LOAN CONTRACTS WITH COLLATERAL
AND CREDIT RATIONING:
A SIGNALING APPROACH

by
MICHAEL LANDSBERGER
and
SHMUEL ZAMIR

Discussion Paper # 26 June 1993

CENTER FOR RATIONALITY
AND INTERACTIVE DECISION THEORY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-584135, [972]-2-584136
E-MAIL: ratio@sunrise.huji.ac.il
FAX: [972]-2-513681Z
Loan Contracts with Collateral
and Credit Rationing:
A Signaling Approach

Michael Landsberger¹
and
Shmuel Zamir²

June 7, 1993

Abstract

Loan contracts with collateral are a common instrument to allocate credit to entrepreneurs who invest in risky projects. Collateral provides lenders with partial insurance against bad outcomes. Since typically, credit is provided under incomplete information about the nature of the project to be undertaken, we investigate in this paper whether collateral can be used as an instrument to identify projects which the bank may consider to be 'bad'. We prove that the ability of collateral to serve as a screening device depends on whether the entrepreneurs and the bank have similar ranking of project quality. If they do not, collateral is less effective as a signal. Within each regime, we identify conditions under which separating and pooling equilibria take place and we characterize the properties of these equilibria. When credit is scarce, we derive equilibria in which credit is rationed. A rationing regime eliminates pooling equilibria and generates more surplus to the bank on every project for which a loan was granted.

¹The University of Haifa, Israel. This research was done in part while the author was a visiting professor at the Economics Department, SUNY at Buffalo.
²The Hebrew University, Jerusalem, Israel. This research was done in part while the author was a visiting professor at the Economics Department, the University of Pittsburgh.
1 Introduction

Loan contracts with collateral are a common financial instrument used by commercial banks to extend credit to households and firms. When credit is required to finance risky projects characterized by probability distributions, collateral allocates risk among the parties so that the bank gets partial insurance against bad outcomes. The major question posed in this paper is whether in environments where banks are not fully informed about projects, loan contracts with collateral support equilibria in which collateral signals project quality. Obviously, while investigating this role of collateral, its risk sharing aspect is fully preserved.

In considering this question, a distinction is made between situations where the bank and the entrepreneur share a common ranking of projects as opposed to another scenario where their rankings differ. From the point of view of the entrepreneur, projects are ranked by the largest collateral he is willing to offer. The bank ranks projects by the smallest acceptable collateral. We show that the ranking of the bank and the entrepreneur need not coincide.

We prove that when the evaluations of the parties coincide, collateral is an effective signal of project quality if the highest collateral offered by an entrepreneur who holds an inferior project is smaller than the smallest collateral the bank is willing to accept not knowing the project type. Under these circumstances, there exists a unique separating equilibrium in which only the better projects are financed. In this equilibrium, the bank breaks even and the entrepreneur who holds the better project takes all the surplus due to private information. If the highest collateral offered by an entrepreneur who holds an inferior project is larger than the smallest collateral the bank is willing to accept not knowing the project type, there are multiple equilibria in some of which both projects are financed.

If the bank and the entrepreneur have conflicting evaluations of projects' quality, the only separating equilibria are those in which the inferior project (by the bank's evaluation) is financed. This equilibrium may not be sustained under complete information. In this sense, collateral is less effective as a signal since when evaluations coincide every separating equilibrium is also sustained under complete information. Pooling equilibria exist if and only if the largest collateral the entrepreneur is willing to offer when he holds a good project exceeds the smallest collateral the bank is willing to accept not knowing the project type.

We also consider a more general scenario in which types are configurations of projects and utilities. Under such bidimensional typology, separating equilibria are less revealing. Separation between types does not assure separation between projects. An important implication of the bidimensional typology is that having been offered collateral $c$, the 'beliefs' of the bank as to the probability that the offer was made by a type who holds, say, a good project, are not monotone in $c$. One consequence of this lack of monotonicity is that the set of equilibria may not be connected in the following sense. For $A < B < C$, there may be an equilibrium in which the threshold collateral for approving a loan is $A$, another equilibrium in which the threshold is $C$ but there is no equilibrium at $B$.

When there is more than one entrepreneur and loans cannot be provided to all applicants, credit rationing takes place. This situation introduces implicit competition between entrepreneurs; by offering collateral, an entrepreneur who holds a 'good' project is trying not only to distinguish himself by signaling his type but also to be a winner in the competition for credit. Under these conditions, the screening effect of collateral is enhanced and consequently, all equilibria are separating where only 'good' projects are undertaken.
In these equilibria, the collateral is strictly higher than when credit need not be rationed. Since the competing parties may be of the same type, the resulting equilibria are in mixed strategies (for the entrepreneurs.) This situation admits much of the characteristics of an auction mechanism.

The performance of loan contracts with collateral when the properties of projects are private information has been a research topic of many papers. To mention just a few, we refer the reader to Stiglitz and Weiss [1981], Bester [1985], Chan and Kanatas [1985] and Besanko and Thakor [1987]. The following remarks relate our paper to this literature.

(i) In these papers, the authors refer to loan contracts with collateral but the emphasis is on the provision of an endogenous explanation for credit rationing rather than investigating the role of collateral as a screening device. To this effect, they consider an environment where lenders vary both the collateral and the interest rates. To focus on the ability of collateral to discriminate among projects the interest rate is fixed in our model.

(ii) In the papers mentioned above, projects are ordered either by Mean Preserving Increase in Risk (Stiglitz and Weiss [1981] and Bester [1985]) or by Monotone Likelihood Ratio (Besanko and Thakor [1987]). These assumptions are crucial in establishing their results. Assuming that the projects are ordered by MPIR implies that as the loan interest rate increases, low-risk borrowers drop out of the market before the high risk ones. The opposite happens when one assumes that projects are ordered by MLR. We consider a more general environment where projects are not required to be ordered by any stochastic order.

(iii) These papers follow the principal agent literature where it is assumed that the principal offers a menu of contracts from which the agents make their choices. This is in spirit of Rothschild and Stiglitz [1976], Wilson [1977] and many others. Such an approach assumes that the principal has the power to make a credible commitment to what he has offered. It is well known that under many types of adverse selection, after the agents have made their choices and have thus revealed themselves, the principal has a motivation to renege on his offer. In order to cope with this problem we adhere to the concept of equilibrium under which contracts are self enforcing; namely, it is in the best interest of all parties not to renege on their offers. The game which underlies our model is also different in the sense that we let the informed player move first whereas in these papers the first move belongs to the uninformed agent (the principal).

2 The Model

In the environment modeled in this paper investment opportunities consist of project types denoted as $G$-project and $B$-project. Projects are assigned to entrepreneurs randomly where the probability of being assigned a $G$-project is $p$. Investment returns induced by projects are represented by random variables $X^G$ and $X^B$, with distributions $G$ and $B$ respectively. To comply with limited liability, both variables are assumed to be non negative.
Entrepreneurs do not have any equity and to undertake an investment project they need to invest one unit of capital. In order to focus attention on loan contracts, we assume that to finance a project the entrepreneur has to borrow one unit from the bank. Loan contracts are characterized by a collateral and an interest rate \( r \), which is determined exogenously. At the stage of applying for a loan the entrepreneur offers a collateral \( c \), and if the loan is granted the contract stipulates that he pays his debt and the interest \((1 + r)\). In case the investment returns and the collateral are insufficient to effect this payment the entrepreneur declares bankruptcy and pays what he can, which is \( c + X \). Consequently, if the loan is granted, the payoffs of the parties are

\[
X^*_e = \max(X^k - (1 + r), -c)
\]

for the entrepreneur, and

\[
Y^*_e = \min(X^k + c, 1 + r)
\]

for the bank; \( k = G, B \).

The entrepreneur and the bank are endowed with von Neumann Morgenstern utilities \( U \) and \( V \) respectively which are normalized by \( U(0) = V(0) = 0 \). If the loan is declined, the project cannot be undertaken, in which case the payoffs of the parties are zero. The sequence of moves is as follows: Nature moves first and assigns to the entrepreneur either a \( G \)-project or a \( B \)-project with probabilities \( p \) and \( 1 - p \) respectively. Knowing his type, the entrepreneur applies for a loan and offers a collateral \( c \). Upon having received the loan application and without knowing whether the entrepreneur holds a \( G \)-project or a \( B \)-project, the bank either approves the loan, in which case the project is undertaken, or rejects it.

This scenario can be presented as a game of incomplete information (a Bayesian game) described in Figure 1.

![Figure 1: The game in extensive form.](image-url)
It is interesting to notice that unlike in many other economic environments modeled as signaling games, the magnitude of the signal (collateral) affects the payoffs of both parties. In general, only the payoff of the sender of the signal is affected by its magnitude. As we shall see when we pursue our analysis, this property affects the conditions under which collateral is an effective screening device.

The strategy sets of the agents are:

\[ \Sigma = [0,1+r] \times [0,1+r] \] for the entrepreneur and \[ T = \{A, R\} \] for the bank, where \( A \) and \( R \) stand for 'Approve' and 'Reject' respectively.

A strategy \( \sigma \in \Sigma \) of the entrepreneur is denoted by \( \sigma = (c_G, c_B) \) and is read as: offer \( c = c_G \) if you hold a \( G \)-project, and offer \( c = c_B \) if you hold a \( B \)-project. A strategy \( \tau \in T \) of the bank is a function \( \tau : [0,1+r] \rightarrow \{A, R\} \) which assigns a reply, \( A \) or \( R \) to each offer \( c \in [0,1+r] \). For notational convenience we shall write 1 for \( A \) and 0 for \( R \) that is, \( \tau(c) = 1 \) means that, according to the strategy \( \tau \) of the bank, the loan application with collateral \( c \) is approved whereas its rejection corresponds to \( \tau(c) = 0 \). A special class of the Bank's strategy, namely the threshold strategies, play the most important role in our analysis: For any \( \alpha \in [0,1+r] \) we denote by \( \tau_\alpha \) the strategy of the bank to approve a loan application only if the collateral is at least \( \alpha \) (i.e. \( \tau_\alpha(c) = 1 \) if \( c \geq \alpha \) and \( \tau_\alpha(c) = 0 \) if \( c < \alpha \)). We shall also need the following variant of this notation: \( \tilde{\tau}_\alpha \) the strategy of the bank to approve a loan application only if the collateral is (strictly) higher than \( \alpha \) (i.e. \( \tilde{\tau}_\alpha(c) = 1 \) if \( c > \alpha \) and \( \tilde{\tau}_\alpha(c) = 0 \) if \( c \leq \alpha \)).

Only pure strategies are included in the strategy sets \( \Sigma \) and \( T \). These are decision rules which involve no randomization. Mixed strategies are defined as usual to be probability distributions on pure strategies. It turns out however, that in the equilibrium points of the above described game, there is no need to consider mixed strategies. The reason is that in equilibrium an agent can apply a mixed strategy only if he is indifferent between outcomes which correspond to the pure strategies in support of the distribution defined by that mixed strategy. In our model, since rejection yields the bank zero payoff, randomization by the bank can occur only in extreme equilibria in which the bank is not making any profit. The entrepreneur also does not randomize in equilibrium between two different levels of collateral since these typically yield different expected utilities (given his project and given the policy of the bank).

Throughout the paper we adhere to the following definitions and notations. For any pair of strategies \( \sigma = (c_G, c_B) \in \Sigma \) and \( \tau \in T \), denote by \( H^\sigma(\sigma, \tau) \) the expected payoff of the entrepreneur and by \( H^\tau(\sigma, \tau) \) the expected payoff of the bank, that is:

\[
\begin{align*}
H^\sigma(\sigma, \tau) &= p \tau(c_G) EU(X^G_{c_G}) + (1-p) \tau(c_B) EU(X^B_{c_B}) \\
H^\tau(\sigma, \tau) &= p \tau(c_G) EU(Y^G_{c_G}) + (1-p) \tau(c_B) EU(Y^B_{c_B}),
\end{align*}
\] (3)

where \( E \) is the appropriate expectation operator.

**Definition 2.1** A pair of strategies \((\sigma^*, \tau^*) \in \Sigma \times T\) is a Nash equilibrium if

(i) \( H^\sigma(\sigma^*, \tau^*) \geq H^\sigma(\sigma, \tau^*) \) for all \( \sigma \in \Sigma \).

(ii) \( H^\tau(\sigma^*, \tau^*) \geq H^\tau(\sigma^*, \tau) \) for all \( \tau \in T \).
Note that for \( \sigma = (c_G, c_B) \) we can rewrite \( H^\sigma(\sigma, \tau) \) as
\[
H^\sigma(\sigma, \tau) = pH^G(c_G, \tau) + (1 - p)H^B(c_B, \tau),
\]
where \( H^G(c, \tau) = \tau(c)EU(X^G_c) \) is the conditional expected payoff of the entrepreneur, if he offers \( c \) and the bank uses \( \tau \), given that he is assigned a G-project, and \( H^B(c, \tau) = \tau(c)EU(X^B_c) \) is his expected payoff given he is assigned a B-project. By virtue of 4, condition (i) in Definition 2.1 holds for \( \sigma^* = (c^*_G, c^*_B) \) if and only if

(i') \( H^G(c^*_G, \tau^*) \geq H^G(c, \tau^*) \) for all \( c \in [0, 1 + r] \).

(i'') \( H^B(c^*_B, \tau^*) \geq H^B(c, \tau^*) \) for all \( c \in [0, 1 + r] \).

In words, \( \sigma^* = (c^*_G, c^*_B) \) is an ex-ante best response of the entrepreneur against \( \tau^* \) if and only if \( c^*_G \) is an ex-post best response of an entrepreneur who holds a G-project and \( c^*_B \) is a best response of an entrepreneur who holds a B-project. This is the basic property of the Bayesian game as a model of a game with incomplete information (see Harsanyi (1967)): In the Bayesian game, the ex-ante equilibrium strategy of the entrepreneur before hearing his type, is also a best response ex-post, given his type.

The constants introduced in the following Definition reflect the evaluations of the two project by the two agents.

**Definition 2.2** The constants \( \gamma, \mu, \beta, b, \) and \( g \) are defined as follows:

(i) \( \gamma = \min\{c \geq 0 \mid EV(Y^G_c) \geq V(1)\} \)

(ii) \( \mu = \min\{c \geq 0 \mid pEV(Y^G_c) + (1 - p)EV(Y^B_c) \geq V(1)\} \)

(iii) \( \beta = \min\{c \geq 0 \mid EV(Y^B_c) \geq V(1)\} \)

(iv) \( b = \max\{0 \leq c \leq 1 + r \mid EU(X^B_c) \geq U(0)\} \)

(v) \( g = \max\{0 \leq c \leq 1 + r \mid EU(X^G_c) \geq U(0)\} \)

The constant \( \gamma \) is the lowest collateral value which makes the G-project acceptable to the bank. \( \mu \) is the lowest collateral value which makes the uncertain project acceptable to the bank. We assume that not knowing which project was assigned to the entrepreneur, the bank maximizes expected utility using his prior \( p \). \( \beta \) is the smallest collateral the bank is willing to take in order to grant a loan for a B-project. From here on we assume without loss of generality that \( \gamma \leq \beta \) (otherwise we interchange the names of the projects.) \( b \) is the highest collateral value which the entrepreneur is willing to offer when he holds a B-project and \( g \) has the same meaning in the case where the entrepreneur holds a G-project.

These constants constitute a natural parameterization of our model and one is tempted to use them in order to apply the terminology good and bad projects where \( \gamma < \beta \) and \( b < g \) indicate that \( G \) is a better project than \( B \). Unfortunately, things are not that simple since it may be that the assessments of the projects by the bank and the entrepreneur do not coincide. Namely, it is possible that \( \gamma < \beta \) and \( b > g \). Since these rankings affect the type of equilibria which emerge, it is interesting to investigate what differences between project types induce the same ranking.
Proposition 2.3

(i) If the distribution \( G \) stochastically dominates the distribution \( B \) then \( \gamma \leq \beta \) and \( g \geq b \), i.e. the assessments of the projects by the bank and the entrepreneurs coincide.

(ii) If the distributions \( G \) and \( B \) are not stochastically ordered, then it is possible that \( \gamma < \beta \) and \( g < b \).

(Recall that \( G \) stochastically dominates \( B \) if \( G(x) \leq B(x) \) for all \( x \).)

By virtue of (i), if projects are ordered by FSD so that \( G \) is the dominating distribution, then \( b < g \) and \( \gamma < \beta \). Under these conditions it is natural to contend that \( G \) is the better project since it is preferred by both parties. However, if projects are not ordered by FSD then by (ii) it may be that \( \gamma < \beta \) and \( g < b \) namely, the bank considers \( G \) to be the better project whereas the entrepreneur prefers \( B \). In this case we cannot use the inequalities between the parameters as indicators of project quality unless we specify whose point of view is taken.

Proof: (i) Since the functions \( U \), \( V \), max and min are monotonically increasing it follows from the stochastic dominance of \( G \) over \( B \) that

\[
EU(X_c^G) \geq EU(X_c^B) \quad \forall c
\]

and

\[
EV(Y_c^G) \geq EV(Y_c^B) \quad \forall c.
\]

By the definition of \( g \) as the maximal \( c \) satisfying \( EU(X_c^G) \geq EU(0) \) we have

\[
EU(X_c^G) \geq EU(0) \Rightarrow c \leq g.
\]

It follows that

\[
EU(X_c^G) \geq EU(X_c^B) \geq EU(0) \text{ and hence } b \leq g.
\]

Similarly

\[
EV(Y_c^G) \geq EV(Y_c^B) \geq V(1),
\]

and since \( \gamma \) is the minimal \( c \) satisfying \( EV(Y_c^G) \geq V(1) \) it follows that \( \gamma \leq \beta \).

(ii) The proof consists of constructing two distribution functions \( G \) and \( B \) and two concave utility functions \( U \) and \( V \) such that \( \gamma < \beta \) and \( b > g \).

- Let \( G \) be any continuous and strictly increasing distribution function on \([0, \infty)\) with \( G(0) = 0 \)
- Let \( V \) be any concave and strictly increasing utility function on \([0, \infty)\) with \( V(0) = 0 \) - this is the bank's utility.
- Define the entrepreneur's utility function by

\[
U(x) = \begin{cases} 
  x & \text{if } x \leq t \\
  t & \text{if } x \geq t 
\end{cases}
\]

where \( t = 1 + \gamma \).
• Define the distribution \( B \) as follows:

\[
B(x) = \begin{cases} 
  G(s) & \text{if } 0 \leq x \leq s \\
  (1 - \varepsilon)G(x) + \varepsilon G(s) & \text{if } s \leq x \leq t \\
  1 & \text{if } t < x < \infty
\end{cases}
\]

where \( s = 1 + r - \gamma \) and \( \varepsilon > 0 \) to be specified later. Note that \( G(s) = B(s), G(t) = B(t), G(x) > B(x) \) for \( s < x < t \) and \( G(x) < B(x) \) for \( 0 < x < s \) or \( x > t \).

• Choose \( \varepsilon > 0 \) small enough so that \( G \) dominates \( B \) in Second Degree Stochastic Dominance (SSD) that is,

\[
\int_{-\infty}^{y} G(x) \, dx \leq \int_{-\infty}^{y} B(x) \, dx \quad \forall y \in (-\infty, \infty).
\]

It is well known (see e.g. Hadar and Russel [1969] or Hanoch and Levy [1969]) that if \( X \) is distributed \( G \) and \( Y \) is distributed \( B \) then this condition is equivalent to:

\[
Ef(X) \geq Ef(Y) \quad \text{for any concave and non decreasing } f. \tag{5}
\]

We shall now show that the above constructed distributions \( G, B \) and utility functions \( U, V \) imply \( \gamma < \beta \) and \( g < b \).

Note first that the bank’s utility \( V(Y^B_c) \) is a concave function of \( X^k \). In fact, \( V(Y^B_c) = V(\min(X^k + c, 1 + r)) \), and since both \( V \) and \( \min(X^k + c, 1 + r) \) are concave and non decreasing, \( V(Y^B_c) = f(X^k) \) for some concave non decreasing \( f \). It follows from inequality (5) (since \( G \) dominates \( B \) in SSD) that \( EV(Y^G_c) > EV(Y^B_c) \) for all values of \( c \) (we may guarantee strict inequality by choosing sufficiently small \( \varepsilon \)), and consequently \( \gamma < \beta \) since \( EV(Y^G_c) \) is increasing in \( c \).

Evaluating \( X^G_c \) for \( c = g \) yields

\[
X^G_g = \max(X^k - (1 + r), g) = \begin{cases} 
  -g & \text{if } 0 \leq X^k \leq s \\
  X^k - (1 + r) & \text{if } X^k \geq t
\end{cases}
\]

and

\[
U(X^G_g) = \begin{cases} 
  -g & \text{if } 0 \leq X^k \leq s \\
  X^k - (1 + r) & \text{if } s \leq X^k \geq t \\
  t - (1 + r) & \text{if } X^k \geq t
\end{cases}
\]

where \( k \in \{G, B\}, X^G \) is distributed \( G \) and \( X^B \) is distributed \( B \). It follows that

\[
EU(X^B_g) = -gB(s) + \int_s^t (x - (1 + r)) dB(x) + (t - (1 + r))(1 - B(t))
\]

and

\[
EU(X^G_g) = -gG(s) + \int_s^t (x - (1 + r)) dG(x) + (t - (1 + r))(1 - G(t)).
\]

Since \( G \) and \( B \) coincide at \( s \) and \( t \), the first and third terms at the right hand side are the same at both equations. As for the middle terms, since \( B(x) < G(x) \) for \( x \in (s, t) \) the
conditional distribution of $B$ given $X \in (s, t)$ stochastically dominates that of $G$ and since $U$ is nondecreasing we have
\[ \int_s^t (x - (1 + r)) dB(x) > \int_s^t (x - (1 + r)) dG(x) \]
(again, the strict inequality is guaranteed by our construction) from which it follows that
\[ EU(X_s^B) > EU(X_s^G) \geq U(0) \]
and finally, since $b$ is the maximal $c$ satisfying $EU(X_c^G) \geq U(0)$, we conclude that $g < b$. This completes the construction and the proof of the Proposition.

3 Equilibrium Loan Contracts with Homogeneous Entrepreneurs

It is well known that many economic models which have the structure of signaling games have many Nash equilibria some of which are quite "unreasonable". Common refinements such as the sequential equilibrium suggested by Kreps and Wilson [1982] usually do not eliminate these equilibria. To cope with this problem many authors suggested to eliminate (weakly) dominated strategies (see Milgrom and Roberts [1986], Bagwell and Ramey [1988] and Kreps [1990] Ch. 12. For different games this notion was advocated by Moulin [1981] as a possible solution concept). In this vein, we make the following assumption which we maintain throughout the paper:

Assumption 3.1

(i) An entrepreneur never offers collateral such that if the loan is granted, his expected utility (given his type) is negative.

(ii) An entrepreneur never offers collateral $c$ which will certainly be rejected, unless (i) is violated by any offer higher than $c$.

Under (i), an entrepreneur with a $B$-project never offers collateral $c > b$, and an entrepreneur with a $G$-project never offers collateral $c > g$. Combined with subgame perfectness this implies that, in equilibrium, the bank never rejects an offer $c > \max(b, \gamma)$ since by (i) it could have been made only by a holder of a $G$-project and hence it should be accepted. Similarly, the bank never rejects an offer $c > \max(g, \beta)$. Subgame perfectness also rules out a rejection of collateral $c > \max(\gamma, \beta)$.

Under (ii), an entrepreneur never offers $c < \min(\gamma, \beta)$ (unless he is forced to do so by (i)). Such an offer is surely rejected and hence it is weakly dominated by any offer $c \geq \min(\gamma, \beta)$ (which, if allowed by (i), yields positive payoff when accepted.)

---

3 By 'homogeneous' entrepreneurs we refer to the case where entrepreneurs have the same utility functions.
The case of unanimous project assessments

This is the situation in which \( \gamma \leq \beta \) and \( b \leq g \) i.e. it is agreed upon both parties which is the good project. By Proposition 2.3 a sufficient (but not necessary) condition for this to hold is that \( G \) stochastically dominates \( B \).

We may assume that \( \gamma \leq g \) since otherwise even under perfect information loans would not be approved.

**Proposition 3.2** If \( b < \mu \) then,

(i) There is no pooling equilibrium (except a trivial 'no deal' equilibrium in the case \( \beta \geq g \)).

(ii) For any \( \varepsilon > 0 \) the strategies \( (\sigma, \tau) = [(\alpha + \varepsilon, b), \tilde{\tau}_a] \), where \( \alpha = \max(b, \gamma) \), support a unique family of separating \( \varepsilon \)-equilibria.

The uniqueness claimed in (ii) is in outcomes (up to an \( \varepsilon \)) and not in strategies.

**Proof**  
(i) Obviously there can not be a pooling equilibrium in which the entrepreneur's strategy is \( (c, c) \) with \( c > b \). This is so because an entrepreneur with a \( B \)-project would deviate from this strategy. Since \( \mu > b \), there is no pooling equilibrium \( [(c, c), \tau] \) with \( c \geq \mu \). It also cannot be that \( c < \mu \) since the bank's best response is to reject such an offer. This last situation in which the entrepreneur always offers \( c < \mu \) and the bank rejects it, can be a trivial Nash equilibrium if and only if \( \beta \geq g \) (since if \( \beta < g \), an entrepreneur with a \( G \)-project can profitably deviate by offering \( \beta + \varepsilon < g \), which the bank should accept.)

(ii) Given \( \sigma = (\alpha + \varepsilon, b) \), the bank can only decrease his expected utility by deviating from \( \tilde{\tau}_a \), either by accepting \( b \) (since \( b < \beta \)), or by rejecting \( \alpha + \varepsilon \) (since \( \alpha + \varepsilon > \gamma \)). Given \( \tilde{\tau}_a \), consider an entrepreneur who holds a \( B \)-project. If \( b \leq \alpha \) then any deviation to a collateral \( c \leq b \leq \alpha \) is rejected, while a deviation to \( c > \alpha \) is not allowed by (i) of Assumption 3.1 since \( \alpha \geq b \). Similarly if \( b = \alpha \), any deviation to a collateral \( c \leq b = \alpha \) is not an improving deviation since it is rejected. A deviation to \( c > b = \alpha \) is not allowed by (i) of Assumption 3.1 since \( \alpha > b \). Therefore \( b \) is a best reply to \( \tilde{\tau}_a \). An entrepreneur who holds a \( G \)-project cannot improve his outcome by more than \( \varepsilon \) while offering \( c > \alpha \) since he decreases his payoff if he offers more than \( \alpha + \varepsilon \) and an offer with collateral less or equal to \( \alpha \) will be declined resulting in payoff \( U(0) \) which is less than \( EU(X_S^G) \).

To see that uniqueness of this equilibrium is in outcome and not in strategies, note that an entrepreneur with a \( B \)-project can offer any collateral \( c < b \) (which yields the same zero payoff).

The limit of the separating \( \varepsilon \)-equilibrium of the last Proposition as \( \varepsilon \to 0 \) is the strategy \( [(\alpha, b), \tau_a] \). A natural question is whether or not this is an equilibrium. It is readily seen that \( [(\alpha, b), \tau_a] \) is in fact an equilibrium if \( b < \gamma = \alpha \) but it is not if \( \gamma \leq b = \alpha \) (since, by the definition of \( \mu \), if both types offer the same collateral \( b < \mu \) it should be rejected.) In this case \( [(\alpha,b-\varepsilon), \tau_a] \) is an equilibrium. We think, however, that the interpretation of this equilibrium is not very appealing since the 'burden' of separation is on the holders of \( B \)-projects who have to distinguish themselves voluntarily by offering strictly less than \( b \) even though they

---

\( ^4 \)A strategy pair is an \( \varepsilon \)-equilibrium if each player cannot gain more than \( \varepsilon \) by a unilateral deviation from his strategy.
could also offer \( b \) which is offered by the holders of \( G \). The \( \varepsilon \)-equilibrium \([(b + \varepsilon, b), \tilde{\tau}] \) is more appealing since it implies that the holders of \( G \)-projects separate themselves by offering \( b + \varepsilon \) which the holders of \( B \)-projects cannot afford.

Another disadvantage of the equilibria \([(a, b - \varepsilon), \tau_n] \) is their lack of continuity when \( b > \gamma \): For any \( \varepsilon > 0 \) the pair \([(b, b - \varepsilon), \tau_n] \) is an equilibrium but \([(b, b), \tau_n] \) is not. Not surprisingly, this discontinuity is inherent to the separation between project types and this is the reason that \( \varepsilon \)-equilibria and ‘discontinuous strategies’ \( \tilde{\tau} \) are natural for this context. On the other hand, in studying pooling equilibria, it is more natural and convenient to look at equilibria rather than \( \varepsilon \)-equilibria, and at ‘continuous strategy’ \( \tau \) rather than \( \tilde{\tau} \). This is done in the following Proposition.

**Proposition 3.3** If \( \mu \leq b \), then for any \( \mu \leq c \leq b \), the point \([(c, c), \tau_n] \) is a pooling equilibrium.

**Proof** Straightforward, since given \( \tau_n \), the entrepreneur can only loose by offering something different than \( c \); a lower collateral will be rejected and a higher one will cost more. Given the entrepreneur’s strategy \((c, c)\), the bank cannot gain from accepting less than \( c \) and will loose if he rejects it.

By Propositions 3.2 and 3.3 the type of equilibria which emerge and the screening properties of collateral depend on whether the highest collateral which the entrepreneur who holds a \( B \)-project is willing to offer is (a) smaller or (b) larger than \( \mu \). Under (a), there exists a unique separating equilibrium in which only \( G \)-projects are financed. In this equilibrium, the bank breaks even in the sense that he receives the smallest collateral he must get in order not to be worse off than he would have been if he refused to finance the project. Since the \( B \)-project is not financed the entrepreneur who holds the \( G \)-project takes all the surplus due to private information. Under (b), there are multiple equilibria in some of which both projects are financed. In such (pooling) equilibria the bank loses money on the \( B \)-project since collateral at which loans are granted is always smaller than the smallest collateral the bank needed to obtain in order to break even when financing the \( B \)-project. Note that the multiplicity of pooling equilibria cannot be refined by a standard notion of refinement. These equilibria are ranked by \( c \) which ranges from \( \mu \) to \( b \), the higher \( c \) the better is the equilibrium for the bank. The only way to get the best equilibrium for the bank (i.e. close to \( b \)) as the unique equilibrium, is to let the bank announce and commit himself to \( \tau_n \). This, however, is a different game which assumes the ability to make credible commitments.

Note that in the separating equilibria considered in Proposition 3.2, a higher collateral is required if the \( B \)-project, which is the inferior among the two, is replaced by a better one (with a higher \( b \)) or the \( G \)-project is replaced by a worse one (with larger \( \gamma \)). An economic interpretation of this result is based on the fact that in our model collateral is a signal by means of which entrepreneurs who hold the better projects want to distinguish themselves from the others. An increase in \( b \) or \( \gamma \) can be interpreted as narrowing the differences between the projects which makes the distinction more costly and harder to achieve.
The case of opposing project assessments

This is a situation in which the bank considers the G-project to be better whereas the entrepreneur prefers the B-project. Properties of equilibria which emerge under these asymmetric evaluations are summarized in the following two Propositions.

Proposition 3.4 Pooling equilibria exist if and only if \( g \geq \mu \) in which case any \([(c, c), \tau_c] \) with \( \mu \leq c \leq \min(g, \beta) \) is an equilibrium.

Notice the similarity between this Proposition and Proposition 3.3. Both stipulate that pooling equilibria take place when the highest collateral the entrepreneur is willing to offer when he holds the project which he considers inferior, exceeds \( \mu \). There is, however, a dissimilarity as well since in these two regimes, pooling equilibria take place under different objective conditions. In the environment considered in Proposition 3.3 the project which the entrepreneur considers inferior was the B-project whereas Proposition 3.4 refers to a situation where the G-project is considered by the entrepreneur as the inferior one.

Proof  By the definition of \( \mu \) if \( c < \mu \) then \([(c, c), \tau_c] \) is not an equilibrium. By (i) of Assumption 3.1 \( c \leq g \) and therefore there is no pooling equilibrium if \( g < \mu \). If \( \mu \leq c \leq g \) then \([(c, c), \tau_c] \) is an equilibrium. In fact, \( c < g < b \) implies that entrepreneurs with either project types can offer \( c \) and, by the usual argument, this is a best reply to \( \tau_c \). As in all cases, the condition \( c \leq \beta \) is required for (subgame) perfection: \( c > \beta \) can never be offered in equilibrium since strictly lower collateral would also be accepted (again by a consequence of (i) in Assumption 3.1).

Note that in this equilibrium the bank finances the project which he considers inferior whereas the better project (in his view) is not financed. This is may be called an 'oddly' separation. In this separating equilibrium as in the one stated in Proposition 3.2 the bank breaks even and the entrepreneur takes all rent due to his private information.

Proposition 3.5 A separating equilibrium exists if and only if \( g < \beta \leq b \) and \( g < \gamma \). In this case the equilibrium is unique and equals to \([(g, \beta), \tau_\beta] \).

Note that the separating equilibrium established in this Proposition separates in a rather unusual way: the bank approves loans only to bad projects (according to his own assessment.) We refer to such equilibria as to 'oddly' separating equilibria. These are equilibria in which only entrepreneurs who hold B-projects offer collateral high enough to be accepted by the bank. Holders of G-projects cannot afford such collateral.

Proof  Let us first show that if \( g < \beta \leq b \) and \( g < \gamma \) \([(g, \beta), \tau_\beta] \) is an equilibrium. \( \tau_\beta \) is obviously a best reply of the bank to \((g, \beta)\) since accepting \( g \) or rejecting \( \beta \) decreases his expected payoff. Given \( \tau_\beta \) an entrepreneur who holds a G-project cannot obtain a positive payoff by deviating from \( g \). The payoff of an entrepreneur with a B-project decreases if he offers more than \( \beta \) and it is 0 if he offers less than \( \beta \) (since his offer would be rejected). Hence, \([(g, \beta), \tau_\beta] \) is an equilibrium. To prove uniqueness observe that to approve a B-project the bank requires collateral of at least \( \beta \). By (i) in Assumption 3.1 the equilibrium collateral cannot be higher than \( \beta \) and therefore, \([(g, \beta), \tau_\beta] \) is the only separating equilibrium.
To see that the conditions \( g < \beta \leq b \) and \( g < \gamma \) are necessary for the existence of separating equilibria consider the negation of those conditions.

If \( g \geq \beta \), holders of either \( G \) or \( B \)-project can offer \( \beta \) which will be accepted. Hence, in any equilibrium both project types are financed. But in our model such an equilibrium cannot be separating since the type who offers the higher collateral can gain by offering the lower collateral, which will also accepted.

If \( g > \gamma \), then by (ii) in Assumption 3.1 an entrepreneur holding a \( G \) project offers \( c_\gamma \geq \gamma \). If the bank knows that \( c_\gamma \) is offered by a holder of a \( G \) project which is the case in a separating equilibrium, he should accept it. Since a holder of a \( B \)-project can offer at least \( c_\beta \), both project types are financed in equilibrium but, as claimed before, such an equilibrium cannot be separating.

Finally, if \( g < \beta > b \) and \( g < \gamma \) then the only equilibrium is the trivial one in which no project is financed even under complete information.

Let us look at the consequences of Propositions 3.4 and 3.5 for various configurations:

**Case 1**

If \( \gamma \leq g < \mu \leq b < \beta \), then there is no equilibrium in which at least one project type is financed. This is in contrast to the complete information case where \( G \)-projects would be financed with some collateral \( c \) satisfying \( \gamma \leq c \leq g \).

```
γ   g   μ   b   β
```

Case 1: No equilibrium exist.

**Case 2**

If \( \gamma < \mu \leq g \leq b < \beta \), then only pooling equilibria exist in which both types of projects are financed with collateral \( \mu \leq c \leq g \). Under complete information case only \( G \)-projects would have been financed.

```
γ   μ   g   b   β
```

Case 2: Only pooling equilibria exist.

**Case 3**

If \( \gamma \leq g < \mu < \beta \leq b \), there is a unique equilibrium. This is a separating equilibrium in which only \( B \) projects are financed. Under complete information both project types would have been financed.

```
γ   g   b   β
```
Case 3: Only ‘oddly’ separating equilibrium exists.

If $\gamma \leq \mu \leq \varrho < \beta \leq b$, then both types of equilibria exist. The corresponding collateral values supporting these equilibria form a disconnected set. The pooling equilibria are supported by collateral $c \in [\mu, \varrho]$ while the separating equilibrium corresponds to $c = \beta$.

In the complete information case both projects would have been financed.

Case 4: Both types of equilibria exist.

Before concluding this section we shall illustrate the role played by Assumption 3.1. To do this, consider the case $b < \gamma$. Without Assumption 3.1, $[(c, b), \tau_c]$, where $\gamma < c \leq \varrho$ is a separating equilibrium. In this equilibrium, the bank rejects any offer of a collateral lower than $c$. In particular, if he gets an offer $c'$ satisfying $\gamma < c' < c$, he rejects it and thus gets a payoff zero. This is hardly reasonable since the chances are that this offer $c'$ was made by an entrepreneur who holds a $G$-project, because one who holds a $B$-project will be loosing money if he took a loan with this collateral (remember, $c' > b$). These considerations refer to behavior of the agents off the equilibrium path, since $c'$ should not be offered in equilibrium. Whether the bank's strategy to reject an offer $c' > b$ is rational or not depends on his beliefs, after he has heard the offer, about the type who made it. If he believes that the offer was made by an entrepreneur with a $G$-project he should accept it, otherwise the offer should be rejected. Such considerations lead to the vast literature on equilibrium refinements which gives rise to the question of whether the unreasonable Nash equilibria in our model (without Assumption 3.1) can be eliminated on the ground of not being perfect or not being sequential. Unfortunately, as it often the case in signaling games, perfect or sequential equilibria do not refine such separating Nash equilibria, as stated in the following proposition.

Proposition 3.6 If Assumption 3.1 is not made, for each $\gamma < c \leq \varrho$ the pair of strategies $[(c, b), \tau_c]$ is a separating equilibrium which is a (‘trembling hand’) perfect and hence sequential equilibrium.

Proof Consider the following strategy of the entrepreneur (a ‘tremble’ around $(c, b)$): If you have a $B$-project, offer $b$ with probability $1 - \epsilon$, and $C'$ with probability $\epsilon$. If you have a $G$-project, offer $c$ with probability $1 - \delta$ and $C'$ with probability $\delta$. In both cases, $C'$ is a random variable with uniform distribution on $[0, 1 + \tau]$. It is readily seen that given this strategy, any collateral $c' \neq c$, offered by the entrepreneur, induces the bank to believe that with probability $\epsilon/(1 + \epsilon)$ he is facing a $G$-project and with probability $1/(1 + \epsilon)$ he is facing a $B$-project. Assuming that $\epsilon$ is sufficiently small, the bank's best reply is then to reject $c'$. This proves the trembling hand perfectness (and hence
the sequentiality) of the equilibrium \([c, b, \tau_0]\).

The proof of Proposition 3.6 illustrates the weakness of these two prominent notions of refinement. Perfectness requires that "there exist a sequence of trembles such that...." Nothing prevents the trembles to be very 'strange', as they in fact are in our case: The entrepreneur who holds a \(B\)-project trembles much more to offer mistakenly \(b < c' < c\) (which can be disastrous for him if accepted) than does a \(G\)-project holder (for whom it can be very profitable if accepted.) The sequential equilibrium requires some off-equilibrium beliefs to sustain it, but the above beliefs according to which 'c' comes most likely from an entrepreneur with a \(B\)-project', does the job. Strangely enough, these are consistent beliefs, and they even pass the test of "the intuitive criterion" (Cho and Kreps [1987]). Actually, this Nash equilibrium also qualifies as a proper equilibrium and probably many other existing refinements. We conclude that our 'unreasonable' equilibria cannot be eliminated by any refinement concept which allows the existence of 'unreasonable' beliefs (off the equilibrium path) or trembles. The way to remove 'unreasonable equilibria' is to remove together the potentially 'unreasonable' actions. This we did in Assumption 3.1.

4 Nonhomogeneous Entrepreneurs

In this section we consider an environment in which nature assigns to each entrepreneur a utility \(U\), which is an element of a certain compact set \(U\) of utility functions, and a project, \(G\) or \(B\). This situation leads naturally to a game of incomplete information on one side (see Harsanyi (1967)) in which the uninformed player is the bank, and the type set of the entrepreneur is \(T = \{G, B\} \times U\) on which there is a probability distribution \(P\).

Denote an entrepreneur's type by \(t = (t_1, t_2)\) where \(t_1 \in \{G, B\}\) and \(U \in U\). For each \(U \in U\) let \(b(U), g(U)\) the corresponding values of \(b\) and \(g\) defined in the previous section. As for the bank, the values of \(\gamma\) and \(\beta\) are, as before, the values of the minimal acceptable collateral for a \(G\)-and \(B\)-project respectively.

The model considered in the previous section is clearly the special case in which \(U\) consists of a single utility function \(U\) and hence the type set consists of two types \((G, U)\) and \((B, U)\) on which the prior probability distribution is given by \(P\{G, U\} = p\) and \(P\{B, U\} = 1 - p\).

Since 'separation' in signaling games applies to types which in this section are pairs of project and utility, separation of project types or of utility types may appear in various ways depending on the specifications of the problem. The following simple examples demonstrates this new phenomenon.

Example Consider a case in which there are two possible utility functions \(U = \{U_1, U_2\}\), with the following configurations of \(b(U_1), b(U_2), g(U_1)\) and \(g(U_2)\):

Case 1

If \(b(U_1) < b(U_2) < g(U_1) < g(U_2)\), then a strategy \(\tau^*\) of the bank with \(b(U_2) < c^* < g(U_1)\) will provide the "usual" separation: Only \(G\)-projects will be financed.

14
Case 2

If $b(U_1) < g(U_1) < b(U_2) < g(U_2)$, then a strategy $\tau_b$ of the bank with $g(U_1) < \hat{c} < b(U_2)$ will separate the more risk averse entrepreneurs which will not obtain the loan no matter whether they hold a $B$ or a $G$-project. In other words it may well happen in equilibrium that the bank will not finance a $G$-project (since it is held by a risk averse entrepreneur who is not willing to provide high enough collateral) while financing a $B$-project (held by less risk averse entrepreneur who is ready to provide high enough collateral).

Case 3

With the same configuration as in the previous case, a strategy $\tau_b$ of the bank with $b(U_2) < \hat{c} < g(U_2)$ will separate both $B$-projects and risk averse entrepreneurs: Only the less risk averse entrepreneurs (with utility $U_2$) who hold $G$-projects will obtain a loan.

Similarly, a strategy $\tau_{B'}$ with $b(U_4) < \hat{c}' < g(U_1)$ eliminates only $B$-projects held by risk averse entrepreneurs. It is easy to construct equilibria with other variants of "separation".

As in the previous sections, we use the notation according to which the $G$ project is preferred by the bank i.e., $\gamma < \beta$. The analysis of the equilibrium is based on the bank's belief of how likely certain proposed collateral is to come from an entrepreneur with a $G$-project and consequently, whether this collateral should be accepted or rejected. In the previous section, when all entrepreneurs had the same utility function, these beliefs were given by the (prior) probability $p$, and by the resulting acceptance threshold $\mu$. In the present more general model, $p$ and $\mu$ are "replaced" by the two functions $p(c)$ and $\mu(c)$ defined as follows:

$$p(c) = \frac{P\{t = (G,U); g(U) > c\}}{P\{t = (G,U); g(U) > c\} + P\{t = (B,U); b(U) > c\}}, \quad (6)$$
and
\[ \mu(c) = \min\{c \geq 0 \mid p(c)EV(Y^G) + (1 - p(c))EV(Y^B) \geq V(1)\}. \] (7)

These functions are defined for values of \( c \) for which the denominator of \( p(c) \) is positive. Thus \( p(c) \) is the conditional probability that an entrepreneur holds a \( G \)-project, given that he can afford collateral \( c \). Given this belief of the bank, \( \mu(c) \) is the minimum collateral he needs to grant a loan which will yield him non-negative expected utility. It is readily verified that \( p(c) \) and \( \mu(c) \) are right continuous and that \( \mu(c) \) depends on \( c \) only through the probability \( p(c) \) that is \( p(c_1) = p(c_2) \) implies \( \mu(c_1) = \mu(c_2) \). Therefore, we shall also write \( \mu(c) = \mu(p(c)) \)
i.e. \( \mu(p) \) is the smallest collateral acceptable by a bank who believes that with probability \( p \) the offer comes from an entrepreneur with a \( G \)-project. Note that \( \mu \) is decreasing in \( p \) and satisfies
\[ \beta = \mu(0) \geq \mu(p) \geq \mu(1) = \gamma, \]
for any \( 0 \leq p \leq 1 \).

As an illustration, assume that \( \mathcal{U} \) consists of a single element \( U \). This is the case of homogeneous entrepreneurs considered in the previous section. Consider the configuration \( b < \gamma < \mu < g \). The functions \( p(c) \) and \( \mu(c) \) are given in Figure 2 and Figure 3.

![Figure 2: The probability p(c) for a homogeneous case.](image)

![Figure 3: The function \( \mu(c) \) for a homogeneous case.](image)
Clearly, for this special case the two functions can be described simply by the two numbers $p$ and $\mu$, as we in fact did in the previous section.

A strategy of the entrepreneur in this incomplete information game is $\sigma : T \rightarrow \mathcal{R}$. So, $\sigma(t)$ is the collateral offered by an entrepreneur of type $t$. A Bayesian equilibrium of the game is a pair $(\sigma, \tau)$ such that for each $t \in T$ the collateral $\sigma(t)$ is a best reply of a $t$-type entrepreneur to $\tau$, and $\tau$ is a best reply of the bank to $\sigma$.

For any $\alpha > 0$ let $\sigma_\alpha$ be the following entrepreneur's strategy: offer collateral $\alpha$ if it provides (when accepted) a positive expected utility. Otherwise, offer the minimum of $b$ and $\gamma$. Formally this is

$$\sigma_\alpha(G, U) = \begin{cases} \alpha & \text{if } g(U) > \alpha \\ \min(g(U), \gamma) & \text{otherwise} \end{cases}$$

and

$$\sigma_\alpha(B, U) = \begin{cases} \alpha & \text{if } b(U) > \alpha \\ \min(b(U), \gamma) & \text{otherwise} \end{cases}$$

**Theorem 4.1** The pair of strategies $(\sigma_\alpha, \tau_\alpha)$ is a Bayesian equilibrium if and only if $\mu(\alpha) \leq \alpha \leq \max(\mu(\alpha), \bar{\varepsilon})$, where $\bar{\varepsilon} = \sup \{ c \mid p(c) < 1 \}$.

**Proof** It is clear that for any $\alpha$, given that the bank is using strategy $\tau_\alpha$, an entrepreneur of type $t$ cannot benefit by deviating from $\sigma_\alpha(t)$ - he will either be rejected instead of getting a loan with positive expected utility, or get a loan with non positive expected payoff. As for the bank, since $\mu(\alpha) \leq \alpha \leq \max(\mu(\alpha), \bar{\varepsilon})$, he will forego profitable loans if he rejects $\alpha$, and will make non profitable if he accepts collateral less than $\gamma$. Thus $(\sigma_\alpha, \tau_\alpha)$ is an equilibrium for this range of $\alpha$. To see that this is also a perfect equilibrium, note that rejecting (out of equilibrium) collateral $\alpha' < \alpha$ can be part of sequential (and hence of perfect) equilibrium. In fact since $\alpha' < \bar{\varepsilon}$, there is a positive proportion of entrepreneurs with $B$-projects who could have made such an offer. Therefore, there are (out of equilibrium) beliefs which are consistent with rejecting $\alpha'$. On the other hand if the entrepreneurs use $\sigma_\alpha$ with $\alpha < \mu(\alpha)$, the bank will have negative expected utility if he accepts $\alpha$, hence he has to reject it and such $(\sigma_\alpha, \tau_\alpha)$ is not an equilibrium. For any collateral $\alpha > \max(\mu(\alpha), \bar{\varepsilon})$ the strategy $\tau_\alpha$ cannot be a perfect equilibrium strategy. To see this, note first that it is irrational for the bank to reject such an $\alpha$ since by definition of $\bar{\varepsilon}$, it could have been offered only by entrepreneurs with $G$-project and it is higher than $\gamma$ (because $\alpha > \bar{\varepsilon}$ implies $p(\alpha) = 1$ and therefore, $\alpha > \mu(\alpha) = \tilde{\mu}(1) = \gamma$.) Next, observe that if $\alpha > \max(\mu(\alpha), \bar{\varepsilon})$ then any $\alpha' < \alpha < \max(\mu(\alpha), \bar{\varepsilon})$ also satisfies $\alpha' > \max(\mu(\alpha'), \bar{\varepsilon})$ since $p(\alpha) = p(\alpha') = 1$ and hence $\mu(\alpha) = \mu(\alpha') = \tilde{\mu}(1) = \gamma$. It follows that $\tau_\alpha$ cannot be a perfect equilibrium strategy since the bank should reject any $\alpha' < \bar{\varepsilon} < \alpha$ (by the definition of $\tau_\alpha$) which is irrational as we have just shown. This concludes the proof of the theorem.

The equilibria established by Theorem 4.1 exhaust, basically, all equilibria of the game. More specifically, any equilibrium can differ from $(\sigma_\alpha, \tau_\alpha)$ only out of the equilibrium path and hence yields the same payoffs. An entrepreneur who cannot afford collateral $\alpha$ may offer something less than $\alpha$, but not necessarily $\min(b(U), \gamma)$ or $\min(g(U), \gamma)$ as specified by $\sigma_\alpha$, since his offer is rejected anyway. Of course, not any such variant of $\sigma_\alpha$ can be an equilibrium strategy.
Note that 'beliefs' of the bank about quality of the project, captured by the conditional probability \( p(c) \) of facing a \( G \)-project, need not be monotone in \( c \). That is, screening via high collateral requirements does not necessarily favor \( G \)-projects. A simple illustration can be obtained from the configuration in Case 2 of our last example if we assume equal probability of \( 1/4 \) to each of the four possible types \( \{G, B\} \times \{U_1, U_2\} \). Applying the definition of \( p(c) \) in equation 6 we obtain:

For \( c \) in the range \( 0 \leq c \leq b(U_1) \):

\[
P\{t = (G, U); g(U) > c\} = P\{(G, U_1), (G, U_2)\} = \frac{1}{2}
\]

and

\[
P\{t = (B, U); b(U) > c\} = P\{(B, U_1), (B, U_2)\} = \frac{1}{2}
\]

Therefore

\[
p(c) = \frac{1}{2} + \frac{1}{2} = \frac{1}{2}
\]

Similarly for other regions of \( c \), and the resulting \( p(c) \) is described in Figure 4.

![Figure 4: The probability \( p(c) \) for Case 2.](image)

A consequence of this lack of monotonicity of \( p(c) \) is that \( \mu(c) \) also need not be monotone and hence the region in which \( c \geq \mu(c) \) need not be connected which means that it may well be that for \( c_1 < c_2 < c_3 \) each of the strategies of the bank \( \tau_{c_1} \) and \( \tau_{c_3} \) is an equilibrium strategy while \( \tau_{c_2} \) is not.

Another new element of the present model, compared to the homogeneous case, is that the payoff of the bank at equilibrium is not merely \( p(c)EV(Y^G_a) + (1 - p(c))EV(Y^B_a) \) but rather this is multiplied by the probability of providing the loan, or the fraction of the entrepreneurs to whom he approves the loan, this is the denominator of \( p(c) \) i.e.

\[
P\{t = (G, U); g(U) > c\} + P\{t = (B, U); b(U) > c\}.
\]

This may have an effect in the variant of the model in which the bank has the possibility of moving first and committing himself to a minimum acceptable collateral. In such a situation,
the bank may be doing better with a lower threshold which will result in a larger volume of
loans. This is also relevant in our standard model (when the entrepreneurs are the first to
make offers) in considering equilibria which differ from \((\sigma_0, \tau_2)\) out of the equilibrium path.

Theorem 4.1 provides the basic tool for finding the perfect equilibria by solving for the
values of \(c\) which satisfy the condition in the Theorem. To illustrate it let us reexamine
the special case of single utility discussed in the previous section. Considering first the
configuration \(b < \gamma < \mu\), the lines \(y = \mu(c)\) and \(y = c\) are given in Figure 5.

![Figure 5: A single U and \(b \leq \gamma < \mu\).](image)

Here, \(c \geq \mu(c)\) holds for \(c \geq \gamma\). Also it is easy to verify that \(\bar{c} = b\) (we omitted the graph
of \(p(c)\) but we have seen in Figure 2 for this case, \(p(c) = 1\) for \(c \geq b\). The condition of
Theorem 4.1 is satisfied therefore only for \(c = \gamma\) providing the only perfect equilibrium and
it is separating. This is precisely the result we had in Proposition 3.2(here \(a = \gamma\).)

The case \(\gamma \leq b < \mu\) is described in Figure 6.

![Figure 6: A single U and \(\gamma \leq b < \mu\).](image)

Here \(\bar{c} = b\) and \(c \geq \mu(c)\) holds for \(c \geq b\). The condition of Theorem 4.1 is satisfied,
therefore, only for \( c = b \) providing a (unique) perfect equilibrium \((\sigma_b, \tau_b)\). This is the result stated in Proposition 3.2 (for this case \( \alpha = b \)).

For the case \( \mu \leq b \) the situation is described in Figure 7.

\[
\begin{align*}
0 & \quad \gamma \quad \mu \quad b \quad g \\
\mu & = \mu(c) \\
\text{\( y = c \)}}
\end{align*}
\]

\text{Figure 7: A single \( U \) and \( \mu \leq b \).}

In this case \( c \geq \mu(c) \) holds for \( c \geq \mu \) and again \( \bar{c} = b \). It follows by Theorem 4.1 that the perfect equilibria are \((\sigma_c, \tau_c)\) where \( c \) is any value in the range \( \mu \leq c \leq b \). These are pooling equilibria since \( b \geq c \) and \( g \geq c \) and hence both types offer \( c \) according to \( \sigma_c \). This is precisely the result stated in Proposition 3.2.

As a second illustration of Theorem 4.1 let us find now the perfect equilibria of the situation described in Case 2 in which there are two possible utility functions \( U_1 \) and \( U_2 \) such that the configuration of \( b(U_1) \) and \( g(U_1) \) is given in Figure 4 i.e.,

\[
b(U_1) < g(U_1) < b(U_2) < g(U_2)
\]

Assume that the prior probability on the type set \( \{G, B\} \times \{U_1, U_2\} \) is uniform i.e. each of the possible four types has a probability \( 1/4 \). The function \( p(c) \) is given in Figure 4. The function \( \mu(c) \) is determined by the three values \( \mu(c; p(c) = 1) = \bar{\mu}(1) = \gamma, \mu(c; p(c) = \frac{1}{2}) = \bar{\mu}(\frac{1}{2}), \) and \( \mu(c; p(c) = \frac{1}{4}) = \bar{\mu}(\frac{3}{4}) \). This function and the diagonal \( y = c \) are graphed in Figure 8.

The values of \( c \) satisfying the \( c \geq \mu(c) \) consist of two disconnected intervals:

\[
\bar{\mu}(\frac{3}{4}) \leq c \leq g(U_1) \text{ and } \bar{\mu}(\frac{1}{2}) \leq c \leq g(U_2). \]

From Figure 4 we have \( \bar{c} = b(U_2) \). By Theorem 4.1 that there are two sets of perfect equilibria:

(i) Equilibria of the form \((\sigma_c, \tau_c)\) where \( \bar{\mu}(\frac{3}{4}) \leq c \leq g(U_1) \). These equilibria separate only the more risk averse entrepreneur who holds a \( B \)-project that is, only type \( (B, U_1) \) does not get the loan.

(ii) Equilibria of the form \((\sigma_c, \tau_c)\) where \( \bar{\mu}(\frac{1}{2}) \leq c \leq b(U_2) \). These equilibria separate types according to utilities. Risk averse entrepreneur (with utility \( U_1 \)) does not get the loan no matter what project he holds. On the other hand the less risk averse entrepreneurs (with utility \( U_2 \)) will offer \( c \) and get the loan, whether they have a \( G \)-or \( B \)-project.
Figure 8: A case of two utility functions.

The rather new phenomenon is that we may have equilibria in which some $B$-projects are financed while some $G$-projects are not. Note also the right extreme point of the second set of equilibria namely $(\sigma, \tau_c)$ with $c = b(U_2)$. This is the only separating equilibrium in which only $G$-projects are financed but not all of them.

5 Credit Rationing

In this section we explore the role collateral as a tool to allocate credit when, as before, there is asymmetric information about project quality and in addition to that, credit is scarce. To capture this element we extend our analysis to the case where there are two entrepreneurs and each of them is assigned, independently, a project in a random fashion. As before, for each entrepreneur the probability of being assigned a $G$-project is $p$. The bank can approve only one loan application. Under these conditions, when offering a collateral, entrepreneurs not only signal their types but they also engage in a competition among themselves.

This is a different setting from the one considered by Stiglitz and Weiss [1981], Bester [1985] and Besanko and Thakor [1987] who derived credit rationing endogenously. We pursue the analysis in this direction since in real life credit markets often operate under conditions where not all demand can be met and the elimination of shortages via higher prices (interest rates) does not take place.

In trying to establish an equilibrium for this scenario we consider first a situation where the entrepreneurs compete among themselves given a fixed strategy of the bank. In the second stage we make the bank 'an active player' in the sense that the policy of granting loans must be a best response to the entrepreneurs' equilibrium strategies established in the first stage.

Consider the game induced by the following strategy $\tau_s$ of the bank: Provide the loan
to the entrepreneur who offers the highest collateral provided it is (strictly) higher \( \theta \), where 
\[
\max(b, \gamma) \leq \theta \leq g.
\]
In the case of a tie, the loan is allocated by a flip of a coin.

Let \( \bar{U}(c) = \mathbb{E}U(X^c) \) stand for the expected utility of an entrepreneur who holds a \( G \)-project and is going to finance it by a loan with collateral \( c \). Define \( \delta \) by:
\[
(1 - p)\bar{U}(\theta) = \bar{U}(\delta),
\]
and note that if \( \theta < g \) then \( \theta < \delta < g \). Consider the distributions \( F_\theta \), parameterized by \( \theta \), of a random variable \( C \) given by:
\[
F_\theta(c) = \Pr\{C \leq c\} = \frac{1 - p}{p} \left( \frac{\bar{U}(\theta)}{\bar{U}(c)} - 1 \right), \tag{8}
\]
for \( \theta \leq c \leq \delta \).

Under very mild assumptions on the utility \( \bar{U} \) it can be shown that \( F_\theta \) is a continuous distribution on \([0, \delta]\).

**Proposition 5.1** Let \( (F_\theta, b) \) be the strategy of an entrepreneur in which he offers a collateral \( b \) if he has a \( B \)-project and offers a randomly chosen collateral, using the distribution \( F_\theta \) if he has a \( G \)-project. Then the pair of strategies, \((\sigma, \sigma)\) is an equilibrium in the game between the entrepreneurs induced by the bank commitment to the strategy \( \tau_\theta \).

**Proof** Note first that since a-priori agents are alike, symmetric equilibria are the natural object to consider. Holding a \( G \)-project, an entrepreneur can not gain by offering collateral \( c \leq \theta \) since it will be rejected resulting in a zero payoff. Any offer higher than \( \delta \) is dominated by an offer \( \gamma \) since offering \( c > \delta \) yields at most \( \bar{U}(c) \) while offering \( \theta \) yields at least \( (1 - p)\bar{U}(\theta) = \bar{U}(\delta) \) and \( c > \delta \) implies \( \bar{U}(c) < \bar{U}(\delta) \). Hence, to prove that holding a \( G \)-project, \( F_\theta \) is the best strategy against the other entrepreneur's strategy \( \sigma \), all that has to be done is to show that payoffs generated by collateral which correspond to all points in the support of \( F_\theta \) are the same. To prove that this is the case note that the expected utility generated by an offer \( c \) is:
\[
[(1 - p) + p\Pr(C < c)]\bar{U}(c) = [(1 - p) + pF_\theta(c)]\bar{U}(c) = (1 - p)\bar{U}(\gamma) = \bar{U}(\delta).
\]

We shall prove now that the equilibria established in Proposition 5.1 are perfect equilibria in the original three player game involving the bank and the two entrepreneurs. Furthermore, these are all the perfect equilibria of this game.

**Theorem 5.2** The set of all perfect equilibria in the game is:
\[
\{(\tau_\theta, \sigma_\theta, \sigma_\theta) \mid \gamma \leq \theta \leq g \text{ and } \theta > b\}
\]

**Remark** In view of the symmetry between the two entrepreneurs, it is not surprising that there is a symmetric equilibrium (i.e. they both use the same strategy \( \sigma_\theta \)). A less obvious matter is the fact that there are only symmetric equilibria. In an unpublished paper, Maskin and Riley [1983] proved a similar result in the context of auctions.
Proof (i) We first prove that for any \( \theta \) satisfying \( \max(b, \gamma) \leq \theta \leq g \) the triple \((\tau_\theta, \sigma_\theta, \sigma_\theta)\) is a perfect equilibrium.

By Proposition 5.1, given the bank's strategy \( \tau_\theta \), the pair \((\sigma_\theta, \sigma_\theta)\) is a perfect equilibrium in the game induced by the bank and played by the entrepreneurs. This implies that \( \sigma_\theta \) is a best reply against \((\tau_\theta, \sigma_\theta)\). It remains to prove that \( \tau_\theta \) is a best reply against \((\sigma_\theta, \sigma_\theta)\). Clearly, any best reply must have to be some \( \tau_\theta \) of the above form i.e. take the higher of two acceptable offers, flip a coin if they are equal and never refuse an offer with \( c \geq \theta' \). First, a strategy \( \tau_\theta \) with \( \theta' < \theta \) is not better than \( \tau_\theta \) given that the entrepreneurs' strategies are \((\sigma_\theta, \sigma_\theta)\) (since the only difference may be financing a \( B \)-project). Next, if \( \theta' > \delta \) then \( \tau_\theta \) is strictly dominated by \( \tau_\theta \) since it yields zero. Finally note that if the bank's strategy is \( \tau_\theta \) with \( c \in [\theta, \delta] \) his expected utility is

\[
\varphi(c) = 2(1 - p) \int_\theta^\gamma v(\alpha) dF_\theta(\alpha) + 2p^2 \int_\gamma^g v(\alpha) F_\theta(\alpha) dF_\theta(\alpha),
\]

where \( v(\alpha) \) is the expected utility of the bank when he approves a loan with collateral \( \alpha \) for a \( G \)-project i.e. \( v(\alpha) = EV(Y^G_\theta) \). Clearly, \( \varphi(c) \) is a strictly decreasing function of \( c \) (except at \( c = \gamma \) where the derivative is 0 since \( v(\gamma) = 0 \)).

(ii) We now prove that any perfect equilibrium is of the form \((\tau_\theta, \sigma_\theta, \sigma_\theta)\) with \( \max(\gamma, \gamma) \leq \theta \leq g \).

Let \((\tau_\theta, \sigma_\theta, \sigma_\theta)\) be a perfect equilibrium in which \( \tau_\theta \) is the strategy of the bank and \( \sigma_i = (F_i^0, F_i^1) \) is the strategy of entrepreneur \( i \) \((i = 1, 2)\) which means: An entrepreneur offers a random collateral \( C \) chosen according to the distribution \( F_i^0 \) if he holds a \( G \)-project and \( F_i^1 \) if he holds a \( B \)-project. Denote by \((L_i, R_i)\) \((\text{resp. } (\ell_i, r_i))\) the smallest interval containing the support of \( F_i^0 \) \((\text{resp. } F_i^1)\) that is,

\[
L_i = \min\{x \mid x \in \text{Supp}(F_i^0)\}, \quad R_i = \max\{x \mid x \in \text{Supp}(F_i^0)\}
\]

\[
\ell_i = \min\{x \mid x \in \text{Supp}(F_i^1)\}, \quad r_i = \max\{x \mid x \in \text{Supp}(F_i^1)\}.
\]

The proof is implied by the following sequence of statements about the property of the equilibrium:

- The strategy \( \tau_\theta \) of the bank must satisfy \( \theta \geq \gamma \) since otherwise it is (at least weakly) dominated by \( \tau_\gamma \).
- For \( \sigma_i \) to be a best reply to \((\tau_\theta, \sigma_j)\), \( i \neq j \) we have \( L_i \geq \min\{\ell_j, L_j\} \) since otherwise player \( i \) who holds a \( G \)-project can strictly increase his payoff by moving the probability of the interval \([L_i, \min\{\ell_j, L_j\}]\) (where he is certain not to get the loan) to \( b + \varepsilon \) (where he has at least a probability \( p \) of getting the loan, for arbitrarily small \( \varepsilon \)).
- It must be that \( \ell_1 = \ell_2 \) since if, say, \( \ell_1 < \ell_2 \) then entrepreneur 1 who holds a \( B \)-project can strictly increase his payoff by moving the probability of the interval \([\ell_1, \ell_2]\) (where he is certain not to get the loan) to \( b \) (where he has at least a probability \( p/2 \) of getting the loan.)
- Note that \( \ell_1 = \ell_2 \) implies that the expected payoff of an entrepreneur who holds a \( B \)-project and offers \( \ell_i \) is zero, since with probability 1 the other entrepreneur will offer higher collateral. Hence, it follows that the expected payoff is zero at all points in the support of \( F_i^1 \). Therefore, \( \ell_i = r_i = b \), for \( i = 1, 2 \).

23
• Given that entrepreneurs who hold $B$-projects offer $c = b$ it is clear that $\theta$ must also satisfy $\theta > b$ to prevent the bank from financing a $B$-project.

• It must be that $L_1 = L_2 = L$ and $R_1 = R_2 = R$ since if, say, $L_1 < L_2$ then entrepreneur 1 who holds a $C$-project can strictly increase his payoff by moving the probability of the interval $[L_1, L_2]$ (where he is certain not to get the loan) to $\theta$ (where he has at least a probability 1/2 of getting the loan.) The argument for $R_1 = R_2$ is similar.

• As we argued in the proof of Proposition 5.1, $L$ and $R$ must satisfy $\theta \leq L \leq R \leq \delta$ (since any collateral smaller than $\theta$ will be rejected and any collateral larger than $\delta$ is dominated by a collateral $\theta + \epsilon$ for sufficiently small $\epsilon$, which will be accepted with probability $(1 - p)$.)

• Collateral $L$ will be accepted with probability $(1 - p)$ (i.e. the probability that the other entrepreneur holds a $B$-project). With this same probability, collateral $\theta + \epsilon$ will be accepted. It follows that

$$(1 - p)\bar{U}(L) \geq (1 - p)\bar{U}(\theta + \epsilon)$$

must hold for all $\epsilon > 0$, hence $\bar{U}(L) \geq \bar{U}(\theta)$ and consequently $L \leq \theta$ (since $\bar{U}(\epsilon)$ decreases in $\epsilon$).

• The expected payoff of collateral $R$ is $\bar{U}(R)$ must be to the expected payoff for any collateral in the support of $F^b_\delta$, in particular to that of $L = \theta$. This implies,

$$\bar{U}(R) = (1 - p)\bar{U}(\theta) = \bar{U}(\delta),$$

and hence $R = \delta$.

We conclude that both $F^b_\delta$ and $F^c_\delta$ have the same support which is included in $[\theta, \delta]$.

• We claim now that the support of $F^b_\delta$ is the whole interval $[\theta, \delta]$. Assume to the contrary that, say, the support of $F^b_\delta$ is not the whole interval. Thus there are $c_1 < c_2$ in the support such that the probability of the open interval $(c_1, c_2)$ is zero. This implies that $F^b_\delta$ must also assign zero probability to this open interval (otherwise entrepreneur 2 could increase his payoff by moving the probability of this interval to $c_2$.) Thus for entrepreneur 1, both $c_1$ and $c_2$ will have the same probability, say $q$, to be accepted and hence they yield expected payoffs $q\bar{U}(c_1)$ and $q\bar{U}(c_2)$ respectively. Noting that $q > 0$ (since $q \geq 1 - p > 0$), this contradicts the fact that $c_1$ and $c_2$ must yield the same payoff being both in the support of the equilibrium strategy.

• Finally, writing the condition that when $F^b_\delta$ and $F^c_\delta$ are used by the two entrepreneurs when they hold a $C$-project, each must have the same expected payoff for all collateral in the support of his mixed strategy namely, in $[\theta, \delta]$, we have:

$$[(1 - p) + pF^b_\delta(c)] = (1 - p)\bar{U}(\gamma),$$

for $i = 1, 2$ and $c \in [\theta, \delta]$. This implies $F^b_\delta = F^c_\delta = F_\delta$, concluding the proof of the Theorem.
An interesting consequence of Theorem 5.2 is that in the above considered environment of credit rationing there are no pooling equilibria; the $B$-projects are always identified and rejected even in the case $b > \mu$ where we found pooling equilibria when no rationing takes place. The reason can be figured out from the proof of the theorem: The competition between entrepreneurs who hold $B$-projects drive the collateral to the maximum they can offer namely $b$ and by that, a collateral $b$ becomes a signal for the bank that the project is bad. To eliminate bad projects the bank sets his minimum collateral requirement higher than $b$.

For the entrepreneurs who hold $G$-projects, the Theorem establishes a family of separating equilibria ranked by $\theta$. The higher $\theta$ is, the more demanding are the collateral requirement to be met if a loan for a $G$-project is to be approved. As $\theta \to g$ the equilibria converge to an equilibrium in which the bank requires (and receives) the maximum collateral entrepreneurs who hold $G$-projects are willing to offer. In such an equilibrium there is no rent to private information since it is eliminated by the competition between the entrepreneurs which is induced by the shortage of credit.

The multiplicity of equilibria cannot be refined by using standard refinement concepts. In particular, the bank's most preferred equilibrium (i.e. $\theta$ close to $g$) cannot be 'selected'. The only way the bank can 'choose' this equilibrium is by credibly committing himself to the strategy $\tau_{2-\varepsilon}$ if he has such a commitment power.
References


