se3], the real tree Y is divided into distinct components, where on each component a subgroup of L_∞ acts according to one of several canonical types of actions. The theorem from [Se3] we present is going to be used extensively in the next sections and its statement uses the notions and basic definitions appear in the appendix of [Ri-Se1]. Hence, we refer a reader who is not yet familiar with these notions to that appendix and to [Be-Fe1] and [Be].

Theorem 1.5 ([Se3],3.1). *Let G be a freely indecomposable f.g. group which admits a stable isometric action on a real tree Y. Assume the stabilizer of each tripod in Y is trivial.*

1) *There exist canonical orbits of subtrees of Y: $Y_1, \dots Y_k$ with the following properties:*
   - (i) *gY_i intersects Y_j at most in one point if $i \neq j$.*
   - (ii) *gY_i is either identical with Y_i or it intersects it at most in one point.*
   - (iii) *The action of stab (Y_i) on Y_i is either discrete or it is of axial type or IET type.*
2) *G is the fundamental group of a graph of groups with:*
   - (i) *Vertices corresponding to orbits of branching points with non-trivial stabilizer in Y.*
   - (ii) *Vertices corresponding to the orbits of the canonical subtrees $Y_1, \dots, Y_k$ which are of axial or IET type. The groups associated with these vertices are conjugates of the stabilizers of these components. To a stabilizer of an IET component there exists an associate 2-orbifold. All boundary components and branching points in this associated 2-orbifold stabilize points in Y. For each such stabilizer we add edges that connect the vertex stabilized by it and the vertices stabilized by its boundary components and branching points.*
   - (iii) *Edges corresponding to orbits of edges between branching points with non-trivial stabilizer in the discrete part of Y with edge groups which are conjugates of the stabilizers of these edges.*
   - (iv) *Edges corresponding to orbits of points of intersection between the orbits of $Y_1, \dots, Y_k$.*

Before concluding our preliminary study of limit groups and their action on the limit real tree, we present the following basic fact which is necessary in the sequel.

Lemma 1.6. *If L_∞ is a limit group acting on a limit tree Y obtained from a converging sequence of homomorphisms from a f.g. group G into a free group F_k and L_∞ is freely-indecomposable, then stabilizers of non-degenerate segments which lie in the complement of the discrete parts of Y are trivial in L_∞. Stabilizers of segments in the discrete components of Y are abelian subgroups of L_∞.*

Proof: Identical to the proof of part (iii) of proposition 1.8 in [Se4].

□

§2. The Canonical Abelian JSJ Decomposition of a Limit Group

In the previous section we defined limit groups and studied their basic algebraic structure. To further study the algebraic structure of a limit group L_∞, and in particular to understand what are all the possible splittings of L_∞ along abelian subgroups, we need to construct the canonical JSJ decomposition of L_∞. A canonical JSJ decomposition was first introduced in [Se2] in the case of hyperbolic groups, then generalized to cyclic splittings of f.p. groups in [Ri-Se2], and to splittings of f.p. groups over more general (slender) groups in [Du-Sa] and [Fu-Pa]. Until this point, we only know that L_∞ is f.g. (lemma 1.3), still the JSJ theory can be applied using acylindrical accessibility [Se3] (cf. ([Se4],2)).

To construct the JSJ decomposition of L_∞ we need to study some basic properties of abelian splittings. We first look how maximal abelian subgroups of L_∞ act on (simplicial) Bass-Serre trees corresponding to abelian splittings of L_∞. Recall that by lemma 1.4 every maximal abelian subgroup of L_∞ is malnormal.

Lemma 2.1. *Let L_∞ be a limit group, let M be a maximal abelian subgroup in L_∞, and let A an abelian subgroup of L_∞. Then:*

(i) *If $L_\infty = U *_A V$ and M is not cyclic then M can be conjugated into either U or V.*

(ii) *If $L_\infty = U*_A$ and M is not cyclic then either M can be conjugated into U, or M can be conjugated to M', so that $L_\infty = U *_A M'$.*

Proof: A non-cyclic abelian subgroup of L_∞ is elliptic in any free decomposition of L_∞, hence, we may assume that A is a non-trivial abelian subgroup of L_∞. To prove (i) suppose that M is not cyclic and M can not be conjugated into U or V in the amalgamated product $L_\infty = U *_A V$. Let T be the Bass-Serre tree corresponding to this amalgamated product. Since M is abelian and is not elliptic when acting on T, it either preserves an axis or it preserves a point in the boundary of T. If it preserves a point in the boundary of T and not an axis in T, then the stabilizers of edges along a ray that approaches the point at the boundary of T that is preserved by M, form an increasing sequence of abelian subgroups of L_∞, $A_1 < A_2 < A_3 < \ldots$, so that the union of these subgroups is M. But all the edge groups in the Bass-Serre tree T are conjugates of A, and M is malnormal in L_∞, hence, in this case $A_1 = A_2 = \ldots = M$, so M must be elliptic. Therefore, we may assume that M preserves an axis in T.
By conjugating M we may assume that A fix an edge in the axis of M. Since M is a maximal non-cyclic abelian subgroup of L_∞, and by lemma 1.4 an abelian

subgroup in L_∞ is contained in a unique maximal abelian subgroup, the abelian subgroup A is a subgroup of M, and A fixes the entire axis preserved by M. This last observation implies that there must exist an element $u \in U$ so that $u \notin A$ and u commutes with A. u centralizes A, so by proposition 1.4 it must be an element of the maximal abelian subgroup containing A, so $u \in M$. But u can not be an element of M, since the only elliptic elements in M are the elements of A, a contradiction.

To prove (ii) suppose M is non-cyclic and M can not be conjugated into the vertex group U in the abelian splitting $L_\infty = U*_A$. Let T be the Bass-Serre tree corresponding to that HNN extension. M is abelian and not elliptic so by the argument used to prove part (i), it must preserve an axis in T, and by conjugating M to M', we may assume the abelian subgroup A fixes that entire axis of M'. By the argument used to prove part (i), the vertex group U intersects M' only in the subgroup A, so by replacing the Bass-Serre generator t by tu for some $u \in U$ we may assume that the Bass-Serre generator t is an element of M'. Since the Bass-Serre generator t centralizes the cyclic subgroup A, $L_\infty = U *_A M'$.

□

By lemma 2.1 if we replace each abelian splitting of L_∞ of the form $L_\infty = U*_A$ in which A is a subgroup of a non-elliptic maximal abelian subgroup M by the amalgamated product $L_\infty = U*_A M'$, we get that every non-cyclic abelian subgroup of L_∞ is elliptic in all the abelian splittings under consideration. This will allow us to use acylindrical accessibility in analyzing all the abelian splittings of L_∞.

Definition 2.2 [Se3]. *A splitting of a group H is called k-acylindrical if for every element $h \in H$ which is not the identity, the fixed set of h when acting on the Bass-Serre tree corresponding to the splitting has diameter at most k.*

If a limit group $L_\infty = V_1 *_{A_1} V_2 *_{A_2} V_3 *_{A_3} V_4$, where A_1, A_2, A_3 are subgroups of a maximal abelian subgroup M that is a subgroup of V_1, then one can modify the corresponding graph of groups to a tripod of groups with V_1 in the center, and V_2, V_3, V_4 at the 3 roots. Since by lemma 1.4 every maximal abelian subgroup of L_∞ is malnormal, the Bass-Serre tree corresponding to this tripod of groups is 2-acylindrical. This sliding operation generalizes to an arbitrary (finite) abelian splitting of a limit group.

Lemma 2.3. *A splitting of L_∞ in which all edge groups are abelian and all non-cyclic abelian groups are elliptic can always be modified (by modifying boundary monomorphisms by conjugations and sliding operations) to be 2-acylindrical.*

Lemma 2.3 shows that if in all abelian splittings of L_∞ under consideration all non-cyclic abelian subgroups are elliptic, these abelian splittings are 2-acylindrical. By lemma 2.1 if we replace abelian splittings of L_∞ having the form $L_\infty = U*_A$ in which A is a subgroup of a non-elliptic maximal abelian subgroup by the amalgamated product $L_\infty = U *_A M$, we get that every non-cyclic abelian subgroup of L_∞ is elliptic in all the abelian splittings under consideration. Hence, we may assume that all abelian splittings of L_∞ under consideration are 2-acylindrical.

Following [Se2] and [Ri-Se2] in order to understand all possible splittings of L_∞ along abelian subgroups, we need to study carefully the "interaction" between any two given one-edge splittings of L_∞ over abelian subgroups.

Let H be a group and T a (simplicial) H-tree. An element $h \in H$ is either *elliptic* in which case it fixes a point in T, or *hyperbolic*, in which case it preserves a line in T and acts along it as a translation. We consider two elementary splittings of L_∞ along abelian subgroups $H_\infty = D_1 *_{A_1} E_1$ (or $L_\infty = D_1 *_{A_1}$ in which case we assume that A_1 is a subgroup of a maximal abelian subgroup M in L_∞ and M is a subgroup of the vertex group D), and $L_\infty = D_2 *_{A_2} E_2$ (or $L_\infty = D_2 *_{A_2}$ under the above conditions on A_2) where A_1 and A_2 are abelian subgroups. Let T_1 and T_2 be the Bass-Serre trees corresponding to the given splittings. The two given splittings are called *elliptic-elliptic* if A_1 is elliptic in T_2 and A_2 is elliptic in T_1, *elliptic-hyperbolic* if A_1 is elliptic in T_2 and A_2 is not elliptic in T_1 and *hyperbolic-hyperbolic* if A_1 is not elliptic in T_2 and A_2 is not elliptic in T_1.

Lemma 2.1 shows that an abelian subgroup of L_∞ which is not cyclic is necessarily elliptic in all abelian splitting of L_∞ under consideration. Following [Se2] we can also exclude the case of elliptic-hyperbolic maximal abelian splittings in case L_∞ is freely indecomposable (the arguments given in [Se2] and [Ri-Se2] for cyclic splittings of groups remain valid for abelian splittings of L_∞).

Theorem 2.4 ([Se2],2.2),([Ri-Se2],2.1). *If L_∞ is freely indecomposable then any two one-edge abelian splittings of L_∞ in which all non-cyclic abelian subgroups are elliptic, are either elliptic-elliptic or hyperbolic-hyperbolic.*

Since by lemma 2.1 every abelian subgroup of L_∞ which is not cyclic is elliptic in all abelian splittings of L_∞ under consideration, pairs of hyperbolic-hyperbolic one edge splittings of L_∞ along abelian subgroups are splittings along cyclic subgroups. Hence, we may borrow the canonical quadratic decomposition constructed in [Ri-Se2] in order to understand all the hyperbolic-hyperbolic splittings of L_∞ along cyclic subgroups.
By a Z-splitting of a group we mean a splitting in which all edge groups are infinite cyclic. For the notion of a CMQ (canonical maximal QH) subgroup, and a weakly essential s.c.c. on a 2-orbifold we refer the reader to section 4 of [Ri-Se2].

Theorem 2.5 ([Ri-Se2],5.6). *Let H be a f.g. group with a single end which is not a 2-orbifold group. There exists a (canonical) reduced Z-splitting of H which we call the quadratic decomposition of H with the following properties:*

(i) *Every canonical maximal QH subgroup (CMQ) of H is conjugate to a vertex group in the quadratic decomposition. In particular, there are only finitely many conjugacy classes of CMQ subgroups. Every edge group is a cyclic boundary subgroup of one of the CMQ subgroups, and every vertex with a non-CMQ vertex group is adjacent only to vertices stabilized by CMQ subgroups in the canonical quadratic decomposition.*

(ii) *an elementary Z-splitting $H = A *_C B$ or $H = A*_C$ which is hyperbolic in another elementary Z-splitting is obtained from the quadratic decomposition of H by cutting a 2-orbifold corresponding to a CMQ subgroup of H along a weakly essential s.c.c.*

(iii) *the edge group of any elementary Z-splitting $H = A *_C B$ or $H = A*_C$ can be conjugated into a vertex group of the quadratic decomposition. In case it can be conjugated into a vertex group which is not a CMQ subgroup, the given elementary Z-splitting is elliptic-elliptic with respect to any other elementary Z-splitting of H.*

(iv) *The quadratic decomposition of H is unique up to sliding, conjugation and modifying boundary monomorphisms by conjugation (see section 1 of [Ri-Se2] for the definition of these notions).*

To construct the JSJ decomposition of L_∞ we need to further refine the quadratic decomposition to include the set of elliptic-elliptic abelian decompositions. In general, such refinement can not be obtained for a f.g. group as was shown by M. Dunwoody [Du]. Fortunately, the abelian splittings of L_∞ we consider are 2-acylindrical according to lemma 2.3, which allows us to apply acylindrical accessibilty in order to complete the construction of the JSJ decomposition.

Theorem 2.6 ([Se3],[We]). *Let H be a f.g. freely indecomposable group. For a given k there exists an integer $\lambda(k,H)$ so that the number of vertices and edges in all k-acylindrical splittings of H does not exceed $\lambda(k,H)$.*

(note that in [We], the bound $\lambda(k,H)$ is given explicitly). To construct the abelian JSJ decomposition of a freely indecomposable limit group L_∞, we start with the collection of all cyclic splittings of L_∞ in which every non-cyclic abelian subgroup is elliptic, from which we construct the cyclic JSJ decomposition of L_∞. To construct this cyclic JSJ decomposition of L_∞, we start with the quadratic decomposition of L_∞, and successively refine the quadratic decomposition using all the cyclic splittings of L_∞ for which their (cyclic) edge groups are elliptic in the quadratic decomposition, and in which every non-cyclic abelian subgroup of L_∞ is elliptic (see section 7 of [Ri-Se2] for a description of this refinement procedure). In the construction of the cyclic JSJ decomposition for f.p. groups presented in [Ri-Se2], this refinement procedure is shown to terminate using the generalized accessibility of M. Bestvina and M. Feighn [Be-Fe2]. The limit group L_∞ is not known to be f.p. so generalized accessibility can not be applied. However, acylindrical accessibility (theorem 2.6) guarantees the termination of the refinement procedure, from which we obtain the cyclic JSJ decomposition of the limit group L_∞ (note that at this point we do not know if the cyclic JSJ decomposition of L_∞ is trivial or not).

Having the cyclic JSJ decomposition of L_∞, we construct the abelian JSJ decomposition of L_∞ by successively refining the non-QH, non-abelian vertex groups in the cyclic JSJ decomposition using the collection of non-cyclic abelian decompositions of L_∞. Again, acylindrical accessibility (theorem 2.6) guarantees that this refinement procedure terminates, and we finally obtain the abelian JSJ decomposition of a freely indecomposable limit group. Note that since a limit group admits a faithful stable action on a real tree, in which segment stabilizers are abelian and stabilizers of tripods are trivial (lemma 1.3), theorem 1.5 implies that if L_∞ is non-cyclic it admits a non-trivial abelian splitting in which all non-cyclic abelian subgroups are elliptic. Hence, if L_∞ is not a surface group or an abelian group, its abelian JSJ decomposition is necessarily non-trivial.

Theorem 2.7 (cf. ([Ri-Se2],7.1)). *Suppose L_∞ is a freely indecomposable limit group. There exists a reduced unfolded splitting of L_∞ with abelian edge groups, which we call an abelian JSJ (Jaco-Shalen-Johannson) decomposition of L_∞ with the following properties:*

(i) *Every canonical maximal QH subgroup (CMQ) of L_∞ is conjugate to a vertex group in the JSJ decomposition. Every QH subgroup of L_∞ can be* *conjugated into one of the CMQ subgroups of L_∞. Every vertex group in the JSJ decomposition which is not a CMQ subgroup of L_∞ is elliptic in any abelian splitting of L_∞ under consideration.*

(ii) *A one edge abelian splitting $L_\infty = D *_A E$ or $H_\infty = D*_A$ under consideration which is hyperbolic in another elementary abelian splitting is obtained from the abelian JSJ decomposition of L_∞ by cutting a 2-orbifold corresponding to a CMQ subgroup of L_∞ along a weakly essential s.c.c..*

(iii) *Let Θ be a one edge splitting along an abelian subgroup $L_\infty = D *_A E$ or $L_\infty = D*_A$ under consideration, which is elliptic with respect to any other one edge abelian splitting of L_∞ under consideration. Then Θ is obtained from the JSJ decomposition of L_∞ by a sequence of collapsings, foldings, and conjugations.*

(iv) *If JSJ_1 is another JSJ decomposition of L_∞, then JSJ_1 is obtained from the JSJ decomposition by a sequence of slidings, conjugations and modifying boundary monomorphisms by conjugations (see section 1 of [Ri-Se2] for these notions)*

Theorem 2.7 gives us the canonical abelian JSJ decomposition of L_∞ which essentially describes all the abelian splittings of L_∞ along abelian subgroups. This decomposition is a basic tool in our approach to obtaining a structure theory for sets of solutions of equations in a free group.

§3. The Canonical Cyclic JSJ Decomposition of a Limit Group

In section 2 we have constructed the (canonical) abelian JSJ decomposition of a limit group L_∞, which includes (in an appropriate sense) all abelian splittings of L_∞. To analyze sets of solutions to systems of equations in a free group we will be interested not in all abelian splittings but rather in cyclic ones. Theorem 1.5 shows that if L_∞ is freely-indecomposable and not abelian it must admit a non-trivial abelian splitting, hence, in case it is not abelian nor a surface group it's abelian JSJ decomposition is not the trivial one. In this section our goal is to prove analogous statements for cyclic splittings of L_∞, i.e., we prove that if L_∞ is freely indecomposable and not abelian it must admit a cyclic splitting, and we define it's non-trivial canonical cyclic JSJ decomposition. We prove the existence of a cyclic splitting of L_∞ by showing that if the JSJ decomposition of L_∞ contains only non-cyclic abelian edge groups, one may further refine the abelian JSJ decomposition of L_∞, which clearly contradicts the canonical properties of the JSJ (theorem 2.7).

With the notation of the first section let G be a f.g. group, let $\{h_n \mid h_n : G \to F_k\}$ be a convergent sequence of homomorphisms, and let L_∞ be the corresponding limit group and (Y, y_0) the corresponding limit tree.

Definition 3.1. *We say that a cyclic splitting of a limit group, $L_\infty = A *_Z B$ or $L_\infty = A*_Z$, is essential, if both vertex groups A and B are not cyclic. We say that it is principal if it is essential, and either A is abelian or the centralizer of the edge group Z in L_∞ is cyclic.*

Theorem 3.2. *Let L_∞ be a non-abelian and freely-indecomposable limit group. Then L_∞ admits a principal cyclic splitting.*

Proof: Since L_∞ is freely-indecomposable it admits a canonical abelian JSJ decomposition by theorem 2.7. Note that since every abelian subgroup in L_∞ is contained in a unique maximal abelian subgroup by lemma 1.4, all vertex groups which are adjacent to an abelian vertex group in the abelian JSJ decomposition of L_∞ are non-abelian. We construct an abelian splitting Λ_{L_∞} of L_∞ in the following way. If the abelian JSJ decomposition of L_∞ contains no cyclic vertex groups we set Λ_{L_∞} to be the abelian JSJ decomposition. Otherwise, for each cyclic vertex group in the abelian JSJ decomposition we collapse an edge which connects the cyclic vertex group to a neighbor. We set Λ_{L_∞} to be the obtained abelian decomposition. Since L_∞ is a limit group, i.e., it is obtained from a convergent sequence of homomorphisms $\{h_n | h_n : G \to F_k\}$, Λ_{L_∞} is a non-trivial abelian decomposition. Indeed, from the sequence of homomorphisms $\{h_n\}$ we can extract a subsequence that converges to an action of L_∞ on some real tree Y. From this action, we get a graph of groups Θ with abelian edge groups and fundamental group L_∞. Since the real tree Y was obtained from a sequence of homomorphisms $\{h_n | h_n : G \to F_k\}$, if C is a cyclic subgroup of L_∞, C fixes a point in Y, and if a non-trivial subgroup $C' < C$ fixes a non-degenerate segment in Y, then C fixes that segment as well. Hence, if L_∞ inherits a cyclic splitting from its action on the real tree Y, then this cyclic splitting is essential. Therefore, there exists either a non-cyclic abelian splitting of L_∞, or an essential cyclic splitting of L_∞, so Λ_{L_∞} is non-trivial.

Clearly, if Λ_{L_∞} contains an edge with cyclic stabilizer, L_∞ admits an essential cyclic splitting. If the centralizer of this edge stabilizer is also cyclic, L_∞ admits a principal cyclic splitting. Hence, we may assume that the centralizers of all edge groups in Λ_{L_∞} are abelian and non-cyclic, and that Λ_{L_∞} contains no vertex with an abelian non-cyclic vertex group A so that all edge stabilizers connected to this vertex generate a cyclic subgroup in A.

Recall that if Δ is a (finite) graph of groups with fundamental group H, and T is a maximal subtree in the graph of groups Δ, then Δ is generated by the vertex groups in the maximal tree T, and elements that correspond to edges in Δ that are not in the maximal tree T, that are called Bass-Serre generators.
Let $g_1, \dots, g_q$ be a set of generators of G. Clearly, $L_\infty = < \eta(g_1), \dots, \eta(g_q) >$. We fix a maximal subtree of the graph of groups Λ_{L_∞}, set $V_\infty^1, \dots, V_\infty^m$ to be the vertex groups in that maximal subtree, $E_\infty^1, \dots, E_\infty^s$ to be the (abelian) edge groups in Λ_{L_∞}, and $t_\infty^1, \dots, t_\infty^b$ to be the set of Bass-Serre generators in Λ_{L_∞} with respect to our fixed maximal subtree. Since we assume that L_∞ admits no principal cyclic splitting, the centralizers of all the edge groups E_∞^j are abelian and non-cyclic and, in particular, the abelian JSJ decomposition of L_∞ contains no CMQ subgroups.
Let $v_1^1, \dots, v_{\ell_1}^1, \dots, v_1^m, \dots, v_{\ell_m}^m, t_1, \dots, t_b$ be elements in G for which $\eta(v_1^i), \dots, \eta(v_{\ell_i}^i) \in V_\infty^i$ for every $1 \leq i \leq m$, $\eta(t_j) = t_\infty^j$ and for every generator g_j of G:

$$g_j = w_j(v_1^1, \dots, v_{\ell_m}^m, t_1, \dots, t_b)$$

Clearly, we may assume that each of the vertex groups V_∞^i is generated by $\eta(v_1^i), \dots, \eta(v_{\ell_i}^i)$ and the edge groups E_∞^j connected to the vertex stabilized by V_∞^i in Λ_{L_∞}. For each j let $e_1^j, e_2^j, \dots \in G$ be a set of elements for which $\eta(e_p^j) \in E_\infty^j$ and in addition $\eta(e_1^j), \eta(e_2^j), \dots$ generate the edge group E_∞^j.

Our strategy in proving theorem 3.2 is to show that if Λ_{L_∞} contains no principal cyclic splitting, then the abelian JSJ decomposition of the limit group L_∞ can be

further refined, which clearly contradicts its canonical properties. If we could have assumed that all the vertex groups in the graph of groups Λ_{L_∞} are f.g. then since a f.g. subgroup of a limit group is a limit group, each of the vertex groups of Λ_{L_∞} is a limit group. Hence, every non-abelian vertex group in Λ_{L_∞} admits a non-trivial abelian JSJ decomposition, and the abelian edge groups of Λ_{L_∞}, that are all assume to have non-cyclic centralizers, are elliptic in these abelian JSJ decompositions. Therefore, the abelian JSJ decompositions of the non-abelian vertex groups in Λ_{L_∞} can be used to construct a proper refinement of the graph of groups Λ_{L_∞}, that gives rise to a proper refinement of the abelian JSJ decomposition of L_∞.
Unfortunately, the edge groups in the abelian JSJ decomposition of Λ_{L_∞} may not be f.g. and so are the vertex groups in Λ_{L_∞}. To construct a proper refinement of the abelian JSJ decomposition of L_∞ in the presence of vertex groups that may not be f.g. we apply the shortening argument presented in section 3 of [Se4] for proving the Hopf property of hyperbolic groups. Using the shortening argument we construct an action of a quotient of the limit group L_∞, Q_∞, on some real tree R, where the vertex groups in the given abelian JSJ decomposition of L_∞ are embedded into Q_∞, and at least one of those vertex groups act non-trivially on the real tree R. This gives us an abelian decomposition of that vertex group, that can be used to obtain a proper refinement of the given abelian JSJ decomposition of the original limit group L_∞.
The shortening argument we apply is rather involved technically because we can not assume that vertex groups are finitely generated. A reader who is not familiar with the shortening argument, may choose to look first at an excellent presentation of the shortening argument in case the vertex groups are f.g., that appears in M. Bestvina's survey paper on R-trees ([Be],7.4).

Our goal is to construct a proper refinement of the abelian JSJ decomposition of L_∞. We start by defining a sequence of finitely presented groups $\{U_n\}$ together with homomorphisms $\tau_n : U_{n-1} \to U_n$ which approximate the f.g. group L_∞. The groups U_n admit homomorphisms into F_k, and the abelian decomposition Λ_{L_∞} of L_∞ can be "lifted" to (f.g. abelian) decompositions of each of the U_n's. This lifting property of the U_n's is crucial in applying our shortening argument ([Se3],[Ri-Se1]) for obtaining further refinements of the abelian JSJ decompositions of L_∞ in case L_∞ does not admit a principal cyclic splitting, and was first introduced in proving the Hopf property for hyperbolic groups [Se4].
We define the groups U_n iteratively. We set U_1 to be a free group generated by the elements:

$$U_1 =< x^1_1, \dots , x^1_{\ell_1}, \dots , x^m_1, \dots , x^m_{\ell_m}, y_1, \dots , y_b, z^1_1, \dots , z^s_1 >$$

Clearly, the homomorphism $h_1 : G \to F_k$ lifts to a homomorphism $\lambda_1 : U_1 \to F_k$ by setting: $\lambda_1(x^i_p) = h_1(v^i_p)$, $\lambda_1(y_r) = h_1(t_r)$ and $\lambda_1(z^j_1) = h_1(e^j_1)$. We define U_2 to be the group generated by:

$$U_2 =< x^1_1, \dots , x^1_{\ell_1}, \dots , x^m_1, \dots , x^m_{\ell_m}, y_1, \dots , y_b, z^1_1, \dots , z^s_1, z^1_2, \dots , z^s_2 >$$

together with the relations $[z^j_1, z^j_2] = 1$ for $j = 1, \dots , s$. Clearly, there exists a natural homomorphism $\tau_1 : U_1 \to U_2$. By part (iii) of lemma 1.3, for a_2 large enough the homomorphism $h_{a_2} : G \to F_k$ lifts to a homomorphism $\lambda_2 : U_2 \to F_k$ by setting: $\lambda_2(x^i_p) = h_{a_2}(v^i_p)$, $\lambda_2(y_r) = h_{a_2}(t_r)$, $\lambda_2(z^j_1) = h_{a_2}(e^j_1)$ and $\lambda_2(z^j_2) = h_{a_2}(e^j_2)$.

Defining U_1 and U_2 we continue by defining the groups U_n, the homomorphisms τ_n and the homomorphisms λ_n iteratively. We first define the group H_n to be the group generated by:

$$H_n =< x_1^1, \ldots, x_{\ell_1}^1, \ldots, x_1^m, \ldots, x_{\ell_m}^m, y_1, \ldots, y_b, z_1^1, \ldots, z_1^s, \ldots, z_n^1, \ldots, z_n^s >$$

together with the relations $[z_{p_1}^j, z_{p_2}^j] = 1$ for $j = 1, \ldots, s$ and $p_1, p_2 = 1, \ldots, n$. The group H_n admits a natural epimorphism σ_n onto L_∞ defined by setting $\sigma_n(x_p^i) = \eta(v_p^i)$, $\sigma_n(y_r) = t_\infty^r$, and $\sigma_n(z_d^j) = \eta(e_d^j)$. We define the group U_n to be a quotient of the group H_n. To the existing set of relations of H_n we add all words w in the given defining generators of H_n for which $\sigma_n(w) = 1$, the length of w in the defining generators of H_n is at most n, and all the generators appear in the word w are mapped by σ_n into the same vertex group V_∞^i in the abelian decomposition Λ_{L_∞} of L_∞.
Clearly, there exists a natural map $\tau_n : U_{n-1} \to U_n$, and by part (iii) of lemma 1.3, there exists some integer $a_n > a_{n-1}$ so that the homomorphism $h_n : G \to F_k$ lifts to a homomorphism $\lambda_n : U_n \to F_k$ defined by: $\lambda_n(x_p^i) = h_{a_n}(v_p^i)$, $\lambda_n(y_r) = h_{a_n}(t_r)$, and $\lambda_n(z_d^j) = h_{a_n}(e_d^j)$.

Since the second set of defining relations of the group U_n are all words in generators which are mapped by σ_n into the same vertex group in Λ_{L_∞}, each of the groups U_n admits an abelian splitting Λ_n which projects by σ_n into the abelian decomposition Λ_{L_∞} of L_∞, i.e., each of the vertex groups V_n^i in Λ_n satisfies $\sigma_n(V_n^i) < V_\infty^i$, each of the edge groups $E_n^j =< z_1^j, \ldots, z_n^j >$ satisfies $\sigma_n(E_n^j) < E_\infty^j$, and each of the Bass-Serre generators in Λ_n satisfies $\sigma_n(y_r) = t_\infty^r$. We will denote by $Mod(U_n)$ the subgroup of $Aut(U_n)$ generated by inner automorphisms and Dehn twists along edges of Λ_n. We set W_n to be the subgroup of U_n generated by the x_p^i's and the y_r's. Clearly, the homomorphism $\tau_n : U_{n-1} \to U_n$ restricts to an epimorphism from W_{n-1} onto W_n, and λ_n restricts to a homomorphism from W_n into F_k. Defining the groups U_n's and W_n's and the homomorphisms between them we have obtained the following diagram:

$$\begin{array}{ccccccc}
W_1 & \xrightarrow{\tau_1} & W_2 & \cdots\xrightarrow{\tau_{n-1}} & W_{n-1} & \xrightarrow{\tau_n} & W_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U_1 & \xrightarrow{\tau_1} & U_2 & \cdots\xrightarrow{\tau_{n-1}} & U_{n-1} & \xrightarrow{\tau_n} & U_n \\
\lambda_1 \downarrow & & \lambda_2 \downarrow & & \lambda_{n-1} \downarrow & & \lambda_n \downarrow \\
F_k & & F_k & & F_k & & F_k
\end{array}$$

where the direct limit of the $\{W_n\}$'s and the maps τ_n's is the limit group L_∞.

At this stage we are finally ready to modify the shortening argument of ([Se3],[Ri-Se1]) in order to further refine the abelian JSJ decomposition of L_∞ in case L_∞ does not admit a principal cyclic splitting. Let T be a maximal tree in Λ_{L_∞}. We may order the m vertices q_i in T, so that q_1 is connected to q_2, q_3 is connected to the subtree of T spanned by q_1 and q_2, and in general q_p is connected to the subtree of T spanned by the pre-chosen vertices $q_1, q_2, \ldots, q_{p-1}$.

W.l.o.g. we may assume that this is the original order defined on the vertex groups V_∞^i of the abelian decomposition Λ_{L_∞} of L_∞. Let X be the Cayley graph of the free group F_k with respect to a (fixed) finite set of generators, and let d_X be the standard simplicial metric on X. For each element $f \in F_k$, each positive integer n, and each automorphism $\varphi \in Mod(U_n)$ we define the following (stretching) constants:

$$\mu_i(n,f,\varphi) = \max\,(d_X(id., f\lambda_n(\varphi(x_1^i))f^{-1}), \ldots, d_X(id., f\lambda_n(\varphi(x_{\ell_i}^i))f^{-1}))$$

$$\chi_r(n,f,\varphi) = d_X(id., f\lambda_n(\varphi(y_r))f^{-1})$$

and the corresponding $(m+b)$-tuple:

$$tup(n,f,\varphi) = (\mu_1(n,f,\varphi), \ldots, \mu_m(n,f,\varphi), \chi_1(n.f,\varphi), \ldots, \chi_b(n,f,\varphi))$$

On the set of $(m+b)$-tuples we define the natural lexicographical order, and for each n we choose $f_n \in F_k$ and $\varphi_n \in Mod(U_n)$ for which $tup(n, f_n, \varphi_n)$ is a minimal $(m+b)$-tuple in the set $\{tup(n,f,\varphi)\}$ with respect to the lexicographical order. We set $disp_n$ to be the sum of the elements in the $(m+b)$-tuple $tup(n, f_n, \varphi_n)$.

If $w_1, w_2 \in W_1$ are a pair of elements for which $\sigma_1(w_1), \sigma_1(w_2) \in V_\infty^i$ for some i, then for some index n_0 and for every $n > n_0$, $\tau_n \circ \ldots \circ \tau_0(w_1)$ and $\tau_n \circ \ldots \circ \tau_0(w_2)$ belong to the same (i-th) vertex group V_n^i in the (f.g. abelian) splitting Λ_n of U_n. Hence, for every $n > n_0$ both $\tau_n \circ \ldots \circ \tau_0(w_1)$ and $\tau_n \circ \ldots \circ \tau_0(w_2)$ are being mapped to their (same) conjugates by every modular automorphism $\varphi \in Mod(U_n)$, and in particular, by the chosen one φ_n. Since in addition at least one of the vertex groups V_∞^i contains abelian non-cyclic subgroups, there could not exist a sequence of indices $n_1 < n_2 < \ldots$ and corresponding elements $f_{n_j} \in F_k$ so that for every index j, f_{n_j} conjugates $\lambda_{n_j} \circ \varphi_{n_j}(x_p^i)$ into $\lambda_{n_1} \circ \varphi_{n_1}(x_p^i)$ for $i = 1, \ldots, m$ and $p = 1, \ldots, \ell_i$ and conjugates $\lambda_{n_j} \circ \varphi_{n_j}(y_r)$ into $\lambda_{n_1} \circ \varphi_{n_1}(y_r)$ for every $r = 1, \ldots, b$. Therefore, in particular, the sequence of displacement constants $\{disp_n\}$ does not contain a bounded subsequence, i.e., $disp_n \to \infty$.

The groups W_n admit a natural action on the Cayley graph X of F_k which we denote $\rho_n : W_n \times X \to X$, by setting $\rho_n(w,x) = \lambda_n \circ \varphi_n(w)(x)$ for every $w \in W_n$ and $x \in X$. Since the sequence of displacements $\{disp_n\}$ is not bounded, and the group W_n is a natural quotient of the group W_1, we may rescale the metric on the Cayley graph X by $disp_n$, and apply proposition 1.1 ([Pa],2.3) for the sequence of actions $\{\rho_n\}$ to obtain a subsequence (still denoted $\{\rho_n\}$) converging into a (pointed) real tree (R, r_0) equipped with a non-trivial isometric action of W_1. We set KW_∞ to be the kernel of the action of W_1 on the real tree R, and the f.g. group Q_∞ to be the quotient $Q_\infty = W_1/KW_\infty$. Q_∞ is also a limit group so it satisfies lemmas 1.2, 1.3 and 1.4 and its action on the pointed limit tree (R, r_0) satisfies the properties given in theorem 1.5.

L_∞ is the direct limit of the groups $\{W_n\}$ and the epimorphisms $\{\tau_n\}$ between them. Since if $w_1, w_2 \in W_0$ are a pair of elements for which $\eta \circ \lambda_1(w_1), \eta \circ \lambda_1(w_2) \in V_\infty^i$ for some i, then for some index n_0 and for every $n > n_0$, $\tau_n \circ \ldots \circ \tau_0(w_1)$ and $\tau_n \circ \ldots \circ \tau_0(w_2)$ belong to the same (i-th) vertex group V_n^i in the (f.g. abelian) splitting Λ_n of U_n, for every $n > n_0$ both $\tau_n \circ \ldots \circ \tau_0(w_1)$ and $\tau_n \circ \ldots \circ \tau_0(w_2)$ are being mapped to their (same) conjugates by the chosen automorphism φ_n. Hence, V_∞^i is naturally embedded in the group Q_∞. By our construction, Q_∞ is generated by the subgroups V_∞^i and the images of the elements $y_r \in W_1$ in Q_∞.

Proposition 3.3. *If all the subgroups $V_\infty^1, \ldots, V_\infty^m$ (the vertex groups in the abelian* *decomposition Λ_{L_∞} of L_∞) fix points in the pointed real tree (R, r_0), then they all fix the base point $r_0 \in R$.*

Proof: By our assumptions V_∞^1 fixes a point $r \in R$. For fixed index n, the conjugating elements $\{f_n\}$ and the modular automorphisms $\{\varphi_n\}$ were chosen to minimize the $(m+b)$-tuple $tup(n, f, \varphi)$, so in particular for fixed n, the chosen elements have to minimize the stretching constant $\mu_1(n, f, \varphi)$.
If V_∞^1 does not fix the base point $r_0 \in R$, then for some n_0 and all $n > n_0$ it is possible to find elements $h_n \in F_k$ for which:

$$\max_{1 \le p \le \ell_1} d_X(id., h_n f_n \lambda_n(\varphi_n(x_p^1)) f_n^{-1} h_n^{-1}) < \max_{1 \le p \le \ell_1} d_X(id., f_n \lambda_n(\varphi_n(x_p^1)) f_n^{-1})$$

Hence, $\mu_1(n, h_n f_n, \varphi_n) < \mu_1(n, f_n, \varphi_n)$, which implies $tup(n, h_n f_n, \varphi_n) < tup(n, f_n, \varphi_n)$, a contradiction to the choice of the pair f_n, φ_n. Therefore, V_∞^1 fixes the base point $r_0 \in R$.

V_∞^1 fixes the base point $r_0 \in R$, so if the claim of proposition 3.2 does not hold there must exist a minimal index i_0, so that $V_\infty^1, \ldots, V_\infty^{i_0 - 1}$ fix the base point $r_0 \in R$ whereas $V_\infty^{i_0}$ does not fix $r_0 \in R$. To prove the proposition we adapt the shortening argument of [Se3] and [Ri-Se1] to show that in this last case it is possible to find a sequence of modular automorphisms $\{\alpha_n \in Mod(U_n)\}$ so that for some index n_0 and every $n > n_0$:

$$tup(n, f_n, \varphi_n \circ \alpha_n) < tup(n, f_n, \varphi_n)$$

Lemma 3.4. *If $V_\infty^1, \ldots, V_\infty^{i_0-1}$ fix the base point $r_0 \in R$ and $V_\infty^{i_0}$ does not fix the point r_0, then there exists some index n_0 so that for all $n > n_0$ there exist modular automorphisms $\alpha_n \in Mod(U_n)$ for which:*

(i) $\alpha_n(V_n^i) = V_n^i$ *for* $i = 1, \ldots, i_0 - 1$.

(ii) $\mu_{i_0}(n, f_n, \varphi_n \circ \alpha_n) < \mu_{i_0}(n, f_n, \varphi_n)$.

Proof: identical to the proof of theorem 6.10 in [Ri-Se1].

□

By lemma 3.4 if i_0 is the first index for which $V_\infty^{i_0}$ does not fix the base point $r_0 \in R$, then for some index n_0 and all $n > n_0$ it is possible to find modular automorphisms $\alpha_n \in Mod(U_n)$ for which:

(1) $\mu_i(n, f_n, \varphi_n \circ \alpha_n) = \mu_i(n, f_n, \varphi_n)$ for $i = 1, \ldots, i_0 - 1$

(2) $\mu_{i_0}(n, f_n, \varphi_n \circ \alpha_n) < \mu_{i_0}(n, f_n, \varphi_n)$.

Hence, $tup(n, f_n, \varphi_n \circ \alpha_n) < tup(n, f_n, \varphi_n)$ in the lexicographical order on the $(m+b)$-tuples $tup(n, f, \varphi)$, which clearly contradicts the way the pair (f_n, φ_n) were chosen. Therefore, all the groups V_∞^i have to fix the base point $r_0 \in R$ and the proof of proposition 3.3 is concluded.

□

Showing that all vertex groups V_∞^i fix the base point $r_0 \in R$, we continue by showing that the images of the elements $y_1, \ldots, y_b$ in Q_∞ fix r_0 as well. Since Q_∞ is generated by the V_∞^i's and the images of the y_j's, we will get that the entire group Q_∞ fixes the base point r_0, a contradiction to the non-triviality of the action of Q_∞ on the real tree R, hence, at least one of the groups $V_\infty^1, \ldots, V_\infty^m$ does not fix a point while acting on the real tree R.

Lemma 3.5. *Suppose that not all the elements $y_1, \ldots, y_b$ fix the base point r_0, and let j_0 be the minimal index for which y_{j_0} does not fix the base point $r_0 \in R$. Then* *there exists some index n_0 so that for all $n > n_0$ there exist modular automorphisms $\beta_n \in Mod(U_n)$ for which:*

(i) $\beta_n(V_n^i) = V_n^i$ *for* $i = 1, \dots, m$.
(ii) $\beta_n(y_j) = y_j$ *for* $j = 1, \dots, j_0 - 1$.
(iii) $\chi_{j_0}(n, f_n, \varphi_n \circ \beta_n) < \chi_{j_0}(n, f_n, \varphi_n)$.

Proof: identical to the proof of theorem 6.13 in [Ri-Se1].

□

By lemma 3.5 if j_0 is the first index for which y_{j_0} does not fix the base point $r_0 \in R$, then for some index n_0 and all $n > n_0$ it is possible to find modular automorphisms $\beta_n \in Mod(U_n)$ for which:

(1) $\mu_i(n, f_n, \varphi_n \circ \beta_n) = \mu_i(n, f_n, \varphi_n)$ for $i = 1, \dots, m$
(2) $\chi_j(n, f_n, \varphi_n \circ \beta_n) = \chi_j(n, f_n, \varphi_n)$ for $j = 1, \dots, j_0 - 1$
(3) $\chi_{j_0}(n, f_n, \varphi_n \circ \beta_n) < \chi_{j_0}(n, f_n, \varphi_n)$.

Hence, $tup(n, f_n, \varphi_n \circ \beta_n) < tup(n, f_n, \varphi_n)$ in the lexicographical order on the $(m+b)$-tuples $tup(n, f, \varphi)$, which clearly contradicts the way the pair (f_n, φ_n) were chosen. Therefore, if all the groups $V_\infty^1, \dots, V_\infty^m$ fix points in the real tree (R, r_0) then all the groups V_∞^i and all the elements y_j have to fix the base point $r_0 \in R$. Since Q_∞ is generated by the groups V_∞^i and the elements y_j, the entire group Q_∞ fixes the base point $r_0 \in R$. This clearly contradicts the non-triviality of the action of Q_∞ on the real tree R, and therefore:

Proposition 3.6. *At least one of the vertex groups in the abelian decomposition Λ_{L_∞} of L_∞, $V_\infty^1, \dots, V_\infty^m$, acts non-trivially on the pointed real tree (R, r_0).*

Since the limit group L_∞ was assumed non-abelian, and since by lemma 1.4 every abelian subgroup of L_∞ is contained in a unique maximal abelian subgroup and every maximal abelian subgroup is malnormal, not all vertex groups in Λ_{L_∞}, $V_\infty^1, \dots, V_\infty^m$ are abelian.

Lemma 3.7. *Let A be an abelian subgroup of one of the groups V_∞^i. Then either A fixes a point in the real tree R, or A can be written as a direct sum $A = \hat{A} + < a_1, \dots, a_c >$ where $< a_1, \dots, a_c > \simeq Z^c$, $\hat{A}$ fixes a point in the real tree R and no non-trivial element $a \in < a_1, \dots, a_c >$ fixes a point in R.*

Proof: The lemma is obvious if A is cyclic, so suppose A is non-cyclic. Since $A < V_\infty^i$ it is naturally embedded in Q_∞. If Q_∞ is freely decomposable A is a subgroup of one of the freely-indecomposable factors of Q_∞, $\hat{Q}_\infty$. The action of $\hat{Q}_\infty$ on the real tree (R, r_0) can be analyzed using theorem 1.5 ([Se3],3.1). It follows from that theorem that an abelian subgroup can either fix a point in (R, r_0), preserve an axial component in (R, r_0), or fix an axis of one of its elements and this axis is not an axial component in (R, r_0).
If A fixes an axial component then A can be expressed as a subgroup $Stab < A$ that pointwise stabilizes this component, direct sum with a f.g. free abelian subgroup of A, where this f.g. free abelian subgroup acts faithfully on the axial component preserved by A. If A preserves a line in (R, r_0) which is not an axial component, A can be expressed as a subgroup $Stab < A$ which fixes this line direct sum with a cyclic subgroup C of A that acts discretely on the line preserved by A.

□

Proposition 3.6 proves that at least one of the vertex groups V_∞^i acts non-trivially on the limit tree (R, r_0). To get a contradiction, we show that if the abelian JSJ decomposition of L_∞ contains no principal cyclic splittings, then each of the non-abelian vertex groups V_∞^i admits a fixed point when acting on the limit tree (R, r_0).

Proposition 3.8. *If Λ_{L_∞} contains no principal cyclic splittings, then every non-abelian vertex group V_∞^i in Λ_{L_∞} fixes a point $r_i \in R$ when acting on the real tree (R, r_0).*

Proof: Suppose V_∞^i is not abelian and it does not fix a point in the real tree (R, r_0). V_∞^i is generated by the abelian edge groups E_∞^j together with the finite set of elements $v_1^i, \dots, v_{\ell_1}^i$. Hence, by lemma 3.7, V_∞^i is generated by finitely many abelian subgroups $A_1^i, \dots, A_u^i$ that admit fix points while acting on (R, r_0) together with a finite set of elements $v_1^i, \dots, v_{\ell'_1}^i$. Therefore, by theorem 1.5 V_∞^i inherits a finite graph of groups from its action on (R, r_0), a graph of groups which is non-trivial and in which all edge stabilizers are abelian.
By lemma 2.1 we can modify this finite graph of groups to get a non-trivial abelian splitting Γ^i of V_∞^i in which all non-cyclic maximal abelian subgroups are elliptic, and by theorem 1.5 if Γ^i contains a cyclic edge group then V_∞^i admits a principal cyclic splitting in which all non-cyclic maximal abelian subgroups are elliptic. Now, we can properly refine the abelian decomposition Λ_∞^i by replacing the vertex stabilized by V_∞^i with Γ^i. If Γ^i contains a cyclic edge group, L_∞ admits a principal cyclic splitting and the theorem follows. Otherwise, the obtained proper refinement can not be obtained from the abelian JSJ decomposition of L_∞ by a sequence of foldings, collapsings, slidings, and modifying boundary monomorphisms which clearly contradicts the canonical properties of the abelian JSJ decomposition of L_∞ (theorem 2.7) and the proposition follows.

□

Since by proposition 3.8 all non-abelian vertex groups V_∞^i in the abelian decomposition Λ_{L_∞} are elliptic when acting on the real tree (R, r_0), and since by proposition 3.6 not all the vertex groups $V_\infty^1, \dots, V_\infty^m$ fix points in (R, r_0), there must exist at least one abelian vertex group $V_\infty^{i_0}$ that does not fix a point in (R, r_0). By lemma 3.7 $V_\infty^{i_0} = A^{i_0} + D^{i_0}$ where A^{i_0} fixes a point in the real tree (R, r_0) and D^{i_0} is a f.g. free abelian group that acts faithfully on the line preserved by $V_\infty^{i_0}$.
Since every maximal abelian subgroup in L_∞ is malnormal, all adjacent vertex groups to an abelian vertex group in Λ_{L_∞} are non-abelian. Since all non-abelian vertex groups fix points in (R, r_0) all edge groups connected to $V_\infty^{i_0}$ are subgroups of A^{i_0}. If A^{i_0} is an infinite cyclic subgroup of $V_\infty^{i_0}$ we have found a principal cyclic splitting of the limit group L_∞, hence, we may assume that A^{i_0} is non-cyclic.

We set L_∞^1 to be the fundamental group of the graph of groups obtained from Λ_{L_∞} by replacing the vertex group $V_\infty^{i_0}$ with its subgroup A^{i_0}. Clearly, L_∞^1 is a subgroups of L_∞, and since $V_\infty^{i_0} = A^{i_0} + D^{i_0}$, L_∞^1 is also a quotient of L_∞. Hence, L_∞^1 is f.g. as well, and the first betti number of L_∞^1 is strictly smaller than that of L_∞: $b_1(L_\infty^1) < b_1(L_\infty)$. If L_∞^1 is freely decomposable, then the abelian subgroup A^{i_0} can be conjugated into one of the free factors, so necessarily L_∞ must be freely decomposable as well, a contradiction to our assumptions. Hence, we may conclude that L_∞^1 is freely-indecomposable.

L_∞^1 is also a limit group, so we may repeat the shortening argument for L_∞^1 and the abelian splitting it inherits from the abelian decomposition Λ_{L_∞} of L_∞ to obtain a new limit group Q^1_∞ acting on a new real tree (R^1, r^1_0). By propositions 3.3-3.8 either one of the non-abelian groups V^i_∞ acts non-trivially on the new limit tree (R^1, r^1_0), in which case it is either possible to obtain a principal cyclic decomposition of L_∞ or to refine the abelian JSJ decomposition of L_∞ which contradicts its canonical properties (theorem 2.7), or we obtain a new limit group L^2_∞ which is a subgroup and a quotient of L^1_∞, and $b_1(L^2_\infty) < b_1(L^1_\infty) < b_1(L_\infty)$. Therefore, after repeating this process finitely many times (depending on the first betti number of L_∞) we must obtain a non-trivial abelian splitting of one of the non-abelian groups V^i_∞, which allows us to either obtain a principal cyclic decomposition of L_∞ or to refine the abelian JSJ decomposition of L_∞ and contradict its canonical properties. This finally concludes the proof of theorem 3.2.

$\square$

Theorem 3.2 proves the existence of a principal cyclic splitting of a freely-indecomposable non-abelian limit group. In [Ri-Se2] a canonical (cyclic) JSJ decomposition that "encodes" all cyclic splittings of a f.p. freely-indecomposable group is being constructed. Since all cyclic splittings can be modified to be 2-acylindrical, we may replace the generalized accessibility of M. Bestvina and M. Feighn [Be-Fe2] used in the construction of the cyclic JSJ decomposition of f.p. groups in [Ri-Se2] by acylindrical accessibility (theorem 2.6) to obtain the canonical cyclic decomposition of a limit group. The cyclic JSJ decomposition is obtained by collapsing all the edges with non-cyclic edge groups in the abelian JSJ decomposition of a limit group. Theorem 3.2 shows it is non-trivial in case the limit group is not abelian.

Theorem 3.9 (cf. ([Ri-Se2],7.1). *Suppose L_∞ is freely indecomposable. There exists a reduced unfolded Z-splitting of L_∞, which we call the cyclic JSJ (Jaco-Shalen-Johannson) decomposition of L_∞ with the following properties:*

(i) *Every canonical maximal QH subgroup (CMQ) of L_∞ is conjugate to a vertex group in the JSJ decomposition. Every QH subgroup of L_∞ can be conjugated into one of the CMQ subgroups of L_∞. Every vertex group in the cyclic JSJ decomposition which is not a CMQ subgroup of L_∞ is elliptic in any Z-splitting of L_∞ under consideration.*

(ii) *A one edge Z-splitting $L_\infty = D *_Z E$ or $H_\infty = D*_Z$ under consideration which is hyperbolic in another elementary Z-splitting is obtained from the cyclic JSJ decomposition of L_∞ by cutting a 2-orbifold corresponding to a CMQ subgroup of L_∞ along a weakly essential s.c.c..*

(iii) *Let Θ be a one edge Z-splitting $L_\infty = D *_Z E$ or $L_\infty = D*_Z$, which is elliptic with respect to any other one edge Z-splitting of L_∞. Then Θ is obtained from the cyclic JSJ decomposition of L_∞ by a sequence of collapsings, foldings, and conjugations.*

(iv) *Let Λ be a general Z-splitting of L_∞. There exists a Z-splitting Λ_1 obtained from the cyclic JSJ decomposition by splitting the CMQ subgroups along weakly essential s.c.c. on their corresponding 2-orbifolds, so that there exists a L_∞-equivariant simplicial map between a subdivision of the Bass-Serre tree T_{Λ_1} to T_Λ.*

(iv) *If JSJ_1 is another cyclic JSJ decomposition of L_∞, then JSJ_1 is obtained from the JSJ decomposition by a sequence of slidings, conjugations and modifying boundary monomorphisms by conjugations (see section 1 of [Ri-Se2] for these notions)*

(vi) *If L_∞ is non-abelian then the cyclic JSJ decomposition of L_∞ is non-trivial. Furthermore, the cyclic JSJ decomposition can be collapsed to give a principal cyclic splitting of L_∞ in this case.*

§4. Finite Presentability of Limit Groups and ω-Residually Free Groups

In the first section we defined limit groups and studied the basic properties of their action on the corresponding limit tree. Since a limit group is a quotient of a f.g. group, it is naturally f.g. as well. To get a structure theory for sets of solutions of equations in a free group, we will have to show limit groups are f.p. and not only f.g. . This will imply, in particular, that a limit group is a ω-residually free group (definition 4.5 below).

To study some of the basic algebraic properties of limit groups we prove that limits groups are strongly accessible. Strong accessibility for certain classes of splittings of f.p. groups was recently proved by T. Delzant and L. Potyagailo [De-Po]. However, limit groups are assumed to be f.g. and we still do not know they are f.p. so the results of [De-Po] can not be applied. To prove that limit groups are strongly accessible, we prove an analogue of a Haken hierarchy for a limit group, and associate with such a group a canonical *analysis* lattice, from which some basic algebraic properties of limit groups can be deduced.

Let L_∞ be a limit group. We construct the *analysis* lattice associated with L_∞ as follows. At the 0-th level of the hierarchy we place $L^0 = L_\infty$. If L_∞ is either a free group, a surface group, or an abelian group this 0-th level is the entire analysis lattice. Otherwise, we first factor $L^0 = L_\infty$ into freely indecomposable groups, then in each factor which is not abelian or a surface group, we define the groups in the 1-st level of the analysis lattice of L_∞, $L^1_1, \dots, L^1_{m_1}$ to be the vertex groups in the canonical cyclic JSJ decompositions of the freely indecomposable factors of L_∞. Note that since L_∞ is f.g. , the groups L^1_i are f.g. as well, hence, they are also limit groups.

To define the next levels we continue iteratively. If a group in the i-th level L^i_j is either a free group, a surface group or an abelian group we don't continue to the $i+1$-st level from this branch. Otherwise, we first factor L^i_j into freely indecomposable factors, and then define the successive groups in the $i+1$ level to be the vertex groups in canonical JSJ decompositions of the freely indecomposable factors of L^i_j.

Our main technical goal in this section is to show that the construction of the analysis lattice of a limit group terminates. To do that we will show that the complexity of limit groups that appear in higher levels in the lattice is smaller than the complexity of limit groups in lower levels. To measure the complexity of the limit groups L^i_j we will use their first betti number which we denote b_1.

Theorem 4.1. *Let L_∞ be a limit group. The construction of the analysis lattice of L_∞ terminates after finitely many steps.*

Proof: Since we are going to use the first betti number to measure the complexity of a limit group, our first step in obtaining theorem 4.1 is showing that the first betti number of a non-cyclic limit group is at least 2.

Lemma 4.2. *Let G be a f.g. group, let h_n be a convergent sequence of homo-*

morphisms of G into F_k, and let L_∞ be the corresponding limit group and Y the corresponding limit tree. If L_∞ is non-cyclic then $b_1(L_\infty) \geq 2$.

Proof: We may assume L_∞ is non-abelian. The image of the homomorphisms $h_n : G \to F_k$ is a subgroup of the free group F_k. If there exists a sequence of indices for which the image is cyclic, then by part (iv) of lemma 1.3 L_∞ is abelian, which contradicts our assumptions. Hence, except for at most finitely many n's the image of the homomorphisms h_n is a non-abelian free subgroup of F_k.

The limit group L_∞ is f.g. and a quotient of the f.g. group G, so let $L_\infty = G/ < r_1, r_2, \ldots >$. We denote by G_i the group $G_i = G/ < r_1, \ldots, r_i >$. Clearly there is a canonical epimorphism from G_i onto G_{i+1}, and the direct limit of the groups $\{G_i\}$ is L_∞. Since for each fixed index i, for all but finitely many n's, the homomorphisms $h_n : G \to F_k$ split through G_i, i.e., they can be written as a composition $h_n = f_n^i \circ p^i$ where p^i is the canonical quotient map from G onto G_i, and f_n^i is a homomorphism from G_i into F_k, G_i admits a non-abelian free quotient, so $b_1(G_i) \geq 2$. Since L_∞ is the direct limit of the groups G_i and the canonical epimorphisms between them:

$$b_1(G) \geq b_1(G_1) \geq b_1(G_2) \geq \ldots \geq b_1(G_i) \geq \ldots \geq b_1(L_\infty)$$

If $A_1, A_2.A_3, \ldots$ is a sequence of f.g. abelian groups so that there exist epimorphisms from A_s to A_{s+1} for every s, then there must exist some s_0 for which $A_{s_0} \simeq A_{s_0+1} \simeq A_{s_0+2} \simeq \ldots$. Applying that for the first homology groups of the G_i's it follows that there exists some index i_0 for which:

$$b_1(G_{i_0}) = b_1(G_{i_0+1}) = b_1(G_{i_0+2}) = \ldots = b_1(L_\infty)$$

so $b_1(L_\infty) \geq 2$.

$\square$

We will prove theorem 4.1 by induction on the first betti number of a limit group. By lemma 4.2 a limit group L_∞ with $b_1(L_\infty) = 1$ is cyclic, the analysis lattice of a cyclic group has only 1 level, so theorem 4.1 is valid for limit groups with first betti number 1. To be able to use the inductive hypothesis for proving the termination of the construction of the analysis lattice, we need the following fact.

Proposition 4.3. *Let L_∞ be a limit group. If L_j^i is a group in the $i-th$ level of the analysis lattice of the limit group L_∞, $L_{j'}^{i+1}$ is connected to L_j^i and $L_{j''}^{i+2}$ is connected to $L_{j'}^{i+1}$ in this lattice, then either $L_{j''}^{i+2}$ is abelian or a surface group or a free group or:*

(i) $b_1(L_{j''}^{i+2}) < b_1(L_j^i)$

(ii) $L_{j'}^{i+1}$ *is a subgroup of another limit group $D_{j'}^{i+1}$ for which:* $b_1(D_{j'}^{i+1}) < b_1(L_j^i)$

(iii) $L_{j''}^{i+2}$ *is a subgroup of another limit group $D_{j''}^{i+2}$ for which:* $b_1(D_{j''}^{i+2}) < b_1(L_j^i)$.

(iv) $L_{j'}^{i+1}$ *is a subgroup of another limit group $M_{j'}^{i+1}$ for which:* $b_1(M_{j'}^{i+1}) = b_1(L_j^i)$*, and the first betti number of the graph corresponding to the canonical cyclic JSJ decomposition of M_j^{i+1} is strictly bigger than the first betti number of the graph corresponding to the canonical cyclic JSJ decomposition of L_j^i.*

Proof: We will assume that $L^{i+2}_{j''}$ is not abelian, nor a surface group nor a free group. Since L^i_j is connected to groups in the $i+1$-level of the analysis lattice, L^i_j, and similarly $L^{i+1}_{j'}$, are not abelian. If $M = A * B$ then $b_1(M) = b_1(A) + b_1(B)$. Since the groups connected to L^i_j in the $i+1$-level are vertex groups in the cyclic JSJ decomposition of freely-indecomposable factors of L^i_j, if $L^{i+1}_{j'}$ is connected to L^i_j in the analysis lattice then $b_1(L^{i+1}_{j'}) \leq b_1(L^i_j)$. If $b_1(L^{i+1}_{j'}) = b_1(L^i_j)$ then L^i_j must be freely-indecomposable, and by lemma 4.2 and a simple homological computation only one vertex in its canonical cyclic JSJ decomposition is non-cyclic. By theorem 3.2 a limit group admits a principal cyclic splitting, so if $b_1(L^{i+1}_{j'}) = b_1(L^i_j)$ the cyclic JSJ decomposition of L^i_j is not a tree of groups with one non-cyclic vertex group. Hence, if we suppose case (i) of the proposition does not hold, i.e., $b_1(L^i_j) = b_1(L^{i+1}_{j'}) = b_1(L^{i+2}_{j''})$, then the cyclic JSJ decompositions of L^i_j and $L^{i+1}_{j'}$ contain one vertex with non-cyclic vertex group, bouquet of circles based on this vertex and, perhaps, some additional vertices with cyclic stabilizers located either in the interior of a circle or connected by a unique edge to the vertex with non-trivial stabilizer.

To simplify notation we set $L_1 = L^i_j$, $L_2 = L^{i+1}_{j'}$ and $L_3 = L^{i+2}_{j''}$. We set Λ_1 to be the cyclic JSJ decomposition of L_1, and Λ_2 to be the cyclic JSJ decomposition of L_2. The cyclic splitting Λ_1 has one vertex group, L_2, bouquet of circles, with cyclic edge groups $< c_1 >, \dots, < c_k >$ and Bass-Serre generators $t_1, \dots, t_k$ so that $t_i c_i t_i^{-1} = c'_i$, and perhaps some additional vertices stabilized by cyclic vertex groups each connected by a single edge to the vertex stabilized by L_2. We set $\hat{L}_2$ to be the subgroup of L_1 generated by L_2 and the cyclic vertex groups connected to the vertex stabilized by L_2. The cyclic splitting Λ_2 has one vertex group, L_3, and bouquet of circles, with cyclic edge groups $< e_1 >, \dots, < e_\ell >$ and Bass-Serre generators $b_1, \dots, b_\ell$ so that $b_j e_j b_j^{-1} = e'_j$, and perhaps some additional vertices stabilized by cyclic vertex groups each connected by a single edge to the vertex stabilized by L_3. Let $e_1, \dots, e_q$ be the entire collection of edge groups in Λ_2. We set $\hat{L}_3$ to be the subgroup of L_2 generated by L_3 and the cyclic vertex groups connected to the vertex stabilized by L_3.

At this point we use the shortening process used in proving theorem 3.2, applied to the cyclic splitting Λ_1 of L_1, and obtain a (new) limit group D_∞, together with a canonical map $\nu : L_1 \to D_\infty$. By construction ν maps the group $\hat{L}_2$ monomorphically into D_∞. If D_∞ is freely decomposable, then either $\hat{L}_2$ is freely decomposable or $b_1(L_2) \leq b_1(\hat{L}_2) < b_1(D_\infty) \leq b_1(L_1)$, so in both cases part (i) of the proposition holds. Hence, we may assume that D_∞ is freely-indecomposable, and D_∞ is clearly not abelian, so D_∞ admits a canonical (non-trivial) cyclic JSJ decomposition. If $D_\infty = U *_C M$ where C is cyclic and M is non-cyclic abelian, then $L_3 < U$ and $b_1(U) \leq b_1(L_\infty) - 1$, so if we set $D^{i+2}_{j''}$ to be the subgroup U of D_∞, part (ii) of the proposition follows. Hence, we may assume that every non-cyclic abelian subgroup of D_∞ is elliptic in its canonical cyclic decomposition. We denote the cyclic JSJ decomposition of D_∞, Λ_D. Note that by lemma 2.3, Λ_D is a 2-acylindrical splitting.

D_∞ is generated by the images (under the homomorphism ν) of L_2, the Bass-Serre generators $t_1, \dots, t_k$ and the cyclic vertex groups in Λ_1. By the way the shortening argument was defined if $\nu(L_2)$ fixes a vertex in Λ_D, or more generally, if all the elements $\nu(c_1), \dots, \nu(c_k)$ fix the same vertex in Λ_D, the cyclic splitting Λ_D is either degenerate or it is a tree of groups with only one non-cyclic vertex group, so D_∞ does not admit a principal cyclic splitting, a contradiction to theorem 3.2. Furthermore, since Λ_1 contains a unique non-cyclic vertex group, if all the elements $\nu(c_1),\dots,\nu(c_k)$ are elliptic in Λ_D, they must fix the same vertex in Λ_D. Hence, $\nu(L_2)$ does not fix a vertex in Λ_D, so it inherits a cyclic decomposition Γ_2 from Λ_D, L_3 fixes a vertex in Γ_2, this vertex is the only one stabilized by a non-cyclic group in Γ_2, and at least some of the elements $\nu(c_i)$ are not elliptic in Γ_2.

Let V_D be the vertex group stabilized by L_3 in Λ_D. Since the cyclic JSJ decomposition of L_2 contains a unique non-cyclic vertex group (L_3), if $\nu(c_i)$ is elliptic in Λ_D for some i, $\nu(c_i)$ fixes either the vertex stabilized by V_D or a vertex connected to V_D and stabilized by a cyclic vertex group in Γ_2.

Since $\nu(c_i)=\nu(t_i)\nu(c'_i)\nu(t_i^{-1})$, if $\nu(c_i)$ is hyperbolic in Λ_D so is $\nu(c'_i)$. Since L_2 is freely-indecomposable, and the cyclic JSJ decomposition of L_2 is Λ_2, each edge in the axis of $\nu(c_i)$ and $\nu(c'_i)$ has to be stabilized by some conjugates (by elements in the subgroup $\nu(L_2)$) of the elements $\nu(e_1),\dots,\nu(e_q)$. $\nu(t_i)$ maps the axis of $\nu(c_i)$ to the axis of $\nu(c'_i)$.

First, suppose that $\nu(t_i)$ maps an edge E stabilized by $\nu(a_1e_ja_1^{-1})$ for some $a_1\in L_2$ in the axis of $\nu(c_i)$, to an edge E' stabilized by $\nu(a_2e_ja_2^{-1})$ for some $a_2\in L_2$. If we denote $a_3=a_2a_1^{-1}$ then the element $d_i=\nu(t_i)\nu(a_3^{-1})$ fixes the edge E. Hence, there exists some element e in the (cyclic) stabilizer of the edge E, for which $d_i=e$ and $\nu(t_i)=e\nu(a_3^{-1})$. Since e and $\nu(a_1e_ja_1^{-1})$ belong to the same cyclic subgroup in D_∞, and $\nu(t_i)$ conjugates $\nu(c_i)$ to $\nu(c'_i)$, $b_1(<e,\nu(L_2)>) \leq b_1(L_2)-1 < b_1(L_1)$. Hence, in this case we set $D^{i+1}_{j'}$ to be the subgroup $<e,\nu(L_2)>$ of D_∞, $b_1(D^{i+1}_{j'}) < b_1(L_1)$, and case (ii) holds.

Let $c_1,\dots,c_{k'}$ be the cyclic edge groups in Λ_1 for which $\nu(c_1),\dots,\nu(c_{k'})$ are not elliptic in Λ_D. Let $E_1,\dots,E_s$ be the orbits of edges in Λ_D under the action of L_2, that intersect non-trivially the axes of $\nu(c_1),\dots,\nu(c_{k'}),\nu(c'_1),\dots,\nu(c'_{k'})$. Each of the (images of the) Bass-Serre generators $\nu(t_i)$ maps the axis of the element $\nu(c_i)$ to the axis of the element $\nu(c'_i)$, for $i=1,\dots,k'$, so each of the elements $\nu(t_i)$ maps edges from the orbits $E_1,\dots,E_s$ to edges from these orbits. We construct a (labeled) finite graph Δ with vertices corresponding to the orbits $E_1,\dots,E_s$, and we connect two vertices E_{j_1},E_{j_2} by an edge, if an element $\nu(t_i)$ maps an edge that is in the orbit of E_{j_1} to an edge that is in the orbit of E_{j_2}. In this case we label the edge E_{j_1},E_{j_2} with the label $\nu(t_i)$. Note that for each label $\nu(t_i)$, there is at least one edge in Δ that is labeled by $\nu(t_i)$.

If there are loops in Δ, then the argument given above implies that L_2 can be embedded in a limit group $D^{i+1}_{j'}$, for which $b_1(D^{i+1}_{j'})) < b_1(L^i_j)$, and case (i) of the proposition holds. Hence, we may assume that Δ contains no loops. If there is a circle in Δ for which the edges in this circle are labeled by distinct labels, then the same argument applies, so we may assume that there are no such circles. Therefore, the number of orbits $E_1,\dots,E_s$ is strictly bigger than the number of labels $\nu(t_1),\dots,\nu(t_{k'})$.

L_2 inherits a graph of groups Γ_2 from Λ_D, that has one non-abelian vertex group and a bouquet of s circles, in which the elements $c_{k'+1},\dots,c_k$ and $c'_{k'+1},\dots,c'_k$ are elliptic. We set the limit group M, that is a subgroup of L^i_j, to be the fundamental group of the graph of groups obtained from Γ_2 by adding loops with edge groups $c_{k'+1},\dots,c_k$ and Bass-Serre generators $t_{k'+1},\dots,t_k$ that conjugate c_i to c'_i in correspondence, for $i=k'+1,\dots,k$.

By construction, $b_1(M)\leq b_1(L^i_j)$ and L_2 is embedded in M. If $b_1(M) < b_1(L^i_j)$

then case (ii) of the proposition holds, hence, we may assume that $b_1(M) = b_1(L^i_j)$. If M is freely decomposable, then either L_2 is freely indecomposable or L_2 can be embedded into a limit group with a strictly smaller betti number, so either case (i) or case (ii) of the proposition hold. Hence, we can assume that M is freely indecomposable. If the cyclic JSJ decomposition of M contains more than one non-cyclic vertex group, then L_3 can be embedded into a limit group with a strictly smaller betti number, and case (iii) of the proposition holds. Hence, we may assume that the cyclic JSJ decomposition of M, that we denote Δ_1, contains a unique non-cyclic vertex group, and at least $k+1$ loops based on the vertex that is stabilized by that non-cyclic vertex group. In this case $b_1(M) = b_1(L^i_j)$, $L_2 = L^{i+1}_{j'}$ is embedded into M, and the betti number of the graph associated with the cyclic JSJ decomposition of M, Δ_1, is strictly bigger than the first betti number of the graph associated with the cyclic JSJ decomposition of L^i_j, hence, case (iv) holds.

□

At this point we are ready to complete the proof of theorem 4.1 by induction on the first betti number of L_∞. If $b_1(L_\infty) = 1$, L_∞ is cyclic by lemma 4.2 and its analysis lattice contains only one level. Suppose $b_1(L_\infty) > 1$. Let $L^2_{j'}, L^3_{j''}$ be the beginning of a branch of the analysis lattice. If $L^2_{j'}$ or $L^3_{j''}$ are abelian, free or surface groups, then they are terminal points in the analysis lattice, and the construction of the analysis lattice terminates along the given branch. If cases (i)-(iii) of proposition 4.3 hold, then $L^2_{j'}$ or $L^3_{j''}$ can be embedded into a limit group with strictly smaller betti number, so the construction of the analysis lattice along our given path terminates after finitely many steps by our inductive hypothesis. If case (iv) of proposition 4.3 holds, then $L^2_{j'}$ can be embedded in a limit group $M^2_{j'}$, with the same betti number as that of L_∞, but for which the betti number of the graph corresponding to the cyclic JSJ decomposition of $M^2_{j'}$ is strictly bigger than the betti number of the graph corresponding to the cyclic JSJ decomposition of L_∞. Since the betti number of the graph corresponding to a graph of groups is bounded by the betti number of the fundamental group of the graph of groups, case (iv) of proposition 4.3 can be iteratively applied only finitely many times before one of the cases (i)-(iii) applies, and our inductive hypothesis can be applied to imply the termination of the construction of the analysis lattice after finitely many steps.

□

The termination of the construction of the analysis lattice is the key technical tool for understanding the algebraic structure of limit groups. In fact it can be viewed as a (canonical) "Haken Hierarchy" for limit groups. The following algebraic properties of limit groups are an immediate corollary. Similar properties of ω-residually free groups (which are shown to be limit groups in the sequel), were proven by O. Kharlampovich and A. Myasnikov (see corollaries 3 and 4 of theorem 6 in [Kh-My]).

Corollary 4.4. *Let L_∞ be a limit group. Then:*

(i) *L_∞ is finitely presented.*
(ii) *An abelian subgroup of L_∞ is a f.g. free abelian group. The rank of a free abelian subgroup is bounded by the first betti number of L_∞.*
(iii) *If L_∞ does not contain a non-cyclic abelian subgroup then L_∞ is (Gromov) hyperbolic.*

Proof: The analysis lattice shows that a limit group can be constructed from abelian, free, and surface groups by a finite sequence of amalgamated products and HNN extensions over cyclic groups and free products, hence, it is finitely presented, and we get (i). Every abelian subgroup of a limit group which is not cyclic or $Z+Z$ is elliptic in its cyclic JSJ decomposition. Therefore, such abelian subgroup of L_∞ must appear as a terminal vertex group in the analysis lattice, so it must be f.g. and its rank is bounded by the first betti number of L_∞. A limit group is torsion-free by lemma 1.3 which gives us part (ii). If L_∞ does not contain non-cyclic abelian groups, it is constructed from free and surface groups by a sequence of amalgamated products and HNN extensions along cyclic groups and free products, and since every cyclic subgroup of a limit group is contained in a unique maximal cyclic one by lemma 1.4 and part (ii), such limit group is a hyperbolic group by the combination theorem of M. Bestvina and M. Feighn [Be-Fe3].

$\square$

The finite presentability of limit groups is a key tool in our analysis of sets of solutions to equations. It is the source for all the finiteness results we will get both for these sets and in dealing with the elementary theory of free groups in our continuation papers. An immediate implication of the finite presentability is equivalence between limit groups and f.g. ω-residually free groups.

Definition 4.5. *Let H be a group. H is called residually free if for every element $h \neq 1$ in H, there exists a free group F_k and a homomorphism $\phi : H \to F_k$ so that $\phi(h) \neq 1$. H is called ω-residually free if for any integer m and every set of m elements $h_1, \ldots, h_m \neq 1$ in H, there exist an integer k and a homomorphsim $\phi : H \to F_k$ so that $\phi(h_1), \ldots, \phi(h_n) \neq 1$.*

Theorem 4.6. *A f.g. group R is a limit group if and only if it is ω-residually free.*

Proof: The theorem is obvious for f.g. free abelian groups, so we may assume the f.g. group R is not abelian.
Suppose R is a limit group. By definition there exists a f.g. group G, an integer k and a sequence of homomorphisms $h_n : G \to F_k$, so that the limit of the actions of G on the Cayley graph of F_k via the homomorphisms h_n is a faithful action of R on some real tree Y.
By corollary 4.4 R is f.p. so by lemma 1.3 for all but finitely indices n, the homomorphisms h_n split through the limit group R, i.e.. $h_n = \psi_n \circ p$ where $p : G \to R$ is the canonical projection map, and the ψ_n's are homomorphisms $\psi_n : R \to F_k$. By lemma 1.3 if $r \neq 1$ in R, then for all but finitely many n's $\psi_n(r) \neq 1$. Hence, for every integer m and every set of m elements $r_1, \ldots, r_m \neq 1$ in R, for all but finitely many indices n, $\psi_n(r_1), \ldots, \psi_n(r_m) \neq 1$, so R is ω-residually free.

To prove the other direction, suppose $R = < r_1, \ldots, r_s >$ is f.g. and ω-residually free. Since R is ω-residually free there exists a sequence of homomorphisms $\phi_n : R \to F_2$, so that ϕ_n maps the elements in a ball of radius m in the Cayley graph of R to distinct elements in F_2. By rescaling the metric on F_2, proposition 1.1 implies that there exists a subsequence of the homomorphisms ϕ_n which converges to an action of a limit group L on a real tree Y. In general, the limit group L is a quotient of the f.g. group R, but since the homomorphisms were chosen so that ϕ_n maps a ball of radius n "monomorphically" into F_2, part (iii) of lemma 1.3 implies that R is isomorphic to L and, therefore, R is a limit group.

□

Clearly, since f.g. ω-residually free groups are limit groups and vice versa, all the algebraic properties proven for limit groups are valid for ω-residually free groups. In particular, they admit a principal cyclic decomposition (theorem 3.2), a canonical cyclic JSJ decomposition (theorem 3.9), a (canonical) analysis lattice, they are f.p. , every abelian subgroup of a f.g. ω-residually free group is a f.g. free abelian group and its rank is bounded by the first betti number of the ambient ω-residually free group, and if a f.g. ω-residually free group does not contain a non-cyclic abelian subgroup it is (Gromov) hyperbolic. Also, since a f.g. ω-residually free group is, in particular, residually finite, it satisfies the Hopf property.

§5. The (Canonical) Makanin-Razborov Diagram

The algebraic properties of limit groups proved in the previous sections introduce tools that initialize the analysis of these groups, but are still short of obtaining any kind of classification of them. i.e., the cyclic JSJ decomposition and the analysis lattice associated with a limit group, are helpful in proving properties of limit groups but they are not enough for getting a necessary and sufficient conditions for a f.p. group to be a limit group (or equivalently an ω-residually free group). To get such criteria, we need to introduce canonical resolutions of limit groups, resolutions which will capture all the possible homomorphisms from a limit group into a free group and which are similar in nature to the ones appear in works of Makanin [Ma] and Razborov [Ra1], obtained there under some additional combinatorial hypothesis (bounded periodicity), and generalized later by Razborov in his dissertation [Ra2], and by Kharlampovich and Myasnikov in [Kh-My].

On the set of limit groups (ω-residually free groups) we define a partial order. Given two limit groups R_1, R_2 we say that $R_1 > R_2$ if R_2 is a proper quotient of R_1. Note that since a limit group is Hopf, if R_2 is a proper quotient of R_1 then R_1 can not be a quotient of R_2. A first implication of the finite presentability of limit groups to their partial order is the termination of decreasing sequences.

Proposition 5.1. *Every sequence of decreasing limit groups $R_1 > R_2 > R_3 > \dots$ terminates.*

Proof: Suppose there is a decreasing sequence which does not terminate. The limit groups $\{R_m\}$ are all quotients of R_1, so we may assume their fixed finite set of generators is the image of a fixed finite set of generators of R_1. The limit groups R_m are ω-residually free by theorem 4.6, so for each fixed m we can define a sequence of homomorphisms $h_n^m : R_m \to F_k$ so that h_n^m maps the n-ball in the Cayley graph of R_m to distinct elements in F_k. From the sequences $\{h_n^m\}$ we extract the diagonal sequence of homomorphisms $\phi_m : R_1 \to F_k$ by setting $\phi_m = h_m^m \circ p_m$ where $p_m : R_1 \to R_m$ is the natural projection. By rescaling the metric on F_k, proposition 1.1 implies that the actions of R_1 on the Cayley graph of F_k via the homomorphisms $\{\phi_m\}$ subconverge into an action of a limit group R on a real tree Y (see section 1).

R is a quotient of all the groups R_m in the decreasing sequence and it is a limit group by construction. By corollary 4.4 R is f.p. and by part (iii) of lemma 1.3 for all but finitely many indices m, the (finite) defining relations of R are mapped to the trivial element by the homomorphisms ϕ_m. Since the homomorphisms h_m^m

were chosen to map the ball of radius m in the Cayley graph of R_m into distinct elements, and since the defining relations of R are mapped to the identity by ϕ_m, for all but finitely many m's, there exists some index m_0, so that $R_{m_0}, R_{m_0+1}, \dots$ are all quotients of the limit group R. But R is a proper quotient of the R_m's, so we obtain a contradiction to the Hopf property of the limit group R, and a decreasing sequence must terminate.

□

To get an understanding of the structure of limit groups (ω-residually free groups) we need to find a way to "encode" all possible homomorphisms from a limit group into free groups. To get such "encoding" we will construct a canonical diagram associated with a limit group. To construct this canonical diagram we need to introduce *shortening quotients* of a limit group. To construct a shortening quotient of a limit group we need to define its associated *modular group*.

Definition 5.2. *Let R be a freely-indecomposable limit group. We define the modular group $Mod(R)$ to be the subgroup of $Aut(R)$ generated by the following families of automorphisms of R:*

(i) *inner automorphisms.*
(ii) *Dehn twists along edges of the cyclic JSJ decomposition of R.*
(iii) *Dehn twists along essential s.c.c. in CMQ (canonical maximal quadratically hanging) vertex groups in the cyclic JSJ decomposition of R.*
(iv) *Let A be an abelian vertex group in the cyclic JSJ decomposition of R, and let $A_1 < A$ be the subgroup generated by all the edge groups connecting A to the other vertex groups in the cyclic JSJ decomposition of R. Every automorphism of A that fixes A_1 (elementwise) can be naturally extended to an automorphism of the ambient limit group R. We call these generalized Dehn twists and they form the fourth family of automorphisms that generate $Mod(R)$.*

At this stage we are ready to modify the shortening argument, presented in ([Be],7.4) and in section in order to obtain the *shortening quotients* of a freely-indecomposable limit group R. Let $\{h_n \mid h_n : R \to F_k\}$ be a sequence of homomorphisms from R into a free group F_k. Let $r_1, \dots, r_m \in R$ be a generating set of R, and let $V_1, \dots, V_\ell$ be the vertex groups in the cyclic JSJ decomposition of R.
Let X be the Cayley graph of the free group F_k with respect to a (fixed) finite set of generators, and let d_X be the standard simplicial metric on X. For each element $f \in F_k$, each positive integer n, and each automorphism $\varphi \in Mod(R)$ we define the following (stretching) constant:

$$\mu(n, f, \varphi) = \max\ (d_X(id., fh_n(\varphi(r_1))f^{-1}), \dots, d_X(id., fh_n(\varphi(r_m))f^{-1}))$$

For each n we choose $f_n \in F_k$ and $\varphi_n \in Mod(R)$ for which $\mu(n, f_n, \varphi_n)$ is minimal in the set $\{\mu(n, f, \varphi)\}$. We set $disp_n = \mu(n, f_n, \varphi_n)$.

The limit group R admits a natural action on the Cayley graph X of F_k which we denote $\rho_n : R \times X \to X$, by setting $\rho_n(r, x) = \tau_{f_n} \circ h_n \circ \varphi_n(r)(x)$ for every $r \in R$ and $x \in X$, where τ_{f_n} is the inner automorphism of F_k defined by f_n. Since X is a simplicial tree, for the n-th action ρ_n we may rescale the metric on the Cayley graph X by $disp_n$, and apply proposition 1.1 ([Pa],2.3) for the sequence of actions

$\{\rho_n\}$ to obtain a subsequence (still denoted $\{\rho_n\}$) converging into a (pointed) real tree (T, t_0) equipped with a non-trivial isometric action of R. We set KS_∞ to be the kernel of the action of R on the real tree T, and the limit group S_∞ to be the quotient $S_\infty = R/KS_\infty$. We say that the limit group S_∞ associated with the canonical quotient map $\eta : R \to S_\infty$ is a *shortening quotient* of the limit group R. Note that the construction of a shortening quotient of R does depend on the sequence of homomorphisms $h_n : R \to F_k$ and the specific generating set $r_1, \ldots, r_m \in R$ pre-chosen for R.

We say that two shortening quotients S_1, S_2 of the limit group R are *equivalent*, if there exists an isomorphism $\tau : S_1 \to S_2$, and the canonical map $\eta_2 : R \to S_2$ can be expressed as $\eta_2 = \tau \circ \eta_1 \circ \varphi$ where $\varphi \in Mod(R)$ and $\eta_1 : R \to S_1$ is the canonical map associated with the (maximal) shortening quotient S_1. The notion of equivalent shortening quotients is clearly an equivalence relation on the set of couples of shortening quotients and their associated canonical maps: $\{(S_i, \eta_i : R \to S_i)\}$ of the limit group R.

Since all the homomorphisms h_n map the limit group R into a free group F_k, a shortening quotient of a limit group R is indeed a quotient of R. In fact, it is a proper quotient of it.

Claim 5.3. *Let R be a freely-indecomposable limit group. Then every shortening quotient of R is a proper quotient of R.*

Proof: Let R be a freely-indecomposable limit group, let $\{h_n : R \to F_k\}$ be a sequence of homomorphisms, suppose the sequence of actions of R on the Cayley graph of F_k via the homomorphisms $\{h_n\}$ converge into an action of a shortening quotient S of R on some real tree Y_S, and suppose each of the homomorphisms h_n is the "shortest" possible under compositions with modular automorphisms of R and inner automorphisms of F_k.
Now, assume that the shortening quotient S is isomorphic to R, so the canonical map $\eta : R \to S$ is an isomorphism. Let Γ_R be the abelian JSJ decomposition of R (see theorem 2.7). By theorem 3.2 R obtains a principal cyclic splitting, so some of the edge groups in Γ_R are cyclic.

To obtain a contradiction to the isomorphism between R and its shortening quotient S, suppose first that all the edge groups in the abelian JSJ decomposition of R are cyclic. By theorem 1.5, the shortening quotient S inherits an abelian splitting from its action on the limit tree Y_S. Since R is assumed isomorphic to its shortening quotient S, R inherits an abelian splitting from its action on Y_S.
The limit action of S on Y_S was obtained from a sequence of homomorphisms $\{h_n : R \to F_k\}$ that were assume the "shortest" possible under compositions with modular automorphisms of R and inner automorphisms of F_k. Therefore, the real tree Y_S can not contain an IET component, since otherwise, for large enough n, it was possible to "shorten" the homomorphism h_n by pre-composing with a modular automorphism of R by the canonical properties of the abelian JSJ decomposition of R (theorem 2.7) and the shortening argument given in ([Ri-Se1],5.1), which contradicts our assumptions on the homomorphisms $\{h_n\}$. By a similar argument Y_S does not contain an axial component. Hence, the action of R on Y_S is discrete. But again, when the action of R on Y_S is discrete, for large enough n, it is possible to "shorten" the homomorphisms h_n by precomposing with a modular automorphism of R by the canonical properties of the abelian JSJ decomposition of R (theorem 2.7)

and the shortening argument given in ([Se3],2.5), a contradiction to our assumptions on the homomorphisms $\{h_n\}$. Hence, the abelian JSJ decomposition of R, Γ_R, must contain some abelian, non-cyclic, edge groups.

Let $E_1, \ldots, E_\ell$ be the abelian non-cyclic edge groups in Γ_R, let $\Lambda_1, \ldots, \Lambda_q$ be the subgraphs of Γ_R which are the connected components of the graph obtained from Γ_R by deleting the edges stabilized by $E_1, \ldots, E_\ell$. Since R admits a principal cyclic splitting by theorem 3.2, the abelian JSJ decomposition Γ_R contains an edge with cyclic edge group which corresponds to a principal cyclic splitting of R, and this edge must be contained in one of the connected subgraphs $\Lambda_1, \ldots, \Lambda_q$, w.l.o.g., Λ_1.

Let R_1 be the subgroup of R which is the fundamental group of the subgraph Λ_1. Let $\{h_n^1 : R_1 \to F_k\}$ be the restrictions of the homomorphisms $\{h_n : R \to F_k\}$ to the subgroup R_1. From the sequence of actions of R_1 on the Cayley graph of F_k via the homomorphisms $\{h_n^1\}$ it is possible to extract a subsequence (still denoted $\{h_n^1\}$) that converge into an action of R_1 on some real tree Y_1. By theorem 1.5 R_1 inherits a (non-trivial) abelian splitting Γ_1 from its action on Y_1.

Let $E_1, \ldots, E_u$ be the abelian non-cyclic edge groups in Γ_R which are the stabilizers of edges connected to Λ_1, i.e., $E_1, \ldots, E_u$ are the edge groups in Γ_R which are also subgroups of R_1. If any of the groups $E_1, \ldots, E_u$ is not elliptic when acting on the limit tree Y_1, it is possible to further refine the abelian JSJ decomposition Γ_R of R, which contradicts its canonical properties (theorem 2.7). Hence, the subgroups $E_1, \ldots, E_u$ can all be conjugated into vertex groups in Γ_1, so Γ_1 can be extended to an abelian splitting on the ambient group R, so by the canonical properties of the abelian JSJ decomposition of R, Γ_1 can be obtained from the connected subgraph Λ_1 of the abelian JSJ decomposition of R, Γ_R.

Now, modifying the "shortening argument" in case all edge groups in Γ_R are cyclic, if the real tree R_1 contains an IET component, for large enough n, it is possible to "shorten" the homomorphisms h_n^1 by pre-composing with a modular automorphism of R that preserve R_1 by the canonical properties of the abelian JSJ decomposition of R (theorem 2.7) and the shortening argument given in ([Ri-Se1],5.1). By construction a composition of these shortening modular automorphisms with the ambient homomorphisms $\{h_n : R \to F_k\}$ make these "shorter", which contradicts our assumptions on the homomorphisms $\{h_n\}$. By a similar argument Y_1 does not contain an axial component. Hence, the action of R_1 on Y_1 is discrete. But again, when the action of R_1 on Y_1 is discrete, for large enough n, it is possible to "shorten" the homomorphisms h_n^1 by precomposing with a modular automorphism of R by the canonical properties of the abelian JSJ decomposition of R (theorem 2.7) and the shortening argument given in ([Se3],2.5). By construction a composition of these shortening modular automorphisms with the ambient homomorphisms $\{h_n : R \to F_k\}$ make these "shorter", a contradiction to our assumptions on the homomorphisms $\{h_n\}$.

$\square$

Let $SQ(R, r_1, \ldots, r_m)$ be the set of shortening quotients of R. On the set $SQ(R, r_1, \ldots, r_m)$ we define a partial order as follows. For given elements $S_1, S_2 \in SQ(R, r_1, \ldots, r_m)$ we say that $S_1 > S_2$ if S_2 is a proper quotient of S_1 and the canonical map $\eta_2 : R \to S_2$ splits as $\eta_2 = \nu \circ \eta_1$ where $\eta_1 : R \to S_1$ is the canonical map associated with S_1 and $\nu : S_1 \to S_2$ is a homomorphism.

Lemma 5.4. *Let R be a freely-indecomposable limit group. Let $S_1 < S_2 < S_3 < \dots$ (where $S_j \in SQ(R, r_1, \dots, r_m)$) be a properly increasing sequence of shortening quotients of R. Then there exists a shortening quotient $S \in SQ(R, r_1, \dots, r_m)$ so that $S > S_j$ for every shortening quotient S_j in the increasing sequence.*

Proof: Let $S_1 < S_2 < S_3 < \dots$ be an infinite properly increasing sequence of shortening quotients of the freely-indecomposable limit group R. Since each of the groups S_j is a shortening quotient of R, for each j there exists a sequence of homomorphisms $h_n^j : R \to F_k$, so that h_n^j can not be shortened by compositions with both a modular automorphism of R and an inner automorphism of F_k, and if $r \in R$ for which $\eta_j(r)$ is not the identity element in S_j and $\eta_j(r)$ is contained in the ball of Radius n in the Cayley graph of S_j, then $h_n^j(r) \neq 1$.

Now, we look at the diagonal sequence $h_j^j : R \to F_k$. By the construction used in section 1, we may choose a converging subsequence (to simplify notation still denoted h_j^j), so that the sequence of homomorphisms $h_j^j : R \to F_k$ converges into an action of a limit group S on some real tree Y. By our assumptions on the homomorphisms $h_n^j : R \to F_k$, the diagonal sequence $h_j^j : R \to F_k$ can not be shortened by compositions with both a modular automorphism of R and an inner automorphism of F_k, so the limit group S is a shortening quotient of R, hence, $S \in SQ(R, r_1, \dots, r_m)$.

The shortening quotient S is in particular a limit group, so by corollary 4.4 it is f.p. . By lemma 1.3 there must exist some index j_0 so that for all indices $j > j_0$, the homomorphisms h_j^j map the (finite set of) defining relations of S to the trivial element in F_k. Since by the way the homomorphisms h_n^j were chosen, h_j^j maps a ball of radius $j/2$ in S_j monomorphically into F_k, all the defining relations of S must be trivial in S_j for large enough j. Therefore, $S > S_j$ for large enough j which clearly implies that $S > S_j$ for all indices j.

$\square$

Lemma 5.4 proves the existence of maximal elements with respect to the partial order on the set of shortening quotients $SQ(R, r_1, \dots, r_m)$. The next lemma shows that there are only finitely many equivalence classes of maximal elements in the set of shortening quotients.

Lemma 5.5. *Let R be a freely-indecomposable limit group. The set of shortening quotients of R, $SQ(R, r_1, \dots, r_m)$, contains only finitely many equivalence classes of maximal elements with respect to its partial order.*

Proof: Suppose there are infinitely many non-equivalent maximal shortening quotients $S_1, S_2, S_3, \dots$ of some freely-indecomposable limit group R. As we did in the proof of lemma 5.4 since each of the groups S_j is a shortening quotient of R, for each j there exists a sequence of homomorphisms $h_n^j : R \to F_k$, so that h_n^j can not be shortened by compositions with both a modular automorphism of R and an inner automorphism of F_k, and if $r \in R$ for which $\eta_j(r)$ is not the identity element in S_j and $\eta_j(r)$ is contained in the ball of Radius n in the Cayley graph of S_j, then $h_n^j(r) \neq 1$.

From the diagonal sequence $h_j^j : R \to F_k$ we can extract a convergent subsequence (still denote h_j^j), so that the sequence of homomorphisms $h_j^j : R \to F_k$

converges into an action of a limit group S on some real tree Y, and by our assumptions on the homomorphisms $h_n^j : R \to F_k$ the limit group S is a shortening quotient of R, i.e., $S \in SQ(R, r_1, \ldots, r_m)$.

The shortening quotient S is in particular a limit group, so by corollary 4.4 it is f.p. . By lemma 1.3 there must exist some index j_0 so that for all indices $j > j_0$, the homomorphisms h_j^j map the (finite set of) defining relations of S to the trivial element in F_k. Since by the way the homomorphisms h_n^j were chosen, h_j^j maps a ball of radius $j/2$ in S_j monomorphically into F_k, all the defining relations of S must be trivial in S_j for some $j > j_1$. Therefore, for every $j > j_1$ either $S > S_j$ or S is equivalent to S_j. But all the shortening quotients S_j are assumed maximal in $SQ(R, r_1, \ldots, r_m)$, so for all $j > j_1$ S is equivalent to S_j, and that contradicts our assumption on the S_j's being from different equivalence classes of shortening quotients of R.

□

The main significance of maximal shortening quotients is the way they "encode" and simplify all homomorphisms from a freely-indecomposable limit group into a free group.

Proposition 5.6. *Let R be a freely-indecomposable limit group. Let $r_1, \ldots, r_m \in R$ be a generating set of R, let $M_1, \ldots, M_k$ be a collection of representatives of the (finite) set of equivalence classes of maximal shortening quotients in $SQ(R, r_1, \ldots, r_m)$, and for $i = 1, \ldots, k$ let $\eta_i : R \to M_i$ be the canonical projection maps.*
Let $h : R \to F_k$ be a homomorphism. Then there exists some index $1 \leq i \leq k$ (not necessarily unique!) and a modular automorphism $\varphi_h \in Mod(R)$ so that $h \circ \varphi_h$ splits through the maximal shortening quotient M_i. i.e., $h \circ \varphi_h = h_{M_i} \circ \eta_i$ where $h_{M_i} : M_i \to F_k$ is a homomorphism.

Proof: Given a homomorphism $h : R \to F_k$ we can choose a modular automorphism $\varphi_h \in Mod(R)$ and an inner automorphism $\tau_f \in Inn(F_k)$ so that the composition $\hat{h} = \tau_f \circ h \circ \varphi_h$ is the shortest possible among all such compositions. The constant sequence of homomorphisms $\hat{h}, \hat{h}, \ldots$ converges into a free limit group F which is a subgroup of F_k. By our assumption on $\hat{h}$, F is a shortening quotient and the map $\hat{h} : R \to F$ is its associated canonical map. By lemmas 5.4 and 5.5, either there exists some maximal element $M_i > F$, or one of the M_i's is equivalent to F. In both cases the homomorphism $\hat{h} : R \to F$ splits through the maximal shortening quotient M_i and the proposition follows.

□

The shortening procedure and the lemmas and propositions proved so far in this section finally allow us to present the main goal of this section, the (canonical) *Makanin-Razborov diagram* associated with a limit group (ω-residually free group). The Makanin-Razborov diagram "encodes" all possible homomorphisms from a limit group into a free group, and as we will later show, some of its properties can be stated as a criteria for a general f.g. group to be a limit group.

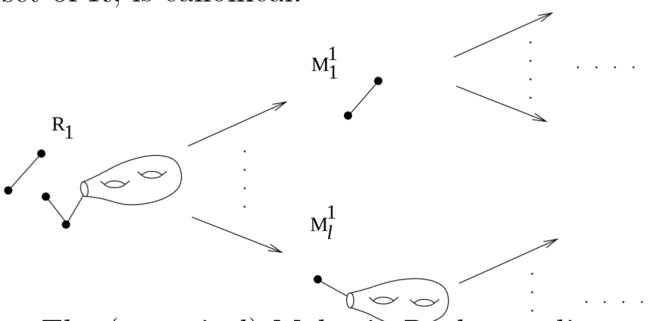
Let R be a limit group and let $R = R_1 * \ldots R_\ell * F_{g_1}$ be a factorization of R, where each of the R_i's is a freely-indecomposable non-cyclic subgroup of R and F_{g_1} is a free group. Let $r_1^1, \ldots, r_{m_1}^1 \in R$ be a generating set of R_1, $r_1^2, \ldots, r_{m_2}^2$ be a generating set of R_2 etc. . By lemma 5.4 the sets of shortening quotients of the freely-indecomposable limit groups $R_1, \ldots, R_\ell$ contain maximal elements (with respect to the partial order defined above) and by lemma 5.5 there are only finitely many equivalence classes of maximal shortening quotients of each of the limit groups $R_1, \ldots, R_\ell$. For $i = 1, \ldots, \ell$ let $M^i_1, \ldots, M^i_{k_i}$ be a collection of representatives of equivalence classes of maximal shortening quotients in $SQ(R_i, r^i_1, \ldots, r^i_{m_i})$, and let $\eta^i_j : R_i \to M^i_j$ be the canonical projection maps.

We define the *Makanin-Razborov diagram* of the limit group R iteratively. We start by factoring R into its freely-indecomposable factors $R_1, \ldots, R_\ell$ and the free factor F_g. From each of the factors R_i we associate k_i directed edges, starting at R_i and terminating at the maximal shortening quotient M^i_j. To each such directed edge we associate the canonical projection η^i_j. Note that we do not proceed from the free factor F_{g_1}.
We proceed iteratively. We factor each of the M^i_j's into freely-indecomposable factors and associate with each such factor representatives for its equivalence classes of maximal shortening quotients. Since each (maximal) shortening quotient of a limit group is a proper quotient of that limit group by claim 5.3, and each sequence of properly decreasing sequence of limit groups terminates by proposition 5.1, the construction of the Makanin-Razborov diagram terminates.

By proposition 5.6 all the homomorphisms from a limit group R into a free group F_k are encoded by the canonical Makanin-Razborov diagram.

Theorem 5.7. *Let R be a limit group. All the homomorphisms $h : R \to F_k$ are given by compositions of modular automorphisms of the limit groups in the diagram with the canonical maps from the limit groups into their maximal shortening quotients and finally with general homomorphisms ("substitutions") of the terminal free groups that appear in the diagram into the target free group.*

By Grushko's theorem the factorization of a f.g. group into freely indecomposable factors is canonical. Once we fix a generating set of the limit group R, the (equivalence classes of) the maximal shortening quotients constructed along the various levels of the Makanin-Razborov diagram are canonical as well, hence, the entire Makanin-Razborov diagram associated with a limit group R and a fixed generated set of R, is canonical.



The (canonical) Makanin-Razborov diagram associated with a limit group, encodes all the homomorphisms from that limit group into free groups. Some specific subdiagrams of it can be viewed as a criteria for a general f.g. group to be a limit group (ω-residually free group).

Definition 5.8. *A subdiagram of the Makanin-Razborov diagram in which for each free factor R_i of the limit group R we choose a unique edge connecting R_i to one of its maximal shortening quotients, and proceed iteratively choosing only one maximal shortening quotient at each stage, is called a Makanin-Razborov resolution.*

To present a criteria for a f.g. group to be a limit group we need to show the existence of special type of Makanin-Razborov resolutions which we call *strict Makanin-Razborov resolutions*.

Definition 5.9. *Let R be a freely-indecomposable limit group. We say that a shortening quotient S of R is a strict shortening quotient if:*

(i) *The subgroups generated by each non-CMQ, non-abelian vertex group together with centralizers of edge groups connected to it in the graph of groups obtained from the cyclic JSJ of R by replacing each abelian vertex group by the direct summand containing the edge groups connected to it, and the cyclic edge groups in the cyclic JSJ decomposition of R are mapped monomorphically into the shortening quotient S by the canonical map $\eta : R \to S$.*

(ii) *each CMQ subgroup of R is mapped to a non-abelian subgroup of S by the canonical map η, and each boundary element of a CMQ subgroup of R is mapped to a non-trivial element in S by η.*

(iii) *Let A be an abelian vertex group in the JSJ decomposition of R, and let $A_1 < A$ be the subgroup generated by all edge groups connected to the vertex stabilized by A. Then A_1 is mapped monomorphically into S by the canonical map η.*

A Makanin-Razborov resolution is called strict if all the maximal shortening quotients appear in it are strict shortening quotients.

Finally, the existence of strict Makanin-Razborov resolutions is a criteria for a f.g. group to be a limit group (ω-residually free group). We first show that a limit group admits a strict Makanin-Razborov resolution.

Proposition 5.10. *Let R be a limit group. Then the (canonical) Makanin-Razborov diagram of R contains a strict Makanin-Razborov resolution.*

Proof: Let R be a freely-indecomposable limit group. R is an ω-residually free group, so there exists a sequence of homomorphisms $h_n : R \to F_k$ so that h_n maps the ball of radius n in R monomorphically into F_k and the sequence of actions of R on the Cayley graph of F_k via the sequence of homomorphisms $\{h_n\}$ converges into a faithful action of R on some real tree Y.

By composing each of the elements h_n with an appropriate modular automorphism $\varphi_n \in Mod(R)$ and an inner automorphism $\tau_{f_n} \in Inn(F_k)$ we obtain homomorphisms $\hat{h}_n : R \to F_k$, $\hat{h}_n = \tau_{f_n} \circ h_n \circ \varphi_n$, which are the shortest with respect to all such compositions. From the sequence $\hat{h}_n$ we can extract a subsequence (still denoted $\hat{h}_n$) that converges into an action of a shortening quotient S on some real tree T. Let $\eta : R \to S$ be the canonical map from R onto its shortening quotient S.

If $r \in R$ is contained in a subgroup generated by a non-CMQ, non-abelian vertex group and centralizers of the edge groups connected to it in the graph of groups obtained from the cyclic JSJ of R by replacing each abelian vertex group by the direct summand containing the edge groups connected to it, or in a cyclic edge group in the cyclic JSJ decomposition of R, then $\varphi(r)$ is conjugate to r for all $\varphi \in Mod(R)$, hence, for all indices $n > n_r$: $h_n(r) \neq 1$ in F_k, so by lemma 1.3 $\eta(r) \neq 1$ in the shortening quotient S and every subgroup generated by a non-CMQ, non-abelian vertex group and centralizers of edge groups connected to it in the graph of groups obtained from the cyclic JSJ of R by replacing each abelian vertex group by the

direct summand containing the edge groups connected to it, and every cyclic edge group in the cyclic JSJ decomposition of R is mapped monomorphically into S by the canonical map η.
Similarly, if A is an abelian vertex group in the cyclic JSJ decomposition of R and $A_1 < A$ is the subgroup generated by all edges connected to the vertex stabilized by A in the cyclic JSJ decomposition, then every modular automorphism $\varphi \in Mod(R)$ maps the subgroup A_1 into a conjugate, hence, the canonical map η maps A_1 monomorphically into the shortening quotient S.

Let V be a CMQ-vertex group in the cyclic JSJ decomposition of R. Since the homomorphism h_n is supposed to map the ball of radius n in R monomorphically into F_k, and since V is non-abelian in R, there must exist some index n_0, so that for all indices $n > n_0$, $h_n(V)$ is a non-abelian subgroup of F_k. Since for every modular automorphism $\varphi \in Mod(R)$, $\varphi(V)$ is conjugate to V, $\hat{h}_n(V)$ is non-abelian subgroup of F_k for all $n > n_0$ as well. By lemma 1.3 this implies that $\eta(V)$ is a non-abelian subgroup of F_k. A modular automorphism of R map the edge groups in the canonical JSJ decomposition of R into their conjugates, hence, the edge groups in the cyclic JSJ decomposition of R are mapped to non-trivial subgroups of S by the canonical map η.

So far we have shown that the shortening quotient S is a strict shortening quotient of R. If S is not a maximal shortening quotient of R then there must exist a maximal shortening quotient M of R so that $M > S$. Clearly, since S is a strict shortening quotient of R, the maximal shortening quotient $M > S$ must be strict as well.

Therefore, we have shown that for every freely-indecomposable limit group there exists a strict maximal shortening quotient. Now, by the way the Makanin-Razborov diagram of a limit group is constructed together with the existence of a strict maximal shortening quotient for every freely-indecomposable limit group, it follows that every Makanin-Razborov diagram contains a strict Makanin-Razborov resolution.

□

Proposition 5.10 shows that a limit group admits a strict Makanin-Razborov resolution. To state a criteria for a f.g. group to be a limit group we need to define an analogue of a strict Makanin-Razborov resolution in the general context of f.g. groups.

Definition 5.11. *Let G be a f.g. group, and let:*

$$G = G_0 \xrightarrow{\nu_0} G_1 \xrightarrow{\nu_1} G_2 \xrightarrow{\nu_2} \ldots \xrightarrow{\nu_{m-2}} G_{m-1} \xrightarrow{\nu_{m-1}} G_m = F_u$$

be a resolution of G. We say that the given resolution of G is a strict MR resolution if the epimorphisms ν_i have the following properties.

First, we start with a (possible) free factorization of G, $G = G^1 * \ldots * G^n * F^s$, where F^s is a free subgroup of G. Each of the factors G^j of $G = G_0$ is mapped by the epimorphism $\nu_0 : G \to G_1$ onto the factor Q^j in a free factorization $G_1 = Q^1 * \ldots * Q^n$ of G_1. We further assume that each G^j admits a non-trivial cyclic splitting Λ_{G^j} with the following properties:

- (i) *each cyclic edge group in Λ_{G^j} is a maximal abelian subgroup in at least one of the vertex groups it is connected to.*
- (ii) *ν_0 maps each of the subgroups generated by a non-QH (quadratically hanging), non-abelian vertex groups and the centralizers of the edges connected*

to it in the graph of groups obtained from the given cyclic splitting by replacing each abelian vertex group with the direct summand containing the edge groups connected to it, and each of the cyclic edge groups in Λ_{G^j} monomorphically into Q^j.

iii) *ν_0 maps each QH vertex group in Λ_{G^j} into a non-abelian subgroup of Q^j.*

(iv) *every abelian vertex group in Λ_{G^j} is non-cyclic free abelian, and if A is an abelian vertex group in Λ_{G^j}, and $A_1 < A$ is the subgroup generated by all edge groups connected to the vertex stabilized by A in Λ_{G^j}, then A_1 is mapped monomorphically into Q^j by the map ν_0.*

Finally, we assume that the epimorphisms ν_i associated with the next levels of the resolution of G satisfy similar conditions to the ones listed for ν_0, and the resolution terminates when the target groups are free (like in the construction of a Makanin-Razborov resolution of a limit group).

Note that a strict Makanin-Razborov resolution of a limit group R is a strict MR resolution of R, so by proposition 5.11 a limit group admits a strict MR resolution. Theorem 5.12 shows that this is also a sufficient condition for a f.g. group to be a limit group.

Theorem 5.12. *A f.g. group G is a limit group (ω-residually free group) if and only if it admits a strict MR resolution.*

Proof: Let:

$$G = G_0 \xrightarrow{\nu_0} G_1 \xrightarrow{\nu_1} G_2 \xrightarrow{\nu_2} \dots \xrightarrow{\nu_{m-2}} G_{m-1} \xrightarrow{\nu_{m-1}} G_m = F_u$$

be a strict MR-resolution of a f.g. group G, and let $\{\Lambda_{G_i}\}$ be the cyclic splittings associated with each of the subgroups G_i. We need to show the existence of sequences of modular automorphsims $\{\varphi_i(n) \in Mod(G_i)\}$ for $i = 0, \dots, m-1$ so that the limit group corresponding to the sequence of homomorphisms:

$$\{h_n : G \to F_k \mid h_n = \nu_{m-1}\varphi_{m-1}(n)\nu_{m-2}\varphi_{m-2}(n) \dots \nu_1\varphi_1(n)\nu_0\varphi_0(n)$$

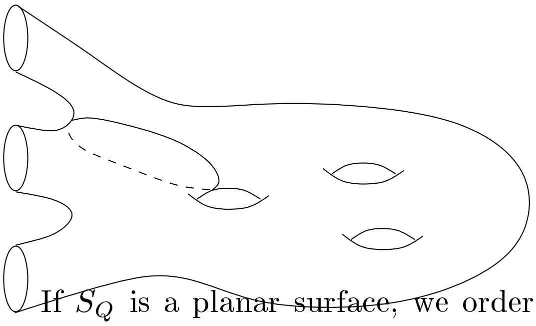
is the group G itself.
If none of the cyclic splittings Λ_{G_i} contains a QH vertex groups, we may pick the automorphisms $\{\varphi_i(n)\}$ to be an increasing sequence of Dehn twists corresponding to the edges in the cyclic splittings Λ_{G_i}. In the presence of QH vertex groups we need the following technical lemma.

Lemma 5.13. *Let Q be the fundamental group of a (possibly punctured) surface S_Q of Euler characteristic at most -2. Let $\mu : Q \to F_k$ be a homomorphism and suppose Q is mapped into a non-abelian subgroup of F_k and the image of every boundary component of Q is non-trivial. Then either:*

(i) *there exists a separating s.c.c. $\gamma \subset S_Q$ such that γ is mapped non-trivially into F_k, and the image in F_k of the fundamental groups of each of the connected components obtained by cutting S_Q along γ is non-abelian.*

(ii) *there exists a non-separating s.c.c. $\gamma \subset S_Q$ such that γ is mapped non-trivially into F_k, and the image of the fundamental group of the connected component obtained by cutting S_Q along γ is non-abelian.*

Proof: First, assume S_Q is orientable. If S_Q is a closed surface or a surface with a single puncture then clearly either (i) or (ii) holds. If S_Q has more than one boundary component and S_Q is not a planar surface, then S_Q is the connected sum of two orientable surfaces S_1 and S_2, so that S_1 has only one boundary component and S_2 is a planar surface. If the fundamental group of S_1 is mapped into a non-abelian subgroup of F_k, then either (i) or (ii) hold as in the case of a surface with at most one puncture. If the fundamental group of S_1 is mapped into a cyclic subgroup of F_k, then the cyclic subgroup corresponding to the boundary of S_1 is necessarily mapped into the trivial subgroup in F_k. If the fundamental group of S_1 is mapped into a non-trivial cyclic subgroup of F_k, then there must exists a non-separating s.c.c. in S_1 that is mapped non-trivially into F_k, so case (ii) holds. If the fundamental group of S_1 is mapped trivially into F_k, then there exists a non-boundary-parallel non-separating s.c.c. in S_Q that is mapped non-trivially into F_k, and again case (ii) holds.



If S_Q is a planar surface, we order its boundary components $b_1, \dots, b_n$. Note that by our assumptions each boundary component b_i is mapped non-trivially into F_k, and not all the images of the b_i's in F_k commute. It is not hard to see that there must exist a couple of consecutive boundary components b_i, b_{i+1} so that the images of b_i and b_{i+1} in F_k do not commute, and the images of the elements $b_1, \dots, b_{i-1}, b_i b_{i+1}, b_{i+2}, \dots, b_n$ generate a nonabelian subgroup in F_k. Hence, the separating s.c.c. in S_Q corresponding to the element $b_i b_{i+1}$ is mapped to a non-trivial element in F_k, and the fundamental groups of each of the connected components obtained by cutting S_Q along this separating s.c.c. are mapped onto non-abelian subgroups of F_k.

Now assume S_Q is non-orientable. We can view S_Q as the connected sum of an orientable surface O (with boundary) and a Mobius band M. If the fundamental group of M is mapped non-trivially into F_k, all the boundary components of O are mapped non-trivially into F_k, and the fundamental group of O is mapped onto a non-abelian subgroup of F_k, so our arguments above apply for the orientable surface O, and there exists a s.c.c. γ with the required properties.

If S_Q is a closed non-orientable surface, then O is an orientable surface with only one boundary component, and the fundamental group of O is mapped to a non-abelian subgroup of F_k, hence, there exists a non-separating s.c.c. $\gamma \subset O$ so that γ is mapped non-trivially into F_k and the fundamental group of the connected component obtained by cutting S_Q along γ is mapped to a non-trivial subgroup of F_k.

If S_Q has boundary, and the fundamental group of M is mapped to the identity in F_k, then there exists a non-separating s.c.c. $\gamma \subset S_Q$ that cuts the Mobius band M, so that γ is mapped non-trivially into F_k and the fundamental group of the connected component obtained by cutting S_Q along γ is mapped to a non-abelian subgroup of F_k.

□

By recursively applying lemma 5.13, for each surface S_{Q_j} corresponding to a QH-vertex group in the cyclic splitting Λ_{G_i}, we can find a finite set of s.c.c. on S_{Q_j}, so that each connected component of the surface obtained by cutting S_{Q_j} along this family of s.c.c. has Euler characteristic -1, and the fundamental group of each of these connected components is mapped monomorphically into G_{i+1} under the map ν_i.
If we extend each of the cyclic splittings Λ_{G_i} by further splitting the QH-vertex groups along the families of s.c.c. chosen according to lemma 5.13, the argument used to prove theorem 5.12 in the absence of QH-subgroups generalizes to a give a proof in their presence.

□

§6. An Example: ω-Residually Free Towers

In the first 5 sections of this paper we studied the structure of ω-residually free groups, using their JSJ decomposition, the analysis lattice and the Makanin-Razborov diagram and resolutions. Before applying the results and techniques used in studying ω-residually free groups to study residually free groups and sets of solutions to equations in a free group which is the goal of this paper, we devote this section to some natural families of examples of ω-residually free groups.

If R_1 and R_2 are ω-residually free groups then $R_1 * R_2$ is clearly an ω-residually free group as well. If R is an ω-residually free group, and $r \in R$ is an element with cyclic centralizer and no non-trivial roots in R, then the "double of R along $< r >$", $D_R = R *_{<r>} R$, admits an obvious strict MR resolution, so by theorem 5.12 it is ω-residually free. In particular, since a free and surface groups are ω-residually free, every double of a free or a surface group along a maximal cyclic subgroup is an ω-residually free group. Another more "geometric" family of ω-residually free groups, which in some way is the motivation for most of the objects we have associated with a general ω-residually free group, is the family of ω-residually free towers.
Note that an ω-residually free tower is similar to the notion of a coordinate group of a non-degenerate quasiquadratic system introduced in [Kh-My].

Definition 6.1. *We say that a f.p. group G is an ω-residually free tower, if it is possible to find a finite set set of generators $G =< \{y^\ell_j\} >$ and defining relations $\{\varphi^\ell_b\}$ for G, where $1 \leq \ell \leq \ell(G)$, $1 \leq j \leq m_\ell$ and $1 \leq b \leq r_\ell$, so that G can be presented using the following hierarchical structure which we define inductively:*

- (i) *G^0 is the trivial subgroup of G.*
- (ii) *with each level ℓ of the hierarchy we associate a group G^ℓ. G^ℓ is the subgroup of G generated by the elements $G^\ell =< \{y^i_j\} >$ where $1 \leq i \leq \ell$ and $1 \leq j \leq m_i$ in G. G^ℓ is the quotient of the free group generated by the elements $\{y^i_j\}$ for $1 \leq i \leq \ell$ and $1 \leq j \leq m_i$ divided by the normal closure of the set of relations φ^i_b where $1 \leq i \leq \ell$ and $1 \leq b \leq r_i$.*
- (iii) *To define the the ℓ-th level, the generators $y^\ell_1, \ldots, y^\ell_{m_\ell}$ are divided into s^ℓ disjoint blocks $y^\ell_1, \ldots, y^\ell_{b^\ell_1}$, $y^\ell_{b^\ell_1+1}, \ldots, y^\ell_{b^\ell_2}$, $\ldots$, $y^\ell_{b^\ell_{s^\ell-1}+1}, \ldots, y^\ell_{m_\ell}$ where to the p-th block of y^ℓ_j's one of the following sets of defining relations is associated:*
  - *(1) a free block. no relation is associated to a free block of generators.*

(2) *an Abelian block. the relations* $[y^\ell_{j_1}, y^\ell_{j_2}] = 1$ *for every* $b^\ell_{p-1}+1 \leq j_1 < j_2 \leq b^i_p$*, and in addition the relation* $[\bar{r}, y^\ell_j] = 1$ *for every* $b^\ell_{p-1}+1 \leq j \leq b^\ell_p$ *and some element* $\bar{r} \in G^{\ell-1}$ *where* $\bar{r}$ *may be the identity element, if* $\bar{r}$ *is non-trivial it can not be conjugated into an Abelian block in lower levels, and* $\bar{r}$ *is not a proper power of an element in* $G^{\ell-1}$*.*

(3) *a quadratic block. A single relation which is in one of the following two forms is being associated with a quadratic block. To save notation we denote the set of generators in the p-th block of* y^ℓ_j *which is a quadratic block by* $y_1, \dots, y_{2g}, z_1, \dots, z_n$ *in the first case and by* $y_1, \dots, y_g, z_1, \dots, z_n$ *in the second. With this notation the relation associated with a quadratic block is either (orientable):*

$$[y_1, y_2] \dots [y_{2g-1}, y_{2g}] z_1 \bar{r}_1 z_1^{-1} \dots z_n \bar{r}_n z_n^{-1} =$$
$$[\bar{y}_1, \bar{y}_2] \dots [\bar{y}_{2g-1}, \bar{y}_{2g}] \bar{z}_1 \bar{r}_1 \bar{z}_1^{-1} \dots \bar{z}_n \bar{r}_n \bar{z}_n^{-1}$$

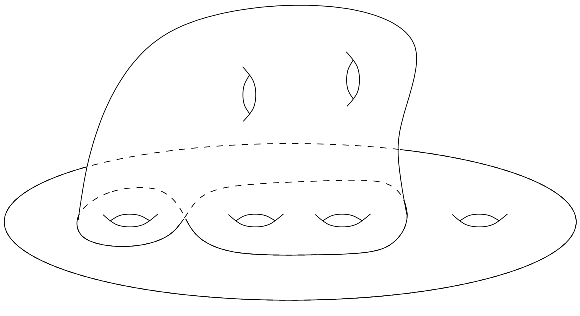
or (non-orientable):

$$y_1^2 \dots y_{2g}^2 z_1 \bar{r}_1 z_1^{-1} \dots z_n \bar{r}_n z_n^{-1} =$$
$$\bar{y}_1^2 \dots \bar{y}_{2g}^2 \bar{z}_1 \bar{r}_1 \bar{z}_1^{-1} \dots \bar{z}_n \bar{r}_n \bar{z}_n^{-1}$$

where $\bar{r}_j$*,* $\bar{y}_j$ *and* $\bar{z}_j$ *are elements in the group* $G^{\ell-1}$*, the subgroups* $< \{\bar{r}_j\}, \{\bar{y}_j\} >$ *and* $< \{\bar{r}_j\}, \{\bar{z}_j\} >$ *are non-abelian subgroups of* $G^{\ell-1}$*, the element* r_i *is conjugate to the element* $\bar{r}_i$ *in* G^ℓ*, and the (punctured) surface corresponding to one of the above relations has Euler characteristic at most -2, or it is a punctured torus.*

Since the tower G can be described as an iterated sequence of amalgamated products, its level groups G^ℓ are naturally embedded in the ω-residually free tower G. From the definition of towers it is clear they admit a strict MR resolution, so by theorem 5.12 they are ω-residually free groups. It is also easy to find their cyclic JSJ decomposition and their analysis lattice.

With an ω-residually free tower G we can naturally associate a 2-complex $T(G)$, constructed hierarchically by adding a bouquet of circles for each free block, a collection of 2-tori for each Abelian block, and a (punctured) surface for each quadratic block. Clearly, $\pi_1(T(G)) = G$.



§7. Residually Free Groups

In the previous 6 sections we have studied the algebraic structure of limit groups which are ω-residually free groups. When we get to study the structure of sets of solutions to equations in a free group in the next section, which we view as the analogue of algebraic varieties, ω-residually free groups are going to serve as the (dual of the) analogue of irreducible varieties.

As we have already pointed out in the first section sets of solutions to a system of equations Φ in a free group F_k are equivalent to the set of all homomorphisms from the naturally associated group $G(\Phi)$ into F_k (if the system Φ is with coefficients, we look only at homomorphisms that send the coefficients to the appropriate elements in F_k). In studying all homomorphisms from $G(\Phi)$ into a free group F_k, we are not really recovering the algebraic structure of the group $G(\Phi)$ itself, but rather the algebraic structure of its *residually free quotient*.

Definition 7.1. *Let G be a f.g. group. We denote by $KF(G)$ the intersection of all the kernels of homomorphisms from G into a free group. We call $RF(G) = G/KF(G)$ the residually free quotient of G.*

Clearly, a residual free quotient is always a residually free group (see definition 4.5) and the residually free quotient of a residually free group H is the group H itself. Therefore, to understand the structure of residually free quotients of f.g groups, which is equivalent to the understanding of sets of solutions to equations in a free group, we need to study the algebraic structure of residually free groups. In this section we will use some of the arguments and results obtained in earlier sections on the structure of ω-residually free groups, to recover the structure of residually free groups.

Let RF be a f.g. freely-indecomposable residually free group. On the set of ordered couples $\{(\tau, R)\}$, where R is an ω-residually free quotient of RF, and τ is an epimorphism $\tau : RF \to R$, we define an equivalence relation. We say that (R_1, τ_1) is equivalent to (R_2, τ_2) if there exists an isomorphism $\nu : R_1 \to R_2$ so that $\tau_2 = \nu \circ \tau_1$.
On the same set of ordered couples we also define a partial order. We say that $(R_1, \tau_1) > (R_2, \tau_2)$ if R_2 is a proper quotient of R_1 and there exists some epimorphism $\lambda : R_1 \to R_2$ so that $\tau_2 = \lambda \circ \tau_1$.

An algebraic variety over an algebraically closed field is a union of irreducible varieties. The following, which is the main result of this section, is the analogue of this statement for varieties of solutions over a free group.

Theorem 7.2. *Let RF be a f.g. residually free group. Then there exists a (canonical) finite collection of ω-residually free quotients of RF: $R_1, \dots, R_\ell$ and for each i an epimorphism $\tau_i : RF \to R_i$ with the following properties:*

(i) *for each homomorphism $h : R \to F_k$ there exists some index i (not necessarily unique!) and a homomorphism $f : R_i \to F_k$ so that h splits as: $h = f \circ \tau_i$.*

(ii) *if R is an ω-residually free quotient of RF, and τ is an epimorphism $\tau : RF \to R$, then there exists some index i and an epimorphism $\lambda : R_i \to R$ so that $\tau = \lambda \circ \tau_i$.*

(iii) *the couples $\{(R_i, \tau_i)\}$ represent distinct equivalence classes, and if $i \neq j$ then the couple (R_i, τ_i) can not be compared with (R_j, τ_j) in the partial order defined on these couples.*

(iv) *if $\hat{R}_1, \dots \hat{R}_u$ and $\{\hat{\tau}_j : RF \to \hat{R}_j\}$ is another collection of ω-residually free quotients of RF and epimorphisms from RF to the $\hat{R}_i$'s that satisfy* *properties (i)-(iii) then $u = \ell$ and up to permutation the couple (R_i, τ_i) is equivalent to the couple $(\hat{R}_i, \hat{\tau}_i)$.*

Proof: The collection of ordered couples $\{(R, \tau)\}$ where R is an ω-residually free quotient of RF and $\tau : RF \to R$ is an epimorphism, has similar properties to the partial order defined on shortening quotients of an ω-residually free group in section 5.

Lemma 7.3. *Let RF be a f.g. residually free group. Let $(R_1, \tau_1), (R_2, \tau_2), (R_3, \tau_3), \ldots$ be a sequence of ω-residually free quotients of RF and suppose $(R_1, \tau_1) < (R_2, \tau_2) < (R_3, \tau_3) < \ldots$ is an infinite properly increasing sequence.*
Then there exists an ω-residually free quotient R of RF and an epimorphism $\tau : RF \to R$ so that $(R, \tau) > (R_j, \tau_j)$ for every ω-residually free quotient (R_j, τ_j) in the increasing sequence.

Proof: identical with the proof of lemma 5.4.

□

Lemma 7.3 proves the existence of maximal elements with respect to the partial order on the set of ω-residually free quotients of the residually free group R. The next lemma shows that there are only finitely many equivalence classes of maximal elements in the set of ω-residually free quotients of RF.

Lemma 7.4. *Let RF be a f.g. residually free group. The set of ordered couples $\{(R, \tau)\}$, where R is an ω-residually free quotient of RF and $\tau : RF \to R$ is an epimorphism, contains only finitely many equivalence classes of maximal elements with respect to its partial order.*

Proof: identical with the proof of lemma 5.5.

□

Now, if $(R_1, \tau_1), \ldots, (R_k, \tau_k)$ are representatives for the distinct equivalence classes of maximal ω-residually free quotients of the residually free group RF, then $(R_1, \tau_1), \ldots, (R_k, \tau_k)$ satisfy the conclusion of theorem 7.2.

□

We will call the groups $R_1, \ldots, R_k$ together with the epimorphisms $\tau_i : RF \to R_i$ that appear in the formulation of theorem 7.2, the (canonical) (maximal) ω-residually free groups associated with the f.g. residually free group RF. By theorem 7.2 these are canonically defined up to equivalence.

In section 5 we have associated a Makanin-Razborov diagram with each ω-residually free group. This diagram was canonically defined and encodes all homomorphisms from an ω-residually free group into free groups. Theorem 7.2 allows us to generalize the construction of the Makanin-Razborov diagram to residually-free groups.
Let RF be a f.g. residually free group. We start with one vertex labeled RF and continue the diagram by adding an edge from this vertex to each of its associated maximal ω-residually free quotients, and from each of the maximal ω-residually free quotients we continue by adding its Makanin-Razborov diagram. By theorem 7.2 and the properties of the Makanin-Razborov diagram of ω-residually free groups (see theorem 5.7), each homomorphism from the residually free group RF into some free group F_k can be "read" from the Makanin-Razborov diagram of RF.

In theorem 5.12 we have shown that a f.g. group is ω-residually free if and only if it admits a strict MR resolution. Theorem 7.2 allows us to get corresponding classification of residually free groups. Note that a similar result is obtained in [Kh-My] (corollary 2).

Claim 7.5. *A f.g. group G is a residually free group if and only if G is a sub-direct product of a finite collection of ω-residually free groups.*

Proof: Clearly, a sub-direct product of a finite collection of residually free groups is residually free, so one direction is obvious. The other direction of the claim follows from theorem 7.2, since a residually free group is a sub-direct product of its associated maximal ω-residually free quotients.

□

Remarks:

(i) Corollary 4.4 shows that ω-residually free groups are f.p. . However f.g. residually-free groups need not be f.p. . If F_g is a free group then $F_g + F_g$ is residually free, so every subgroup of $F_g + F_g$ is residually free, and $F_g + F_g$ contains f.g. subgroups which are not f.p. .

(ii) There is no global bound for the length of a sequence of proper residually-free quotients of a residually-free group.

§8. Sets of Solutions to Equations in a Free Group

In the first section we have pointed out that the sets of solutions to a system of equations Φ in a free group is equivalent to the sets of homomorphisms from a corresponding f.p. group $G(\Phi)$ to a free group.
If the system of equations is without coefficients, then the sets of homomorphisms from $G(\Phi)$ into a free group can be "read" from the Makanin-Razborov diagram corresponding to $G(\Phi)$ as was shown in the previous section. If the system Φ is with coefficient one needs to modify the whole theory developed in the previous sections to the presence of prescribed coefficients. The rest of this section is devoted to this modification.

Let Φ be a finite system of equations defined over a free group $F_k =< a_1, \dots , a_k >$. Recall that with Φ it is natural to associate a f.p. group $G(\Phi)$. If the system Φ is defined by the coefficients $a_1, \dots , a_k$, the unknowns $x_1, \dots , x_n$ and the equations:

$$w_1(a_1, \dots , a_k, x_1, \dots , x_n) = 1$$
$$\vdots$$
$$w_s(a_1, \dots , a_k, x_1, \dots , x_n) = 1$$

we set the associated f.p. group $G(\Phi)$ to be:

$$G(\Phi) \;=<\; a_1, \dots , a_k, x_1, \dots , x_n \,|\, w_1, \dots , w_s \;>$$

Clearly, every solution of the system Φ corresponds to a homomorphism $h : G(\Phi) \to F_k$ for which $h(a_i) = a_i$, and every such homomorphism corresponds to a solution of the system Φ. Therefore, the study of sets of solutions to systems of equations in a free group is equivalent to the study of all homomorphisms from a fixed f.p. group G into a free group F_k for which a given prescribed set of elements in G is mapped to a fixed basis of the free group F_k.

For the rest of this section, we set G to be a f.g. group $G =< g_1, \ldots, g_m >$, $\gamma_1, \ldots, \gamma_k \in G$ to be a prescribed set of elements in G, and F_k to be a free group with a fixed basis $F_k =< a_1, \ldots, a_k >$. With these notation our main goal is to get a structure theory for understanding the set of all homomorphisms:

$$Hom((G, \{\gamma_i\}), (F_k, \{a_i\})) \ = \ \{ h \mid h : G \to F_k \, , \, h(\gamma_i) = a_i \}$$

which for brevity we will denote $Hom(G, F_k)$. We set X to be the Cayley graph of F_k with respect to its given basis $a_1, \ldots, a_k$. Given any homomorphism $h \in Hom(G, F_k)$, G admits a natural action λ_h on X given by $\lambda_h(g, x) = h(g)(x)$ for every $g \in G$ and $x \in X$.

Let $\{h_j\} \subset Hom(G, F_k)$ be a set of distinct homomorphisms from G to F_k satisfying the prescribed conditions. We set μ_j to be: $\mu_j = \max_{1 \leq u \leq m} d_X(id., h_j(g_u))$ (note that unlike what we did in section 1, here, due to the presence of coefficients, we do not compose with an inner automorphism of F_k). Since the homomorphisms in the sequence $\{h_j\} \subset Hom(G, F_k)$ are distinct, the sequence of stretching factors $\{\mu_j\}$ does not contain a bounded subsequence. We set $\{(X_j, x_j)\}_{j=1}^{\infty}$ to be the pointed metric spaces obtained by rescaling the metric on the Cayley graph of F_k, $(X, id.)$, by μ_j. (X_j, x_j) is endowed with a left isometric action of our f.g. group G via the homomorphisms $\{h_j\}$. This sequence of actions of G on the metric spaces $\{(X_j, x_j)\}_{j=1}^{\infty}$ allows us to obtain an action of G on a real tree (Y, y_0) by passing to a Gromov-Hausdorff limit of a convergent subsequence (proposition 1.1).

With the limit tree we obtained by using the Gromov-Hausdorff topology we associate natural algebraic objects, the *kernel of the action* of G on this (limit) real tree and the quotient of G by this kernel which we call the *restricted limit group*. We will denote the restricted limit group by RL_∞, and the canonical quotient map by $\eta : G \to RL_\infty$. Note that since a restricted limit group is a limit group in the sense of section 1, all the algebraic properties of limit groups hold for restricted limit groups. Still, for the purpose of understanding the set of all *restricted* homomorphisms from G to the (fixed) free group F_k we will need to modify some of the algebraic objects associated in the previous sections with limit groups, to study restricted limit groups.

The abelian JSJ decomposition of a freely-indecomposable limit group "encodes" all the possible splittings of such limit group with abelian edge groups. In studying restricted limit groups we will be interested only abelian splittings in which the subgroup $< \eta(\gamma_1), \ldots, \eta(\gamma_k) >$ is elliptic. The construction of the abelian JSJ decomposition of a limit group described in section 2, naturally generalizes to a *restricted* abelian JSJ decomposition of a restricted limit group.

Theorem 8.1 (cf. theorem 2.7). *Suppose RL_∞ is a restricted limit group which does not split to a non-trivial free decomposition in which the subgroup $\eta(\Gamma) =< \eta(\gamma_1), \ldots, \eta(\gamma_k) >$ is contained in one of the factors. Suppose that $\eta(\Gamma)$ is a proper subgroup of RL_∞. Then there exists a reduced unfolded splitting of RL_∞ with abelian edge groups, which we call an abelian restricted JSJ (Jaco-Shalen-Johannson) decomposition of RL_∞ with the following properties:*

(i) *$\eta(\Gamma)$ is elliptic in the restricted abelian JSJ decomposition.*

(ii) *Every (restricted) canonical maximal QH subgroup (CMQ) of RL_∞ is conjugate to a vertex group in the JSJ decomposition. Every (restricted) QH subgroup of RL_∞ can be conjugated into one of the CMQ subgroups of RL_∞. Every vertex group in the JSJ decomposition which is not a CMQ subgroup of RL_∞ is elliptic in any abelian splitting of RL_∞ under consideration (i.e., in one in which $\eta(\Gamma)$ is elliptic).*

(iii) *A one edge abelian splitting $RL_\infty = D *_A E$ or $RL_\infty = D*_A$ in which $\eta(\Gamma)$ is elliptic which is hyperbolic in another such elementary abelian splitting is obtained from the restricted abelian JSJ decomposition of RL_∞ by cutting a 2-orbifold corresponding to a CMQ subgroup of RL_∞ along a weakly essential s.c.c..*

(iv) *Let Θ be a one edge splitting along an abelian subgroup A, $RL_\infty = D *_A E$ or $RL_\infty = D*_A$ in which $\eta(\Gamma)$ is elliptic, so that A is elliptic with respect to any other one edge abelian splitting of RL_∞ in which $\eta(\Gamma)$ is elliptic. Then Θ is obtained from the restricted JSJ decomposition of RL_∞ by a sequence of collapsings, foldings, and conjugations.*

(v) *If JSJ_1 is another restricted abelian JSJ decomposition of RL_∞, then JSJ_1 is obtained from the restricted abelian JSJ decomposition by a sequence of slidings, conjugations and modifying boundary monomorphisms by conjugations (see section 1 of [Ri-Se2] for these notions)*

From the canonical properties of the abelian JSJ decomposition of a limit group we were able to show in section 3 that a non-abelian, freely indecomposable limit group admits a principal cyclic splitting. The arguments used in proving theorem 3.2 naturally generalize to prove the existence of a (restricted) principal cyclic splitting of a restricted limit group.

Theorem 8.2. *Let RL_∞ be a restricted limit group which does not split to a non-trivial free product in which the subgroup $\eta(\Gamma)$ is contained in one of the factors. Suppose that $\eta(\Gamma)$ is a proper subgroup of RL_∞. Then there exists a principal cyclic splitting of RL_∞ in which $\eta(\Gamma)$ is elliptic.*

The restricted abelian JSJ decomposition together with the existence of a restricted principal splitting, allow us to generalize the arguments of section 3 to construct the restricted cyclic JSJ decomposition of a restricted limit group (cf. theorem 3.9).

Since a restricted limit group is in particular a limit group, it is finitely presented (in fact, it is also possible to modify the construction of the analysis lattice to the restricted setup). Like in the case of limit groups, the finite presentability of restricted limit groups also shows that restricted limit groups are restricted ω-residually free groups. i.e., that for every finite set of elements in a restricted limit group RL_∞ there exists a restricted homomorphism $h : RL_\infty \to F_k$ that maps the elements in this finite set to distinct elements in F_k.
In section 5 we have used the canonical cyclic JSJ decomposition of a limit group to construct its canonical Makanin-Razborov diagram. We continue by modifying the notions and arguments appear in section 5 to construct the restricted Makanin-Razborov diagram of a restricted limit group.

Definition 8.3. *Let RL_∞ be a restricted limit group which does not split to a non-trivial free product in which the subgroup $\eta(\Gamma)$ is contained in one of the factors.*

Suppose that $\eta(\Gamma)$ is a proper subgroup of RL_∞. We define the restricted modular group $RMod(RL_\infty)$ to be the subgroup of $Aut(R)$ generated by the following families of automorphisms of R:

(i) *Dehn twists along edges of the restricted cyclic JSJ decomposition of RL_∞. The Dehn twists are assumed to fix (elementwise) the vertex stabilized by $\eta(\Gamma)$.*

(ii) *Dehn twists along essential s.c.c. in CMQ (canonical maximal quadratically hanging) vertex groups in the restricted cyclic JSJ decomposition of RL_∞. Again, these Dehn twists are assume to fix (elementwise) the vertex stabilized by $\eta(\Gamma)$.*

(iii) *Let A be an abelian vertex group in the restricted cyclic JSJ decomposition of RL_∞. Let $A_1 < A$ be the subgroup generated by all edge groups connected to the vertex stabilized by A in the cyclic JSJ decomposition of R. Every automorphism of A that fixes A_1 (elementwise) can be naturally extended to an automorphism of the ambient limit group RL_∞ that fixes the vertex stabilized by $\eta(\Gamma)$. We call these generalized Dehn twists and they form the third family of automorphisms that generate $RMod(R)$.*

Given a sequence of homomorphisms of a limit group into a free group, we used the modular group of automorphisms of that limit group to modify the sequence of homomorphisms and obtain a shortening quotient of the limit group we started with. Given a restricted limit group, and a sequence of homomorphisms of that restricted limit group into F_k, we may use the restricted modular group of automorphisms of that restricted limit group to modify the sequence of homomorphisms and obtain a *restricted shortening quotient* of a restricted limit group. By the same argument used to prove claim 5.3 a restricted shortening quotient of a restricted limit group is a proper quotient of it.

On the set of restricted shortening quotients of a restricted limit group we can naturally define a partial order and an equivalence relation, similar to the ones defined in section 5 for shortening quotients of limit groups. By the same argument used to prove lemma 5.4 the set of restricted shortening quotients of a restricted limit group contains maximal elements with respect to the partial order, and by the same argument used to prove lemma 5.5 there are only finitely many equivalence classes of maximal restricted shortening quotients of a restricted limit group. Like in proposition 5.6 and using an identical argument, for every restricted homomorphism $RL_\infty \to F_k$ there exists some index $1 \leq i \leq k$ (not necessarily unique!) and a restricted modular automorphism $\varphi_h \in RMod(R)$ so that $h \circ \varphi_h$ splits through the maximal restricted shortening quotient RM_i. i.e., $h \circ \varphi_h = h_{RM_i} \circ \eta_i$ where $h_{RM_i} : RM_i \to F_k$ is a restricted homomorphism.

These basic facts concerning restricted limit groups and their restricted shortening quotients allow us to present the main goal of this section, the (canonical) *restricted Makanin-Razborov diagram* associated with a restricted limit group (restricted ω-residually free group). The restricted Makanin-Razborov diagram "encodes" all possible restricted homomorphisms from a limit group into the (fixed) free group F_k, and like for limit groups some of its properties can be stated as a criteria for a general f.g. group to be a restricted limit group.

Let RL_∞ be a restricted limit group (a restricted ω-residually free group). By definition in RL_∞ there exists a (distinguished) finite set of elements $< \gamma_1, \dots, \gamma_k >$

so that for any finite set U of elements in RL_∞ there exists a (restricted) homomorphism $h : RL_\infty \to F_k$ that sends γ_i to a_i, the i-th basis element of F_k, and sends the elements of U to distinct elements of F_k. Let $\Gamma = < \gamma_1, \ldots, \gamma_k >$.
Let $RL_\infty = RL_1 * \ldots RL_\ell * F_{g_1}$ be a factorization of R, where $\Gamma < RL_1$, RL_1 can not be written as a free product in which Γ is contained in one of the factors, each of the RL_i's for $2 \leq i \leq \ell$ is a freely-indecomposable non-cyclic subgroup of RL and F_{g_1} is a free group. It follows from Grushko's theorem that such a free decomposition of RL_∞ is canonical. By an analogue of lemma 5.4 to the restricted case, the set of restricted shortening quotients of the restricted limit group RL_1 contain maximal elements (with respect to the partial order defined above) and by an analogue of lemma 5.5 to the restricted case there are only finitely many equivalence classes of maximal restricted shortening quotients of RL_1. Let $RM_1, \ldots, RM_s$ be a collection of representatives of equivalence classes of maximal restricted shortening quotients of RL_1, and let $\eta_j : RL_1 \to M_j$ be the canonical projection maps.

We define the *restricted Makanin-Razborov diagram* of the restricted limit group RL_∞ iteratively. We start by factoring RL_∞ as above. From each of the factors $RL_2, \ldots, RL_\ell$ we continue by adjoining its Makanin-Razborov diagram presented in section 5. From RL_1 we continue along s edges going from RL_1 to each of its maximal restricted quotients $RM_1, \ldots, RM_s$. To each such directed edge we associate the canonical projection η_j.
We proceed iteratively. Since a restricted shortening quotient of a restricted limit group is a proper quotient of it, the construction terminates. Each terminal vertex is either a free group (as in the Makanin-Razborov diagram presented in section 5) or it is the (image of the) "distinguished" free group $\Gamma = < \gamma_1, \ldots, \gamma_k >$. At least one of the terminal vertices in a restricted Makanin-Razborov diagram must be labeled with an image of the "distinguished" subgroup Γ. By construction, up to the equivalence relation of restricted shortening quotients, the restricted Makanin-Razborov diagram of a restricted limit group is canonical. By an analogue of theorem 5.7 to the restricted case, all the restricted homomorphisms from a restricted limit group RL_∞ into the free group F_k are encoded by the canonical restricted Makanin-Razborov diagram. i.e., they are all given by compositions of (restricted) modular automorphisms of the (restricted) limit groups in the diagram with the canonical maps from the (restricted) limit groups into their maximal (restricted) shortening quotients and finally with either the fixed map from (an image of) the distinguished subgroup Γ onto F_k, or general homomorphisms ("substitutions") of the terminal (non-distinguished) free groups appear in the diagram into the target free group F_k.

The (canonical) restricted Makanin-Razborov diagram associated with a restricted limit group, encodes all the restricted homomorphsims from that restricted limit group into a free group F_k. Like in the non-restricted case (definitions 5.8 and 5.9) using the restricted Makanin-Razborov diagram we can define restricted Makanin-Razborov resolutions and restricted strict Makanin-Razborov resolutions of a restricted limit group. Like in the non-restricted case (theorem 5.12) the existence of a restricted strict MR resolution for a f.g. group G with a distinguished subgroup $\Gamma = < \gamma_1, \ldots, \gamma_k >$ is a criteria for G to be a restricted limit group (restricted ω-residually free group). The proofs are all direct modifications of the ones given in section 5 in the non-restricted case.

Like restricted limit groups (restricted ω-residually free groups) we may define in a similar way restricted residually free groups. The study of sets of solutions to equations with coefficients is naturally equivalent to study of all restricted homomorphisms from a f.g. restricted residually free group into the free group generated by the coefficients. For studying this collection of restricted homomorphisms we have an analogue of theorem 7.2. The proof is a direct modification.

Theorem 8.4. *Let RF be a f.g. restricted residually free group. Then there exists a (canonical) finite collection of restricted ω-residually free quotients of RF: $R_1, \ldots, R_\ell$ and for each i an epimorphism $\tau_i : RF \to R_i$ with the following properties:*

- (i) *for each restricted homomorphism $h : R \to F_k$ there exists some index i (not necessarily unique!) and a restricted homomorphism $f : R_i \to F_k$ so that h splits as: $h = f \circ \tau_i$.*
- (ii) *if R is a restricted ω-residually free quotient of RF, and τ is an epimorphism $\tau : RF \to R$, then there exists some index i and an epimorphism $\lambda : R_i \to R$ so that $\tau = \lambda \circ \tau_i$.*
- (iii) *the couples $\{(R_i, \tau_i)\}$ represent distinct equivalence classes, and if $i \neq j$ then the couple (R_i, τ_i) can not be compared with (R_j, τ_j) in the partial order defined on these couples.*
- (iv) *if $\hat{R}_1, \ldots \hat{R}_u$ and $\{\hat{\tau}_j : RF \to \hat{R}_j\}$ is another collection of restricted ω-residually free quotients of RF and epimorphisms from RF to the $\hat{R}_i$'s that satisfy properties (i)-(iii) then $u = \ell$ and up to permutation the couple (R_i, τ_i) is equivalent to the couple $(\hat{R}_i, \hat{\tau}_i)$.*

We have already modified the construction of the Makanin-Razborov diagram of a ω-residually free group to the restricted case. Having such diagram for restricted ω-residually free group, theorem 8.4 allows us to modify the construction of the Makanin-Razborov diagram associated with a residually free group presented in section 7 to the restricted case. This restricted Makanin-Razborov diagram of restricted residually free groups finally concludes our study of the structure of sets of solutions to systems of equations with coefficients.

§9. Graded Limit Groups

In the previous sections we have analyzed the structure of sets of solutions to systems of equations in a free group by studying the algebraic structure of limit groups which turned out to be ω-residually free groups. To understand the elementary theory of free groups, and in particular, to study the structure of sets defined by elementary predicates (*elementary sets*), we will need to study the structure of sets of solutions to systems of equations with parameters, i.e., we will examine how the (restricted) Makanin-Razborov diagrams and resolutions vary with a change of the values of the defining parameters.

With a finite system of equations with parameters Φ over a free group $F_k = < a_1, \ldots, a_k >$ it is natural to associate a f.p. group $G(\Phi)$. If the system Φ is defined by the coefficients $a_1, \ldots, a_k$, the parameters $p_1, \ldots, p_u$, the unknowns $x_1, \ldots, x_n$

and the equations:

$$w_1(a_1,\dots,a_k,p_1,\dots,p_u,x_1,\dots,x_n)=1$$
$$\vdots$$
$$w_s(a_1,\dots,a_k,p_1,\dots,p_u,x_1,\dots,x_n)=1$$

we set the associated f.p. group $G(\Phi)$ to be:

$$G(\Phi) \ = < \ a_1,\dots,a_k,p_1,\dots,p_u,x_1,\dots,x_n \,|\, w_1,\dots,w_s \ >$$

Clearly, every solution of the system Φ corresponds to a homomorphism $h: G(\Phi) \to F_k$ for which $h(a_i)=a_i$, and every such homomorphism corresponds to a solution of the system Φ. Therefore, the study of sets of solutions to systems of equations in a free group is equivalent to the study of all homomorphisms from a fixed f.p. group G into a free group F_k for which a given prescribed set of elements in G is mapped to a fixed basis of the free group F_k.

In the previous sections we studied the global set of (restricted) homomorphisms from a f.g. group G into a free group F_k, and later showed the existence of effective tools which enabled us to study the structure of this global set. In studying parametric systems of equations we will analyze the variation of this structure with a change of the defining parameters. Our whole study of the global set of restricted homomorphisms was based on understanding the algebraic structure of limit groups (ω-residually free groups). To analyze the variation in this structure with a change of the parameters we will need to look at a special class of limit groups which we call *graded limit groups*. Our study of graded limit groups is based on the tools we developed for studying (restricted) limit groups. However, as we will see in the sequel, not all the results we have obtained for (restricted) limit groups have analogues for graded limit groups, and the treatment of the new phenomenas that appear in studying graded limit groups, require new notions and techniques.

We set $G=< x_1,\dots,x_k,p_1,\dots,p_u,a_1,\dots,a_k >$ to be a f.g. group. For brevity we will denote this (fixed) generating set by $G=< g_1,\dots,g_m >$, and F_k to be a free group with a fixed basis $F_k=< a_1,\dots,a_k >$. With this notation our goal is to get a structure theory for understanding the set of all homomorphisms:

$$Hom((G,\{p_j\},\{a_i\}),(F_k,\{a_i\})) \ = \ \{\, h \,|\, h: G \to F_k \, , \, h(a_i)=a_i \,\}$$

with a change in the values of the parameters $\{p_j\}$. We set X to be the Cayley graph of F_k with respect to its given basis $a_1,\dots,a_k$. Given any homomorphism $h \in Hom(G,F_k)$, G admits a natural action λ_h on X given by $\lambda_h(g,x)=h(g)(x)$ for every $g \in G$ and $x \in X$.

Definition 9.1. *Let G be a f.g. group and let $h_n: G(x,p,a) \to F_k$ be a sequence of homomorphisms. If $\{\lambda_{h_n}\}$, a sequence of actions of G on X, the Cayley graph of F_k, that corresponds to the sequence of homomorphisms $\{h_n\}$, converges in the Gromov-Haussdorff topology, we say that the sequence $\{h_n\}$ converges. We call the obtained limit group, a graded limit group.*

A graded limit group is in particular a limit group, so every f.g. graded limit group is f.p. and every f.g. graded limit group is ω-residually free by theorem 4.6.

To construct the (canonical) graded Makanin-Razborov diagram of a graded limit group which is the fundamental tool in our analysis of the variation of sets of solutions to systems of equations with parameters, we need to study the algebraic structure of graded limit groups "relative" to their grading. To do that we need to construct the graded abelian JSJ decomposition of a graded limit group, and the graded analysis lattice of a graded limit group. These constructions are all generalizations of their restricted analogues presented in section 8.

For the rest of this section let $G(x,p,a)$ be a f.g. group, let $\{h_m : G(x,p,a) \to F_k\}$ be a convergent graded sequence of homomorphisms, and let $Glim(x,p,a)$ be the corresponding graded limit group. We set $\eta : G(x,p,a) \to Glim(x,p,a)$ to be the canonical epimorphism and $AP < Glim(x,p,a)$ to be the subgroup: $AP =< \eta(p_1),\dots,\eta(p_u),\eta(a_1),\dots,\eta(a_k) >$. The abelian JSJ decomposition of a freely-indecomposable limit group "encodes" all the possible splittings of such limit group with abelian edge groups. In studying graded limit groups we will be interested only in abelian splittings in which the subgroup AP is elliptic. The construction of the abelian JSJ decomposition of a limit group described in section 2, naturally generalizes to a *graded abelian* JSJ *decomposition* of a graded limit group.

Theorem 9.2 (cf. 8.1). *Suppose $Glim(x,p,a)$ is a graded limit group which does not split to a non-trivial free decomposition in which the subgroup AP is contained in one of the factors. Then there exists a (relatively) reduced unfolded (perhaps trivial) splitting of $Glim(x,p,a)$ with abelian edge groups, which we call a graded abelian JSJ (Jaco-Shalen-Johannson) decomposition of $Glim(x,p,a)$ with the following properties:*

- (i) *AP is elliptic in the graded abelian JSJ decomposition.*
- (ii) *Every (graded) canonical maximal QH subgroup (CMQ) of $Glim(x,p,a)$ is conjugate to a vertex group in the graded JSJ decomposition. Every (graded) QH subgroup of $Glim(x,p,a)$ can be conjugated into one of the CMQ subgroups of $Glim(x,p,a)$. Every vertex group in the graded JSJ decomposition which is not a CMQ subgroup of $Glim(x,p,a)$ is elliptic in any abelian splitting of $Glim(x,p,a)$ under consideration (i.e., in one in which AP is elliptic).*
- (iii) *A one edge (graded) abelian splitting $Glim(x,p,a) = D*_A E$ or $Glim(x,p,a) = D*_A$ in which AP is elliptic which is hyperbolic in another such elementary abelian splitting is obtained from the graded abelian JSJ decomposition of $Glim(x,p,a)$ by cutting a 2-orbifold corresponding to a (graded) CMQ subgroup of $Glim(x,p,a)$ along a weakly essential s.c.c..*
- (iv) *Let Θ be a one edge (graded) splitting along an abelian subgroup $Glim(x,p,a) = D*_A E$ or $Glim(x,p,a) = D*_A$ in which AP is elliptic, which is elliptic with respect to any other one edge (graded) abelian splitting of $Glim(x,p,a)$ in which AP is elliptic. Then Θ is obtained from the graded JSJ decomposition of $Glim(x,p,a)$ by a sequence of collapsings, foldings, and conjugations.*
- (v) *If JSJ_1 is another graded abelian JSJ decomposition of $Glim(x,p,a)$, then JSJ_1 is obtained from the graded abelian JSJ decomposition by a sequence of slidings, conjugations and modifying boundary monomorphisms by conjugations (see section 1 of [Ri-Se2] for these notions)*

§10. Graded Makanin-Razborov Resolutions and Diagram

In section 5 we have used the canonical cyclic JSJ decomposition of a limit group to construct its canonical Makanin-Razborov diagram. In section 8 we modified this construction to obtain the restricted Makanin-Razborov diagram for analyzing sets of solutions to equations with coefficients. To understand the variation of the restricted Makanin-Razborov diagram with a change of the defining parameters we need to further modify the construction of the Makanin-Razborov diagram to obtain the *graded Makanin-Razborov diagram* of a graded limit group. As we have already indicated, some of the results that were essential in the construction of the (ungraded) Makanin-Razborov diagram do not hold when graded limit groups and their graded decompositions are concerned. Hence, to construct the graded Makanin-Razborov diagram of a graded limit group, some new types of graded limit groups that do not have analogues in the ungraded case are presented, and the structure of the graded diagram as a whole requires far more delicate analysis than in the ungraded case. This analysis is presented in the rest of this paper and in the third paper of this sequence.

Definition 10.1. *Let $Glim(x,p,a)$ be a graded limit group which does not split to a non-trivial free product in which the subgroup $AP = < p_1,\dots,p_u,a_1,\dots,a_k >$ is contained in one of the factors. Suppose that the principal graded abelian JSJ decomposition of $Glim(x,p,a)$ is non-trivial. We define the graded modular group $GMod(Glim(x,p,a))$ to be the subgroup of $Aut(Glim(x,p,a))$ generated by the following families of automorphisms of $Glim(x,p,a)$:*

- (i) *Dehn twists along edges of the principal graded abelian JSJ decomposition of $Glim(x,p,a)$. The Dehn twists are assumed to fix (elementwise) the vertex stabilized by AP.*
- (ii) *Dehn twists along essential s.c.c. in (graded) CMQ (canonical maximal quadratically hanging) vertex groups in the principal graded abelian JSJ decomposition of $Glim(x,p,a)$. Again, these Dehn twists are assume to fix (elementwise) the vertex stabilized by AP.*
- (iii) *Let A be an abelian vertex group in the principal graded abelian JSJ decomposition of $Glim(x,p,a)$. Let $A_1 < A$ be the subgroup generated by all edge groups connected to the vertex stabilized by A in the principal graded abelian JSJ decomposition of $Glim(x,p,a)$. Every automorphism of A that fixes A_1 (elementwise) can be naturally extended to an automorphism of the ambient limit group $Glim(x,p,a)$ that fixes the vertex stabilized by AP. We call these generalized Dehn twists and they form the third family of automorphisms that generate $GMod(Glim(x,p,a))$.*

We say that a homomorphism $h: Glim(x,p,a) \to F_k$ is in the graded modular class of the homomorphism $\hat{h}: Glim(x,p,a) \to F_k$ if $\hat{h} = h \circ \varphi$, where φ is a graded modular automorphism of $Glim(x,p,a)$, i.e., $\varphi \in GMod(Glim(x,p,a))$. Clearly, a graded modular class is an equivalence class of homomorphisms from the graded modular group $Glim(x,p,a)$ to the free group F_k.

Let $Glim(x,p,a)$ be a graded limit group that admits a non-trivial graded abelian JSJ decomposition. Once we fix a generating set for the graded limit group

$Glim(x,p,a)$, given a homomorphism $h : Glim(x,p,a) \to F_k$ we can choose the shortest homomorphisms in the graded modular class of h. A graded limit group obtained from a convergent sequence of homomorphisms $h : Glim(x,p,a) \to F_k$ which are the shortest in their graded modular class, is called a *graded shortening quotient* of the graded limit group $Glim(x,p,a)$.

On the set of graded shortening quotients of a graded limit group we can naturally define a partial order and an equivalence relation, similar to the ones defined in section 5 for shortening quotients of limit groups. By the same argument used to prove lemma 5.4 the set of graded shortening quotients of a graded limit group contains maximal elements with respect to the partial order, and by the same argument used to prove lemma 5.5 there are only finitely many equivalence classes of maximal graded shortening quotients of a graded limit group.

So far we have modified the construction of maximal shortening quotients presented in section 5 to the graded setup. However, since in general the graded modular group of a graded limit group does not contain the (ungraded) modular group of the graded limit group (as a limit group), graded shortening quotients of a graded limit group need not be proper quotients of it. Furthermore, to get graded shortening quotients we naturally require that the graded abelian JSJ decomposition is non-trivial. However, in general the graded abelian JSJ decomposition of a graded limit group may be trivial. To view some of these phenomenas we present the following two examples:

(1) Let $G(x,p)$ be a graded limit group that is isomorphic to a non-abelian free group, and suppose that the graded abelian JSJ decomposition of $G(x,p)$ is: $G(x,p) = P *_C B$, where C is cyclic and P is the parameter subgroup. Since $G(x,p)$ is isomorphic to a free group, there is clearly an embedding: $h : G(x,p) \to F_k$, for which the image of the generator of the cyclic group C has no non-trivial roots in F_k. In this case for every graded modular automorphism $\varphi \in GMod(G(x,p))$, $h \circ \varphi$ is an embedding of $G(x,p)$ into F_k. Hence, $G(x,p)$ has a shortening quotient that is isomorphic to it.

(2) Let $G(x,p)$ be a surface group, and suppose that $G(x,p) = P *_C B$, where C is cyclic and P is the parameter subgroup. Then it is possible to find a sequence of homomorphisms $h_n : G(x,p) \to F_k$, for which the corresponding sequence of graded shortest homomorphisms $h_n \circ \varphi_n$, converge to a group that is isomorphic to $G(x,p)$.

As we have already indicated, unlike (ungraded) limit groups, to understand the variation of the set of restricted homomorphisms from a graded limit group to F_k with a change of the defining parameters, we will need to study the structure of graded limit groups with a trivial principal graded abelian JSJ decomposition.

Definition 10.2. *Let $Glim(x,p,a)$ be a graded limit group that does not split into a free product in which AP can be conjugated into one of the factors. If $Glim(x,p,a)$ has a trivial graded abelian JSJ decomposition it is called rigid graded limit group. If $Glim(x,p,a)$ has a non-trivial graded abelian JSJ decomposition, and a (unique) maximal graded shortening quotient $GMSQ(x,p,a)$ that is isomorphic to $Glim(x,p,a)$, $Glim(x,p,a)$ is called solid graded limit group.*

To study *rigid* and *solid* graded limit groups we need to introduce the notion of *flexible graded limit groups.*

Definition 10.3. *Let $Glim(x,p,a)$ be either a rigid or a solid graded limit group. A sequence of homomorphisms $\{h_m : Glim(x,p,a) \to F_k\}$ is called flexible graded sequence if one of the following holds:*

(i) *each homomorphism h_m factors as $h_m = \nu_m \circ \tau_m$, where $\tau_m : Glim(x,p,a) \to F_k * < v >$, τ_m is an epimorphism that maps the subgroup AP onto the factor F_k, and $\nu_m : F_k * < v > \to F_k$ restricts to the identity map on F_k. Furthermore, τ_m can not be shortened (in $F_k * < v >$) by an element from the graded modular group $GMod(Glim(x,p,a))$.*

(ii) *each homomorphism h_m does not factor through a free product in which AP is mapped into one of the factors, and can not be shortened by an element from the graded modular group $GMod(Glim(x,p,a))$. In addition, for each index m:*

$$\max_{g \in B_m} d_X(h_m(g), id.) > 2^m \cdot (1 + \max_{1 \leq j \leq u} d_X(h_m(p_j), id.))$$

where B_m is the ball of radius m in the Cayley graph of $Glim(x,p,a)$.

A graded limit group which is the limit of a flexible graded sequence is called a flexible graded quotient of the rigid or solid graded limit group $Glim(x,p,a)$.

Lemma 10.4. *Let $FlxGlim(x,p,a)$ be a flexible graded quotient of the rigid or solid graded limit group $Glim(x,p,a)$. Then:*

(i) *$FlxGlim(x,p,a)$ is not a rigid graded limit group.*

(ii) *$FlxGlim(x,p,a)$ is a proper quotient of $Glim(x,p,a)$.*

Proof: If $FlxGlim(x,p,a)$ splits into a free product in which AP can be conjugated into one of the factors, then $FlxGlim(x,p,a)$ is not rigid nor solid, and it can not be isomorphic to $Glim(x,p,a)$. Hence, we may assume that $FlxGlim(x,p,a)$ does not split into a free product in which AP can be conjugated into one of the factors. By the way a flexible graded sequence of homomorphisms of $FlxGlim(x,p,a)$ is defined, it has a subsequence that converges into a stable action of $FlxGlim(x,p,a)$ on a real tree Y (see section 1), so that the subgroup AP fixes a point in Y.
By (([Se3],3.1), see theorem 1.5) $FlxGlim(x,p,a)$ admits a (non-trivial) graph of groups Λ from its action on the real tree Y. By that same theorem if the action of $FlxGlim(x,p,a)$ on Y contains an IET component, then $FlxGlim(x,p,a)$ admits a graded abelian splitting. If the action contains an axial component, $FlxGlim(x,p,a)$ inherits a graded abelian splitting and if the stabilizer of some (non-degenerate) segment in Y is non-trivial, it is maximal abelian by lemma 1.3, so $FlxGlim(x,p,a)$ splits over a maximal abelian subgroup, hence, $FlxGlim(x,p,a)$ admits a graded abelian splitting.
Therefore, in all cases $FlxGlim(x,p,a)$ admits a (non-trivial) graded abelian splitting, so the graded abelian JSJ decomposition of $FlxGlim(x,p,a)$ is non-trivial, and we get part (i) of the lemma.

If $Glim(x,p,a)$ is rigid it has a trivial graded abelian JSJ decomposition. Hence, by part (i) $FlxGlim(x,p,a)$ is a proper quotient of it. If $Glim(x,p,a)$ is solid and $FlxGlim(x,p,a)$ is isomorphic to $Glim(x,p,a)$, then for large enough m, it is possible to shorten the homomorphisms $h_m : Glim(x,p,a) \to F_k$ or the homomorphism $\tau_m : Glim(x,p,a) \to F_k * < v >$ by pre-composing with elements of the graded modular group $GMod(Glim(x,p,a))$, which contradicts the properties of homomorphisms in a flexible sequence. Hence, a flexible graded quotient is a proper quotient of a graded limit group.

$\square$

The following is an example of a flexible quotient of a solid limit group. Let $G(x,p) =< x_1, x_2, p \,|\, p = x_1^3 x_2^{-2} >$. $G(x,p)$ is a graded limit group and its graded abelian decomposition is the amalgamated product:

$$G(x,p) =< p > *_{<p=x_1^3 x_2^{-2}>} < x_1, x_2 > .$$

$G(x,p)$ is isomorphic to a free group of rank 2, hence, as in the first example presented above it is a solid limit group.
The solid limit group $G(x,p)$ admits a flexible quotient. Its flexible quotient is a limit group of the form

$$Flx(x,p) =< x_1, x_2, p \,|\, p = x_1^3 x_2^{-2},\ [x_1,x_2] = 1 >=< p > *_{<p=x_1^3 x_2^{-2}>} < x_1, x_2 \,|\, [x_1,x_2] = 1 > .$$

Let $Glim(x,p,a)$ be a rigid or solid graded limit group. On the set of flexible graded quotients of $Glim(x,p,a)$ we can naturally define a partial order and an equivalence relation, similar to the ones defined on graded limit groups. By the same argument used to prove lemma 5.4, the set of flexible graded quotients of $Glim(x,p,a)$ contains maximal elements with respect to the partial order, and by the same argument used to prove lemma 5.5 there are only finitely many equivalence classes of maximal flexible graded quotients of $Glim(x,p,a)$. In the sequel we will need the following definitions associated with rigid and solid graded limit groups and their associated maximal flexible limit quotients.

Definition 10.5. *Let $Rgd(x,p,a)$ be a rigid limit group and let:*

$$Flx_1(x,p,a), \ldots, Flx_v(x,p,a)$$

be the maximal flexible graded quotients of it. A homomorphism $h : Rgd(x,p,a) \to F_k$ that does not factor through any of the maximal graded flexible quotients

$$Flx_1(x,p,a), \ldots, Flx_v(x,p,a)$$

of $Rgd(x,p,a)$ is called a rigid homomorphism (solution, specialization)of the rigid graded limit group $Rgd(x,p,a)$. A homomorphism that does factor through one of the maximal flexible quotients is called a $flexible$ homomorphism (solution, specialization).

Definition 10.6. *Let $Sld(x,p,a)$ be a solid graded limit group and let*

$$Flx_1(x,p,a), \ldots, Flx_v(x,p,a)$$

be the maximal flexible quotients of it. A homomorphism $h : Sld(x,p,a) \to F_k$ for which $h = h' \circ \varphi$ where h' factors through one of the maximal graded flexible *quotients of $Sld(x,p,a)$, and φ is a graded modular automorphism of $Sld(x,p,a)$, is called a flexible homomorphism (solution, specialization) of the solid graded limit group $Sld(x,p,a)$. Note that every homomorphism $h: Sld(x,p,a) \to F_k$ that factors through a free product of the form $F_k * < v >$ so that AP is mapped into F_k, is flexible. A non-flexible homomorphism $h: Sld(x,p,a) \to F_k$ is called a solid homomorphism (solution, specialization) of the solid graded limit group $Sld(x,p,a)$.*

Flexible quotients of rigid graded limit groups and flexible quotients of solid graded limit groups, contain all the "generic infinite families" of specializations of these graded limit groups. Rigid solutions of rigid graded limit groups are the exceptional single solutions, and solid solutions of solid graded limit groups are the exceptional families of solutions.

Proposition 10.7. *Let $Rgd(x,p,a)$ be a rigid graded limit group, let $Sld(x,p,a)$ be a solid graded limit group, and let p_0 be a specialization of the defining parameters p. Then:*

- (i) *there are at most finitely many rigid solutions: $h: Rgd(x,p,a) \to F_k$ for which $h(p) = p_0$.*
- (ii) *up to pre-composing with a graded modular automorphism of the solid graded limit group $Sld(x,p,a)$ there are at most finitely many solid homomorphisms: $h: Sld(x,p,a) \to F_k$ for which $h(p) = p_0$.*

Proof: Suppose that for some specialization p_0 of the defining parameters p there are infinitely many rigid homomorphisms of $Rgd(x,p,a)$. From this infinite set of homomorphisms it is possible to extract a flexible subsequence, from which it is possible to extract a convergent flexible subsequence of homomorphisms. Now, this last flexible sequence converges into a flexible graded quotient of $Rgd(x,p,a)$. Hence, all but at most finite of the rigid homomorphisms of the convergent subsequence factors through a flexible graded quotient of $Rgd(x,p,a)$, a contradiction to their rigidity, and we get part (i) of the proposition. Part (ii) follows by the same argument starting with an infinite family of solid homomorphisms that are the shortest possible under the action of the graded modular group $GMod(Sld(x,p,a))$.

□

We will call the (ungraded) Makanin-Razborov diagram of a graded limit group obtained using the (ungraded) abelian JSJ decompositions of the various limit groups that appear along the various resolutions, the *ungraded* Makanin-Razborov diagram. We call each resolution of the ungraded Makanin-Razborov diagram of a graded limit group, an *ungraded* resolution. The above basic facts concerning graded limit groups and their graded shortening quotients and graded flexible quotients allow us to present the main goal of this section, the (canonical) *graded Makanin-Razborov diagram* associated with a graded limit group. The graded Makanin-Razborov diagram "encodes" all possible homomorphisms from a graded limit group into the (fixed) free group F_k in a form "compatible" with the parametric description of the set of solutions.

Let $Glim(x,p,a)$ be a graded limit group. Let $Glim(x,p,a) = Glim_1 * \ldots Glim_\ell * F_{g_1}$ be a factorization of $Glim(x,p,a)$, where $AP < Glim_1$, $Glim_1$ can not be written as a free product in which AP is contained in one of the factors, each of the $Glim_i$'s for $2 \leq i \leq \ell$ is a freely-indecomposable non-cyclic subgroup of

$Glim(x,p,a)$ and F_{g_1} is a free group. It follows from Grushko's theorem that such a free decomposition of $Glim(x,p,a)$ is canonical.

Suppose first that $Glim_1$ is not rigid or solid. By an analogue of lemma 5.4 the set of graded shortening quotients of the graded limit group $Glim_1$ contains maximal elements (with respect to the partial order defined above) and by an analogue of lemma 5.5 in the graded case, there are only finitely many equivalence classes of maximal graded shortening quotients of $Glim_1$. Since $Glim_1$ is not a solid graded limit group each maximal graded shortening quotient of it is a proper quotient. Let $GM_1, \ldots, GM_s$ be a collection of representatives of equivalence classes of maximal graded shortening quotients of $Glim_1$, and let $\eta_j : Glim_1 \to GM_j$ be the canonical projection maps.
Now, suppose $Glim_1$ is rigid or solid. In this case, $Glim_1$ has finitely many equivalence classes of maximal flexible graded quotients. Let $FlxGlim_1, \ldots, FlxGlim_v$ be a collection of representatives of equivalence classes of maximal flexible graded quotients of $Glim_1$, and let $\nu_j : Glim_1 \to FlxGlim_j$ be the canonical projection maps.

We define the *graded Makanin-Razborov diagram* of the graded limit group $Glim(x,p,a)$ iteratively. We start by factoring $Glim(x,p,a)$ as above. From each of the factors $Glim_2, \ldots, Glim_\ell$ we continue by adjoining its (ungraded) Makanin-Razborov diagram. If $Glim_1$ is not rigid we continue along s edges going from $Glim_1$ to each of its maximal graded shortening quotients $GM_1, \ldots, GM_s$. To each such directed edge we associate the canonical projection η_j.
If $Glim_1$ is rigid or solid we continue along v edges going from $Glim_1$ to each of its maximal flexible graded quotients $FlxGlim_1, \ldots, FlxGlim_v$. To each such directed edge we associate the canonical projection ν_j.

We proceed iteratively. Since a graded shortening quotient of a non-solid graded limit group, and a flexible shortening quotient of a solid or rigid graded limit groups are proper quotients of the graded limit group, the construction terminates by proposition 5.1. Each terminal vertex is either a free group (as in the Makanin-Razborov diagram presented in section 5), or it is a rigid graded limit group $RgdGlim(x,p,a)$ for which for every value of the parameters p_0, there are at most finitely many possible homomorphisms $h : RgdGlim(x,p,a) \to F_k$ that satisfy $h(p) = p_0$, or a solid graded limit group $Sld(x,p,a)$ with no flexible quotients. By construction, up to the equivalence relation of graded and restricted shortening quotients, and flexible graded quotients, the graded Makanin-Razborov diagram of a graded limit group is canonical.

As in the case of (ungraded) limit groups (definition 5.8), we can use the graded Makanin-Razborov diagram to define graded Makanin-Razborov resolutions of a graded limit group. The main purpose of defining graded Makanin-Razborov diagrams and resolutions is the fact these "encode" all the (ungraded) restricted resolutions for every specialization of the defining parameters of the graded limit group. A better understanding of this "encoding" requires a better understanding of rigid and solid limit groups and will be obtained in the following sections of this paper and the following papers in the sequel.

As we pointed out in section 8, the study of sets of solutions to equations with coefficients is naturally equivalent to the study of all restricted homomorphisms from a f.g. restricted residually free group into the free group generated by the coefficients. For studying this collection of homomorphisms we have an analogue of theorems 7.2 and 8.4 in the graded case. The proof is a direct modification.

Theorem 10.8. *Let $RF(x,p,a)$ be a f.g. (graded) residually free group. Then there exists a (canonical) finite collection of graded limit group quotients of $RF(x,p,a)$: $Glim_1(x,p,a),\dots,Glim_\ell(x,p,a)$, and for each i an epimorphism $\tau_i : RF(x,p,a) \to Glim_i(x,p,a)$ with the following properties:*

(i) *for each restricted homomorphism $h : RF(x,p,a) \to F_k$ there exists some index i (not necessarily unique!) and a restricted homomorphism $f : Glim_i(x,p,a) \to F_k$ so that h splits as: $h = f \circ \tau_i$.*

(ii) *if $Glim(x,p,a)$ is a graded quotient of $RF(x,p,a)$, and τ is an epimorphism $\tau : RF(x,p,a) \to Glim(x,p,a)$, then there exists some index i and an epimorphism $\lambda : Glim_i(x,p,a) \to Glim(x,p,a)$ so that $tau = \lambda \circ \tau_i$.*

(iii) *the couples $\{(Glim_i(x,p,a),\tau_i)\}$ represent distinct equivalence classes, and if $i \neq j$ then the couple $(Glim_i(x,p,a),\tau_i)$ can not be compared with $(Glim_j(x,p,a),\tau_j)$ in the partial order defined on these couples.*

(iv) *if $G\hat{lim}_1(x,p,a),\dots G\hat{lim}_u(x,p,a)$ and $\{\hat{\tau}_j : RF(x,p,a) \to G\hat{lim}_j(x,p,a)\}$ is another collection of graded quotients of $RF(x,p,a)$ and epimorphisms from $RF(x,p,a)$ to the graded limit groups $G\hat{lim}_i(x,p,a)$'s that satisfy properties (i)-(iii) then $u = \ell$ and up to permutation, the couple $(Glim_i(x,p,a),\tau_i)$ is equivalent to the couple $(G\hat{lim}_i(x,p,a),\hat{\tau}_i)$.*

We have already modified the construction of the Makanin-Razborov diagram of a limit group to the graded case. Having such diagram for a graded limit group, theorem 10.9 allows us to modify the construction of the Makanin-Razborov diagram associated with a residually free group presented in section 8 to the graded case.

§11. The Singular Locus

The main purpose for modifying limit groups and their Makanin-Razborov diagrams and resolutions to the graded category is the ability to use the graded objects in order to understand the variation of sets of solutions with a change in the values of the defining parameters. Such analysis is crucial in studying the structure of elementary sets and elementary predicates as we will see later in the sequel.

Since the collection of graded resolutions of a graded limit group Glim(x,p,a) "encodes" all the restricted homomorphisms of the group $Glim(x,p_0,a)$ for any possible specialization p_0 of the set of defining parameters p, to understand the variation in the set of solutions with a change of the defining parameters we need to look closely at all the possible degenerations in the graded resolutions of $Glim(x,p,a)$ for all possible values of the defining parameters. Our aim in this section is to associate finitely many *singular graded limit groups* with each rigid and solid limit group in a graded resolution of the graded Makanin-Razborov diagram. The finite collection of singular limit groups associated with a rigid (or solid) limit group in a graded resolution describe all the possible degenerations in the part of the graded resolution that terminates at the rigid (or solid) graded limit group.

Let $Glim(x,p,a)$ be a graded limit group, let $GRes_i$ be a graded resolution in the graded Makanin-Razborov diagram of $Glim(x,p,a)$ and let $RgdGlim(y,p,a)$

be a rigid limit group in this resolution.

Definition 11.1. *We say that the specialization (y_0, p_0, a) of the rigid limit group $RgdGlim(y,p,a)$ belongs to the singular locus of the tuple $(RgdGlim(y,p,a), GRes_i)$ if for all homomorphisms $h : Glim(x,p,a) \to F_k$ that factor through the resolution $GRes_i$ and the specialization (y_0,p_0,a) at least one of the graded abelian JSJ decompositions of the graded limit groups along the graded resolution $GRes_i$ remains degenerate, i.e., at least one of the following possible degenerations in the graded abelian JSJ decompositions occurs:*

(i) *an edge group is always mapped to the trivial group in F_k.*
(ii) *a non-abelian vertex group is always mapped to a cyclic subgroup in F_k.*
(iii) *a QH vertex group is always mapped to a cyclic subgroup in F_k.*

(note that (iii) can be viewed as a special case of (ii), but we prefer to list it as an additional separate case).

Clearly, there are only finitely many ways in which one of the graded abelian JSJ decompositions of the graded limit groups along the graded resolution $GRes_i$ can get degenerate, so the singular locus of $(RgdGlim(x,p,a), GRes_i)$ can be stratified according to the degenerate parts of these graded abelian JSJ decompositions.

Theorem 11.2. *There exist finitely many systems of equations $\Sigma_1(y,p,a), \dots, \Sigma_q(y,p,a)$ so that a specialization (y_0,p_0,a) of the rigid limit group $RgdGlim(y,p,a)$ belongs to the singular locus of the tuple $(RgdGlim(y,p,a), GRes_i)$ if and only if it satisfies one of the systems $\Sigma_1, \dots, \Sigma_q$.*

Proof: Given the resolution $GRes_i$ and elements in the graded modular groups of each of the graded limit groups in the resolution $GRes_i$, a degeneration in one of the graded abelian JSJ decomposition of one of the graded limit groups can be expressed as an equation in the variables (y,p). Hence, a specialization (y_0,p_0,a) of $RgdGlim(y,p,a)$ belongs to the singular locus of the tuple $(RgdGlim(y,p,a), GRes_i)$ if (y_0,p_0,a) satisfies an infinite system of equations parameterized by the product of the graded modular groups of the different graded limit groups in the graded resolution $GRes_i$. Since by Guba's theorem [Gu] every infinite system of equations in a free group is equivalent to a finite one, the theorem follows.

□

Definition 11.3. *We say that a graded limit group $SngGlim(y,p,a)$ is a singular graded limit group of the tuple $(RgdGlim(y,p,a), GRes_i)$ if it is a maximal graded limit group corresponding to one of the systems of equations $\Sigma_1, \dots \Sigma_q$ constructed in theorem 11.2. Clearly, this associates canonically finitely many singular limit groups with every tuple $(RgdGlim(x,p,a), GRes_i)$.*

In a similar way we associate (canonically) finitely many singular limit groups with every tuple of a graded resolution and a solid graded limit group in it. Let $Glim(x,p,a)$ be a graded limit group, let $GRes_i$ be a graded resolution in the graded Makanin-Razborov diagram of $Glim(x,p,a)$ and let $SldGlim(y,p,a)$ be a rigid limit group in this resolution.

Definition 11.4. *We say that the specialization (y_0,p_0,a) of the solid limit group $SldGlim(y,p,a)$ belongs to the singular locus of the tuple $(SldGlim(y,p,a), GRes_i)$*

if for all homomorphisms $h: Glim(x,p,a) \to F_k$ that factor through the resolution $GRes_i$ and the specialization (y_0,p_0,a), at least one of the abelian graded JSJ decompositions of the graded limit groups along the graded resolution $GRes_i$ remains degenerate (note that by all homomorphisms that factor through graded resolution $GRes_i$ and the specialization (y_0,p_0,a) we allow a possible pre-composition with a graded modular automorphism of the solid graded limit group $SldGlim(y,p,a)$ itself).

Like in the rigid case there exist finitely many systems of equations $\Sigma_1(y,p,a),\dots,\Sigma_q(y,p,a)$ so that a specialization (y_0,p_0,a) belongs to the singular locus of the tuple $(SldGlim(y,p,a),GRes_i)$ if and only if it satisfies one of the systems $\Sigma_1,\dots,\Sigma_q$.
We say that a graded limit group $SngGlim(y,p,a)$ is a singular graded limit group of the tuple $(SldGlim(y,p,a),GRes_i)$ if it is a maximal graded limit group corresponding to one of the systems of equations $\Sigma_1,\dots\Sigma_q$. Clearly, this associates canonically finitely many singular limit groups with every tuple $(SldGlim(x,p,a),GRes_i)$.

§12. Multi-graded Makanin-Razborov Diagrams

In the previous sections 9-11 we have analyzed the structure of parameterized sets of solutions to systems of equations in a free group, where the sets of solutions were parameterized by some subgroup of parameters $P=<p_1,\dots,p_u>$. The goal of this section is a generalization of the objects introduced in the previous sections, to study sets of solutions to systems of equations parameterized by one subgroup, to the study of sets of solutions to systems of equations parameterized by finitely many subgroups. As we will see in the sequel such parameterization appears in the analysis of projections of varieties (Diophantine sets), that is crucial in the analysis of sentences and elementary sets defined over a free group.

With a finite system of equations with parameters Φ over a free group $F_k=<a_1,\dots,a_k>$ it is natural to associate a f.p. group $G(\Phi)$. If the system Φ is defined by the coefficients $a_1,\dots,a_k$, the parameters $p_1,\dots,p_u$, and $r_1^1,\dots,r_{m_1}^1,\dots,r_1^\ell,\dots,r_{m_\ell}^\ell$, the unknowns $x_1,\dots,x_n$ and the equations:

$$w_1(a_1,\dots,a_k,p_1,\dots,p_u,r_1^1,\dots,r_{m_\ell}^\ell,x_1,\dots,x_n)=1$$
$$\vdots$$
$$w_s(a_1,\dots,a_k,p_1,\dots,p_u,r_1^1,\dots,r_{m_\ell}^\ell,x_1,\dots,x_n)=1$$

we set the associated f.p. group $G(\Phi)$ to be:

$$G(\Phi) \ = <\ a_1,\dots,a_k,p_1,\dots,p_u,r_1^1,\dots,r_{m_\ell}^\ell,x_1,\dots,x_n \,|\, w_1,\dots,w_s\ >$$

Clearly, every solution of the system Φ corresponds to a homomorphism $h: G(\Phi)\to F_k$ for which $h(a_i)=a_i$, and every such homomorphism corresponds to a solution of the system Φ.

In sections 9-11 we studied the variation of the structure of the set of solutions with a change of the defining parameters. In this section we study the variation of the set of solutions with a change of the specialization of defining parameters $P=<p_1,\dots,p_u>$, and the conjugacy classes of the specializations of the subgroups $R_1=<r_1^1,\dots,r_{m_1}^1>,\dots,R_\ell=<r_1^\ell,\dots,r_{m_\ell}^\ell$. To analyze the variation in the

structure of the set of solutions with a change of the specialization of the defining parameters P and the conjugacy classes of the specializations of the subgroups $R_1, \dots, R_\ell$ we will need to look at a special class of graded limit groups which we call *multi-graded limit groups*.

We set $G =< x_1, \dots, x_k, p_1, \dots, p_u, r_1^1, \dots, r_{m_\ell}^\ell, a_1, \dots, a_k >$ to be a f.g. group. For brevity we will denote this (fixed) generating set by $G =< g_1, \dots, g_m >$, and F_k to be a free group with a fixed basis $F_k =< a_1, \dots, a_k >$. With this notation our goal is to get a structure theory for understanding the set of all homomorphisms:

$$Hom((G, \{p_j\}, \{r_q\}, \{a_i\}), (F_k, \{a_i\})) \ = \ \{ h \,|\, h : G \to F_k \, , \, h(a_i) = a_i \, \}$$

with a change in the specializations of the parameters $\{p_j\}$ and the conjugacy classes of the specializations of the subgroups $R_1, \dots, R_\ell$. We set X to be the Cayley graph of F_k with respect to its given basis $a_1, \dots, a_k$. Given any homomorphism $h \in Hom(G, F_k)$, G admits a natural action λ_h on X given by $\lambda_h(g, x) = h(g)(x)$ for every $g \in G$ and $x \in X$.

Definition 12.1. *Let G be a f.g. group and let $h_n : G(x, r, p, a) \to F_k$ be a sequence of homomorphisms. If $\{\lambda_{h_n}\}$, a sequence of actions of G on X, the Cayley graph of F_k, that corresponds to the sequence of homomorphisms $\{h_n\}$, converges in the Gromov-Haussdorff topology, we say that the sequence $\{h_n\}$ converges. We call the obtained limit group, a multi-graded limit group.*

A multi-graded limit group is in particular a limit group, so every f.g. multi-graded limit group is f.p. and every f.g. graded limit group is ω-residually free by theorem 4.6. To construct the (canonical) multi-graded Makanin-Razborov diagram of a multi-graded limit group, which is the fundamental tool in our analysis of the variation of sets of solutions to systems of equations with a change in the specialization of the parameters P and the conjugacy class of the specializations of the subgroups $R_1, \dots, R_m$, we need to study the algebraic structure of multi-graded limit groups "relative" to their multi-grading. To do that we need to construct the multi-graded abelian JSJ decomposition of a multi-graded limit group. These constructions are all generalizations of their graded analogues presented in sections 9-11.

For the rest of this section let $G(x, r, p, a)$ be a f.g. group, let $\{h_m : G(x, r, p, a) \to F_k\}$ be a convergent multi-graded sequence of homomorphisms, and let $MGlim(x, r, p, a)$ be the corresponding graded limit group. We set $\eta : G(x, r, p, a) \to MGlim(x, r, p, a)$ to be the canonical epimorphism and $AP < MGlim(x, r, p, a)$ to be the subgroup: $AP =< \eta(p_1), \dots, \eta(p_u), \eta(a_1), \dots, \eta(a_k) >$. The abelian JSJ decomposition of a freely-indecomposable limit group "encodes" all the possible splittings of such limit group with abelian edge groups. In studying multi-graded limit groups we will be interested only in abelian splittings in which the subgroup AP and the subgroups $R_1, \dots, R_m$ are elliptic. The construction of the abelian JSJ decomposition of a limit group described in section 2, naturally generalizes to a *multi-graded abelian* JSJ *decomposition* of a multi-graded limit group.

Theorem 12.2 (cf. 9.2). *Suppose $MGlim(x, r, p, a) = L_0 * \dots * L_v$ is the most refined free decomposition in which $AP < L_0$ and each of the subgroups R_j can be conjugated into one of the factors L_i. With each of the factors we associate a multi-graded abelian JSJ decomposition. If none of the subgroups R_j can be conjugated* *into L_i we associate with it its abelian JSJ decomposition. If L_i is L_0 or it contains a conjugate of one of the subgroups R_j then there exists a (relatively) reduced unfolded (perhaps trivial) splitting of the factor L_i with abelian edge groups, which we call a $multi-graded$ abelian JSJ (Jaco-Shalen-Johannson) decomposition of the factor L_i with the following properties:*

(i) *AP is elliptic in the multi-graded abelian JSJ decomposition of L_0. If a subgroup R_j can be conjugated into the factor L_i, then R_j is elliptic in the multi-graded abelian JSJ decomposition of L_i.*

(ii) *Every (multi-graded) canonical maximal QH subgroup (CMQ) of L_i is conjugate to a vertex group in the multi-graded JSJ decomposition. Every (multi-graded) QH subgroup of $MGlim(x,r,p,a)$ can be conjugated into one of the CMQ subgroups of L_i. Every vertex group in the multi-graded JSJ decomposition which is not a CMQ subgroup of L_i is elliptic in any abelian splitting of L_i under consideration (i.e., in one in which AP and the R_j's are elliptic).*

(iii) *A one edge (multi-graded) abelian splitting $MGlim(x,r,p,a) = D *_A E$ or $L_i = D*_A$ in which AP and the R_j's are elliptic which is hyperbolic in another such elementary abelian splitting is obtained from the multi-graded abelian JSJ decomposition of L_i by cutting a 2-orbifold corresponding to a (multi-graded) CMQ subgroup of L_i along a weakly essential s.c.c..*

(iv) *Let Θ be a one edge (multi-graded) splitting along an abelian subgroup $L_i = D *_A E$ or $L_i = D*_A$ in which AP and the R_j's are elliptic, which is elliptic with respect to any other one edge (multi-graded) abelian splitting of L_i in which AP and the R_j's are elliptic. Then Θ is obtained from the multi-graded abelian JSJ decomposition of L_i by a sequence of collapsings, foldings, and conjugations.*

(v) *If JSJ_1 is another multi-graded abelian JSJ decomposition of L_i, then JSJ_1 is obtained from the multi-graded abelian JSJ decomposition by a sequence of slidings, conjugations and modifying boundary monomorphisms by conjugations (see section 1 of [Ri-Se2] for these notions)*

In section 5 we have used the canonical cyclic JSJ decomposition of a limit group to construct its canonical Makanin-Razborov diagram. In section 10 we modified this construction to obtain the graded Makanin-Razborov diagram for analyzing the variation of the restricted Makanin-Razborov diagram with a change of the defining parameters. To understand the variation of the (ungraded) Makanin-Razborov diagram with a change of the specialization of the parameter subgroup P, and the conjugacy classes of the specializations of the subgroups R_j, we need to further modify the construction of the graded Makanin-Razborov diagram to obtain the *multi-graded Makanin-Razborov diagram* of a multi-graded limit group.

Definition 12.3. *Let $MGlim(x,r,p,a)$ be a multi-graded limit group and let $MGlim(x,r,p,a) = L_0 * \ldots * L_v$ be the most refined free decomposition in which $AP < L_0$ and the subgroups R_j can be conjugated into the different factors L_i. Let L_i be one of those factors and suppose that the multi-graded abelian JSJ decomposition of L_i is non-trivial. We define the $multi-graded$ $modular$ $group$ $MGMod(L_i)$ to be the subgroup of $Aut(L_i)$ generated by the following families of automorphisms of the factor L_i:*

(i) *Dehn twists along edges of the multi-graded abelian JSJ decomposition of L_i. The Dehn twists are assumed to fix (elementwise) the vertex stabilized by AP in case L_i is the factor L_0*

(ii) *Dehn twists along essential s.c.c. in (multi-graded) CMQ (canonical maximal quadratically hanging) vertex groups in the multi-graded abelian JSJ decomposition of L_i. Again, these Dehn twists are assume to fix (elementwise) the vertex stabilized by AP in case L_i is the factor L_0.*

(iii) *Let A be an abelian vertex group in the multi-graded abelian JSJ decomposition of L_i. Let $A_1 < A$ be the subgroup generated by all edge groups connected to the vertex stabilized by A in the multi-graded abelian JSJ decomposition of L_i. Every automorphism of A that fixes A_1 (elementwise) can be naturally extended to an automorphism of the ambient factor L_i that fixes the vertex stabilized by AP and conjugates of the other subgroups R_j. We call these generalized Dehn twists and they form the third family of automorphisms that generate $MGMod(L_i)$.*

We say that a homomorphism $h : MGlim(x,r,p,a) \to F_k$ is the same multi-graded modular class as the homomorphism $\hat{h} : MGlim(x,r,p,a) \to F_k$ if $\hat{h} = h \circ \varphi$, and φ is a multi-graded modular automorphism of $MGlim(x,r,p,a)$, i.e., $\varphi \in MGlim(x,r,p,a)$.

Let L_i be a factor in the most refined free decomposition of the multi-graded limit group $MGlim(x,r,p,a)$ in which the subgroups AP and R_j's can be conjugated into the various factors. Once we fix a generating set for the factor L_i, given a homomorphism $h : L_i \to F_k$ we can choose the shortest homomorphisms in the multi-graded modular class of h. A multi-graded limit group obtained from a convergent sequence of homomorphisms $h : L_i \to F_k$ which are the shortest in their multi-graded modular class, is called a *multi-graded shortening quotient* of the factor L_i.

On the set of multi-graded shortening quotients of the factor L_i we can naturally define a partial order and an equivalence relation, similar to the ones defined in section 5 for shortening quotients of limit groups. By the same argument used to prove lemma 5.4 the set of multi-graded shortening quotients of L_i contains maximal elements with respect to the partial order, and by the same argument used to prove lemma 5.5 there are only finitely many equivalence classes of maximal multi-graded shortening quotients of the factor L_i.

So far we have modified the construction of maximal shortening quotients presented in section 5 to the multi-graded setup. However, multi-graded shortening quotients of a multi-graded limit group need not be proper quotients of it. Furthermore, to get multi-graded shortening quotients we naturally require that the multi-graded abelian JSJ decomposition of the factor L_i is non-trivial. Unlike (ungraded) limit groups, to understand the variation of the set of restricted homomorphisms from a graded limit group to F_k with a change of the specialization of the defining parameters P and the conjugacy classes of the specializations of the subgroups R_j, we will need to study the structure of multi-graded limit groups with a trivial principal graded abelian JSJ decomposition.

Definition 12.4. *Let L_i be a factor in the most refined free decomposition of a multi-graded limit group $MGlim(x,r,p,a)$ in which the subgroups AP and $R_1,\dots,R_m$ can be conjugated into the various factors. If the factor L_i has a trivial multi-graded* *abelian JSJ decomposition it is called rigid multi-graded limit group. If L_i has a non-trivial multi-graded JSJ decomposition, and a (unique) maximal multi-graded shortening quotient $GMSQ(x,r,p,a)$ that is isomorphic to L_i, L_i is called solid multi-graded limit group.*

To study *rigid* and *solid* multi-graded limit groups we need to introduce the notion of *flexible multi-graded limit groups.*

Definition 12.5. *Let L_i be either a rigid or a solid factor of a multi-graded limit group $MGlim(x,r,p,a)$. A sequence of homomorphisms $\{h_m : L_i \to F_k\}$ is called flexible multi-graded sequence if one of the following holds:*

- (i) *each homomorphism h_m factors as $h_m = \nu_m \circ \tau_m$, where $\tau_m : L_i \to F_k * < v >$ and τ_m is an epimorphism that maps the subgroup AP onto the factor F_k and the subgroups R_j into a conjugate of F_k in $F_k * < v >$, and $\nu_m : F_k * < v > \to F_k$ restricts to the identity on F_k. Furthermore, τ_m can not be shortened (in $F_k * < v >$) by an element from the graded modular group $MGMod(L_i)$.*
- (ii) *each homomorphism h_m does not factor through a free product as in (i), and can not be shortened by an element from the multi-graded modular group $MGMod(L_i)$. In addition, if $L_i = L_0$ then for each index m:*

$$\max_{g \in B_m} d_X(h_m(g), id.) >$$

$$2^m \cdot (1 + \max_{1 \le j \le u} d_X(h_m(p_j), id.) + \max_{1 \le j \le \ell} \min_{f \in F_k} (d_X(f h_m(r_1^j) f^{-1}, id.) + \ldots + (d_X(f h_m(r_{m_j}^j) f^{-1}, id.))$$

where B_m is the ball of radius m in the Cayley graph of L_i. If L_i is not the factor L_0, i.e., if AP is not a subgroup of L_i, then for each index m:

$$\min_{f \in F_k} \max_{g \in B_m} d_X(f h_m(g) f^{-1}, id.) >$$

$$2^m \cdot (1 + \max_{1 \le j \le \ell} \min_{f \in F_k} (d_X(f h_m(r_1^j) f^{-1}, id.) + \ldots + (d_X(f h_m(r_{m_j}^j) f^{-1}, id.)).$$

A multi-graded limit group which is the limit of a flexible multi-graded sequence is called a flexible multi-graded quotient of the rigid or solid multi-graded limit group L_i.

Lemma 12.6. *Let $Flx(x,r,p,a)$ be a flexible multi-graded quotient of the rigid or solid graded limit factor L_i. Then:*

- (i) *$FlxGlim(x,p,a)$ is not a rigid multi-graded limit group.*
- (ii) *$FlxGlim(x,p,a)$ is a proper quotient of L_i.*

Proof: Identical with the proof of lemma 10.5.

□

Let $MGlim(x,r,p,a)$ be a rigid or solid multi-graded limit group. On the set of flexible multi-graded quotients of $MGlim(x,r,p,a)$ we can naturally define a partial order and an equivalence relation, similar to the ones defined on multi-graded limit groups. By the same argument used to prove lemma 5.4, the set of flexible graded quotients of $MGlim(x,r,p,a)$ contains maximal elements with respect to the partial order, and by the same argument used to prove lemma 5.5 there are only finitely many equivalence classes of maximal flexible multi-graded quotients of $MGlim(x,r,p,a)$. For the sequel we will need the following definitions associated with rigid and solid multi-graded limit groups and their associated maximal flexible limit quotients.

Definition 12.7. *Let $Rgd(x,r,p,a)$ be a rigid limit group and let*

$$Flx_1(x,r,p,a),\dots,Flx_v(x,r,p,a)$$

be the maximal flexible multi-graded quotients of it. A homomorphism $h: Rgd(x,r,p,a) \to F_k$ that does not factor through any of the maximal multi-graded flexible quotients $Flx_1(x,r,p,a),\dots,Flx_v(x,r,p,a)$ of $Rgd(x,r,p,a)$ is called a rigid homomorphism (solution)of the rigid multi-graded limit group $Rgd(x,r,p,a)$. A homomorphism that does factor through one of the maximal flexible quotients is called a flexible homomorphism (solution).

Definition 12.8. *Let $Sld(x,r,p,a)$ be a solid multi-graded limit group and let $Flx_1(x,r,p,a),\dots,Flx_v(x,r,p,a)$ be the maximal flexible quotients of it. A homomorphism $h: Sld(x,r,p,a) \to F_k$ for which $h = h' \circ \varphi$ where h' factors through one of the maximal multi-graded flexible quotients of $Sld(x,r,p,a)$, and φ is a multi-graded modular automorphism of $Sld(x,r,p,a)$, is called a flexible homomorphism (solution) of the solid multi-graded limit group $Sld(x,r,p,a)$. Note that every homomorphism of $Sld(x,r,p,a) \to F_k$ that factors through a free product of the form $F_k * < v >$ so that AP and the subgroups R_j are mapped into F_k, is flexible. A non-flexible homomorphism $h: Sld(x,r,p,a) \to F_k$ is called a solid homomorphism (solution) of the solid multi-graded limit group $Sld(x,r,p,a)$.*

Flexible quotients of rigid multi-graded limit groups and flexible quotients of solid multi-graded limit groups, contain all the "generic infinite families" of specializations of these multi-graded limit groups. Rigid solutions of rigid multi-graded limit groups are the exceptional single solutions, and solid solutions of solid multi-graded limit groups are the exceptional families of solutions.

Proposition 12.9. *Let $Rgd(x,r,p,a)$ be a rigid multi-graded limit group, let $Sld(x,r,p,a)$ be a solid multi-graded limit group, and let p_0 be a specialization of the defining parameters p. Then:*

(i) *there are at most finitely many rigid solutions: $h: Rgd(x,r,p,a) \to F_k$ for which $h(p) = p_0$ and each of the subgroups R_j are mapped to a conjugate of $h(R_j)$ (elementwise).*
(ii) *up to pre-composing with a graded modular automorphism of the solid graded limit group $Sld(x,r,p,a)$ there are at most finitely many solid homomorphisms: $h : Sld(x,r,,p,a) \to F_k$ for which $h(p) = p_0$ and each of the subgroups R_j are mapped to a conjugate of $h(R_j)$ (elementwise).*

Proof: Identical with the proof of proposition 10.7.

□

The above basic facts concerning multi-graded limit groups and their multi-graded shortening quotients and multi-graded flexible quotients allow us to present the main goal of this section, the (canonical) *multi-graded Makanin-Razborov diagram* associated with a multi-graded limit group.

Let $MGlim(x,r,p,a)$ be a multi-graded limit group. Let $MGlim(x,r,p,a) = L_0 * \ldots * L_v$ be the most refined free decomposition of $MGlim(x,r,p,a)$ so that $AP < L_0$ and each of the subgroups R_j can be conjugated into one of the L_i's. Suppose first that a factor L_i contains either the subgroup AP or a conjugate of one of the subgroups R_j. Suppose L_i is not rigid or solid. Then by claim 13.2 the set of multi-graded shortening quotients of the factor L_i contains maximal elements (with respect to the partial order defined above) and by an analogue of lemma 5.5 to the multi-graded case there are only finitely many equivalence classes of maximal multi-graded shortening quotients of L_i. Since L_i is not a solid multi-graded limit group each maximal multi-graded shortening quotient of it is a proper quotient. Let $MGM_1, \ldots, MGM_s$ be a collection of representatives of equivalence classes of maximal multi-graded shortening quotients of the factor L_i, and let $\eta_q : L_i \to MGM_q$ be the canonical projection maps. Note that MGM_q is a multi-graded limit group with respect to some subset of the subgroups P and $R_1, \ldots, R_\ell$.

Now, suppose L_i is rigid or solid. In this case, L_i has finitely many equivalence classes of maximal flexible multi-graded quotients. Let $Flx_1, \ldots, Flx_t$ be a collection of representatives of equivalence classes of maximal flexible graded quotients of L_i, and let $\nu_q : L_i \to Flx_q$ be the canonical projection maps.

We define the *multi-graded Makanin-Razborov diagram* of the multi-graded limit group $MGlim(x,r,p,a)$ iteratively. We start by factoring $MGlim(x,r,p,a)$ as above. From each of the factors L_i that does not contain the subgroup AP or a conjugate of one of the subgroups R_j we continue by adjoining its (ungraded) Makanin-Razborov diagram presented in section 5. If L_i contains the subgroup AP or a conjugate of one of the R_j's, and L_i is not rigid or solid we continue along s edges going from L_i to each of its maximal multi-graded shortening quotients $MGM_1, \ldots, MGM_s$. To each such directed edge we associate the canonical projection η_q.

If L_i is rigid or solid we continue along t edges going from L_i to each of its maximal multi-graded flexible quotients $Flx_1, \ldots, Flx_t$. To each such directed edge we associate the canonical projection ν_q.

We proceed iteratively. Since a multi-graded shortening quotient of a non-solid multi-graded limit group, and a flexible shortening quotient of a solid or rigid multi-graded limit groups are proper quotients of the multi-graded limit group, the construction terminates by proposition 5.1. Each terminal vertex is either a free group (as in the Makanin-Razborov diagram presented in section 5), or it is a rigid multi-graded limit group $Rgd(x,r,p,a)$ for which for every value of the parameters P and given conjugacy classes of the subgroups R_j, there are at most finitely many possible homomorphisms $h : Rgd(x,r,p,a) \to F_k$ that obtaining these values and conjugacy classes, or a solid multi-graded limit group $Sld(x,r,p,a)$ with no flexible quotients. By construction, up to the equivalence relation of multi-graded and restricted shortening quotients, and multi-graded flexible quotients, the multi-graded Makanin-Razborov diagram of a multi-graded limit group is canonical. As in the case of (ungraded) limit groups (definition 5.8), we can use the multi-graded Makanin-Razborov diagram to define multi-graded Makanin-Razborov resolutions of a multi-graded limit group.

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