THE GAME FOR THE SPEED
OF CONVERGENCE IN REPEATED GAMES
OF INCOMPLETE INFORMATION

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THE GAME FOR THE SPEED OF CONVERGENCE
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IRIT NOWIK AND SHMUEL ZAMIR

ABSTRACT. We consider an infinitely repeated zero-sum two-person game with incomplete information on one side, in which the maximizer is the (more) informed player. Such games have value \( v_{\infty}(p) \) for all \( 0 \leq p \leq 1 \). The informed player can guarantee that all along the game, the average payoff per stage will be greater or equal to \( v_{\infty}(p) \), (and will converge from above to \( v_{\infty}(p) \) if the minimizer plays optimally). Thus there is a conflict of interest between the two players regarding the speed of convergence of the average payoffs, to the value \( v_{\infty}(p) \). In the context of such repeated games, we define a game, denoted as \( SG_{\infty}(p) \), for the speed of convergence, and a value for this game. We prove that the value exists for games with the highest error term, namely games in which \( v_{\infty}(p) - v_{\infty}(p) \) is of the order of magnitude of \( \frac{1}{n^{1/2}} \). In that case the value of \( SG_{\infty}(p) \) is also of the order of magnitude of \( \frac{1}{n^{1/2}} \). Then we show that in another class of games, the value does not exist.

For our first result we define for any infinite martingale \( X = (X_n)_{n=1}^{\infty} \), the variation of it: \( V_n(X) := E \sum_{k=1}^{n} |X_{k+1} - X_k| \), and prove that the variation of a uniformly bounded, infinite martingale \( X^{\infty} \), can be of the order of magnitude of \( n^{1/2} \), for arbitrarily small \( \epsilon > 0 \).

1. INTRODUCTION

In this paper we treat a two-person 0-sum repeated game, with incomplete information on one side (see e.g. p.116 of [10]): Let \( A_1, A_2 \) be \( 2 \times 2 \) matrices, each corresponding to the payoff of a two person zero-sum game, with elements: \( a_{ij}^k \), where \( k \in \{1, 2\} \) represents the number of the matrix and \( i \in I = \{T, B\} \) and \( j \in J = \{L, R\} \) are the pure strategies of PI (the Maximizer) and PII (the minimizer) respectively. For each \( p \), \( 0 \leq p \leq 1 \), we consider the \( n \)-stage repeated game \( G_n(p) \) defined as follows:

- At stage 0, chance chooses \( k = 1 \) with probability \( p \), and \( k = 2 \) with probability \( p' = 1 - p \). Both players know \( p \), but (only) PI is informed also about the chosen value of \( k \).
- At stage 1, PI chooses \( i_1 \in I \), PII chooses \( j_1 \in J \), and \( (i_1, j_1) \) is publicly announced.
- At stage \( m, m = 2, 3, \ldots \) knowing \((i_1, j_1), \ldots, (i_{m-1}, j_{m-1})\), PI (resp. PII) chooses \( i_m \in I \) (resp. \( j_m \in J \)) and then \((i_m, j_m)\) is announced.

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• After stage \( n \), PI receives from PII the following amount:

\[
\frac{1}{n} \sum_{m=1}^{n} a_{imjm}^n
\]

(We divide by \( n \) in order to compare payoffs in \( G_n(p) \) for different values of \( n \).)

When \( n \) is finite, \( G_n(p) \) is a finite game and therefore has a (minmax) value (in mixed strategies.), which we denote by \( v_n(p) \).

The strategies in \( G_n(p) \):

Denote by \( h_m \), the r.v that represents the history of announcements up to stage \( m \):

\((i_1, j_1) \ldots (i_{m-1}, j_{m-1})\) and by \( H_m = (I \times J)^{m-1} \), the set of all \( m \)–stage histories.

\((H_1 = \emptyset)\)

A (behavioral) strategy for PI is: \( \sigma_n = (\sigma_n^1, \sigma_n^2) \) where for each \( k \in \{1, 2\} \): \( \sigma_n^k = (s_n^1, \ldots s_n^k) \), and for all \( m, 1 \leq m \leq n \), \( s_n^k \) is a function from \( H_m \) into the set of probability distributions on \( I \). The interpretation of \( \sigma_n \) is as follows: If \( k \) was chosen, then at stage \( m \), given \( h_m \), PI will choose \( T \) with probability \( s_n^k(h_m) \).

A (behavioral) strategy for PII is: \( \tau_n = (t_1 \ldots t_n) \) where for all \( m, 1 \leq m \leq n \), \( t_m \) is a function from \( H_m \) into the set of probability distributions on \( J \).

**Remark 1.1.** The difference in the structure of the strategies of the two players is due to the fact that only PI knows the chosen value of \( k \), and therefore can play differently in each of the two matrices.

We define now the infinitely repeated game \( G_\infty(p) \), as follows:

A strategy for PI in \( G_\infty(p) \) is: \( \sigma = (\sigma^1, \sigma^2) \), where for all \( k \in \{1, 2\} \), \( \sigma^k \) is an infinite sequence \( \{s_n^k : n \geq 1\} \), and each \( s_n^k \) is a function from \( H_n \) into the set of probability distributions on \( I \). A strategy for PII in \( G_\infty(p) \), is: \( \tau \), where \( \tau = \{t_n : n \geq 1\} \), and \( t_n \) is a function from \( H_n \) into the set of probability distributions on \( J \).

\( G_\infty(p) \) is a model for a game with a very large number of stages, in which the players do not know the exact number of stages that are going to be played.

In \( G_\infty(p) \) the natural definition of payoff would be:

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{m=1}^{n} a_{imjm}^n \right)
\]

but this limit may fail to exist for any pair of strategies. (see e.g Aumann and Maschler p.188 of [1].) Nevertheless the value \( v_\infty(p) \), can be defined without defining the payoff: (see e.g p.187 in [1].)

For any pair of strategies, \( \sigma, \tau \), let \( \gamma_n(\sigma, \tau) \) be the average expected payoff for the \( n \) first stages in \( G_\infty(p) \) (or in any \( G_l(p) \), \( l \geq n \)) when \( \sigma \) and \( \tau \) are played. that is:

\[
\gamma_n(\sigma, \tau) = \mathbb{E}_{p, \sigma, \tau} \left( \frac{1}{n} \sum_{m=1}^{n} a_{imjm}^n \right).
\]
(E_{p,\sigma,\tau} \text{ is the expectation with respect to the probability measure on } H_{n+1} \text{ induced by } p, \sigma, \tau.) \text{ From now on, we use } \sigma \text{ and } \tau \text{ to denote strategies of PI and PII respectively, in the game } G_\infty(p).$

**Definition 1.2.**

- We say that PI can guarantee \( f(p) \) in \( G_\infty(p) \), if for any \( \varepsilon > 0 \) there is \( \sigma_\varepsilon \) and \( N_\varepsilon \), such that:
  \[
  \gamma_n(\sigma_\varepsilon, \tau) - f(p) \geq -\varepsilon \quad \forall n > N_\varepsilon.
  \]

- We say that PII can guarantee \( g(p) \) in \( G_\infty(p) \), if for any \( \varepsilon > 0 \) there is \( \tau_\varepsilon \) and \( N_\varepsilon \), such that:
  \[
  \gamma_n(\sigma, \tau_\varepsilon) - g(p) \leq \varepsilon \quad \forall n > N_\varepsilon.
  \]

- We say that \( G_\infty(p) \) has a value \( v_\infty(p) \) if both players can guarantee \( v_\infty(p) \).

**Remark 1.3.** An alternative definition for the value of an infinitely repeated game would be the limit of the values of the n-stage games \( G_n(p) \), namely: \( v_\infty(p) := \lim_{n \to \infty} v_n(p) \) (if this limit exists.) (see Zamir [9]).

Denote: \( D(p) \) as the game with payoff-matrix of \( pA^1 + p'A^2 \) (see e.g definition 3.10 p.123 of [10]). \( D(p) \) can be interpreted as the 1-stage game, in which both players are not informed of the matrix chosen. Let \( u(p) = \text{val}D(p) \), and \( Cau u(p) \) be the smallest concave function, that is greater or equal to \( u(p) \) on \([0,1] \).

**Theorem (Aumann and Maschler):** \( v_\infty(p) \) and \( \lim_{n \to \infty} v_n(p) \) both exist, and:

\[
\begin{align*}
  v_\infty(p) &= \lim_{n \to \infty} v_n(p) = Cau u(p).
\end{align*}
\]

(see e.g proposition A, p.187 and Theorem C, p.191 of [1]).

In other words: in this model as long as we are interested only in the value of \( G_\infty(p) \), both concepts: 'the value of the limit-game', and the 'limit of the values' (of the finite games), lead to the same result.

Aumann and Maschler constructed a strategy \( \sigma^* \) in \( G_\infty(p) \) (namely the splitting strategy; see e.g p.126 of [10].) such that:

\[
\inf_{\tau} \gamma_n(\sigma^*, \tau) \geq Cau u(p), \quad \forall n.
\]

This implies: \( v_n(p) \geq Cau u(p), \quad \forall n \).

The difference: \( e_n(p) := v_n(p) - Cau u(p) \) is called the error-term of the game.

The error-term is the extra gain that PI can guarantee (over \( Cau u(p) \)) in \( G_n(p) \).

In analogy with the error-term, given \( \sigma \) in \( G_\infty(p) \), let us look at:

\[
\inf_{\tau} \gamma_n(\sigma, \tau) - Cau u(p).
\]

This is the extra gain that PI can guarantee by using \( \sigma \), if the game ends after \( n \) stages. (and similarly for PII: \( \sup_{\tau} \gamma_n(\sigma, \tau) - Cau u(p) \).)
Because as said before PI can guarantee to get at least $\text{Cav}_u(p)$ for all $n$, then PI and PII have contradicting interests (as PI gets the payoff, and PII is the Fayer): PI is interested in a slow speed of convergence of $\gamma_n(\sigma, \tau)$ to $\text{Cav}_u(p)$, and PII is interested in a quick speed of convergence of $\gamma_n(\sigma, \tau)$ to $\text{Cav}_u(p)$. It is therefore natural to define a new $0$-sum game denoted as $SG_\infty(p)$: The Game for the speed of convergence of $\gamma_n(\sigma, \tau) - \text{Cav}_u(p)$ to zero. This is the topic of this note. We will now define formally the game $SG_\infty(p)$ and its value:

- The payoff matrices and the strategy-spaces for each player, are the same as in $G_\infty(p)$.
- As in $G_\infty(p)$ defining the payoff for every $\sigma, \tau$ is problematic and we will define a value for this game without it.

This value would have to represent the sequence of $\gamma_n(\sigma, \tau) - \text{Cav}_u(p)$, In order to do that, we need to use zamir's following definition (see definition 2 of [11]):

**Definition 1.4.**

- Two sequences of non-negative numbers $f(n)$ and $g(n)$ are said to be of the same order of magnitude, if there are constants $c_1, c_2 > 0$, and $N$, such that:

  $$c_2 g(n) \leq f(n) \leq c_1 g(n),$$

  $\forall n \geq N$.

  This will be denoted by $f(n) = O^*(g(n))$, or $g(n) = O^*(f(n))$.

- If there is a $c > 0$, and $N$, such that: $cg(n) \leq f(n)$, $\forall n \geq N$, we write that:

  $$g(n) = O^*(f(n)).$$

**Definition 1.5.**

- We say that PI can guarantee $f(n, p) \geq 0$ in $SG_\infty(p)$, if for any $\epsilon > 0$, there exists a sequence $f_\epsilon(n, p)$ s.t.:

  1. $n^\epsilon f_\epsilon(n, p) \geq f(n, p)$, $\forall n$.
  2. There is a strategy $\sigma_f(\epsilon, p)$ and a const $c_f(p) > 0$ (independent of $\epsilon$), such that for all $\tau$:

     $$(\gamma_n(\sigma_f(\epsilon, \tau), \tau) - \text{Cav}_u(p)) \geq c_f(p) f_\epsilon(n, p).$$

- We say that PII can guarantee $g(n, p) \geq 0$ in $SG_\infty(p)$, if for any $\epsilon > 0$, there exists a sequence $g_\epsilon(n, p)$ s.t.:

  1. $g_\epsilon(n, p) \leq g(n, p)n^\epsilon$, $\forall n$.
  2. There is a strategy $\tau_g(\epsilon, p)$ and a const $c_g(p) > 0$ (independent of $\epsilon$), such that for all $\sigma$:

     $$(\gamma_n(\sigma, \tau_g(\epsilon, \tau), p) - \text{Cav}_u(p)) \leq c_g(p) g_\epsilon(n, p).$$
We say that the game $SG_\infty(p)$ has a value, if there are sequences such that PI and PII can guarantee $f(n, p)$ and $g(n, p)$ respectively in $SG_\infty(p)$ and $f(n) = O^*(g(n))$.

We then define: $v_s(p) := O^*(f(n)) = O^*(g(n))$.

Note that for $SG_\infty(0)$ (or $SG_\infty(1)$), there is always a value, and this value is $O^*(0)$, (where 0 is the constant zero sequence), since this is a game with complete information.

2. Definitions and Preliminary Results

**Proposition 2.1.** For all $\sigma, \tau$, strategies in $G_\infty(p)$:

$$\inf_\tau \gamma_\infty(\sigma, \tau) \leq v_s(p) \leq \sup_\sigma \gamma_\infty(\sigma, \tau).$$

This proposition expresses the intuition that both, PI and PII, have more freedom in their choices of strategies in $G_\infty(p)$ than in $G_n(p)$, since a strategy in $G_\infty(p)$ can depend on $n$, and a strategy in $G_n(p)$ (where the end of the game is not known to the players) can not depend on $n$. In other words, any strategy available to any of the players in $G_\infty(p)$, is available to them also in $G_n(p)$ (as its $n$-truncation), hence they can not ‘do better’ in $G_\infty(p)$, than in $G_n(p)$.

**Proof.** Let: $\sigma_n^*, \tau_n^*$ be respectively, optimal strategies of PI and PII in $G_n(p)$, and for any pair of strategies $\sigma$ and $\tau$ of PI and PII in $G_\infty(p)$, we denote by $\sigma_n, \tau_n$, the $n$-truncation of $\sigma, \tau$ respectively.

$$\inf_\tau \gamma_\infty(\sigma, \tau) = \inf_\tau \gamma_\infty(\sigma_n, \tau_n) \leq \gamma_\infty(\sigma_n, \tau_n^*) \leq \gamma_\infty(\sigma_n^*, \tau_n^*) = v_n(p).$$

For the first inequality, note that: $\tau_n^*$ can not do better than be the best reply to $\sigma$. Similarly: $\sup_\sigma \gamma_\infty(\sigma, \tau) \geq v_n(p)$. □

**Proposition 2.2.** If there is a value $v_s(p)$ for $SG_\infty(p)$, then it satisfies:

$$v_s(p) = O^*(e_n(p)).$$

**Proof.** from proposition 2.1 we get that for any $\sigma, \tau$:

$$\inf_\tau \gamma_\infty(\sigma, \tau) - C(a) u(p) \leq e_n(p) \leq \sup_\sigma \gamma_\infty(\sigma, \tau) - C(a) u(p), \quad \forall n.$$

Now if there is a value $v_s(p)$ for $SG_\infty(p)$, then by definition 1.5, there are sequences $f(n, p), g(n, p)$, such that:

1. $v_s(p) = O^*(f(n, p))$.
   and for all $\varepsilon > 0$, there are strategies: $\sigma_\varepsilon(\varepsilon, p), \tau_\varepsilon(\varepsilon, p)$, and sequences: $f_\varepsilon(n, p), g_\varepsilon(n, p)$ such that:
2. $n^\varepsilon f_\varepsilon(n, p) \geq f(n, p), \quad \forall n$.
3. $g_\varepsilon(n, p) n^\varepsilon \leq g(n, p) n^\varepsilon, \quad \forall n$.

and there are cons $c_\varepsilon > 0, c_\varepsilon(\varepsilon, p) > 0$ such that:
4.
\[ c_f(p) f_n(n, p) \leq \inf_{\tau} \gamma_n(\sigma_f(\varepsilon, p), \tau) - C a v u(p) \leq e_n(p) \]
\[ \leq \sup_{\sigma} \gamma_n(\sigma, \tau_p(\varepsilon, p)) - C a v u(p) \leq c_g(p) g_n(n, p). \]

Letting \( \varepsilon \) go to 0, we get that: \( c_f(p) f(n, p) \leq e_n(p) \leq c_g(p) g(n, p) \). Hence: \( e_n(p) = O^*(f(n)) = O^*(g(n)) \), and therefore: \( v_\varepsilon(p) = O^*(e_n(p)) \).

The main result of this note is that although \( v_\varepsilon(p) \) (unlike \( e_n(p) \)), does not always exist, it does exist for the special class of games, in which: \( e_n(p) = O^* \left( \frac{1}{\sqrt{n}} \right) \quad \forall \varepsilon \in (0, 1) \), and hence equals to \( O^* \left( \frac{1}{\sqrt{n}} \right) \) by proposition 2.2.

The games for which \( e_n(p) = O^* \left( \frac{1}{\sqrt{n}} \right) \quad \forall \varepsilon \in (0, 1) \), were characterized by Mertens and Zamir as the games for which: \( \sqrt{n}(u_n(p) - C a v u(p)) \rightarrow \Phi(p) \), where \( \Phi(p) \) is an appropriately scaled Normal density function (see Mertens and Zamir [6]). We shall refer to this class as the class of "Normal Games".

For any strategy \( \sigma \) of PI in \( G_\infty(p) \), (or in \( G_n(p) \)) let us define a sequence of r.v (see e.g. p.189 of [4] and p.122 of [10]):

\[ P_1 = p \]
\[ n > 1 \quad P_n := P_{p, \pi, i}(K = 1 | h_n). \]

That is \( P_n \) is the conditional probability, given \( h_n \) and given \( \sigma, \tau \), that at stage 0 chance chose \( k = 1 \). Although \( P_n \) is not known to PI since he does not necessarily know \( \sigma \), it plays a central role in the analysis. Basically this is because, by the minimax theorem any optimal (minimax) strategy \( \sigma \) of PI, guarantees the value even when it is known to PI, who can then compute \( P_n \).

It is easily seen that the distribution \( P_n \) is conditionally independent of \( \tau \) (since \( \tau \) is independent of \( K \)) and using Bayes' law we get that:

\[ P_{n+1} = \begin{cases} \frac{\delta_n(\varepsilon_n)}{s_n(h_n)} P_n, & \text{if } i_n = T \\ \frac{\delta_n(\varepsilon_n)}{\overline{s}_n(h_n)} P_n, & \text{if } i_n = B \end{cases} \]

where \( s_n(h_n) = P_n s_n^T(h_n) + P_n s_n^B(h_n) \)
and \( s_n^T(h_n) = 1 - s_n(h_n) \). (see in the Introduction.)

It is easy to see that \( \mathcal{P}^\infty = \{ P_n \}_{n=1}^\infty \) is a Martingale.

Hence any \( \sigma \), a strategy of PI in \( G_\infty(p) \), yields an infinite Martingale \( \mathcal{P}^\infty \) of probabilities, namely of random variables satisfying: \( 0 \leq P_n \leq 1, \quad \forall n \).

For all \( n \) and all \( m \geq n \) let:

\[ V_n(\mathcal{P}^m) = E \sum_{k=1}^{n} |P_{k+1} - P_k| \]
be the variation of $\mathcal{P}^m = \{P_i\}_{i=1}^m$. The variation is a measure for the expected amount of information revealed by PI up to (and including) stage $n$, when using the strategy $\sigma$. Note that the definition for $V_n(\mathcal{P}^m)$ holds also for $m = \infty$.

The variation $V_n(\mathcal{P}^m)$ serves as a key role in the analysis, since the extra gain of PI (beyond $Cav u(p)$) is constrained by the amount of information he reveals. More precisely: it is proven (see p.224 of [8]), that there is $c > 0$, such that:

$$\inf_{\tau} \gamma_n(\sigma, \tau) \leq \frac{1}{n} \sum_{k \geq n} E u(P_k) + \frac{c}{n} V_n(\mathcal{P}^m) \leq Cav u(p) + \frac{c}{n} V_n(\mathcal{P}^m)$$

for all $n$ and all $\sigma$ (a strategy in $G_m(p)$, $m \geq n$).

Using Cauchy-Schwartz inequality, and the fact that $\mathcal{P}^\infty$ is a uniformly bounded Martingale, it is shown (see e.g proposition 3.8 p.122 of [10]), that:

$$V_n(\mathcal{P}^m) \leq a(p)\sqrt{n} \quad \text{for some } a(p) > 0$$

Remark 2.3. It follows from (1) and (2), that: $e_n(p) \leq O^*\left(\frac{1}{\sqrt{n}}\right)$.

It was shown (see Zamir [8]) that $O^* \left(\frac{1}{\sqrt{n}}\right)$ is the least upper bound for the order of magnitude of $e_n(p)$. Namely there exists a game in which: $e_n(p) \geq \frac{a(p)}{\sqrt{n}}$, $\forall p \in (0, 1)$.

Hence from (1) it follows, that for each $n$, there is a uniformly bounded martingale $\mathcal{P}^n$, s.t:

$$V_n(\mathcal{P}^n) \geq c p^\epsilon \sqrt{n}. \quad \text{for some } c > 0.$$ 

Mertens and Zamir also showed (see Theorem 2.4 p. 255 of [5]) that for the infinite uniformly bounded martingale $\mathcal{P}^\infty$, $\lim_{n \to \infty} \left\{\frac{V_n(\mathcal{P}^\infty)}{\sqrt{n}}\right\} = 0$.

From this point we proceed as follows:

- Although there isn't any infinite uniformly bounded martingale $\mathcal{W}^\infty$, satisfying:
  $$V_n(\mathcal{W}^\infty) \geq a\sqrt{n}, \quad \forall n \quad \text{for some } a > 0,$$
  we prove in part 3 that $\sqrt{n}$ can be reached asymptotically, that is: For every $\epsilon > 0$, we will construct an infinite martingale $\mathcal{X}^\infty$, satisfying:
  $$V_n(\mathcal{X}^\infty) \geq c n^\frac{\epsilon}{2}, \quad \forall n$$
  for some $c > 0$.

- In Part 4 we construct for any $\epsilon > 0$, a strategy $\sigma$ in $G_{\infty}(p)$, for PI, that yields an infinite $\mathcal{P}^\infty$, that coincides with $\mathcal{X}^\infty$ in some interval $(t, u)$ in $[0, 1]$.

- In Part 5 : We prove that in the Normal Games, there is a value $\nu_\epsilon(p)$ for $SG_{\infty}(p)$, $\forall p \in (0, 1)$, and that: $\nu_\epsilon(p) = O^*\left(\frac{1}{\sqrt{n}}\right)$.

- We conclude by showing in Part 6 a class of games that for all $p$, do not have a value for $SG_{\infty}(p)$. 

3. THE VARIATION OF UNIFORMLY BOUNDED INFINITE MARTINGALES

Our first result states that although the \( n \)-stage variation of a uniformly bounded infinite martingale is smaller than \( O^* \left( \sqrt{n} \right) \), it can be of \( O^* \left( n^{\frac{1}{2}-\varepsilon} \right) \), for arbitrarily small \( \varepsilon > 0 \).

**Theorem 3.1.** For any \( \varepsilon > 0 \), \( \eta > 0 \) and \( 0 < l < p < u \leq 1 \), there is \( c > 0 \) and a Martingale \( \mathcal{X}_\varepsilon = \{X_n\}_{n=1}^\infty \), with: \( E X_1 = p \) \( \forall n \), that satisfies:

(a) \( P \left( l < X_n < u, \forall n \right) > 1 - \eta \)

(b) For all \( n \): \( V_n(\mathcal{X}_\varepsilon) \geq c n^{\frac{1}{2} - \varepsilon} \).

Furthermore, we will prove that:

\[
P \left\{ \sum_{k=1}^{n} |X_{k+1} - X_k| \geq c n^{\frac{1}{2} - \varepsilon}, \forall n \right\} = 1.
\]

**Proof.** We construct a Martingale that satisfies (a) and (b).

For a given \( 0 < \theta < 1 \), let \( Y_k, k = 1, 2, \ldots \) be i.i.d. random variables, defined by:

\[
P(Y_k = \theta) = \theta' \quad \text{and} \quad P(Y_k = \theta') = \theta
\]

(where \( \theta' = 1 - \theta \).

The required martingale \( \mathcal{X}_\varepsilon \) is now defined as follows:

\[
X_1 \equiv p \quad \text{and for all } n > 1:
X_n := X_{n-1} + \frac{Y_n}{(n_0 + n)^{\frac{1}{2} + \varepsilon}},
\]

where \( n_0 = n(\varepsilon, p, l, u) \) is a constant that we choose so that \( \mathcal{X}_\varepsilon \) satisfies (a) and (b).

To choose \( n_0 \) so that \( \mathcal{X}_\varepsilon \) satisfies (a), we proceed as follows:

\[
\text{Var} \left( X_n \right) = \text{Var} \left\{ \sum_{k=n_0+1}^{n_0+n} \frac{Y_k}{(k+1)^{\frac{1}{2} + \varepsilon}} \right\} = \sum_{k=n_0+1}^{n_0+n} \frac{\theta \theta'}{k^{1+2\varepsilon}} \leq M \text{,}
\]

where \( M = \sum_{k=1}^{\infty} \frac{\theta \theta'}{k^{1+2\varepsilon}} \). Note that \( E|X_n| \leq \text{Var} \left( X_n \right) + 1 \leq M + 1 \).

By the Martingale Convergence Theorem (see e.g. p.244 of [7]), we get that (since \( E|X_n| \leq M + 1, \forall n \)): \( \mathcal{X}_\varepsilon = \lim_{n \to \infty} X_n \) exists and is finite, and: \( EX_\infty = p \).

Using Egoroff's theorem (see e.g. p.88 of [2]), we get that the convergence of \( X_n \) to \( X_\infty \) is even stronger, namely almost uniformly. That is: For all \( \eta, \delta > 0 \), there is an \( N \), such that:

\[
P \left( |X_n - X_\infty| < \delta, \forall n > N \right) \geq 1 - \eta.
\]

**Lemma 3.2.** For any \( \delta, \eta > 0 \), there is an \( N(\delta, \eta) \), such that for any \( n_0 > N \), the process \( \mathcal{X}_\varepsilon \) defined with \( n_0 \), will satisfy:

\[
P \left( |X_n - X_\infty| < \delta, \forall n \right) \geq 1 - \eta.
\]
Proof. Consider the above defined process with \( n_0 = 0 \), that is for each \( n > 1 \):

\[
\bar{X}_n = p + \sum_{k=2}^{n} \frac{Y_k}{k^{\frac{1}{2} + \varepsilon}}
\]

\[
\bar{X}_\infty = p + \sum_{k=0}^{\infty} \frac{Y_k}{k^{\frac{1}{2} + \varepsilon}}.
\]

By Egoroff's theorem we have, that for any \( \delta > 0 \) and \( \eta > 0 \), there is an \( N = N(\delta, \eta) \), such that:

\[
P\left( |\bar{X}_n - \bar{X}_\infty| < \delta, \ \forall n > N \right) \geq 1 - \eta.
\]

Now for any \( n_0 \): \( X_\infty - X_n = \bar{X}_\infty - \bar{X}_{n+n_0} \) so we have:

\[
P\left( |X_\infty - X_n| < \delta, \ \forall n \right) = P\left( |\bar{X}_\infty - \bar{X}_{n+n_0}| < \delta, \ \forall n \right) = P\left( |\bar{X}_\infty - \bar{X}_n| < \delta, \ \forall n > n_0 \right).
\]

and so for any such process defined with \( n_0 > N \), where \( N \) satisfies (5), we get:

\[
P\left( |X_\infty - X_n| < \delta, \ \forall n \right) = P\left( |\bar{X}_\infty - \bar{X}_n| < \delta, \ \forall n > n_0 \right) \geq P\left( |\bar{X}_\infty - \bar{X}_n| < \delta, \ \forall n > N \right)
\]

\[
\geq 1 - \eta,
\]

and with that we proved lemma 3.2. \( \square \)

To complete the definition of the martingale \( \mathcal{X}_2^\infty \), we now define \( n_0 \) as follows:

Given \( \varepsilon, p, l \) and \( u \), s.t.: \( 0 \leq l < p < u \leq 1 \), define \( \delta^* = \min \left\{ \frac{u-1}{3}, \frac{u-2}{4} \right\} \).

For fixed \( \eta > 0 \), let \( n_0 \) be the minimal \( N(\delta^*, \eta) \), that satisfies the inequality of lemma (3.2) for \( \delta = \delta^* \).

* We now prove that the martingale \( \mathcal{X}_2^\infty \), satisfies (a):

\[
P\left( l < X_n < u, \ \forall n \right) \geq P\left( |X_n - p| < 2\delta^*, \ \forall n \right)
\]

\[
\geq P\left( |X_n - X_\infty| + |X_\infty - p| < 2\delta^*, \ \forall n \right)
\]

\[
\geq P\left( |X_n - X_\infty| < \delta^* \ and \ |X_\infty - X_1| < \delta^*, \ \forall n \right)
\]

\[
= P\left( |X_n - X_\infty| < \delta^*, \ \forall n \right) \geq 1 - \eta.
\]

* To prove that \( \mathcal{X}_2^\infty \) satisfies (b):

Denote: \( \alpha = \min(\theta, \theta') \).

\[
\sum_{k=1}^{n} |X_{k+1} - X_k| = \sum_{k=n_0+2}^{n_0+n} \frac{|Y_k|}{k^{\frac{1}{2} + \varepsilon}} \geq \sum_{k=n_0+2}^{n_0+n} \frac{\alpha}{k^{\frac{1}{2} + \varepsilon}}
\]

\[
\geq cn^{\frac{1}{2} - \varepsilon}
\]
for some $c > 0$, hence:

$$
P \left\{ \sum_{k=1}^{n} |X_{k+1} - X_k| \geq cn^{1/2}, \quad \forall n \right\} = 1.
$$

In particular, $V_n(X_n^n) \geq cn^{1/2}, \forall n$, concluding the proof of Theorem 3.1.

\[\square\]

4. CONSTRUCTING FOR PI A STRATEGY WITH MAXIMAL VARIATION

As mentioned earlier:

$$\inf_{\tau} \gamma_n(\sigma, \tau) \leq C \text{av} u(p) + \frac{C}{n} V_n(P_\epsilon^\infty), \quad \forall n.$$  

Therefore a strategy for PI, that will give him highest $O^*(\inf_{\tau} \gamma_n(\sigma, \tau))$, must have maximal $O^*(V_n(P_\epsilon^\infty))$, where $P_\epsilon^\infty$ is the Martingale of the conditional probabilities derived from $\sigma$.

We shall use the Martingale $X_\epsilon^\infty$ constructed in the proof of Theorem 3.1, in the following way:

For any $\epsilon > 0$, we will construct a strategy $\sigma_{\epsilon}$ for PI, such that inside an interval $(l, u)$, the sequence of conditional probabilities will be the same as $X_\epsilon^\infty$ for that $\epsilon$.

The only information about the history that PI will use at stage $n$, is the conditional probability $P_n$, and so we can denote: $s^k_n(P_n)$ as the probability that PI will choose $T$ at stage $n$, given $K = k$, and given $P_n$. In part 3, we defined the constant $n_0 = n(\epsilon, p, l, u)$. Since $\epsilon$ and $p$ are fixed, we abbreviate this by $n_0(l, u)$. We now define the following sequence: $\varphi(n) = (n_0(l', u') + n)^{1/2}$, where $l' = \frac{1}{\sqrt{n_0(0,1)}}$ and $u' = 1 - \frac{1}{\sqrt{n_0(0,1)}}$.

Definition of $\sigma_{\epsilon}$: given $0 < \theta < 1$, for any stage $n = 1, 2, \ldots$:

- If $l' < P_n < u'$, then:

  $$s^k_n(P_n) = \theta' \frac{\theta' \varphi(n)P_n}{\varphi(n) P_n + \theta' P_n}.$$

- Otherwise: $s^k_n(P_n) = s(P_n)$ for $k = 1, 2$, where $s(P_n)$ is an optimal strategy of PI in $D(P_n) = P_nA_1 + (1 - P_n)A_2$.

Remark 4.1. The reason for choosing $l', u'$ as above is for $s^k_n(P_n)$ to be well defined. that is, for $k = 1, 2$, and $n$, the $P_n$ will satisfy: $0 \leq s^k_n(P_n) \leq 1$.

Remark 4.2. Note that for any two intervals s.t. $(a, b) \subseteq (c, d)$, we have: $n_0(a, b) \geq n_0(c, d)$, since by the definition of $\delta^*$, we get that: $\delta^*(a, b) \leq \delta^*(c, d)$, (where $\delta^*(x, y)$ is $\delta^*$ that correspond to the case of interval $(x, y)$). Hence any $N$ that satisfies (5) for $\delta^*(a, b)$, satisfies (5) for $\delta^*(c, d)$. Since we defined $n_0(c, d)$ to be the minimal $N$ that satisfies (5) for $\delta^*(c, d)$, we get that: $n_0(a, b) \geq n_0(c, d)$.

Lemma 4.3. The $\sigma_{\epsilon}$ defined above satisfies:
THE SPEED OF CONVERGENCE IN REPEATED GAMES

1. For all \( k, n \), the \( P_n \) satisfies: \( 0 \leq s_n^k(P_n) \leq 1 \).
2. The strategy \( \sigma_n \) yields a Martingale \( P_n^\infty = \{P_n\}_{n=1}^\infty \) which coincides with \( X_n^\infty \) (defined in part 3) inside \((l', u')\), and is absorbed outside \((l', u')\).

Proof:
1. If \( l' < P_n < u' \) then:
   \[
   s_n^1(P_n) = \theta + \frac{\theta u'}{\varphi(n) P_n}.
   \]
   We have to prove that: \( \theta + \frac{\theta u'}{\varphi(n) P_n} \leq 1 \). This is equivalent to: \( \frac{\theta u'}{\varphi(n) P_n} \leq 0' \), which is equivalent to: \( \theta \leq \varphi(n) P_n \). Now \( \varphi(n) = (n_0(l', u') + n) \frac{1}{1+e} \), so:
   \[
   \varphi(n) P_n \geq (n_0(l', u') + n) \frac{1}{1+e} > \left( \frac{n_0(l', u')}{n_0(0, 1)} \right) \frac{1}{1}.
   \]
   By Remark 4.2: \( \frac{n_0(l', u')}{n_0(0, 1)} \geq 1 \), so we get:
   \[
   \varphi(n) P_n \geq 1 > \theta.
   \]
   In the same way we prove that \( 0 \leq s_n^2(P_n) \leq 1 \).

2. \( \{P_n\}_{n=1}^\infty \) is a Martingale that satisfies:

   \[
   P_{n+1} = \begin{cases} 
   s_n^1(P_n) P_n, & \text{if } i_n = T \\
   s_n^2(P_n) P_n, & \text{if } i_n = B
   \end{cases}
   \]
   (recall \( s_n(P_n) = P_n s_n^1(P_n) + P_n s_n^2(P_n) \)) so:
   \[ P(i_n = T) = \bar{s}_n(P_n) \text{ and } P(i_n = B) = \bar{s}_n(P_n) \text{ and hence:} \]

   - If \( P_n \notin (l', u') \) then by definition:
     \[
     \bar{s}_n(P_n) = P_n \left( \theta + \frac{\theta u'}{\varphi(n) P_n} \right) + P_n' \left( \theta - \frac{\theta u'}{\varphi(n) P_n} \right) = \theta.
     \]
     And so by (6), if \( P_n \in (l', u') \), then:
     \[
     P_{n+1} = \begin{cases} 
     \left( \frac{(s_n^1 P_n)}{\theta} \right) P_n, & \text{if } i_n = T \\
     \left( \frac{(s_n^2 P_n)}{\theta} \right) P_n, & \text{if } i_n = B.
     \end{cases}
     \]
     That is:
     \[
     P \left( P_{n+1} = P_n + \frac{\theta}{\varphi(n)} \right) = P(i_n = T) = \theta \text{ and: } P \left( P_{n+1} = P_n - \frac{\theta}{\varphi(n)} \right) = P(i_n = B) = \theta'.
     \]
In other words, if $P_n \in (l', u')$, then:

$$P_{n+1} = P_n + \frac{Y_n}{(n_0(l', u') + n)^{1+\varepsilon}}.$$

- If $P_n \notin (l', u')$ then $s_k^*(P_n) = s(P_n), \quad k = 1, 2$, that is $s_2$ is then non-revealing, and therefore: $P_{n+1} = P_n$ with probability 1, i.e. $P_n$ is absorbed outside $(l', u')$, concluding the proof of lemma 4.3.

\[\square\]

Lemma 4.4. For the Martingale $\mathcal{P}_x^\infty$ derived from $s_2$: $V_n(\mathcal{P}_x^\infty) \geq cn^{1-\varepsilon}, \quad \forall n$.

Proof. Because in $(l', u')$ $\mathcal{P}_x^\infty = \{P_n\}_n^\infty$ coincides with $X_x^\infty = \{X_n\}_1^\infty$ defined in part 3, then:

$$P(l' < P_n < u', \quad \forall n) = P(l' < X_n < u', \quad \forall n) \geq 1 - \eta.$$

In particular, for all $n$: $P(l' < P_m < u', \quad \forall m \leq n) \geq 1 - \eta$.

And so:

$$V_n(\mathcal{P}_x^\infty) = E \sum_{k=1}^{n} |P_{k+1} - P_k| \geq$$

$$E \left\{ \sum_{k=1}^{n} |P_{k+1} - P_k| \quad l' < P_m < u', \quad \forall m \leq n \right\} P(l' < P_m < u', \quad \forall m \leq n) \geq$$

$$\geq E \sum_{k=n_0+1}^{n_0+n} \frac{|Y_k|}{(n_0(l', u') + k)^{1+\varepsilon}} \cdot (1 - \eta) \geq \sum_{k=n_0+1}^{n_0+n} \frac{a(1 - \eta)}{(n_0(l', u') + k)^{1+\varepsilon}} \geq cn^{1-\varepsilon},$$

for some $c > 0$. \[\square\]

5. The Speed of Convergence in the Normal Games

We will now use $s_2$ that we constructed in section 4, for the Normal Games. We will show that by using it, PI guarantees maximal speed of convergence. That is there is a $c > 0$, such that for all $\tau$: $\gamma_n(s_2, \tau) - Cav u(p) \geq \frac{c}{n^{\frac{1}{1+\varepsilon}}}, \quad \forall n$.

Theorem 5.1. In the Normal Games there is a value $v_*(p)$ for $SG_{\infty}(p)$, for all $0 < p < 1$, and $v_*(p) = O^{*} \left( \frac{1}{n^2} \right)$.

The Normal games were characterized by Mertens and Zamir who showed that a Normal game has a presentation of (see p.289 of [3]):

$$A_1 = \begin{pmatrix} \theta' a & \theta' a' \\ -\theta a & \theta a' \end{pmatrix}, \quad A_2 = \begin{pmatrix} \theta' b & -\theta' b' \\ -\theta b & \theta b' \end{pmatrix}$$

$0 < \theta, a, b < 1$ and without loss of generality $a > b$.

We will prove the Theorem by proving two Lemmas:
Lemma 5.2. In the Normal Games: PI can guarantee \( O^* \left( \frac{1}{\sqrt{n}} \right) \) in \( SG_\infty(p) \), for all \( 0 < p < 1 \).

Proof. For \( \epsilon > 0 \), use \( \sigma_\epsilon \) which was defined in part 4.

At stage \( n \):

- If \( P_n = p_n \), such that \( l' < p_n < u' \), and PII is using \((t, t')\), then the payoff for this stage is:

\[
\begin{align*}
&\left[ p_n \begin{pmatrix} s_n^1(p_n), & s_n^1(p_n) \end{pmatrix} \begin{pmatrix} \theta' a & -\theta' a' \\ -\theta a & \theta a' \end{pmatrix} + p_n' \begin{pmatrix} s_n^2(p_n), & s_n^2(p_n) \end{pmatrix} \begin{pmatrix} \theta' b & -\theta' b' \\ -\theta b & \theta b' \end{pmatrix} \right] \begin{pmatrix} t \\ t' \end{pmatrix} \\
&= \left[ p_n \begin{pmatrix} \theta, & \theta' \end{pmatrix} \begin{pmatrix} \theta' a & -\theta' a' \\ -\theta a & \theta a' \end{pmatrix} \begin{pmatrix} t \\ t' \end{pmatrix} + p_n' \begin{pmatrix} \theta, & \theta' \end{pmatrix} \begin{pmatrix} \theta' b & -\theta' b' \\ -\theta b & \theta b' \end{pmatrix} \begin{pmatrix} t \\ t' \end{pmatrix} \\
&+ \frac{\theta' \theta'}{\varphi(n)} \left[ \begin{pmatrix} 1, & -1 \end{pmatrix} \begin{pmatrix} \theta' a & -\theta' a' \\ -\theta a & \theta a' \end{pmatrix} + \begin{pmatrix} -1, & 1 \end{pmatrix} \begin{pmatrix} \theta' b & -\theta' b' \\ -\theta b & \theta b' \end{pmatrix} \right] \begin{pmatrix} t \\ t' \end{pmatrix} \\
&= \frac{\theta' \theta'}{\varphi(n)} \left[ (\theta' a + \theta a, & -\theta' a' - \theta a') + (\theta' b - \theta b, & \theta' b' + \theta b') \right] \begin{pmatrix} t \\ t' \end{pmatrix} \\
&= \frac{\theta' \theta'}{\varphi(n)} (a - b, & -a' + b') \begin{pmatrix} t \\ t' \end{pmatrix} \\
&= \frac{\theta' \theta'}{\varphi(n)} (a - b)(t + t') = \frac{\theta' \theta'(a - b)}{\varphi(n)}
\end{align*}
\]

This is true for all \( t \), and for all \( l' < p_n < u' \), so when \( \sigma_\epsilon \) is used by PI, the payoff \( g_n \) for stage \( n \) satisfies:

\[
E(g_n \mid l' < P_n < u') = \frac{\theta' \theta'(a - b)}{\varphi(n)}, \quad \forall t,
\]

- Now, if \( P_n \notin (l', u') \) then \( s_n^1(P_n) = s(P_n) = \theta \), so:

\[
E(g_n \mid P_n \notin (l', u')) = 0 \quad \forall t.
\]

Hence for all \( n \), and all \( \tau \):

\[
\gamma_n(\sigma_\epsilon, \tau) = \frac{1}{n} \sum_{k=1}^{n} E(g_k(\sigma_\epsilon, \tau)) \geq \frac{1}{n} E \left\{ \sum_{k=1}^{n} (g_k(\sigma_\epsilon, \tau)) \right\} \quad l' < P_n < u', \quad \forall m \leq n \}
\]

\( (1 - \eta) \)
and so by equation (7):

\[ \gamma_n(\sigma, \tau) \geq \left[ \frac{1}{n} \sum_{k=1}^{n} \frac{\theta^2(a - b)}{\varphi(k)} \right] (1 - \eta) = \frac{(1 - \eta)\theta^2(a - b)}{n} \sum_{k=1}^{n} \frac{1}{\varphi(k)} \geq \frac{c}{n^{\theta + \epsilon}}, \]

for some \( c > 0 \).

Hence by definition 1.5, this concludes the proof of lemma 5.2.

\[ \square \]

Lemma 5.3. \( PII \) can guarantee \( O^* \left( \frac{1}{\sqrt{n}} \right) \) in \( SG_\infty(p) \) for all \( 0 < p < 1 \).

Proof. \( PII \) has a strategy \( \tau_B \), based on Blackwell’s approachability theorem, (see e.g. Aumann and Maschler p.225 of [3]) , which guarantees him in any \( G_\infty(p) \), (not just in the Normal Games,) not to pay more than: \( Ca_v u(p) + \frac{g(p)}{\sqrt{n}} \), for all \( 0 < p < 1 \), for some \( 0 < a(p) < \infty \).

\[ \square \]

Proof of Theorem 5.1: By the definition of \( v_\pi(p) \), Lemmas 5.2 and 5.3 imply that there is a value \( v_\pi(p) \) for \( SG_\infty(p) \) in the Normal Games, and that:

\[ v_\pi(p) = O^* \left( \frac{1}{\sqrt{n}} \right) \]

\( \forall p \in (0, 1) \).

6. A Case in which the game \( SG_\infty(p) \), does not have a value

Since for all \( p \), \( v_\pi(p) \) is a non-decreasing function in \( n \), (see proposition 3.19 in [10]) and \( v_\pi(p) \geq Ca_v u(p) \) \( \forall n \), then \( v_1(p) = Ca_v u(p) \), implies \( v_n(p) = Ca_v u(p) \) \( \forall n \), and thus \( e_\pi(p) = 0, \forall n \). Hence in this special case PI cannot gain any benfit from his extra knowledge and therefore we consider this game to be trivial:

Definition 6.1. If \( v_1(p) = Ca_v u(p) \), then we say that \( G_\infty(p) \) is a trivial Game.

Note that \( G_\infty(0) \) and \( G_\infty(1) \) are always trivial. It is easy to see that if \( G_\infty(p) \) is trivial, then \( v_\pi(p) \) exists and \( v_\pi(p) = O^*(0) \).

Theorem 6.2. If \( u(p) \) is strictly concave on \([0, 1]\) and \( G_\infty(p) \) is not trivial, then the game \( SG_\infty(p) \) does not have a value.

This Theorem expresses the following intuition: Games with strictly Concave \( u(p) \) represent cases in which PI prefers the situation that none of players PI and PII knows which is the game played, rather than the situation that both of them do know. To see that, note that when none of the players know which game is being played then they play \( D(p) \) and the value is \( u(p) \). If both players know which game is played, then both can play optimal in that game, so the value is:

\[ p v_1 + p' v_2. \]
where \( v_1, v_2 \) are the values of \( A_1, A_2 \) respectively. (Note that: \( u(1) = v_1 \) and \( u(0) = v_2 \).) By the strict concavity of \( u(p) \), we have:

\[ u(p) > pu(1) + p'u(0) = pv_1 + p'v_2, \quad \text{for all } 0 < p < 1. \]

So in such games we would expect PI to be conservative in his use of information, in order not to reveal it to PII. It turns out that in \( G_\infty(p) \), PI should never use his information:

**Lemma 6.3.** If \( u(p) \) is strictly concave on \([0,1] \), then if PI guarantees \( f(n,p) \) in \( SG_\infty(p) \), then \( f(n,p) = O^*(0) \).

**Proof.** Since \( u(p) \) is strictly concave, then for all \( p \):

\[ \text{Cav } u(p) = u(p) \]

1. If \( \sigma \) is a non-revealing (NR) strategy, that is for all \( n \): \( s_n^1(h_n) \equiv s_n^2(h_n) \) and then for all \( n \):

\[ \inf_{\tau} \gamma_n(\sigma, \tau) \leq u(p) = \text{Cav } u(p). \]

Thus the NR optimal strategy in \( G_\infty(p) \), consisting of playing repeatedly an optimal strategy in \( D(p) \), guarantees \( O^*(0) \) in \( SG_\infty(p) \).

2. We claim that any other strategy \( \sigma \), is not (even) optimal in \( G_\infty(p) \). We do that by proving that there is an \( N \), such that: \( \inf_{\tau} \gamma_n(\sigma, \tau) < \text{Cav } u(p) \) \( \forall n > N \).

Let \( \hat{n} \) be the first stage such that: \( s_n^1(h_n) \neq s_n^2(h_n) \).

For all \( n \):

\[ \inf_{\tau} \gamma_n(\sigma, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} E u(P_i) + \frac{c}{\sqrt{n}} \]

(see (1) and (2) in part 2).

- For all \( n \leq \hat{n} \): \( P_i \equiv p \)
- For \( n = \hat{n} \): \( P_{\hat{n} + 1} \neq P_{\hat{n}} \).

So by the strict concavity of \( u(p) \), given \( P_n \):

\[ E(u(P_{\hat{n} + 1} | P_{\hat{n}})) < u(E(P_{\hat{n} + 1} | P_{\hat{n}})) = u(P_{\hat{n}}) = u(p) \]

- denote: \( -\delta = E u(P_{\hat{n} + 1}) - u(p) \).

\( \{u(P_n)\} \) is a Super-Martingale, since using Jensen’s inequality:

\[ E(u(P_n) | u(P_{n-1})) \leq u((EP_n | P_{n-1})) = u(P_{n-1}). \]
So: For all $n > n$: $E(u(P_n)) - u(p) \leq -\delta$. Hence:

\[
\inf_{\tau} \gamma_n(\sigma, \tau) \leq \frac{1}{n} \left[ \sum_{i=1}^{n} E(u(i)) + \sum_{i=n+1}^{2n} E(u(i)) \right] + \frac{c}{\sqrt{n}}
\]

\[
\inf_{\tau} \gamma_n(\sigma, \tau) \leq \frac{1}{n} \sum_{i=1}^{n} u(p) + \sum_{i=n+1}^{2n} (u(p) - \delta) \right] + \frac{c}{\sqrt{n}}
\]

\[
\inf_{\tau} \gamma_n(\sigma, \tau) \leq u(p) - \delta \cdot \frac{n - \hat{n}}{n} + \frac{c}{\sqrt{n}}
\]

Thus:

\[
\inf_{\tau} \gamma_n(\sigma, \tau) - u(p) \leq -\delta \left( 1 - \frac{\hat{n}}{n} \right) + \frac{c}{\sqrt{n}}
\]

For $n$ large enough, the left side of the last inequality is strictly negative, so $\sigma$ is not an optimal strategy in $G_{\infty}(p)$, concluding the proof of Lemma 6.3.

\[\square\]

**Lemma 6.4.** If $u(p)$ is concave on $[0, 1]$ and $G_{\infty}(p)$ is not trivial, then if PII guarantees $g(n, p)$ in $SG_{\infty}(p)$, then $g(n, p) \geq O^{*} \left( \frac{1}{n^2} \right)$.

**Proof.** If the game is not trivial, then PI can play $\sigma_n^*$ in $G_n(p)$ defined as follows: For first $n - 1$ stages play for every realization of $h_m$, $m \leq n$, optimal in $D(p)$. So up to stage $(n - 1)$:

\[
\inf_{\tau} \gamma_{n-1}(\sigma_n^*, \tau) = u(p).
\]

At stage $n$, play optimal in $G_1(p)$, to guarantee in this stage: $v_1(p) > u(p)$. denote: $c(p) = v_1(p) - u(p)$, then PI can guarantee:

\[
\inf_{\tau} \gamma_n(\sigma_n^*, \tau) = u(p) + \frac{c(p)}{n}.
\]

So then for any strategy $\tau_n$ of PII in $G_n(p)$:

\[
\gamma_n(\sigma_n^*, \tau_n) \geq u(p) + \frac{c(p)}{n}.
\]

Since $u(p)$ is concave, then for all $p$: $u(p) = Cav u(p)$, and so:

\[
\gamma_n(\sigma_n^*, \tau_n) \geq Cav u(p) + \frac{c(p)}{n}.
\]

Now PII cannot play better in $G_{\infty}(p)$ than in any $G_n(p)$ (see proposition 2.1 in part 2), which implies that if PII guarantees $g(n, p)$ in $SG_{\infty}(p)$ then:

\[
[g(n, p)] \geq \left[ \frac{1}{n} \right].
\]

\[\square\]

**Proof of Theorem 6.2:** By Lemmas 6.3 and 6.4 and by the definition of $v_*(p)$, we get that the game $SG_{\infty}(p)$ does not have a value. 

\[\square\]
We conclude with an example of a game in which for all $0 < p < 1$, $v_s(p)$ does not exist and the gap between any $f(n, p), g(n, p)$ that PI and PII can guarantee respectively in $SG_\infty(p)$, is bounded away from zero by $\frac{ln n}{n}$.

Let:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This game was presented by Aumann and Maschler and it was proved by Zamir, (see Theorem 4 of [8]) that $e_n(p) = O^\star(\frac{ln n}{n})$, $\forall p \in (0, 1)$. For this game $u(p) = p(1-p)$, which is a strict concave function and hence by lemma 6.3, if PI can guarantee $f(n, p)$ in $SG_\infty(p)$, then: $f(n, p) = O^\star(1)$.

On the other hand PII can not do better in $G_\infty(p)$ than in any $G_\infty(n, p)$, hence: if PII can guarantee $g(n, p)$ in $SG_\infty(p)$ then: $g(n, p) \geq O^\star(\frac{ln n}{n})$.

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