Primärbericht

A Two-Period Material Safeguards Game

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ABSTRACT

The game model proposed in FWA 123/84 and discussed in FWA 108/85 is fully analyzed under assumptions similar to those made in the statistical treatment of the problem. It is proved that the game has a unique Nash Equilibrium. Furthermore, except for some extreme configurations of covariance structure of \( \text{MU}_1 \) and \( \text{MU}_2 \) and of extremely low critical mass \( M \), the CUMUF test is exactly the "inspector"'s strategy in this equilibrium. Consequently the N.E. provides a deterrence CUMUF test in which the false alarm probability and the critical value are determined endogenously by the combined effect of all payoff functions involved (unlike the statistical test which assumes the false alarm probability to be a given parameter).
INTRODUCTION

The subject of our work is the problem of accounting of nuclear material. Consider a well defined closed box at given time into which material enters and from which material leaves during a given interval of time \([t_0,t_1]\). Let us refer to this box as the material balance area (for example, a nuclear material processing plant).

The material contained in the material balance area at time \(t_0\) is the real inventory \(I_0\). The algebraic sum of the amounts of material that enter and leave the material balance area in the interval of time \([t_0,t_1]\) is the net flow \(D\). The sum \(B = I_0 + D\) is called the book inventory. This is the amount of material that should be in the material balance area at time \(t_1\). In other words if the real inventory at time \(t_1\) is \(I_1\) then if all material contained, and passing through the material balance area is carefully accounted for, if there were no measurement errors, and if no material has disappeared or has been diverted we should have \(B = I_1\). However, since not all of these conditions are satisfied, the difference between these two quantities which for historical reasons was given the name material not accounted for (MUF),

\[
MUF = B - I_1 = I_0 + D - I_1,
\]

is usually not zero. The problem therefore is that of finding out the various causes of this difference being non zero. More specifically, in the framework of international nuclear material safeguards, the purpose of material accountancy is to decide whether or not an operator behaves legally. So the very existence of safeguards authorities is a consequence of the assumption that the operator eventually may divert material, and furthermore, if he will divert material, he will do that in a way which is most favorable to him, i.e.
that it will be most difficult for the inspector to detect it. So in a very fundamental way the problem of material safeguards involves two decision makers; the inspector and the operator. Therefore this is not a statistical decision problem but rather a game. The use of game theory for material safeguards is what we do in this paper.

Our main interest is the sequential character of the problem. That is assume that the procedure extends over n periods in each of which the MUF is observed:

\[ \text{MUF}_1 = I_0 + D_1 - I_1 \]
\[ \text{MUF}_2 = I_1 + D_2 - I_2 \]
\[ \vdots \]
\[ \text{MUF}_n = I_{n-1} + D_n - I_n \]

How should the inspector make his decisions whether or not to declare violation, based on the observed MUF_i? How does the operator carry out the diversion if he decides to do so? Eventually we hope to be able to describe the time aspect of the problem: The inspector's clear interest to detect a violation as short as possible after its occurrence. This is an aspect which is not taken care of in the existing statistical procedures commonly used in material accounting. (See e.g. [1], [2], [3], [5]).

To get an insight into the problem we formulate a game for a sequence of two inventory periods. This model is general enough to accommodate elements of the problem ignored or suppressed in the traditional statistical treatment such as:

1. General "payoff" functions to evaluate a certain procedure (not just by "detection" and "false alarm" probabilities).
2. "Behavioral" nature of the strategies of the inspector and the operator:
Both may make their decision at each stage depend on the observed values of MUF in previous periods.

3. By appropriate choice of the payoff functions we can introduce the time aspect: For the operator it is better to accomplish his diversion as early as possible and for the inspector early detection of a diversion is less harmful than a late one.

The results we present in this paper after describing the game may be viewed as "calibration" of the model. First we prove that under quite mild assumptions the strategies of the inspector may be taken to be threshold strategies which are: Choose a first critical value \( s_1 \) and declare an alarm if \( \text{MUF}_1 > s_1 \). If alarm was not declared at first period choose \( s_2 \) (which may depend on \( \text{MUF}_1 \)) and declare an alarm in second period if \( \text{MUF}_2 > s_2 \). This result which is actually a generalization of the Neyman-Pearson Lemma to our context simplifies the game model considerably. But beyond that and more importantly it establishes the natural and most intuitive statistical test as part of our game theoretical model.

Next, the well known Nash Equilibrium concept is defined. Imposing assumptions similar to those underlying the statistical treatment we prove that there is a unique equilibrium point (disregarding the uninteresting "conflict equilibrium" in which the operator openly diverts and is caught by the inspector). Furthermore, except for some extreme configurations of covariance structure of MUF1 and MUF2 and of extremely low critical mass \( M \), the \( \text{CUMUF test} \) is exactly the equilibrium strategy of the inspector. That is he never calls an alarm at first period while he does so in the second period if \( \text{MUF}_1 + \text{MUF}_2 \) is greater than some critical value. Making this critical value slightly smaller than its equilibrium value yields a \text{deterrence CUMUF test}. When used
by the inspector the best reply of the operator is not to divert. In contrast
to the statistical model in which the critical value is determined uniquely by
the false alarm probability which is taken to be a given parameter of the game,
in the game theoretical model both the false alarm probability and the critical
value are endogenously determined by the combined effect at all payoff
functions involved.

Another by-product of the game theoretical treatment is the diversion
probability \( q \) in equilibrium. Although diversion "should not come into action"
if the inspector moves from his equilibrium strategy to his deterrence
strategy, the probability \( q \) may be interpreted as a measure for the stability
of the safeguard system. Furthermore, knowing how \( q \) is determined by the
parameters of the game it can be controlled to some extent by the inspector,
for instance by increasing the penalty for a detected diversion.

In some numerical computations we show the dependence of the false alarm
probability and the diversion probability on the parameters of the game, mainly
on the critical mass of significant diversion and the disutility of alarm to
the inspector and the operator.

1. The Game

We assume that the measurements of material are subject to normally
distributed errors. Also note that since \( \text{MUF}_1 = I_0 + D_1 - I_2 \) and \( \text{MUF}_2 = I_1 + D_2 - I_2 \),
\( \text{MUF}_1 \) and \( \text{MUF}_2 \) are not independent therefore we shall assume:
\( \text{var}(\text{MUF}_i) = \sigma_i^2 \), \( i = 1, 2 \); \( \rho = \cos(\text{MUF}_1, \text{MUF}_2) \). Later we shall also assume that \( \rho \leq 0 \) which is
the case if for instance \( I_1 \) and \( I_2 \) are statistically independent measurements.
This implies that if \( m_1 \) is diverted in first period and \( m_2 \) in second period
then the distributions of \( \text{MUF}_i \) are:
\[ \text{MUF}_1 \sim N(m_1, \sigma_1^2) \]
\[ \text{MUF}_2 \sim N(m_2 + \rho \frac{\sigma_2}{\sigma_1} (x_1 - m_1), \sigma_2^2 (1 - \rho^2)) \]

where \( x_1 \) is the realized value of \( \text{MUF}_1 \).

The game can now be described as follows:

**Step 1:** Player 0 (the operator) chooses \( m_1 \in [0, M] \) (the amount of material he diverts in the first period).

**Step 2:** A chance move chooses \( x_1 \), an observation of the random variable \( X_1 \) normally distributed \( N(m_1, \sigma_1^2) \) (this is the observation of \( \text{MUF}_1 \)).

**Step 3:** Player 1 (the inspector) observing \( x_1 \) (and not knowing \( m_1 \)) chooses

- either A (alarm) in which case the game stops with payoffs \( -a_1(m_1) \) to I and \( -b_1(m_1) \) to O, or
- C (continue) in which case the game proceeds to step 4.

**Step 4:** Player 0 chooses \( m_2 \in [0, M] \) (the amount of material he diverts in the second period).

**Step 5:** A chance move chooses \( x_2 \), an observation of the random variable \( X_2 \) normally distributed \( N(m_2 - \rho \frac{\sigma_2}{\sigma_1} (x_1 - m_1), \sigma_2^2 (1 - \rho^2)) \) (This is the observation of \( \text{MUF}_2 \)).

**Step 6:** Player 1, observing \( x_2 \) (and recalling \( x_1 \)) chooses:

- either A (alarm) in which case the game ends with payoffs \( -a_2(m_1, m_2) \), \( -b_2(m_1, m_2) \) to I and O respectively.
- or C (clear) in which case the game ends with payoffs \( -c(m_1, m_2) \), \( d(m_1, m_2) \) to I and O respectively.
$a_1$ and $b_1$ are functions on $[0,M]$ and $a_2$, $b_2$, $c$ and $d$ are functions on $[0,M]^2$. All these functions are nonnegative and nondecreasing in each variable.

The extensive form of the game looks as follows:

![Diagram](image)

Fig. 1

*The game in extensive form*
2. The Game in Strategic Form

In order to study the solution of the game defined in the previous section it will be more convenient to describe it in strategic (normal) form. In view of the complexity of the strategy spaces and the payoff functions we shall do this first for the restricted case needed here.

The main purpose of this paper is to study the relations of the solution of our game to the commonly used CUMUF test. Therefore let us consider now only strategies of the operator in which he completely determines his diversion plan at the beginning of the game. This is the underlying assumption in the nonsequential approach leading to the CUMUF test.

More precisely we assume:

Assumption 1. There is a critical quantity $M$ (e.g. the quantity needed to produce an atomic bomb) such that any total diversion $m_1 + m_2$ less than $M$ is worthless to the operator and harmless to the inspector. Any total diversion greater than $M$ is equivalent for both players to total diversion of $M$.

Assumption 2. The strategy space for the operator is:

$$(1) \quad T = \{(q,m) | 0 \leq q \leq 1, 0 \leq m \leq M\}$$

A strategy $r = (q,m)$ is that in which with probability $(1-q)$ no diversion takes place and with probability $q$ the operator divert $m$ in first period and $M-m$ in the second period.

A most general behavioral strategy for the inspector is:

Choose a critical region $X_c^1$ for the first period, i.e., if $x_1 \in X_c^1$ declare an alarm. In view of the first observation $x_1$, if you decide not to declare an
alarm (i.e. if $x_1 \notin X_c^1$) choose a critical region $X_c^2$ (depending on the value of $x_1$) and declare an alarm in the second period if $x_2 \in X_c^2$.

Since our interest is in Nash-Equilibrium points (N.E.) we need only to make the strategy space of I big enough so as to contain a best reply to any strategy of O. Now given a strategy $r=(q,m)$ of O, the payoff to the inspector depends only on his actions and on the probability of completing a diversion $M$. It is easily verified that:

- The conditional probability of (eventual) diversion given $\text{MUF}_1 = x_1$ increases in $x_1$.
- For any $x_1$, the conditional probability of diversion given $\text{MUF}_2 = x_2$ increases in $x_2$.

If we make the natural assumption: $m_1 + m_2 \geq M > c(m_1, m_2) > a(m_1, m_2)$ (that is the inspector prefers calling alarm when significant diversion takes place on declaring o.k.) then it follows that:

- For any $x_1$, the conditional expected payoff for the inspector given $x_1$, increases in $x_2$ if he chooses A (alarm) and decreases in $x_2$ if he chooses C (clear).
- For any critical region $X_c^2$ of the second stage, the conditional expected payoff of the inspector given $x_1$ increases in $x_1$ if he chooses A and decreases in $x_1$ if he chooses C.

From this we conclude:

**Proposition 1.** If the operator is restricted to the strategy set $T$ given by

(1) then the best replies of the inspector are threshold strategies, that is his strategy space may be taken to be:

(2) $S = \{(s_1, s_2) | s_1 \in \mathbb{R}, s_2 : \mathbb{R} \to \mathbb{R}\}$
So a threshold strategy $s=(s_1,s_2)$ consists of a critical value $s_1$ for the first period (i.e. call alarm if $x_1 > s_1$) and a critical value $s_2$ which is a function of $x_1$ (i.e. call an alarm if $x_2 > s_2(x_1)$).

**Remark.** Proposition 1 may be viewed as a variant of the Neyman-Pearson Lemma.

It turns out that it holds under much less restrictive assumptions on the strategy set $T$ of the operator. (See e.g. [8]). However the proof is no longer immediate. In view of the great practical advantage in being able to restrict the inspector's strategy set to threshold strategies, the generalizations of proposition 1 are now being studied as part of this research project.

The payoff functions $I:S \times T \to \mathbb{R}$ for the inspector and $O:S \times T \to \mathbb{R}$ for the operator can now be written explicitly.

If $s=(s_1,s_2)$, $r=(q,m)$ then:

(3) \quad I(s,r) = (1-q)I(s,0) + q I(s,m)

(4) \quad O(s,r) = (1-q)O(s,0) + q I(s,m),

The functions $I(s,0)$, $I(s,m)$, $O(s,0)$ and $O(s,m)$ are the expected payoffs under 0 diversion and $(m,M-m)$ diversion. For instance:

\[
I(s,0) = -a_1(0) \cdot p_{m_1=0} (\text{MUF}_1 > s_1) \\
- a_2(0) \cdot p_{m_1=0} (\text{MUF}_1 \leq s_1, \text{MUF}_2 > s_2) \\
\quad \quad \quad \quad \quad \quad m_2=0 \\
- c(0) \cdot p_{m_1=0} (\text{MUF}_1 \leq s_1, \text{MUF}_2 \leq s_2) \\
\quad \quad \quad \quad \quad \quad m_2=0
\]
Introducing the appropriate distributions we obtain:

(5) \( I(s,0) = -a_1(0)(1-\phi(s_1/s_1)) \)

\[-\frac{a_2(0)}{\sigma_1^2 2\pi} \int_{-\infty}^{s_1} \left[ 1-\phi\left( \frac{s_2(x_1)-\rho \sigma_2/s_1}{\sigma_2 \sqrt{1-\rho^2}} \right) \right] e^{-x_1^2/2\sigma_1^2} \, dx_1 \]

\[-\frac{c(0)}{\sigma_1^2 2\pi} \int_{-\infty}^{s_1} \phi\left( \frac{s_2(x_1)-\rho \sigma_2/s_1}{\sigma_2 \sqrt{1-\rho^2}} \right) e^{x_1^2/2\sigma_1^2} \, dx_1 \]

(6) \( I(s,m) = -a(m)(1-\phi((s_1-m)/\sigma_1)) \)

\[-\frac{a_2(M)}{\sigma_1^2 2\pi} \int_{-\infty}^{s_1} \left[ 1-\phi\left( \frac{s_2(x_1)-(M-m)-\rho \sigma_2^2}{\sigma_2 \sqrt{1-\rho^2}} \right) \right] e^{-x_1^2/2\sigma_1^2} \, dx_1 \]

\[-\frac{c(M)}{\sigma_1^2 2\pi} \int_{-\infty}^{s_1} \phi\left( \frac{s_2(x_1)-(M-m)-\rho \sigma_2^2}{\sigma_2 \sqrt{1-\rho^2}} \right) e^{x_1^2/2\sigma_1^2} \, dx_1 \]

where: \( \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du \) is the cumulative standard normal distribution.

\( O(s,0) \) and \( O(s,m) \) are obtained from (5) and (6) by changing \( a_1, a_2, (-c) \) to \( b_1, b_2 \) and \( d \) respectively.

The game in strategic form can now be defined as the ordered quartet \( G=(S,T,I,O) \) where \( S \) and \( T \) are the strategy sets given by (2) and (1).
respectively and I, O, are the payoff functions given by (3) and (4) respectively.

**Definition.** The pair of strategies \((s, r)\) is said to be a Nash Equilibrium (NE) if

\[
I(s, r) \geq I(s', r) \quad \forall s' \in S.
\]

\[
O(s, r) \geq O(s, r') \quad \forall r' \in T.
\]

3. **CUMUF as a N.E. Strategy**

To see the relations between our game theoretical model and the CUMUF test we have to impose on the model the assumptions underlying the test. These are basically that the payoffs in the various events are the false alarm probability and the probability of undetected diversion (of amount \(M\) or more). Technically this is captured by the following assumption (which implies our previous assumptions 1 and 2).

**Assumption 3:**

(i) \(a_1(m) = a_2(m_1, m_2) = a\) for all \((m_1, m_2) \in \mathbb{R}^2\).

(ii) \(b_1(m_1) - b_2(m_1, m_2) = b\) for all \((m_1, m_2) \in \mathbb{R}^2\).

(iii) \(c(m_1, m_2) = d(m_1, m_2) = \begin{cases} 0 & m + m < M \\ 1 & 2 \end{cases}
\]

where \(0 < a < 1\) and \(0 < b\) are constants.

In other words this assumption states that:

An alarm yields a damage \(a\) to \(I\) and \(b\) to \(O\) independently of the amount of material diverted.
- When no diversion takes place and no alarm is called the payoff for both players is 0.
- Any diversion which totals less than M gives no utility to 0 and causes no damage to 1.
- An undetected diversion totaling M or more gives a fixed positive utility to 0 and a fixed damage to 1 greater than a.
- The utility scales of the players are such that the payoffs to undetected diversion of M is -1 for 1 and 1 for 0.

Remark. Most of the results of this paper hold for much less restrictive assumption as long as we keep the assumption $a_2(m_1 m_2) = a_1(m_1 m_2)$ and $b_2(m_1 m_2) = b_1(m_1 m_2)$.

With this assumption the payoff functions becomes (recall that $s=(s_1, s_2)$ and $r=(q, m)$ are the strategies of 1 and 0 respectively):

(7) \[ I(s; (q, m)) = (1-q)I(s; 0) + q I(s; m) \]
(8) \[ O(s; (q, m)) = (1-q)O(s; 0) + q O(s; m), \]
where:

(9) \[ I(s; 0) = -a + \frac{a}{\sigma_1^2 \pi} \int_{-\infty}^{\infty} \phi\left(\frac{x_1 \mu - \nu}{\sigma_1^2 (1-\rho)}\right) e^{-\frac{x_1^2}{2\sigma_1^2}} dx_1 \]

(10) \[ O(s; 0) = -b + \frac{b}{\sigma_1^2 \pi} \int_{-\infty}^{\infty} \phi\left(\frac{x_1 \mu - \nu}{\sigma_1^2 (1-\rho)}\right) e^{-\frac{x_1^2}{2\sigma_1^2}} dx_1 \]

(11) \[ I(s; m) = -a - \frac{1-a}{\sigma_1^2 \pi} \int_{-\infty}^{\infty} \phi\left(\frac{x_1 \mu + m - \nu}{\sigma_1^2 (1-\rho)}\right) e^{-\frac{(x_1 + m)^2}{2\sigma_1^2}} dx_1 \]
\begin{equation}
(12) \quad O(s; m) = -b + \frac{1+b}{\sigma_1 \sqrt{2\pi}} \int_{-\infty}^{s_1} \frac{\sigma_2 (x_1-m)}{\sigma_1 (1-\rho)} e^{-\frac{(x_1-m)^2}{2\sigma_1^2}} \end{equation}

We conclude:

**Definition.** The game in strategic (normal) form is the ordered quadruple \((S, T, I, O)\) where \(T\) is given by (1), \(S\) is given by (2) and \(I, O\) are the payoff functions given by (7)-(11).

**Remarks**

1) \(I(s; 0)\) and \(O(s; 0)\) increase both in \(s_1\) and \(s_2\) reflecting the fact that if no diversion is committed it is better for both players to avoid false alarm. As \(s_1 \to \infty\) both payoffs go to 0, highest payoff possible in these circumstances.

2) \(I(s; m)\) is a decreasing function both of \(s_1\) and \(s_2\) and goes to \(-a\) as \(s_1 \to \infty\) reflecting the fact that if a diversion of \(M\) was committed the best thing that can happen to \(I\) is to declare an alarm and obtain \(-a\), his highest possible payoff in these circumstances.

3) There exists no equilibrium with \(q=0\), i.e., \(O\) certainly does not divert. This is because a best response of \(I\) to that is \(s_1 = s_2 = -\infty\) namely, never call an alarm. But then \(O\) can profit by diverting safely.

4) \(s_1 = -\infty\) and \(q=1\) consists of NE in which \(O\) surely diverts and \(I\) declares violation independently of the MUF observations. This is, of course, a non-interesting equilibrium of an open conflict between \(I\) and \(O\).

**Proposition 2:** There is no NE with \(s_1 = -\infty\) except for that with \(q=1\) (the conflict NE mentioned in Remark 4).

**Proof:** We claim that \(s_1 = -\infty\) is not a best reply against any strategy of \(O\) with \(q < 1\). In fact, \(s_1 = -\infty\) gives \(I\) a payoff \(-a\) while a finite \(s_1\) yields
\[ I(s; (q, m)) = -a + \frac{a(1-q)}{\sigma_1^2\pi}\int_{-\infty}^{s_1} \phi\left(\frac{\sigma_2^2}{\sigma_1^2} x_1 \right) e^{-\frac{x_1^2}{2\sigma_1^2}} dx_1 \]

\[ - \frac{q(1-a)}{\sigma_1^2\pi}\int_{-\infty}^{s_1} \phi\left(\frac{\sigma_2^2}{\sigma_1^2} (x_1 - m) \right) e^{-\frac{(x_1 - m)^2}{2\sigma_1^2}} dx_1 \]

**Case 1:** If \( m = 0 \) (i.e., all diversion is made at second period) choosing \( s_2 = 0 \) we have:

\[ I(s; (q, m)) = -a + \frac{1}{\sigma_1^2\pi}\int_{-\infty}^{s_1} \left[ a(1-q)\phi(-\rho x_1 / \sigma_1^2) + q(1-a)\phi(-\rho x_1 / \sigma_1^2 - m / \sigma_2^2) \right] e^{-\frac{x_1^2}{2\sigma_1^2}} dx_1 \]

Recalling that \( \rho \leq 0 \) we have in the integrand an expression of the form: \( a(1-q)\phi(y) - q(1-a)\phi(y-c) \) for \( c > 0 \). Since \( \lim_{y \to c} \phi(y-c) = 0 \) (as can be seen by L'Hospital's Rule), for \( s_1 \) sufficiently small but finite the integrand is positive on the range of integration \((0, s_1)\), since \( q < 1 \). Hence, for this \( s_1 \), \( I(s; (q, m)) > -a \) which means that \( s_1 = \infty \) is not an optimal response as claimed.

**Case 2:** \( m > 0 \). Changing variables we obtain
\[ I(s; (q, m)) = -a + \frac{a(1-q)}{\sigma_1 2\pi} \int_{-\infty}^{s_1} \varphi \left( \frac{s_2 - \rho x_1}{\sigma_2 \sqrt{1 - \rho^2}} \right) e^{-\frac{x_1^2}{2\sigma_1^2}} \, dx_1 \]

\[ - \frac{q(1-a)}{\sigma_1 2\pi} \int_{-\infty}^{s_1 - m} \varphi \left( \frac{s_2 - \rho x_1 - (M-m)}{\sigma_2 \sqrt{1 - \rho^2}} \right) e^{-\frac{x_1^2}{2\sigma_1^2}} \, dx_1 \]

Again, by the same technique one shows that the ratio of the second integral to the first one tends to 0 as \( s_1 \to -\infty \). Thus for \( s_1 > -\infty \), sufficiently small the difference becomes negative, i.e. \( I(s; (q, m)) > -a \) showing that \( s_1 = -\infty \) is not a best reply of \( I \) when \( q < 1 \).

**Conclusion 2:** Except for the conflict point mentioned in Remark 4, any NE \( (s; (q, m)) \) must satisfy \( 0 < q < 1, \ s_1 > -\infty \).

**Proposition 3.** When \( \rho = 0 \) there is no NE with \( m = M \).

**Proof.** When \( \rho = 0 \) and \( m = M \) the payoff to the inspector is (see (12)):

\[ I(s; (q, M)) = -a + \frac{1}{\sigma_1 2\pi} \int_{-\infty}^{s_1} \varphi \left( \frac{s_2(x_1)}{\sigma_2} \right) [a(1-q)] \, e^{-\frac{x_1^2}{2\sigma_1^2}} - q(1-a) e^{-\frac{(x_1-M)^2}{2\sigma_1^2}} \, dx_1 \]

Since \( 0 < q < 1 \) and \( 0 < a < 1 \) it is readily verified that if we denote
\[
f(x_1) = a(1-q)e^{-x_1^2/2\sigma_1^2}/(q(1-a)e^{-(x_1-M)^2/2\sigma_1^2})
\]

then \( f(x_1) \) monotonically decreases from \(+\) at \( x_1 = -\) to \(-\) at \( x_1 = +\). It follows that there exists \( x_1^* \) s.t. the integrand is positive for \( x_1 < x_1^* \) and negative for \( x_1 > x_1^* \). Therefore \( I(s;(q,m)) \) is maximized for \( s_1 = x_1^* \) and

\[
z_2(x_1) = \begin{cases} 
+ & x_1 \leq x_1^* \\
- & x_1 > x_1^*
\end{cases}
\]

Now clearly against this \( s \), diverting \( m=M \) in first period is not a best reply. In fact any \( m' < M \) (with diversion \( M-m' \) at the second period) will have strictly lower probability of detection and higher probability for undetected diversion of \( M \).

**Conclusion 1.** In any \( NE \) \( M-m(1+\rho \sigma_2/\sigma_1) > 0 \). (Recall that \( \rho \leq 0 \).

**Equations determining \( NE \)**

Any \( NE \) is determined by three real numbers: \( s_1, q \) and \( m \) and one real function \( z_2: X \to \mathbb{R} \).

Let us first impose the condition that \( (q,m) \) is a best reply for \( (s_1, z_2) \). Ignoring the conflict \( NE \) of \( \text{(Remark 4)} \) we know by our last conclusion that \( 0 < q \leq 1 \), hence \( O(s;(q,m)) \) has a local maximum at \( q \). But this is a linear function in \( q \) (see (4)) therefore it must be constant in \( q \), i.e.,

\[
O(s;m) = O(s;0)
\]
Given \( q \) and \((s_1, s_2)\), \( O(s;(q, \hat{q})) \) must be maximized at \( m \), i.e.,

\[
(1-q)O(s;0) + qO(s;\hat{q}) \quad \text{for} \ 0 \leq \hat{q} \leq m \quad \text{has a global maximum at} \ m.
\]

(15)

We now impose the conditions that \((s_1, s_2)\) is a best reply to \((q, m)\). This says first that \( I((s_1, s_2); (q, m)) \) as a function of \( s_1 \) has a global maximum at \( s_1 \). By Conclusion 2 and Remark 4, \(-\infty < s_1 \leq \infty\), therefore either \( s_1=\infty \) or this is a local maximum in which case:

\[
\frac{\partial}{\partial s_1} I((s_1, s_2); (q, m)) = 0.
\]

(16)

The last condition is that \( s_2 \) is the best reply to \((q, m)\) given \( s_1 \). This means that:

\[
I((s_1, s_2 + \delta); (q, m)) \leq I((s_1, s_2); (q, m))
\]

for any infinitesimal integrable function \( \delta: X_1 \to \mathbb{R} \).

This yields the variational equation

\[
\frac{a(1-q)}{\sigma_2^2 \pi^2} \int_{s_1}^{s_2} \frac{d}{ds_1} \phi\left(\frac{s_1 - x_1}{\sigma_1 \sqrt{1-\rho}}\right) e^{-\frac{(x_1-m)^2}{2\sigma_1^2}} \delta(x_1)dx_1 \leq 0.
\]

This is a condition of the form:

\[
\int_{-\infty}^{\infty} F(x_1; s_1, q, m) \delta(x_1)dx_1 \leq 0 \quad \text{for any infinitesimal} \ \delta: X_1 \to \mathbb{R}. \quad \text{A necessary condition for this is that (as a function of } x_1 \text{) } F \text{ must be identically 0 for } -\infty < x_1 \leq s_1, \text{ which yields:}
\]
\begin{equation}
\alpha(1-q) \exp \left( -\frac{1}{2} \left[ \frac{s_2 - \rho \sigma_1 x_1}{\sigma_2 \sqrt{1 - \rho^2}} \right]^2 + \frac{x_2^2}{\sigma_2^2} \right)
\end{equation}

\begin{equation}
= (1-a)q \exp \left( -\frac{1}{2} \left[ \frac{s_2 - M + m - \rho \frac{\sigma_2}{\sigma_1} (x_1 - m)}{\sigma_2 \sqrt{1 - \rho^2}} \right]^2 + \frac{(x_1 - m)^2}{\sigma_1^2} \right).
\end{equation}

By taking logarithms and simplifying we finally obtain:

\begin{equation}
\sigma_2 = C - \delta x_1
\end{equation}

\begin{equation}
\delta = \frac{\sigma_2^2}{\sigma_1^2} \frac{(1 - \rho^2)m}{M - m(1 + \rho \sigma_2 / \sigma_1)} - \rho \frac{\sigma_2}{\sigma_1}
\end{equation}

\begin{equation}
C = \frac{1}{2} \left[ M - m(1 + \rho \sigma_2 / \sigma_1) - \frac{\sigma_2^2 (1 - \rho^2)(A - m^2 / \sigma_1^2)}{M - m(1 + \rho \sigma_2 / \sigma_1)} \right];
\end{equation}

\begin{equation}
A = 2 \ln\left( \frac{q}{1-q} \right) \cdot \frac{1-a}{a}
\end{equation}

Remark that all denominators are strictly positive by conclusions 2 and 3.

Noting that since \( \rho \leq 0, \delta > 0 \), equation (17) says that the strategy \((s_1, s_2)\)
of I is actually a CUMUF type test since it calls an alarm if \( x_2 \geq s_2 \), i.e. \( \delta x_1 + x_2 \geq C \). Furthermore, for \( \delta = 1 \) this is precisely the CUMUF test based on
the statistics $x_1 + x_2 = MUF_1 + MUF_2$. Our main objective in the rest of the paper is to prove that this is in fact the case in (the unique) Nash Equilibrium, provided the parameters of the game satisfy certain conditions to be specified later.

**Theorem**

(i) Generically, the game analyzed has a unique NE.

(ii) In the NE, $s_1 = \omega$.

(iii) Provided some conditions on the parameters are satisfied, in the NE, $\theta = 1$.

That is, no test is made at first period and under the conditions mentioned in (iii), (which will be derived in the proof), the statistics for the second period test is $x_1 + x_2 = MUF_1 + MUF_2$. In other words, in the only NE of the game the inspector's strategy consists of the CUMUF test. Generically means here for any set of parameters that do not satisfy certain equalities to be specified later.

**Proof.** In view of (17), a NE point is determined by the three quantities $s_1$, $q$ and $m$. These have to satisfy conditions (14), (15) and (16).

**Step 1.** Condition (16) implies $s_1 = \omega$.

To prove this we show now that given $(q, m)$ the inspector's payoff $I(s, (q, m))$ strictly increases in $s_1$ therefore in NE we must have $s_1 = \omega$.

By differentiating (7) with respect to $s_1$ using (9), (10) and (17) we have:
\[
\frac{\delta l}{\delta s_1} = \frac{a(1-q)}{\sigma_1 2\pi} \phi\left(\frac{C-r s_1-s}{\sigma_1 1-\rho^2}\right) e^{-s_1^2/2\sigma_1^2}
\]
\[
\quad - \frac{a(1-q)}{\sigma_1 2\pi} \phi\left(\frac{C-r s_1-s}{\sigma_1 1-\rho^2}\right) e^{-\left(s_1-m\right)^2/2\sigma_1^2} \cdot \left(s_1-m\right) \cdot \left(M-m\right)
\]

\[
\frac{\delta l}{\delta s_1} > 0 \text{ is equivalent to:}
\]

\[
\frac{C-r s_1-s}{\sigma_1 1-\rho^2} \phi\left(\frac{C-r s_1-s}{\sigma_1 1-\rho^2}\right) \exp\left(-\frac{s_1^2}{2\sigma_1^2}\right) > \frac{a(1-q) \exp \delta}{a(1-q) \exp \frac{\delta}{2}}
\]

(21)

\[
\quad \frac{C-r s_1-s}{\sigma_1 1-\rho^2} \phi\left(\frac{C-r s_1-s}{\sigma_1 1-\rho^2}\right) \exp\left(-\frac{s_1^2}{2\sigma_1^2}\right) \left(s_1-m\right)^2/2\sigma_1^2 > \frac{a(1-q) \exp \delta}{a(1-q) \exp \frac{\delta}{2}}
\]

Let:

\[
\delta = \frac{M-m(1+\rho)}{\sigma_2 \sqrt{1-\rho^2}} > 0, \quad \overline{c} = \frac{C}{\sigma_2 \sqrt{1-\rho^2}}
\]

(22)

\[
\mu = \exp\left(\frac{1}{2} \left(A - \frac{m^2}{\sigma_2^2}\right) + \delta \overline{c}\right); \quad y = \delta \overline{c} - \frac{m}{\delta \sigma_2^2} s_1.
\]

In equality (21) can then be written as:
(23) \[ \frac{\phi(y)}{\phi(y-\delta)} e^{\delta y} > \mu \]

To show that (23) holds for all \( y \in (-\infty, \infty) \) we first show that:

(24) \[ \lim_{y \to \infty} \frac{\phi(y)e^{\delta y}}{\phi(y-\delta)} = \mu \]

In fact:

\[
\lim_{y \to \infty} \frac{\phi(y)e^{\delta y}}{\phi(y-\delta)} = \lim_{y \to \infty} \frac{\delta \sqrt{2\pi} \phi(y) + \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{\delta y}}{1 e^{-(y-\delta)^2/2}} = \lim_{y \to \infty} \frac{\delta \sqrt{2\pi} \phi(y) + \frac{1}{\sqrt{2\pi}} e^{-y^2/2}}{e^{-(y^2+\delta^2)/2}} = e^{\delta^2/2} - \mu
\]

The last equality follows from \( \delta^2 = 2 \ln \mu \) which is readily verified using the definitions (19) and (22).

Next denote by \( f(y) \) the left hand side of (23). The proof of (21) will be completed by proving that \( f'(y) > 0 \) \( \forall y \). To do that we make use of the Mill's Ratio:

\[ R(y) = \frac{1 - \phi(y)}{\phi'(y)} = \frac{\phi(-y)}{\phi(y)} \] and its property (see [4] and [6]).

(25) \[ 0 < \frac{d}{dy} \left( \frac{1}{R(y)} \right) < 1 \quad \forall \ y \in (-\infty, \infty) \]
In fact $f'(y) > 0$ is equivalent $(\ln f(y))' > 0$ which is:

$$(\ln \phi(y))' - (\ln \phi(y-\delta))' + \delta > 0$$

Noting that $(\ln \phi(y))' = -\frac{1}{\phi(y)}$ this is equivalent to:

$$\frac{1}{R(-y)} \cdot \frac{1}{R(-y+\delta)} + \delta > 0$$

or

$$\int_{y}^{y+\delta} \frac{d}{du} \left( \frac{1}{R(u)} \right) \, du < \delta$$

which is true by (25).

This concludes the proof of step 1.

**Step 2.** The values of $m$ and $q$ in NE.

Let us now write condition (15) which after substituting (13), (17) and $s_{1}=∞$

yields:

$$m = \arg \max \int_{-\infty}^{\infty} \phi\left( \frac{C-\delta x_{1}-M-m-\rho x_{2}}{\sigma_{1}} \right) \frac{\sigma_{2}^{2}}{\sigma_{1}^{1+\rho^{2}}} \left( x_{1} \cdot m \right)^{2}/2\sigma_{1} \, dx$$

$$= \arg \max \int_{-\infty}^{\infty} \phi\left( \frac{C-\delta (\sigma_{1} x_{1}+m)-M-m-\rho x_{2}}{\sigma_{2}} \right) \frac{\sigma_{2}^{2}}{\sigma_{1}^{1+\rho^{2}}} \, dx$$

Let $\lambda = \sigma_{2}/\sigma_{1}$ and measure $M$, $m$ and $C$ in units of $\sigma_{2}$ (i.e. replace $M/\sigma_{2}$, $m/\sigma_{2}$ and $C/\sigma_{2}$ by $M$, $m$ and $C$ respectively) we obtain:
\[ m = \arg \max \int_{-\omega}^{\omega} \phi \left( \frac{C \cdot (\theta + \rho)x - (\theta - 1)m + M}{\sqrt{1 - \rho^2}} \right) \exp \left( -\frac{x^2}{2} \right) \, dx \]

Use now the fact (See Owen [2] equation 10.010.8 page 403):

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi (\alpha + \beta x) \exp \left( -\frac{x^2}{2} \right) dx = \phi \left( \frac{\alpha}{\sqrt{1 + \beta^2}} \right) \]

to obtain

\[ m = \arg \max \phi \left( \frac{C - M + m(1 - \theta)}{\sqrt{1 - \rho^2 + (\frac{1}{\lambda} + \rho)^2}} \right) \]

which is equivalent to:

\[ m = \arg \max \left( \frac{C - M + m(1 - \theta)}{\sqrt{1 - \rho^2 + (\frac{1}{\lambda} + \rho)^2}} \right) \]

where

\[ \theta = \frac{(1 - \rho^2) \frac{\lambda^2 m}{M - (1 - \lambda \rho) m} - \lambda \rho}{(1 - \rho^2) \frac{\lambda^2 m}{M - (1 - \lambda \rho) m} - \frac{(1 - \rho^2) (A + \lambda^2 m^2)}{M - (1 - \lambda \rho) m}} \]

\[ A = \frac{1}{2} \ln \left( \frac{1 - q}{1 - a} \cdot \frac{\frac{1 - a}{a}}{1 - q} \right) \]

\[ \frac{A}{2} = \ln \left( \frac{1 - q}{1 - a} \cdot \frac{\frac{1 - a}{a}}{1 - q} \right) \]
Denote also:

\[ \beta = \frac{M}{2\sqrt{1-\rho^2}}, \quad \gamma = \frac{\lambda M}{2\sqrt{1-\rho^2}}, \quad \eta = \frac{1+\lambda \rho}{1+2\lambda \rho + \lambda^2} \]

When considering \( f(m, \lambda) \) it will be convenient to refer to \( \lambda \) as a parameter and \( m \) as the (only) variable. For two real numbers \( x \) and \( z \) we denote by \( x \wedge z \) and \( x \vee z \) the numbers \( \max(x, z) \) and \( \min(x, z) \) respectively.

The following observations which are needed later are proved by direct verification.

**Proposition 4**

(i) If \( 0 < 4\rho^2 \) then:

- \( m_2 \) has no real solution, \( y^2(m) > (1-\rho^2)A \), \( \forall m \in [0, M] \) and the (global) maximum of \( f \) is at the (global) minimum of \( y(m) \).

(ii) If \( 4\rho^2 < A < 4\beta^2 \vee 4\gamma^2 \), then:

- a) If \( 0 \leq \eta \leq 1 \), \( m_2 \) has two real solutions in \([0, M] \) each of which consist of a global maximum for \( f \) with \( f(m_2) = -\sqrt{\Lambda} \).
- b) If \( \eta < 0 \), i.e. \( \rho < -1/\lambda \), \( m_2 \) has two negative solutions and \( f(m) \) is maximized at \( m = 0 \).
- c) If \( \eta > 1 \), i.e. \( \rho < -\lambda \), \( m_2 \) has two solutions larger than \( M \) and \( f(m) \) is maximized at \( m = M \).

(iii) If \( 4\beta^2 \vee 4\gamma^2 < A < 4\beta^2 \vee 4\gamma^2 \) then:

- \( m_2 \) has one real solution in \([0, M] \) which is the maximum of \( f \) at value \(-\sqrt{\Lambda} \).

(iv) If \( 4\beta^2 \vee 4\gamma^2 < A \) then:

- \( y^2(m) < (1-\rho^2)A \) for \( m \in [0, M] \) and the maximum of \( f \) is at \( m = 0 \) if \( \lambda < 1 \) or at \( m = M \) if \( \lambda > 1 \).
The second and last condition for the NE is (14), that is for the inspector’s strategy and \( m \) in equilibrium (since \( 0 < q < 1 \)), \( O(s,m) = O(s,m) \) must be satisfied. Substituting in (10) and (12), \( s_1 = m, s_2 = C - \delta x_1 \) and applying (25) condition (14) becomes:

\[
\phi \left( \frac{C - (M + (\delta - 1)m)}{\sqrt{1 - \rho^2 + (\delta - 1)^2}} \right) - \phi \left( \frac{C}{\sqrt{1 - \rho^2 + (\delta - 1)^2}} \right) = \frac{b}{1 + b}
\]

(38)

Since the only candidates for \( m \) in NE are in the set \( \{0, m_1, m_2, M\} \), let us write (38) for each of these values. Also let \( B = b/1 + b \) (so \( 0 < B < 1 \)). The results are given in the following table:

<table>
<thead>
<tr>
<th>Value of ( m )</th>
<th>The equation determining ( A_1 )</th>
<th>( f(m_i) ) for ( \hat{A} = A_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_0 = 0 )</td>
<td>( \phi \left( - \frac{A_0}{4\beta} - \beta \right) / \phi \left( - \frac{A_0}{4\beta} + \beta \right) = B )</td>
<td>( -\frac{A_0}{4\beta} - \beta )</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>( \phi \left( - \frac{A_1}{4\alpha} - \alpha \right) / \phi \left( - \frac{A_1}{4\alpha} + \alpha \right) = B )</td>
<td>( -\frac{A_1}{4\alpha} - \alpha )</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>( \phi \left( - A_2 \right) = B/2 )</td>
<td>( -A_2 )</td>
</tr>
<tr>
<td>( m_3 = M )</td>
<td>( \phi \left( - \frac{A_3}{4\gamma} - \gamma \right) / \phi \left( - \frac{A_3}{4\gamma} + \gamma \right) = B )</td>
<td>( -\frac{A_3}{4\gamma} - \gamma )</td>
</tr>
</tbody>
</table>

**TABLE 1**

Recall that determining \( A \) means determining \( q \) since by the definition of \( A(20) \):

\[
q = q(A) = \frac{a e^{A/2}}{1 - a + a e^{A/2}}
\]

(39)
The conclusion of our discussion can be written as:

**Proposition 3**

(1) The only candidates for NE strategies for the operator are \((m_i, A_i)\) for \(i = 0, 1, 2, 3\) of Table 1.

(ii) A necessary and sufficient condition for \((m_i, A_i)\) to be in NE is \(f(m_i) \geq f(m) \quad \forall m \epsilon [0, M]\).

To proceed in our proof we need now the following:

**Lemma** For any \(d > 0\), the function \(F(x) = \phi(x+d)/\phi(x+d)\) satisfy:

(i) \(F(x)\) is strictly increasing in \(x\).

(ii) \(\lim_{x \to \infty} F(x) = 0\)

(iii) \(\lim_{x \to -\infty} F(x) = 1\)

**Proof** (iii) immediately follows from \(\phi(x) = 1\). (ii) is verified by l'Hopital's rule.

\[
\lim_{x \to \infty} F(x) = \lim_{x \to \infty} e^{xd} = 0.
\]

As for (i):

\[
(40) \quad \frac{dF}{dx} = \frac{e^{-(x-d)^2/2}}{2\pi \phi^2(x+d)} g(x)
\]

where

\[
(41) \quad g(x) = \phi(x+d) - e^{-2xd} \phi(x-d)
\]

To prove (i) we shall show

\[
(42) \quad g(x) > 0 \quad \forall x \epsilon (-\infty, \infty).
\]
In fact:

\[
\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{\phi(x-d)}{e^{2xd}} = \lim_{x \to -\infty} \frac{e^{-\frac{(x-d)^2}{2}}}{2\sqrt{2\pi}e^{2xd}} - \lim_{x \to -\infty} \frac{1}{2d\sqrt{2\pi}} e^{-\frac{(x+d)^2}{2}} = 0
\]

Also: \( \frac{dg}{dx} = 2d e^{-2xd} \phi(x-d) > 0 \), completing the proof of the lemma.

**Corollary 1.** The quantities \( A_i \; i=0,1,2,3 \) in Table 1 are uniquely determined.

**Corollary 2.** For fixed \( B \) if \( x \) is defined by:

\[
\frac{\phi(x-d)}{\phi(x+d)} = B, \text{ then } (x-d) \text{ is strictly increasing in } d.
\]

To see that, let

\[
\frac{\phi(x_1 \cdot d_1)}{\phi(x_1 + d_1)} = B = \frac{\phi(x_2 \cdot d_2)}{\phi(x_2 + d_2)} \quad \text{where } d_1 > d_2, \text{ then: let } z = x_2 \cdot d_2 + d_1.
\]

We have:

\[
\frac{\phi(z \cdot d_1)}{\phi(z + d_1)} = \frac{\phi(x_1 \cdot d_1)}{\phi(x_1 + d_1)} < \frac{\phi(x_2 \cdot d_2)}{\phi(x_2 + d_2)} = B.
\]

It follows from (1) of the lemma that \( x_1 > z \). That is, \( x_1 \cdot d_1 > x_2 \cdot d_2 \).

**Proposition 4.** \( -\frac{A_1}{4a} - \alpha \leq -\frac{A_0}{4\beta} - \beta \) and equality holds only if \( \rho = -\frac{1}{\lambda} \).

\[
-\frac{A_1}{4a} - \alpha \leq -\frac{A_3}{4\gamma} - \gamma \quad \text{and equality holds only if } \rho = -\lambda.
\]
Proof. From (36) and (37) it follows that: \( \alpha \leq \beta \) and equality holds only for \( \rho = -1/\lambda \) and \( \alpha \leq \gamma \) with equality only when \( \rho = -\lambda \). The result now follows from Corollary 2.

**Proposition 7.** In each of the following the three inequalities listed are equivalent.

(i) \( \gamma \geq A_3 - \frac{A_0}{A_3} - \beta \); \( \alpha \geq \beta ; \lambda \geq 1 \).

(ii) \( A_2 \geq \frac{A_1}{4\alpha} - \alpha \); \( A_2 \geq 4\alpha^2 - A_1 \geq 4\alpha^2 \).

(iii) \( A_4 \geq \frac{A_0}{4\beta} - \beta \); \( A_4 \geq 4\beta^2 - A_0 \geq 4\beta^2 \).

(iv) \( A_4 \geq \frac{A_3}{4\gamma} - \gamma \); \( A_4 \geq 4\gamma^2 - A_3 \geq 4\gamma^2 \).

**Proof.** (i) Follows from Corollary 2 and (36).

(ii) If \( A_2 \geq 4\alpha^2 \) then:

\[
\frac{\phi((-\sqrt{A_2} + \alpha) - \alpha)}{\phi((2\alpha - \sqrt{A_2}) - \alpha)} = \frac{\phi((-\sqrt{A_2}) - \alpha)}{1/2} = B
\]

Therefore by (i) of the Lemma and by the definition of \( A_1 \):

\(-\sqrt{A_2} + \alpha \geq A_2/4\alpha \), that is \( -\sqrt{A_2} \geq \frac{A_2}{4\alpha} - \alpha \). That this implies \( A_2 \geq 4\alpha^2 \) proved in the same way. To prove that \( A_2 \geq 4\alpha^2 \) is equivalent to \( A_1 \geq 4\alpha^2 \):

\( A_2 \geq 4\alpha^2 \rightarrow -\sqrt{A_2} \geq \frac{A_1}{4\alpha} - \alpha = A_1 \geq 4\alpha \) \( \phi((-\sqrt{A_2} - \alpha)) \geq 4\alpha^2 \).
Similarly, in the other direction, (iii) and (iv) are proved in the same way.

Note that $A_2$ may be taken as a parameter of the game since it is a well defined function of $B = b/(1-b)$ (which satisfies $0 < B < 1$):

$$A_2 = \left(\phi^{-1}(B/2)\right)^2. \tag{43}$$

We intend to prove that except for three specific values of $A_2$ there is a unique NE.

**Proposition 8 (Generic Uniqueness of the Equilibrium).**

If $A_2 \neq (4\alpha^2,4\beta^2,4\gamma^2)$ then there is exactly one $(m_1,A_1)$ which consists of the (unique) NE strategy of the operator.

**Proof.** The proof consists of straightforward verification based on Propositions 4 and 7 which we shall refer briefly as p4(iii), p7(iv), etc. A solution to the NE conditions is $(m_1,A_1)$ such that $f(m,A_1)$ is maximized at $m=m_1$.

**Case 1.** $4\beta^2 < 4\gamma^2 < A_2$.

(a) If $\gamma < \beta$ (i.e. $A < 1$) then by p7(iii), $4\beta^2 < 4\gamma^2 < A_0$ and by p4(iv) $f(m,A_0)$ is maximized at $m=0$, i.e. $(0,A_0)$ is a solution.

- By p7(iv) $A_3 > 4\gamma^2$ so either $A_3 \geq 4\beta^2$ or $4\gamma^2 < A_3 < 4\beta^2$ which implies (by p4(iv)) that $f(m,A_3)$ is maximized at $m=0$ or $4\beta^2 < 4\gamma^2 < A_3 < 4\beta^2$ implying that $f(m,A_3)$ is not a solution.

- $(m_2,A_2)$ is not a solution since by p4(iv) $f(m,A_2)$ is maximized either at $m=0$ or at $m=M$. 
- By p7(iii) and $4\alpha^2 < 4\beta^2 \land 4\gamma^2$, $A_1 > 4\alpha^2$ and so $(m_1, A_1)$ is not a solution since by p4, $m_1$ can be the maximizer of $f(m, A)$ only if $A_1 \leq 4\alpha^2$.

We conclude that in this case $(0, A_0)$ is the only solution. Similarly it is proved that:

(b) If $\beta < \gamma$, $(M, A_3)$ is the only solution.

**Case 2.** $4\beta^2 \land 4\gamma^2 < A_2 < 4\beta^2 \land 4\gamma^2$.

We claim that $(m_2, A_2)$ is the only solution. In fact, by p4(iii) it is a solution.

- $(m_1, A_1)$ is not a solution since $A_0 > 4\alpha^2$ (by p7(ii)) and hence by p4 $f(m_1, A_1)$ is not maximized at $m_1$.

(a) Assume $4\beta^2 < A_2 < 4\gamma^2$ (implying $\lambda > 1$ and hence $\eta < 0$). By p7 we have $A_3 < 4\gamma^2$ and $A_0 > 4\beta^2$, then either $4\beta^2 < A_3 < 4\gamma^2$ in which case $f(m, A_3)$ is maximized at $m_2$ or $A_3 < 4\beta^2 < 4\gamma^2$ and $f(m, A_3)$ is not maximized at $M$ (by p4(ii)). So in any case $(M, A_3)$ is not a solution. $(0, A_0)$ is also not a solution since either $4\beta^2 < 4\gamma^2 < A_0$ in which case $f(m, A_0)$ has a maximum at $M$ or $4\beta^2 < A_0 < 4\gamma^2$ in which case it has a maximum at $m_2$.

(b) The case $4\gamma^2 < A_2 < 4\beta^2$ is treated in the same way.

**Case 3.** $0 \leq A_2 < 4\beta^2 \land 4\gamma^2$ and $\gamma < 0$.

We claim that in this case $(0, A_0)$ is the only solution. First note that $\gamma < 0$ implies $\lambda > 1$, i.e., $4\beta^2 < 4\gamma^2$. By p7, $0 \leq A_0 < 4\beta^2 \land 4\gamma^2$ and by p4 (i) (ii) $f(m, A_0)$ is maximized at $m = 0$, i.e., $(0, A_0)$ is a solution.

- $(M, A_3)$ is not a solution since $A_3 < 4\gamma^2 - 4\beta^2 \lor 4\gamma^2$ (by p7) so either $4\beta^2 \land 4\gamma^2 < A_3 < 4\beta^2 \land 4\gamma^2$ or $0 \leq A_3 < 4\beta^2 \land 4\gamma^2$. In both cases $M$ is not a maximizer of $f(m, A_3)$, by p4.

- Finally $(m_1, A_1)$ is not a solution since by p4, $m_1$ is not a maximizer of $f(m, A_1)$ unless $0 \leq \eta \leq 1$. 
Case 4. $0 \leq A_2 < 4\beta^2 \vee 4\gamma^2$ and $\eta > 1$.

Exactly as in case 3, interchanging $\beta$ and $\gamma$ it is proved that $(M,A_3)$ is the only solution.

Case 5. $4\alpha^2 < A_2 < 4\beta^2 \vee 4\gamma^2$ and $0 \leq \eta \leq 1$.

We claim that in this case $(m_2,A_2)$ is the only solution. In fact, it is a solution by p4(ii). None of the $(0,A_0)$ or $(M,A_3)$ is a solution since by p4, when $0 \leq \eta < 1$, the maximum of $f(m,A)$ can be at $m=0$ or $M$ only if $A > 4\beta^2 \vee 4\gamma^2$ but by p7 both $A_0$ and $A_3$ are smaller than $4\beta^2 \vee 4\gamma^2$. Finally, $(m_1,A_1)$ is not a solution by p4 since $A_2 > 4\alpha^2$ implies $A_1 > 4\alpha^2$.

Case 6. $0 \leq A_2 < 4\alpha^2$ and $0 \leq \eta \leq 1$.

We claim that in this case $(m_1,A_1)$ is the only solution. In fact it is a solution by p4(i). By the same reason $(m_2,A_2)$ is not a solution (since $f(m,A_2)$ is maximized at $m_1$). Neither of $(0,A_0)$ or $(M,A_3)$ is a solution for the same reasons as in Case 5.

This completes the proof of Proposition 8.
The Unique Nash Equilibrium Strategies

The Operator's NE strategy is summarized from the proof of Proposition 8 by Table 2:

<table>
<thead>
<tr>
<th>$A_2 &lt; 4a^2$</th>
<th>$\eta &lt; 0$</th>
<th>$0 \leq \eta \leq 1$</th>
<th>$\eta &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4a^2 &lt; A_2 &lt; 4\beta^2 \wedge 4\gamma^2$</td>
<td>$(0, A_0)$</td>
<td>$(m_1, A_1)$</td>
<td>$(M, A_3)$</td>
</tr>
<tr>
<td>$4\beta^2 \wedge 4\gamma^2 &lt; A_2 &lt; 4\beta^2 \wedge 4\gamma^2$</td>
<td>$(m_2, A_2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4\beta^2 \wedge 4\gamma^2 &lt; A_2$</td>
<td>$(0, A_0)$ if $\lambda &lt; 1$</td>
<td>$(M, A_3)$ if $\lambda &gt; 1$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2

The NE point for various configurations of the parameters

The probability of diversion is obtained by substituting the value of $A$ in $q(A)$ given by (39).

The inspector's equilibrium strategy is:

- Never call alarm in first period ($s_1 = \infty$).
- Call alarm in second period if $\theta MUF_1 + MUF_2 > C$, where $\theta$ and $C$ are obtained by substituting the values of $(m, A)$ from Table 2 into (28) and (29).

The resulting expressions are given in Table 3:
Table 3

Equilibrium strategies

For the inspector: \( s_1 = \infty; \ s_2 = C - \theta \ MUF \).

For the Operator: \((m_1, q_1);\) Where \( q_1 = q(A_1) \) is given by (39).

Remark. It is readily checked from our proofs that both the diversion probability \( q \) and the false alarm probability are continuous functions of the parameters of the games including the values for which \( A_2 < (4\alpha^2, 4\beta^2, 4\gamma^2) \).

Conclusion: When does CUMUF become the unique NE of the game.

From our proof of proposition 8 it follows that CUMUF is the unique NE strategy of the operator if and only if the parameters of the game satisfy two conditions (see Table 2).

\[(\lambda, \rho) \geq \max \{-\alpha, -1/\lambda\} \]
(45) \[ \lambda_2 \leq \Delta^2. \]

Recalling that \( \lambda = \sigma_2 / \sigma_1 \), the domain in the \((\sigma_2 / \sigma_1, \rho) \) space in which \((\Delta) \) is satisfied is shown by the shaded area in Figure 2.

Fig.2
Condition (44) on \( \rho, \sigma_1 \) and \( \sigma_2 \)

The situations excluded by these conditions are those in which \( \text{MUF}_1 \) and \( \text{MUF}_2 \) are measured in different accuracies and they are strongly negatively correlated. It is clear that in such situations the operator's best strategy, if he is determined to divert, is to make the total diversion of \( M \) in the period in which \( \text{MUF} \) is less accurately measured. This is exactly what our result says (Case 4). Note that when \( \sigma_1 = \sigma_2 \) there is no restriction on \( \rho \) and then \( \rho = 0 \) there is no restriction on \( \lambda \).

As for condition (45), using (35) and (43) it can be written as

(46) \[ \frac{\lambda M}{1 + 2 \lambda + \rho^2 \lambda} \geq | \phi^{-1} \left( \frac{b}{2(1+\rho)} \right) | \]
Recall that the utility scale of the operator is such that his benefit from undiscovered diversion of $M$ is 1 and his damage from an alarm called by the inspector is $b$. Condition (46) can now be interpreted as relation between $b$, the damage of an alarm, and $M$ which may be thought of as a measure of the risk needed to be taken for a significant diversion: When alarm is not very frightening ($b+0$ implying $\phi^{-1}(b/2(1+b)) = -\infty$) only if the significant amount to be diverted is very large, the CUMUF test is still a good one. If $M$ is not large enough the operator will tend to use different diversion patterns and therefore the inspector also has to modify his test.

On top of this relation between $M$ and $b$ there is also the effect of $\lambda = \sigma_2/\sigma_1$: If $\lambda$ is too small the operator will tend to divert more in first period since $\sigma_1$ is much larger than $\sigma_2$ and the result may be that $\text{MUF}_1 + \text{MUF}_2$ is no longer the right statistics for the inspector's test.

4. Deterrence Strategy

Conclusion 2 asserts that except for the uninteresting conflict equilibrium, in the only NE point of the game there is a positive probability of diversion. This is the mathematical expression of a very fundamental feature of the situation under consideration: If the inspector is certain that no violation is taking place he can increase his utility by relaxing his safeguard procedure. But very low level of safeguard makes diversion, which is profitable for the operator more likely. Any deterrence strategy of the inspector which guarantees zero probability of diversion must be inefficient. If we assume that the inspector is willing to pay this price of inefficiency then the unique NE strategies that we established provide the required level needed for deterrence. To see that consider the situation in which the (unique)
NE point consists of CUMUF test for the inspector with threshold C for \( \text{MUF}_1 + \text{MUF}_2 \) and a strategy \((q,m)\) for the operator (see Table 3).

**Proposition.** If the inspector commits himself to threshold \( C-\varepsilon \) for some small \( \varepsilon > 0 \) then the best reply of the operator is \( q=0 \) i.e. not to divert.

**Proof.** Notice that by (27) since \( \theta-1 \), the expected payoff for the operator in case of diversion is independent of \( m \). This is also the case if \( C \) is replaced by \( C-\varepsilon \). Now \( C \) is determined by (38) (with \( \theta-1 \)). This equation says that the operator is indifferent between diversion and nondiversion. Since the left-hand side of (38) increases in \( C \) (see Lemma on page 27), it follows that:

\[
\phi \left( \frac{C-q-M}{1-\rho^2+(\frac{1}{\lambda+\rho})^2} \right) / \phi \left( \frac{C-\varepsilon}{1-\rho^2+(\frac{1}{\lambda+\rho})^2} \right) < \frac{b}{1+b}
\]

From (10) and (13) with \( s_1 = 0, s_2 = C-\varepsilon, \lambda_1 \) this implies \( O(s,0) > O(s,m) \) i.e. the operator's best reply is not to divert.

Another way to see this deterrence strategy is the following: Consider a variant of the game in which the inspector announces in advance the threshold \( C \) of his CUMUF test. Then the inspector chooses his diversion strategy so as to maximize his expected utility. For this game the NE of the original game \((C;(q,m))\) is not a NE. In fact since \( q>0 \), the inspector strictly prefers \( C-\varepsilon \) for \( \varepsilon \) small enough since this induces nondiversion. However this \( C-\varepsilon \) and nondiversion is also not NE since \( C-\varepsilon/2 \) and nondiversion is even better for the inspector. Nevertheless this is almost NE or \( \varepsilon \)-NE in the sense that no player
can, by unilateral deviation, increase his expected payoff by more than $\epsilon$. We therefore conclude:

**Conclusion.** If $C$ is the threshold of the CUMP NE strategy of the inspector in the original game then for any $\epsilon > 0$, a threshold $C - \epsilon$ and nondiversion is an $\epsilon$-NE of the game in which the inspector announces his threshold in advance.

5. **Concluding Remarks**

The main purpose of this paper was to provide a game theoretical sequential model for material accounting. Although the game was formulated with very general strategy spaces and payoffs, in the first stage of the analysis we imposed restrictions very similar to those assumed in the statistical treatment of the problem. The results were:

1. The game has a unique Nash Equilibrium (except the open conflict situation.)

2. If the significant amount $M$ is not too small, the damage of an alarm is not too small the precision of measurement is not very different for $MU_{F1}$ and $MU_{F2}$, or if it is different then they are not strongly and negatively correlated then:

   **The CUMP test is the only equilibrium strategy for the inspector.**

In addition to the very clear relation established here between the games and the well known statistical procedure the game theoretical analysis provide the following new results:

- In the deterrence CUMP tests of the inspector both the false alarm probability and the critical value are endogenously determined, as part of the equilibrium, by the combined effect of all payoff functions involved. This is
in clear contrast to the statistical analysis in which the critical value is
determined only by the false alarm probability which is taken to be a given
parameter.

- The diversion probability \( q \), though "not coming into action" if the
  inspector uses deterrence strategies, is an informative value about the
  stability of the safeguards system. The inspector may wish to decrease \( q \) for
  instance by increasing the penalty on caught diversion (i.e. roughly increasing
  \( b \)).

The results of this paper may be viewed as calibration of the same
theoretical model. Very loosely speaking we showed that the game can produce
the statistical procedure as a special case. Now we have to study what are the
results of this game in more general cases not covered by the statistical
paradigm. The main directions of generalization we intend to study are:

(i) To say something about the time effect by choosing payoff functions
reflecting the fact that early detection of diversion is more desirable to the
inspector and less desirable to the operator.

(ii) To relax the not very realistic constraints imposed on the operator's
strategies. In particular he may be allowed to use mixed strategies i.e.
randomization on several diversion plans \((q,m)\). It is quite clear that this
may be more difficult for the inspector to counter. Or the operator may use
behavior strategies in the full sense of word: He does not commit himself, as
assumed so far, to complete the diversion of the totality of \( M \) but rather
reconsiders it after the observation of \( M_U \): whether to complete the
diversion or give up the idea because it became too risky.
APPENDIX - Some numerical results

We made some numerical computations to get some feeling about the dependence of the solution on the parameters of the game.

Fig. A1 gives the validity region of CUMUF that is the region of parameters for which CUMUF is the unique NE. This region is defined by inequalities (44) and (46). We fixed λ=1 (hence (44) is automatically fulfilled) and drew the equality lines of (46) in the M-b space for ratios values of ρ. For each of the lines the region North-west of the line is the region where CUMUF applies for that value of ρ. As expected this Fig. A1 shows that CUMUF is no longer the solution when both M and b are very small.

The rest of the numerical computations consist of computing the NE for fixed set of parameters, λ, ρ, M, a as a function of b. At each point the value of A2 was solved from (43) then the appropriate solution was chosen according to Table 2. The appropriate equation for A was solved (Table 1). The diversion probability q was then computed by (39) and the false alarm probability (FAP) is one minus the denominator of the equation determining A (Table 1). For Fig. A2 we inverted the procedure and computed b for given value of FAP and all other parameters of the game: λ=1, ρ=0.5. Note that FAP is independent of a.

The two lines denoted by 4 define a region in which FAP = 1/2 and the NE solution is (m2, A2). To the right of this region (lines 1, 2, 3) we have the CUMUF solution and to the left of it (lines 5, 6) we have the extreme solutions (0, A2) and (M, A3).

Figures A3, A4, A5 give the diversion probability q as a function of b for various values of M and a. As expected the diversion probability decreases in b (this disadvantage from an alarm), decreases in M (the critical mass) and increases in a: The more embarrassed the inspector is from calling an alarm the more likely the operator is to divert.
REFERENCES


[9] Zamir, S. (1986), "Toward the Solution of the Two-Period Material Accounting Game", KFK Pragerbericht 04.01.10F44D.
Validity Region of CUMUF

Fig. A1
False Alarm Probabilities

$\lambda = 4$
$\beta = 0.5$

Fig. A2
Diversion probability

\[ q \]

- \( M = 1 \)
- \( \lambda = 1 \)
- \( \rho = -0.5 \)

\( a = 0.2 \)
\( a = 0.5 \)
\( a = 0.8 \)

Region of CUMUF test

Fig. A3
Divergence probability

\[ M=2 \]
\[ \lambda=1 \]
\[ \rho=-0.5 \]

Fig. A4
Region of CUMUF test

Fig. A5