INSPECTION GAMES

by

RUDOLF AVENHAUS
BERNHARD VON STENGEL
and
SHMUEL ZAMIR

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CENTER FOR RATIONALITY
AND INTERACTIVE DECISION THEORY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-584135, [972]-2-584136
E-MAIL: ratio@sunrise.huji.ac.il
FAX: [972]-2-513681
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Rudolf Avenhaus*  Bernhard von Stengel*
i81avenh@rz.unibw-muenchen.de  i81bbvs@rz.unibw-muenchen.de

Shmuel Zamir†
zamir@shum.cc.huji.ac.il

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*Informatik 5, University of the Federal Armed Forces at Munich, 85577 Neubiberg, Germany.
†Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, Givat Ram Campus, 91904 Jerusalem, Israel.
Abstract

Starting with the analysis of arms control and disarmament problems in the sixties, inspection games have evolved into a special area of game theory with specific theoretical aspects, and, equally important, practical applications in various fields of human activities where inspection is mandatory.

In this contribution, a survey of applications is given first. Then, the general problem of inspection is presented in a game theoretic framework as an extension of a statistical hypothesis testing problem. Using this framework, two important models are solved: material accountancy and data verification. A second important aspect of inspection games are limited inspection resources that have to be used strategically. This is presented in the context of sequential inspection games, where many mathematically challenging models have been studied. Finally, the important concept of leadership, where the inspector becomes a leader by announcing and committing himself to his strategy, is shown to apply naturally to inspection games.
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1. Introduction

Inspection games form an applied field of game theory. An inspection game is a mathematical model of a situation where an inspector verifies that another party, called inspectee, adheres to certain legal rules. This legal behavior may be defined by an arms control treaty, for example, and the inspectee has a potential interest in violating these rules. Typically, the inspector's resources are limited so that verification can only be partial. A mathematical analysis should help in designing an optimal inspection scheme, where it must be assumed that an illegal action is executed strategically. This defines a game theoretic problem, usually with two players, inspector and inspectee; in some cases, several inspectees are considered as individual players.

Game theory is not only adequate to describe an inspection situation, it has also been actually applied. In the early 1960's, game theoretic studies of arms control inspections were commissioned to the Mathematica company by the United States Arms Control and Disarmament Agency (ACDA). The International Atomic Energy Agency (IAEA) performs inspections under the Nuclear Non-Proliferation Treaty. The decision rules of IAEA inspectors for detecting a deviation of nuclear material are saddle point strategies in a zero-sum game with the detection probability as payoff function. In these applications, game theory has proved itself useful as a technical tool in line with statistics and other methods of operations research. Since these models should be accessible to practitioners, the classical concepts of game theory are used, like zero-sum games or games in extensive form. Nevertheless, as we want to demonstrate, the solution of these games is mathematically challenging, and leads to interesting conceptual questions as well.

In this contribution, we will start with a survey of applications of inspection games. After that, we will present in mathematical detail a number of inspection games that are of methodological interest.

In Section 2, we survey applications of inspection games to arms control, auditing and accounting and economics, and other areas like environmental regulatory enforcement or crime control. Some of this work is scattered in technical reports and across various journals. Moreover, some interesting publications (like Diamond 1982) seem to have been largely overlooked because inspection games are not the topic of a single community of game theorists. With the present paper, we hope to contribute to a greater unity of the discipline.

In order to delimit the topic, we distinguish it from certain related areas. We will not consider inspection problems that are exclusively statistical, since our emphasis is on games. In the same way, we also exclude industrial inspections for quality control and maintenance. However, there is an interesting worst-case analysis of timely inspections in this area that we will discuss below in Section 4.2.
Furthermore, we do not consider search games modeling pursuit and evasion in war. (In this Handbook, see O'Neill 1994 for a detailed survey of applications of game theory to problems of peace and war, and Gal 1983 for a survey of search games.) Search games are distinguished from inspection games by a certain symmetry between the two players whose actions are usually equally legitimate. In contrast, inspection games are fundamentally asymmetrical, because of the standard that is set by legal behavior. To a certain extent, the inspector can define the rules of the game: for example, he may announce his optimal inspection strategy, but the inspectee may not announce an optimal violation scheme; as we will show in Section 5, this is indeed advantageous to the inspector.

In Section 3, we provide a general game theoretical framework to the inspection problem. We view this as an extension of a statistical hypothesis testing problem, when the distribution of the observed random variable is strategically manipulated by the inspectee. We show how the equilibrium of the general non-zero-sum game is found using an auxiliary zero-sum game in which the inspectee chooses a violation procedure and the inspector chooses a statistical test with a given false alarm probability. We then illustrate this by solving two specific important models, namely material accountancy and data verification.

These inspection games extend statistical methods used in practice. Other games are methodologically interesting as well. In Section 4, we present several models of sequential inspections, where the game evolves over time. There, the information of the players about the actions of their respective opponent is very important, which is best understood if the game is represented in extensive form. If the payoffs have a comparatively simple linear structure (in terms of the number of successful violations, for example), then the games can sometimes be solved analytically. These sequential games can often be represented recursively. The mathematically difficult part lies in solving certain recurrence equations explicitly. In another ‘timeliness’ game, the optimal inspection times, continuously chosen from an interval, are determined by differential equations.

In Section 5 we discuss the inspector leadership principle. This says that the inspector may commit himself to his inspection strategy in advance, and thereby gains an advantage compared to the symmetrical situation where both players choose their actions without knowledge of the other. This concept is particularly applicable to inspection games. We believe that it represents a promising direction of further research, since there are a number of related ideas in economics, like principal-agent problems, that should be compared more closely with leadership as presented here.
2. Applications

The largest group of publications on inspection games concerns arms control and disarmament, usually relating to one or the other arms control treaty that has been signed. We will survey these models first. Some of them will be described in mathematical detail in later sections of this article. Subsequent and parallel to these studies in arms control, inspection games have been applied to problems in economics, particularly in accountancy and auditing, in enforcement of environmental regulations, and in crime control and related areas. There are also some papers that treat inspection games in an abstract setting, rather than modeling a particular application. We will summarize these contributions at the end of this section.

2.1. Arms Control and Disarmament

There are roughly three phases where inspection games have been applied to arms control and disarmament. The first phase comprises the time from about 1961 to 1967. These studies analyze inspections for a nuclear test ban treaty, which was then under negotiation. The second phase, about 1988 to 1995, comprises work stimulated by the Non-Proliferation Treaty for Nuclear Weapons. Under that treaty, proper use of nuclear material is verified by the International Atomic Energy Agency (IAEA) in Vienna. For inspections of nuclear plants, very detailed and practically oriented work is published in conferences organized by the IAEA and other institutions. The third phase lasts from about 1986 until today. The end of the Cold War brought about new disarmament treaties. Verification of these treaties has also been analyzed using game theory.

In the 1960’s, a test ban treaty was under negotiation between the United States and the Soviet Union, and verification procedures were discussed. Tests of nuclear weapons above ground can be detected by satellites. Underground tests can be sensed by seismic methods, but in order to discriminate them safely from earthquakes, on-site inspections are necessary. The two sides never agreed on the number of such inspections that they would allow to the other side, so eventually they decided not to ban underground tests in the treaty.

While the test ban treaty was being discussed, the problem arose how an inspector should use a certain limited number of on-site inspections, as provided by the treaty, for verifying a much larger number of suspicious seismic events. In a report by the RAND corporation, Drescher (1982) modeled this problem as a recursive game. The estimated number of events to be inspected is a fixed parameter, and so is the number of inspections that can be used. Each event of the game represents a stage where each player has two possible actions. The inspectee may conduct a test, which is a treaty violation, or behave legally; in the first case, the inspectee in fact generates the event, in the second case the event has natural causes, but the inspector does not know. The inspector, in turn, may or may not use an inspection
for the event. For the first stage, this results in a two-by-two game with recursively defined payoffs. Its solution is defined by a recurrence equation, which Dresher solved explicitly. We will define this game in Section 4.1; it formed the basis for much subsequent work.

With political backing for scientific disarmament studies, the United States Arms Control and Disarmament Agency (ACDA) commissioned game theoretic analyses of inspections to Mathematica, Inc., in the years 1963 to 1968. Game theoretic researchers involved in this or related work were Aumann, Dresher, Kuhn, Maschler, Selten, among others. Within that group, publications on inspection games are Kuhn (1963) and, in general, the reports to ACDA edited by Anscombe et al. (1963, 1965), as well as Maschler (1966, 1967). A related study is Saaty (1968), which however is not in the line of game theoretical methodology. All these papers more or less extend Dresher's game in various ways, for example by generalizing the payoffs, or by assuming an uncertainty in the signals given by detectors. We describe some of these extensions, as well as more recent publications of that kind, in Section 4.

The Non-Proliferation Treaty (NPT) for Nuclear Weapons was inaugurated in 1968. This treaty divided the world into weapons states, who promised to reduce or even eliminate their nuclear arsenals, and non-weapon states who promised never to acquire such weapons. Nevertheless, all states which became parties to the treaty agreed that the International Atomic Energy Agency (IAEA) in Vienna verifies the nuclear material contained in the peaceful nuclear fuel cycles of all states.

The verification principle of the IAEA is material accountancy, that is, the comparison of book and physical inventories for a given material balance area at the end of an inventory period. Furthermore, the verification is organized so that the plant operators report their material balance data via their national organizations to the IAEA, and that IAEA inspectors verify these reported data with the help of independent measurements on a random sampling basis.

Two kinds of sampling procedures are considered in all situations, depending on the nature of the problem. Attribute sampling is used to test or to estimate the percentage of items in the population containing some characteristic or attribute of interest. In inspections, the attribute of interest is usually if a safeguards measure has been violated. This may be a broken seal of a container, or an unquestioned decision that a data item has been falsified. The inspector uses the rate of tampered items in the sample to estimate the population rate or to test a hypothesis.

The second kind of procedure is variable sampling. This is designed to provide an estimate of or a test on an average or total value of material. Each observation, instead of being counted as falling in a given category, provides a value which is totaled or averaged for the sample. This is described by a certain test statistic, like the total of Material Unaccounted For (MUF). Based on this statistic, the
inspector has to decide if nuclear material has been diverted or if the result is due to measurement errors. This decision depends on the probability of a false alarm chosen by the inspector.

Game theoretic work in this area was started by Bierlein (1968, 1969), who emphasized that payoffs should be expressed by detection probabilities only. In contrast, inspection costs are parameters that are fixed externally. This is an adequate model of the IAEA which has a certain budget limiting its inspection effort; the agency has no intent to minimize that effort further, but instead wants to use it most efficiently.

Since 1969, international conferences on nuclear material safeguards are regularly held by the following institutions. The IAEA organizes about every five years conferences on Nuclear Safeguards Technology and publishes proceedings under that title. The European Safeguards Research and Development Association (ESARDA) as well as the Institute for Nuclear Material Management (INMM) in the United States meet every year and publish proceedings on that subject as well.

The bulk of work published in these proceedings is concerned with practical matters, for example measurement technology, data processing, and safety. However, decision theoretical approaches, including game theoretic methods, were presented throughout the years. Monographs which emphasize the theoretical aspects are Jaech (1973), Avenhaus (1986), and Bowen and Bennett (1988).

Concluding our summary on nuclear material safeguards, we should also mention studies that are not related to the NPT. The U.S. Nuclear Regulatory Commission (NUREG) is in charge of safeguarding nuclear plants against theft and sabotage to guarantee their safe operation, not in fulfillment of an international treaty. In a study for that commission, Goldman (1984) investigates the possible use of game theory (under the name of ‘strategic analysis’) and its potential role in safeguards. This work presents a large number of references. It also discusses basic issues of the use of game theory in inspections, like its understandability and possible difficulties in defining payoffs or using mixed strategies, which are very relevant in practice.

In the mid-eighties, when the Cold War ended, new disarmament treaties were signed, like the treaty on Intermediate Nuclear Forces in 1987, or the treaty on Conventional Forces in Europe in 1990 (see Altmann et al. 1992 on verification issues). The verification of these new treaties was investigated in game theoretic terms by Brams and Davis (1987), by Brams and Kigour (1988), and by Kilgour (1992). Variants of recursive games have been described by Rucke (1992), who evaluated the resulting recurrence equations for the game value numerically and discussed these values for some parameter constellations. In part, these papers extend the work done before, in particular Dresher (1962).
2.2. Accounting and Auditing in Economics

In economic theory, inspection games have been studied for the auditing of accounts. In insurance, inspections are used against the 'moral hazard' that the client may abuse his insurance by fraud or negligence. Our summary of these topics is based on Borch (1990). We will also consider models of tax inspections.

Accounting is usually understood as a system for keeping track of the circulation of money. The monetary transactions are recorded in accounts. These may be checked in full detail by an inspector, which is called an audit. Auditing of accounts is often based on sampling inspection, simply because it is unnecessarily costly to check every voucher and entry in the books. The possibility that any transaction may be checked in full detail is believed to have a deterring effect likely to prevent irregularities.

A theoretical analysis of problems in this field naturally leads to a search for suitable sampling methods; for an outline see Kaplan (1973). The concepts of attribute and variable sampling described above for material accountancy apply to the inspection of monetary accounts as well. In particular, there are tests to validate the reasonableness of account balances, without classifying a particular observation as correct or falsified. These may be considered as variable measurement tests and are sometimes termed 'dollar value' samples in the literature.

These theoretical investigations include the use of game theoretic methods. An early contribution that employs both noncooperative and cooperative game theory is given by Klages (1968), who describes quite detailed models of the practical problems in accountancy, and discusses the merits of a game theoretic approach.

Borch (1982) formulates a zero-sum inspection game between an accountant and his employer. The accountant, how is the inspector, may either record transactions faithfully or cheat to embezzle some of the company's profits, and the employer may either trust the accountant or audit his accounts. If the inspector is honest, he receives payoff zero irrespective of the actions of the inspector. In that case, the inspector gets payoff zero if he trusts and has a certain cost if he audits. If the accountant steals while being trusted, he has a gain that is the employer's loss. If the inspector catches an illegal action, he has the same auditing cost as before but the inspector must pay a penalty. Borch interprets the unique mixed strategies as 'fractions of opportunities' in a repeated situation, but does not formalize this further.

The employer may buy 'fidelity guarantee insurance' to cover losses caused by dishonest accountants. The insurance may require a strict auditing system that is costly to the employer. Borch (1982) considers a three-person game where the insurance company inspects (or trusts) the employer if he audits properly. The employer is both inspectee, of the insurance company, and inspector, of his accountant. No interaction is assumed between insurance company and accountant, so this game is
fully described by two bimatrix games; for general ‘polymatrix’ games of this kind see Howson (1972).

Borch (1990) sees many potential applications of games to the economics of insurance: In any situation where the insured has undertaken to take special measures to prevent accidents and reduce losses, there is a risk - usually called moral hazard - that he may neglect his obligations. The insurance company reserves the right to inspect that the insured really carries out the safety measures foreseen in the insurance contract. This inspection costs money, which the insured must pay for by an addition to the premium.

Moral hazard has its price. Therefore, inspections of economic transactions raise the question of the most efficient design of such a system, for example so that surveillance is minimized or even unnecessary. These problems are closely related to a variety of economic models known as principal-agent problems. They have been extensively studied in the economic literature, where we refer to the surveys by Baiman (1982), Kanodia (1985), and Dye (1986). Agency theory focuses on the optimal contractual relationships between two individuals whose roles are asymmetric. One, the principal (or employer) delegates work or responsibility to the other, the agent (or employee). The principal chooses, based on his own interest, the payment schedule that best exploits the agent’s self-interested behavior. The agent chooses an optimal level of action contingent on the fee schedule proposed by the principal. One important issue in agency theory is the asymmetry of information available to the principal and the agent. In inspection games, a similar asymmetry exists with respect to defining the rules of the game, see Section 5.

We conclude this section with applications to tax inspections. Schleicher (1971) describes an interesting recursive game for detecting tax law violations, which extends Maschler (1966).

Rubinstein (1979) analyzes the problem that it may be unjust to penalize illegal actions too hard since the inspectee might have committed them unintentionally. For example, if a tax authority discovers that a taxpayer has failed to report a certain part of his income, he may have done this intentionally or by an innocent oversight. The innocently generated illegal act has a certain probability \( \alpha \). In a one-shot game, there is no alternative to the inspector but to use a high penalty, although its potential injustice has a disutility. Rubinstein shows that if this game is repeated, a more lenient policy also induces the inspectee to legal behavior. Namely, he is only punished after an apparent illegal act (intentional or not) if he has done so in the past more than a certain fraction of times beyond the assumed ‘error rate’ \( \alpha \). That fraction converges to zero with the number of rounds, so that the inspectee has to abstain from repeated violations.

Another model of tax inspections, also using repeated games, has been proposed by Greenberg (1984). In that model, a tax function is assumed to be given
defining the tax to be paid by an individual with a certain income. Furthermore, a
given penalty function defines a penalty to an audited individual who did not report
its income properly, and there is a limited percentage of individuals that the tax au-
thorities can audit. Under reasonably weak assumptions about these functions and
the individuals' utility functions on income, Greenberg proposes an auditing scheme
that achieves an arbitrary small percentage of tax evaders. In that scheme, the
individuals are partitioned into three groups that are audited with different proba-
bilities, and individuals are moved among these groups after an audit depending on
whether they cheated or not. A similar scheme of auditing individuals with different
probabilities depending on their compliance history is proposed by Landsberger and
Meilijson (1982). However, their analysis does not use game theory explicitly.

Reinganum and Wilde (1986) describe a model of tax compliance where they
apply the sequential equilibrium concept. In that model, the income reporting
process is considered explicitly as a signalling round. The tax inspector is only
aware of the overall income distribution and reacts to the reported income. He
cannot precommit to a strategy. The analysis shows that in equilibrium, efforts at
verification are lower the greater the reported income is, since such a reported income
is more likely to be truthful. If the tax inspector has additional information about
the inspeclee's likely true income, like if he knows the profession of the individual,
which determines an 'audit class', then this rule applies only within the particular
audit class, not across classes.

2.3. Environmental Control

Obviously, environmental control problems call for a game theoretical treatment: On
one hand, there is a firm which produces some pollution of air, water or ground, and
which can save abatement costs by illegal emission, that is, emission beyond some
agreed level. On the other hand, there is a monitoring agent whose responsibility is
to detect, or better, to prevent such illegal pollution. Both agents are assumed to
act strategically.

Indeed, various analyses for problems of this kind have been performed that
will be discussed subsequently. However, contrary to arms control and disarmament,
they have not yet addressed concrete cases taken from the practical domain.

First, we mention several papers which present game theoretic analyses of pol-
lution problems but deal only marginally with monitoring problems. Bird and Ko-
rtanek (1974) explore various concepts in order to aid the formulation of regula-
tions of sources of pollutant in the atmosphere, related to given least cost solutions.
Höpfinger (1979) models the problem of how to determine and adapt global emission
standards for carbon dioxide as an infinite stage game with three players: regula-
tor, producer, and population. Kilgour, Okada, and Nishikori (1988) describe the
load control system for regulating chemical oxygen demand in water bodies; they
formulate a cost sharing game and solve it in some illustrative cases.
In the last years, pollution control problems have been analyzed with game theoretic methods. Russell (1990) characterizes current enforcement of U.S. environmental laws as very likely inadequate, while admitting that proving this proposition would be extremely difficult, exactly because there is so little information about the actual behavior of regulated firms and government activities. One path to improvement is explored with the help of a one-stage game between a polluter and an environmental protection agency. This game forms the benchmark for discussing a multiple-stage game in which the source's past record of discovered violations determines its future probabilities of being monitored. It is shown that this approach can yield significant savings in limiting the extent of violations to a particular frequency in the population of polluters.

Weissing and Ostrom (1991) examine how irrigation institutions affect equilibrium rates of stealing and enforcement. It is assumed that irrigators come periodically into the position of a turntaker. The latter may choose between taking a legal amount of water, and taking more water than authorized (stealing). The other irrigators are turnwaiters who must decide whether to expand resources for monitoring the behavior of the turntaker or not. There is no combination of parameters where the rate of stealing by the turntaker drops to zero. In other words, in equilibrium some stealing is always going on.

Gäth and Pethig (1992) consider a polluting firm that can save abatement costs by illegal waste emission, and a monitoring agent (controller) whose job it is to prevent such pollution. When deciding on whether to dispose of its waste legally or illegally the firm does not know for sure whether the controller is sufficiently qualified or motivated to detect the firm's illegal releases of pollutant. The firm has the option of undertaking a small-scale (deliberate) 'exploratory pollution accident' to get a hint about the controller's qualification before deciding on how to dispose of its waste. The controller may or may not respond to that 'accident' by a thorough investigation, thus perhaps revealing his type to the firm. This sequential decision process along with the asymmetric distribution of information constitutes a signaling game whose equilibrium points may (but do not have to) signal the type of the controller to the firm.

Avonhaus (1995) considers a decision theoretic problem. The management of an industrial plant may be authorized to release some amount of pollutant per unit time into the environment. An environmental agency may decide with the help of randomly sampled measurements whether or not the real releases are larger than the permitted ones. The 'best' inspection procedure can be determined as follows: For a given value of the false alarm probability, only a zero-sum game has to be considered where the probability of detecting illegal behavior is the payoff to the inspector. The solution of this game is then determined by the use of the Neyman Pearson Lemma; see Section 3 below.
2.4. Miscellaneous Models

In this subsection, we survey a number of papers that do not belong to the above categories. Some of these deal with smuggling or crime control, other papers treat inspection games in an abstract setting.

Thomas and Nisgav (1976) consider a game where a smuggler tries to cross a strait in one of \( M \) nights. The inspecting border police has a speedboat for patrolling in \( k \) of these nights. In a patrolled night, the smuggler has some risk of being caught. The game is described recursively in the same way as the game by Dresher (1962) discussed in Section 4.1 below. The only difference is that the smuggler must traverse the strait even if there will a patrol at every night, or otherwise receive the same worst payoff as if he is caught. The resulting recurrence equation for the game has a very simple solution where the game value is a linear function of \( k/M \).

Baston and Bostock (1991) generalize the paper by Thomas and Nisgav (1976). They clarify some implicit assumptions made by these authors, in particular the full information of the players about past events, and the detection probability associated with a night patrol. Then, they study the case of two boats and derive explicit solutions that depend on the detection probabilities if one boat or both are on patrol. In their ‘generalized inspection game’, the smuggler may choose not to violate at all. This defines the game and solution by Dresher (1962), a paper that Baston and Bostock are unaware of.

Goldman and Pearl (1976) study inspections in an abstract setting, where the inspector has to select among several sites where an inspectee can cheat, and only a limited number of sites can be inspected in total. A simple model is successively refined to study these questions and the effect of penalty levels and inspection resources.

Diamond (1982) describes a timeliness game and solves it by differential equations; we cover this model in detail in Section 4.2 below.

Feichtinger (1983) considers a differential game with a suggested application to crime control. The dynamic control variables of police and thief are the ‘rate of law enforcement’ and the ‘pilfering rate’, respectively.

Filar (1985) applies the theory of stochastic games to a generic ‘traveling inspector model’. There are a number of sites, each with an inspectee that acts as an individual player. Each site is a state of the stochastic game. That state is deterministically controlled by the inspector who chooses the site to be inspected at the next time period. The inspector has (unspecified) costs associated with inspection levels, travel, and if he fails to detect a violation. The players’ payoffs are either sums up to a finite stage, or limiting averages. In this model, all inspectees can be aggregated into a single player without changing equilibria. For finite horizon payoffs, the game has equilibria in Markov strategies, which depend on the time and
the state but not on the history of the game. For limiting average payoffs, stationary strategies suffice, which only depend on the state.

3. Extending Statistical Decision Theory

Many practical inspection problems are based on random sampling procedures. Furthermore, measurement techniques are often used which inevitably produce random and systematic errors. Then, it is appropriate to think of the inspection problem as being an extension of a statistical decision problem. The extension is game-theoretic since the inspector has to decide whether the inspectee has behaved illegally, and that action is strategic and not random.

In this section, we first present a general framework were we extend a classical statistical testing problem to an inspection game. The ‘illegal part’ of that game, where the inspectee has decided to violate, is equivalent to a two-person zero-sum game with the non-detection probability as payoff to the violator. If that game has a value, then the Neyman Pearson Lemma can be used to determine an optimal inspection strategy. In this framework, we then discuss two important inspection models that have emerged from statistics: material accountancy and data verification.

3.1. General Game and Analysis

In a classical hypothesis testing problem, the statistician has to decide, based on an observation of a random variable, between two alternatives ($H_0$ or $H_1$) regarding the distribution of the random variable. To make this an inspection game, assume that the distribution of the random variable is strategically controlled by another ‘player’ called the inspectee. More specifically, the inspectee can behave either legally or illegally. If he behaves legally, the distribution is according to the null hypothesis $H_0$. If he chooses to act illegally, he also decides on a violation procedure which we denote by $\omega$. Thus, the distribution of the random variable $Z$ under the alternative hypothesis $H_1$ depends on the procedure $\omega$ which is also a strategic variable of the inspectee. The statistician (who, for obvious reasons, will be called the inspector), has to decide, based on the observation $z$ of the random variable $Z$, between two actions: calling an alarm (rejecting $H_0$) or no alarm (accepting $H_0$). Note that the random variable $Z$ can well be a vector, for instance in a multi-stage inspection. This, rather general, inspection game is described in Figure 3.1.

The pairs of payoffs to inspector and inspectee, respectively, are also shown in Figure 3.1. The status quo is legal action and no alarm, represented by the payoff 0 to both players. The payoffs for undetected illegal behavior are $-1$ for the inspector and 1 for the inspectee. In case of a detected violation, the inspector receives $-a$ and the inspectee $-b$. Finally, if a false alarm is raised, the inspector gets $-\varepsilon$ and
Figure 3.1. Inspection game extending the statistical decision problem of the inspector, who has to decide between the null hypothesis $H_0$ and alternative hypothesis $H_1$ about the distribution of the random variable $Z$. $H_0$ means legal action and $H_1$ means illegal action of the inspectee with violation procedure $\omega$. The inspectee is informed about the observation $z$ of $Z$, but not if $H_0$ or $H_1$ is true. There is a separate information set for each $z$. The inspector receives the top payoffs, the inspectee the bottom payoffs. The parameters $a, b, e, h$ are subject to (3.1).

the inspectee $-h$. These parameters are subject to the restrictions

$$0 < e < 1, \quad 0 < a < 1, \quad 0 < h < b,$$  \hspace{1cm} (3.1)

since the worst event for the inspector is an undetected violation, an alarm is undesirable for everyone, and the worst event for a violator is to be caught. Sometimes it is also assumed that $e < a$, which means that for the inspector a detected violation, representing a 'failure of safeguards', is worse than the inconvenience of a false alarm.

A pure strategy of the inspectee consists of a choice between legal and illegal behavior, and a violation procedure $\omega$. A mixed strategy is a probability distribution on these pure strategies. Since the game under consideration is of perfect recall, such a mixed strategy is equivalent to a behavior strategy given by the probability $q$ for acting illegally and a probability distribution on violation procedures $\omega$ given that the inspectee acts illegally. Since the set $\Omega$ of violation procedures may be
infinite, we assume that it includes, if necessary, randomized violations as well, which are therefore also denoted by \( \omega \). That is, a behavior strategy of the inspectee is represented by a pair \((q, \omega)\).

A pure strategy of the inspector is an alarm set, that is, a subset of the range of \( X \), with the interpretation that the inspector calls an alarm if and only if the observation \( x \) is in that set. A mixed strategy of the inspector is a probability distribution on pure strategies. A strategy of the inspector (pure or mixed) is also called a statistical test and will be denoted by \( \delta \). The alarm set or sets used in such a test are usually determined by first considering the error probabilities of falsely rejecting or accepting the null hypothesis, as follows.

A statistical test \( \delta \) and a violation strategy \( \omega \) determine two conditional probabilities. The probability of an error of the first kind, that is, of a false alarm, is the probability \( \alpha(\delta) \) of an alarm given that the inspectee acts legally, which is independent of \( \omega \). The probability of an error of the second kind, that is, of non-detection, is the probability \( \beta(\delta, \omega) \) that no alarm is raised given that the inspectee acts illegally.

The inspection game has then the following normal form. The set of strategies \( \delta \) of the inspector is \( \Delta \). The set of (behavior) strategies \((q, \omega)\) of the inspectee is given by \([0,1] \times \Omega\). The payoffs to inspector and inspectee are denoted \( I(\delta,(q,\omega)) \) and \( V(\delta,(q,\omega)) \), respectively, where the letter \('V'\) indicates that the inspectee may potentially (but not necessarily) violate. In terms of the payoffs in Figure 3.1, these payoff functions are

\[
\begin{align*}
I(\delta,(q,\omega)) &= (1-q)(e \alpha(\delta)) + q(-a-(1-a)\beta(\delta,\omega)), \\
V(\delta,(q,\omega)) &= (1-q)(-h \alpha(\delta)) + q(-b+(1+b)\beta(\delta,\omega)).
\end{align*}
\]

\( (3.2) \)

We are looking for an equilibrium of this noncooperative game. This is a strategy pair \((\delta^*, (q^*, \omega^*))\) so that

\[
\begin{align*}
I(\delta^*, (q^*, \omega^*)) &\geq I(\delta, (q^*, \omega^*)) \quad \text{for all } \delta \in \Delta, \\
V(\delta^*, (q^*, \omega^*)) &\geq V(\delta^*, (q, \omega)) \quad \text{for all } q \in [0,1], \quad \omega \in \Omega.
\end{align*}
\]

\( (3.3) \)

Usually, there is no equilibrium in which the inspectee acts with certainty legally \((q^* = 0)\) or illegally \((q^* = 1)\). Namely, if \( q^* = 0 \), then by \((3.2)\), the inspector would choose a test \( \delta^* \) with \( \alpha(\delta^*) = 0 \) excluding a false alarm. However, then the equilibrium condition for the inspectee in \((3.3)\) requires \(-b+(1+b)\beta(\delta^*, \omega^*) \leq 0\), which means that the non-detection probability \( \beta(\delta^*, \omega^*) \) has to be sufficiently low. This is usually not possible with a test \( \delta^* \) that has false alarm probability zero; on the contrary, such a test usually has a non-detection probability of one. Similarly, if \( q^* = 1 \), then the inspector could always raise an alarm irrespective of his observation.
to maximize his payoff. However, then the equilibrium choice of the inspectee would not be $q^* = 1$ since $h < b$ by (3.1). Thus, in equilibrium,

$$0 < q^* < 1.$$  \hfill (3.4)

According to our discussion, this also implies

$$-h \alpha(\delta^*) = -b + (1 + b) \beta(\delta^*, \omega^*).$$  \hfill (3.5)

With respect to the non-detection probability $\beta$, the payoff function in (3.2) is monotonically decreasing for the inspector and increasing for the inspectee. Thus, given any test $\delta$, the inspectee will choose a violation mechanism $\omega$ so as to maximize $\beta(\delta, \omega)$. In equilibrium, the inspectee will therefore determine $\omega^*$ (which depends on $\delta$) so that $\beta(\delta, \omega^*) = \max_{\omega \in \Omega} \beta(\delta, \omega)$. For simplicity, we assume that this maximum can be achieved.

On the side of the inspector, it is useful to choose first a fixed false alarm probability $\alpha$ and then consider only those tests that result in this error probability. Denote this set of tests $\{\delta \in \Delta \mid \alpha(\delta) = \alpha\}$ by $\Delta_\alpha$. For these tests, the inspector’s payoff in (3.2) depends only on the non-detection probability. Thus, he will choose that test $\delta^*$ in $\Delta_\alpha$ that minimizes the worst-case non-detection probability $\beta(\delta^*, \omega^*)$. It follows that in equilibrium, the non-detection probability is represented by

$$\beta(\alpha) := \min_{\delta \in \Delta_\alpha} \max_{\omega \in \Omega} \beta(\delta, \omega).$$  \hfill (3.6)

That is, the inspector’s strategy is a minmax strategy with respect to the non-detection probability.

This suggests that, given a false alarm probability $\alpha$, the equilibrium non-detection probability $\beta(\alpha)$ is the minmax value of a certain zero-sum game $G_\alpha$ which arises naturally. Namely, the set of strategies of the inspector in this game $G_\alpha$ is $\Delta_\alpha$, that is, the set of all statistical tests with false alarm probability $\alpha$. The set of strategies $\Omega$ of the inspectee consists of all violation strategies $\omega$. The payoff to the inspectee is the non-detection probability $\beta(\delta, \omega)$, which he tries to minimize and the inspector tries to maximize.

In other words, the game $G_\alpha$ captures one part of our original game, namely the situation resulting after the inspectee has decided to behave illegally and the inspector has decided to use tests with false alarm probability $\alpha$. Thus, $G_\alpha$ does not depend directly on the parameters $a, b, e, h$ of the original game, but only through the fact that these parameters determine the value of $\alpha$ in equilibrium.

In order to find an equilibrium as in (3.3), we make the following assumptions: For each false alarm probability $\alpha$, the game $G_\alpha$ has a value $\beta(\alpha)$, given by (3.6). Furthermore, the corresponding function $\beta : [0, 1] \rightarrow [0, 1]$ fulfills

$$\beta(0) = 1, \quad \beta(1) = 0, \quad \beta \text{ is convex, and continuous at } 0.$$  \hfill (3.7)
These conditions imply that \( \beta \) is continuous and monotonically decreasing. An extreme example is the function \( \beta(\alpha) = 1 - \alpha \) which applies if the inspector ignores his observation \( z \) and calls an alarm with probability \( \alpha \), independently of the data. The properties (3.7) are standard for statistical tests; in particular, \( \beta \) is convex since the inspector may randomize.

These assumptions imply the following solution of the inspection game in Figure 3.1. The game has an equilibrium \( (\delta^*, q^*, \omega^*) \) in which the false alarm probability \( \alpha^* = \alpha(\delta^*) \) is, by (3.5), the solution of

\[
-h \alpha^* = -b + (1 + b) \beta(\alpha^*). \tag{3.8}
\]

The probability \( q^* \) of illegal behavior is determined as follows. The inspector's payoff in (3.2) is a convex combination of two concave functions of \( \alpha \), one decreasing, the other increasing. For \( q = q^* \), the resulting concave function must have its maximum at \( \alpha^* \) (see also Figure 5.5 in Section 5 below). Therefore,

\[
q^* = \frac{e}{e - (1 - a)\beta'(\alpha^*)} \tag{3.9}
\]

if the derivative \( \beta'(\alpha^*) \) of the function \( \beta \) at \( \alpha^* \) exists; otherwise, \( \beta'(\alpha^*) \) represents the derivative of any tangent supporting the convex function \( \beta \) at its argument \( \alpha^* \). Finally, the strategy \( \delta^* \) and the violation procedure \( \omega^* \) are optimal strategies in the game \( G_{\alpha^*} \).

Usually, this solution is also unique. By (3.7), equation (3.8) has a unique solution \( \alpha^* \) in \( (0, 1) \). If \( \beta \) is differentiable at \( \alpha^* \), then \( q^* \) is also a unique solution to (3.9). The optimal strategies in the game \( G_{\alpha^*} \) are often unique as well; at any rate, they are equivalent since the value \( \beta(\alpha^*) \) of the game is unique.

In effect, we have decomposed the inspection game into two parts. In the 'outer' part of the game, the inspector chooses a false alarm probability \( \alpha^* \), and the inspectee selects a probability \( q^* \) of acting illegally. Both depend on the payoff parameters of the respective other player: By (3.8), the false alarm probability is chosen so that the inspectee is indifferent between legal and illegal behavior. By (3.9), the violation probability is chosen so that \( \alpha^* \) is the optimal false alarm probability of the inspector. Both \( \alpha^* \) and \( q^* \) can be regarded as 'political' components of the solution and may be subject to controversy because the payoff parameters are. However, the 'inner', illegal part of the game is just a zero-sum game \( G_{\omega^*} \) with the non-detection probability as payoff. That part and its solution \( \delta^*, \omega^* \) is usually much less controversial.

Solving the game \( G_{\alpha} \) is often the technically more challenging part. (In fact, many inspection games in the literature deal only with zero-sum models of this kind.) Operationally, the most important tool for solving the game \( G_{\alpha} \) is the Neyman
**Pearson Lemma** (see, for example, Lehmann 1959). We *assume* first that the game $G_\alpha$ has a value, which can therefore be written as

$$\beta(\alpha) = \max_{\omega \in \Omega} \min_{\delta \in \Delta_\alpha} \beta(\delta, \omega).$$

Hereby, the non-detection probability $\beta(\delta, \omega)$ for a given false alarm probability $\alpha$ is minimized, assuming that $\omega$ (and thus the alternative hypothesis) is fully known. The corresponding *best test* $\delta^*$ is described by the Neyman Pearson Lemma. This lemma provides a test procedure to decide between two *simple hypotheses*, that is, hypotheses which determine the alternative probability distributions completely. With $\delta^*$ thus found, it is often possible to describe an optimal violation procedure $\omega^*$ against this test. Then, it remains to show that these strategies form a saddle point of the game $G_\alpha$. In the following, we will demonstrate the use of the Neyman Pearson Lemma with two important applications.

### 3.2. Material Accountancy

Material accountability procedures are designed to control materials with particular properties — rare, unpleasant, dangerous, or precious. In particular, the material balance concept is fundamental to IAEA safeguards, where the inspector watches for a possible diversion of nuclear material and the inspectee is the operator of a nuclear plant. We consider the case of periodic inventory measurements; more details of this model are described in Avenhaus (1986, Section 3.3).

Consider a certain physical material balance area and a sequence of $n$ inventory periods. At the beginning of the first period, the amount $I_0$ of material under control is measured in the area. During the $i$th period, $1 \leq i \leq n$, the known net amount $S_i$ of material enters the area, and at the end of that period, the amount $I_i$ is measured in the area. The quantity

$$Z_i = I_{i-1} + S_i - I_i, \quad 1 \leq i \leq n,$$

is called the material balance test statistic for the $i$th inventory period. Under $H_0$, its expected value is zero (because of the preservation of matter),

$$E_0(Z_i) = 0, \quad 1 \leq i \leq n. \quad (3.10)$$

Under $H_1$, its expected value is $\mu_i$,

$$E_1(Z_i) = \mu_i, \quad 1 \leq i \leq n, \quad \sum_i \mu_i = 1. \quad (3.11)$$

where $\mu_i$ is the amount of material diverted in the $i$th period, and where 1 is the normalized total amount of material to be diverted throughout all periods.
Given normally distributed measurement errors, the random (column) vector \( \mathbf{Z} = (Z_1, \ldots, Z_n)^T \) is multivariate normally distributed with covariance matrix \( \Sigma \), which is the same for \( H_0 \) and \( H_1 \). The elements of this matrix are given by

\[
\text{cov}(Z_i, Z_j) = \begin{cases} 
\text{var}(Z_i) & \text{for } i = j \\
-\text{var}(I_i) & \text{for } |i - j| = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

The two hypotheses to be tested by the inspector are defined by the expectation vectors \( E_0(\mathbf{Z}) \) and \( E_1(\mathbf{Z}) \) as given by (3.10) and (3.11).

According to our general model, the "best" test procedure is defined as the equilibrium strategy of the inspector in the two-person zero-sum game \( G_\alpha = (\Delta_\alpha, \Omega, \beta) \). In this game, the inspector's set of strategies \( \Delta_\alpha \) is the set of test procedures with a fixed false alarm probability \( \alpha \). The inspectee's set of strategies \( \Omega \) is the set of all diversion vectors \( \mu = (\mu_1, \ldots, \mu_n)^T \) with fixed total diversion \( \mu \), and the non-detection probability \( \beta \) is the payoff to the inspectee. We assume – and will show later – that this game has a saddlepoint. Furthermore, this saddlepoint involves only pure strategies of the players.

For fixed diversion \( \mu \), the alarm set of the best test (the set of observations where the null hypothesis is rejected) is described by the Neyman Pearson Lemma. It is given by the set of observations \( \mathbf{z} = (z_1, \ldots, z_n)^T \)

\[
\{ \mathbf{z} \mid \frac{f_1(\mathbf{z})}{f_0(\mathbf{z})} > \lambda \},
\]

where \( f_1 \) and \( f_0 \) are the densities of the random vector \( \mathbf{Z} = (Z_1, \ldots, Z_n)^T \) under \( H_1 \) respectively \( H_0 \). These are

\[
f_1(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n \cdot |\Sigma|}} \exp\left(-\frac{1}{2} \cdot (\mathbf{z} - \mu)^T \Sigma^{-1} (\mathbf{z} - \mu)\right),
\]

and the density \( f_0 \) has the same form with \( \mu \) replaced by the zero vector. Finally, \( \lambda \) is a parameter that is determined according to the false alarm probability \( \alpha \). This alarm set is equivalent to the set

\[
\{ \mathbf{z} \mid \mu^T \Sigma^{-1} \mathbf{z} > \lambda' \}
\]

for some \( \lambda' \), which means that the test statistic is the linear expression \( \mu^T \Sigma^{-1} \mathbf{z} \).

Since \( \mathbf{Z} \) is multivariate normally distributed, a linear combination \( \mathbf{a}^T \mathbf{Z} \) of its components, for any \( n \)-vector \( \mathbf{a} \), is univariate normally distributed with expectation 0 under \( H_0 \) and \( \mathbf{a}^T \mu \) under \( H_1 \), and variance \( \mathbf{a}^T \Sigma \mathbf{a} \). Thus, the expected value of \( \mu^T \Sigma^{-1} \mathbf{Z} \) is

\[
E(\mu^T \Sigma^{-1} \mathbf{Z}) = \begin{cases} 
0 & \text{under } H_0 \\
\mu^T \Sigma^{-1} \mu & \text{under } H_1
\end{cases}
\]

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and its variance
\[ \text{var} (\mu^T \Sigma^{-1} Z) = \mu^T \Sigma^{-1} \mu. \]

Therefore, the probability of detection as a function of the false alarm probability is given by
\[ 1 - \beta = \Phi(\sqrt{\mu^T \Sigma^{-1} \mu} - q_{1-\alpha}) \]
where \( \Phi \) denotes the Gaussian or normal distribution function and \( q_{1-\alpha} \) the corresponding quantile (given by the inverse of \( \Phi \)).

The optimal strategy \( \mu \) of the inspector that minimizes the probability of detection, subject to the boundary condition \( 1^T \mu = \mu \) (with \( 1 = (1, \ldots, 1)^T \)), is
\[ \mu^* = \Sigma 1 \cdot \frac{\mu}{1^T \Sigma 1}. \]

The minimum probability of detection is given by
\[ 1 - \beta^* = \Phi(\frac{\mu}{\sqrt{1^T \Sigma 1}} - q_{1-\alpha}). \]

The optimal strategy of the inspector can be represented by the test statistic
\[ 1^T Z = \sum_{i=1}^{n} Z_i = J_0 + \sum_{i=1}^{n} S_i - J_n \]
which is the overall material balance for all periods. This means that the intermediate inventories are ignored. Note also that both players use pure strategies. In particular, the inspector does not randomize over diversion plans.

Finally, we show that these optimal strategies are indeed equilibrium strategies. This means that they fulfill the saddlepoint criterion
\[ \beta(\delta^*, \mu) \leq \beta(\delta^*, \mu^*) \leq \beta(\delta, \mu^*) \quad \text{for all } \delta \in \Delta, \mu \in \Omega. \]

Since we have constructed the Neyman Pearson test for any diversion strategy \( \mu \), the right inequality is just the Neyman Pearson Lemma applied to \( \mu^* \). Note that in that way, we did not have to consider all possible tests \( \delta \) of the inspector explicitly; even defining the set of these strategies would be complicated. The left inequality holds for all \( \mu \) as equality since the optimal test \( \delta^* \) is based on the statistics \( 1^T Z \) which is normally distributed with mean \( 1^T \mu \) and variance \( 1^T \Sigma 1 \), that is, this distribution, and hence the detection probability, does not depend on the diversion vector \( \mu \), as long as \( 1^T \mu = \mu \).

It should be mentioned that the same result is obtained if one considers the problem of subdividing a plant into several material balance areas: For the general payoff structure used so far, the equilibrium strategy of the inspector does not require subdividing the plant.
This example demonstrates the power of the Neyman Pearson Lemma of statistics in connection with the game theoretic concept of a saddlepoint. We should emphasize that, as in many game theoretic problems, finding a pair of saddlepoint strategies can be rather difficult, while it is usually easy to verify its properties.

3.3. Data Verification

As already mentioned, IAEA safeguards is based on material accountancy, which derives from the physical principle of the preservation of matter. A second, operational principle of the IAEA is data verification. The inspector has to compare the material balance data reported by the plant operators (via their national authorities) with his own findings in order to verify that these data are not falsified for the purpose of concealing diversion of nuclear material. Since both sets of data, those of the operators and of the inspector, are based on measurements, statistical errors cannot be avoided; furthermore, since one has to assume that the operators will – if at all – falsify their reported data strategically, decision problems arise which are very similar to those analyzed before. Again, the Neyman Pearson Lemma has to be applied, and again very interesting solutions, both from a theoretical and practical point of view, are obtained. We present an analytical model taken from Avenhaus (1986). It has a number of variations which have been solved in the literature or which are a topic of current research, as outlined below.

Assume that a nuclear plant is shut down and that the plant operator, the inspectee, has collected the nuclear material in \( K \) classes with \( N_i \) batches in the \( i \)-th class, \( i = 1, \ldots, K \). Typically, each class represents a particular type of measurement method. The operator has measured all contents of all batches and has reported the results to an inspector sent by the IAEA. The inspector, in turn, measures some of the material contents in order to compare them with the reported ones. His decision problem will be how to allocate his limited inspection effort. Before we can describe this, we have to make assumptions about the measurement statistics, and, as we will see, about the possible falsification of the data.

Let \( u_{ij} \) be the true value of the content of the \( j \)-th batch of the \( i \)-th class, and let \( e_{ij} \) be the operator's random measurement error. The measurement result \( x_{ij} \) can therefore be written as

\[
x_{ij} = u_{ij} + e_{ij} \quad \text{for } i = 1, \ldots, K, \quad j = 1, \ldots, N_i.
\]  

(3.13)

We assume that the random variables \( E_{ij} \) associated to the errors are independent and normally distributed with zero expectation and variances \( \sigma^2_{ij} \), that are specific to the class \( i \) and to the operator's instruments, that is,

\[
E_{ij} \sim N(0, \sigma^2_{ij}) \quad \text{for } i = 1, \ldots, K, \quad j = 1, \ldots, N_i.
\]

Under the null hypothesis \( H_0 \), no data are falsified. Then, the data reported by the operator are given by (3.13). Under the alternative hypothesis \( H_1 \), the operator is
assumed to falsify all batch data of one class by a class specific positive amount \( \mu_i \), so that each batch in that class is reported to contain \( \mu_i \) more material than it actually does. In that case, the reported data are given by

\[ x_{ij} = \nu_{ij} + \mu_i + \epsilon_{ij} \quad \text{for } i = 1, \ldots, K, \quad j = 1, \ldots, N_i. \]

Of course, this is a very restrictive assumption about the falsification that takes place. It may be justified with practical arguments, for example manipulation of the calibration of the measuring instruments. More general assumptions are very difficult to handle; we will mention approaches in that direction below.

The inspector verifies \( n_i \) randomly chosen items of the \( N_i \) batch data of the \( i \)th class with the help of own measurements. Without loss of generality, let these be the first \( n_i \) ones. Then, the inspector’s findings \( y_{ij} \) are

\[ y_{ij} = \nu_{ij} + d_{ij} \quad \text{for } i = 1, \ldots, K, \quad j = 1, \ldots, n_i. \]

The random variables \( D_{ij} \) associated to the inspector’s errors \( d_{ij} \) are independent and normally distributed with zero expectation and variances \( \sigma_i^2 \),

\[ D_{ij} \sim N(0, \sigma_i^2) \quad \text{for } i = 1, \ldots, K, \quad j = 1, \ldots, n_i. \]

The inspector is not interested in estimating the true values \( \nu_{ij} \) but only in verifying the reported data. Therefore, he uses the differences

\[ z_{ij} = x_{ij} - y_{ij} \quad \text{for } i = 1, \ldots, K, \quad j = 1, \ldots, n_i \]

for constructing his verification test. According to our assumptions, we have

\[ Z_{ij} \sim N(\mu_{it}, \sigma_i^2) \quad \text{for } i = 1, \ldots, K, \quad j = 1, \ldots, n_i, \quad t = 0, 1, \]

where \( \sigma_i^2 = \sigma_{0i}^2 + \sigma_{1i}^2 \), and where

\[ \mu_{it} = \begin{cases} 0 & \text{for } t = 0 \ (H_0), \\ \mu_i & \text{for } t = 1 \ (H_1). \end{cases} \]

The inspector uses the Neyman Pearson test. As above in (3.12), its alarm set is with \( z = (z_{11}, \ldots, z_{KnK}) \) given by

\[ \{ z \mid \frac{f_1(z)}{f_0(z)} > \lambda \}, \]

where for \( t = 0, 1 \)

\[ f_t(z) = \left( \frac{1}{\sqrt{2\pi\sigma_i^2}} \right)^n_t \cdot \exp \left( -\frac{1}{2} \sum_{i=1}^K \sum_{j=1}^{n_i} \frac{(x_{ij} - \mu_{it})^2}{\sigma_i^2} \right). \]
Therefore, the alarm set is explicitly given by

\[ \{ z \mid \sum_{i=1}^{K} \frac{\mu_i}{\sigma_i^2} \sum_{j=1}^{n_i} z_{ij} > \lambda' \}. \]

Now we have under \( H_i, t = 0, 1, \)

\[ \sum_{i=1}^{K} \frac{\mu_i}{\sigma_i^2} \sum_{j=1}^{n_i} z_{ij} \sim N\left( \sum_{i=1}^{K} n_i \frac{\mu_i^2}{\sigma_i^2}, \sum_{i=1}^{K} n_i \frac{\mu_i^2}{\sigma_i^2} \right). \]

Thus, the probabilities of error of the first and second kind are given by

\[ \alpha = 1 - \Phi\left( \frac{\lambda'}{\sqrt{\sum_{i=1}^{K} n_i \frac{\mu_i^2}{\sigma_i^2}}} \right) \quad \text{and} \quad \beta = \Phi\left( \frac{\lambda' - \sum_{i=1}^{K} n_i \frac{\mu_i^2}{\sigma_i^2}}{\sqrt{\sum_{i=1}^{K} n_i \frac{\mu_i^2}{\sigma_i^2}}} \right). \]

If we eliminate \( \lambda' \) with the help of \( \alpha, \) then we get the non-detection probability

\[ \beta = 1 - \Phi\left( \sqrt{\sum_{i=1}^{K} n_i \frac{\mu_i^2}{\sigma_i^2} - q_{1-\alpha}} \right). \quad (3.14) \]

In order to describe the strategy sets of the players, let us assume that the operator wants to falsify all data by the fixed total amount \( \mu. \) Furthermore, let one measurement by the inspector in the \( i \)th class, \( i = 1, \ldots, K \) cost the effort \( \varepsilon_i, \) where the total effort available, measured in monetary terms or inspection hours, is \( \varepsilon. \) This means, according to the result in Section 3.1, that we consider a zero-sum game \( G_\varepsilon \) where the sets of strategies for inspector and inspectee are, respectively,

\[ \{ \mathbf{n} = (n_1, \ldots, n_K) \mid \sum_{i=1}^{K} \varepsilon_i n_i = \varepsilon \} \quad \text{and} \quad \{ \mathbf{\mu} = (\mu_1, \ldots, \mu_K) \mid \sum_{i=1}^{K} N_i \mu_i = \mu \} \]

and where the payoff to the inspectee is \( \beta(\mathbf{n}, \mathbf{\mu}) \) according to (3.14).

For simplicity, we consider \( \mathbf{n} \) as a vector of continuous variables. Then this game has a saddle point \( \mathbf{n}^*, \mathbf{\mu}^*, \) that is, for all \( \mathbf{n}, \mathbf{\mu} \) in the strategy sets we have

\[ \beta(\mathbf{n}^*, \mathbf{\mu}) \leq \beta(\mathbf{n}^*, \mathbf{\mu}^*) \leq \beta(\mathbf{n}, \mathbf{\mu}^*). \quad (3.15) \]

This saddle point is given by

\[ n_i^* = \frac{\varepsilon_i}{\sum_{j=1}^{K} N_j \sigma_j \sqrt{\varepsilon_j}} \cdot \frac{N_i \sigma_i}{\sqrt{\varepsilon_i}} \quad \text{for } i = 1, \ldots, K, \]

\[ \mu_i^* = \frac{\mu}{\sum_{j=1}^{K} N_j \sigma_j \sqrt{\varepsilon_j}} \cdot \sigma_i \sqrt{\varepsilon_i} \quad \text{for } i = 1, \ldots, K, \quad (3.16) \]

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and the value of the game is

$$
\beta(n^*, \mu^*) = 1 - \Phi \left( \frac{\mu \sqrt{E}}{\sum_{j=1}^{K} N_j \sigma_j \sqrt{\sigma_j^2}} - n - \alpha \right).
$$

The proof is easy, observing that (3.15) is equivalent to

$$
\sum_{i=1}^{K} n_i^* \frac{\mu_i^2}{\sigma_i^2} \geq \sum_{i=1}^{K} n_i^* \frac{\mu_i^2}{\sigma_i^2} \geq \sum_{i=1}^{K} n_i \frac{\mu_i^2}{\sigma_i^2} \quad \text{for all } n_i, \mu_i.
$$

Here, the right inequality is identically fulfilled. Since \( \sum_{i=1}^{K} N_i \mu_i = \mu \), the left inequality is just a special case of the Cauchy-Schwarz inequality \((\sum a_i^2)(\sum b_i^2) \geq (\sum a_i b_i)^2\) with \( \sigma_i^2 = N_i \mu_i^2 / \sigma_i \sqrt{\sigma_i^2} \) and \( b_i^2 = N_i \sigma_i \sqrt{\sigma_i^2} \).

It should be mentioned that the sampling design defined by (3.16) is known in the statistical literature as Neyman-Chuprov sampling (see, for example, Cochran 1963). The above model provides a game theoretic justification for this sampling design.

This data verification problem has a lot of ramifications if one assumes more general falsification strategies. It may be unrealistic to assume that all batches in class \( \hat{i} \) are falsified by the same amount \( \mu_i \). In particular, this is the case for a high total diversion \( \mu \) that can no longer be hidden in measurement errors for the individual batches. In that case, it turns out that the inspector will concentrate his falsification on as few batches as possible, in the hope that these will not be measured by the inspector. A theoretical analysis of the general case is very difficult. Some approaches in this direction have been made for unstratified problems, that is, only a single class of batches; see Avenhaus, Battenberg and Falkowski (1991), Mitzrotsky (1993), and Avenhaus and Piehlmeier (1994).
4. Sequential Inspection Games

The topic of this section are inspection games that evolve over a number of stages, where the inspector may use a limited number of inspections. His problem is to distribute his limited inspection effort optimally. Bounded inspection resources were already an aspect of the data verification model described in Section 3.3 above. In the following, no statistical errors are assumed, although they could be introduced in more refined models. The sequential inspection games presented here vary with respect to the information of the players about the course of the game, the rules for detecting a violation, and the resulting payoffs.

These games are complex enough to be mathematically challenging, yet they are still quite canonical. They are easily specified, and, pleasantly, also permit comparatively simple explicit solutions. These solutions are obtained by elementary methods as well as using involved generating functions for recurrence relations and differential equations. One gets a lot from a little. This is the appeal of these games to the mathematician, beyond their applications.

4.1. Recursive Inspection Games

We consider sequential games that are defined by a finite number of inspection stages. The inspector has to distribute a certain number of inspections over these stages, and the inspector can decide whether to violate at each stage or not. These games can be defined recursively and their solution is given by a recurrence equation, which in some cases can be solved explicitly. The recursive description implies that each player learns about the action of the other player after each stage, which is not always justified. This is not problematic if the inspectee can violate at most once, so we consider this case first.

The most basic recursive inspection game was introduced by Dresher (1962). That game has \( n \) stages. At each stage, the inspector may or may not use an inspection, with a total of \( m \) inspections permitted for all stages. The inspectee may decide at a stage to act legally or illegally, and will not perform more than one illegal act throughout the game. Illegal action is detected if and only if there is an inspection at the same stage.

Dresher modeled this game recursively, assuming zero-sum payoffs. For the inspector, this payoff is one unit for a detected violation, zero for legal action throughout the game, and a loss of one unit for an undetected violation. The value of the game, the equilibrium payoff to the inspector, is denoted by \( I(n, m) \) for the parameters \( 0 \leq m \leq n \). For \( m = n \), the inspector will inspect at every stage and the inspectee will act legally, and similarly the decision is unique for \( m = 0 \) where the inspectee can safely violate, so that

\[
I(n, n) = 0 \quad \text{for} \quad n \geq 0 \quad \text{and} \quad I(n, 0) = -1 \quad \text{for} \quad n > 0.
\]
For $0 < m < n$, the game is represented by the recursive payoff matrix as shown in Figure 3.1. The rows denote the possible actions at the first stage for the inspector and the columns those for the inspectee. If the inspectee violates, then he is either caught, where the game terminates and the inspector receives 1, or not, after which he will act legally throughout, so that the game eventually terminates with payoff $-1$ to the inspector. After a legal action of the inspectee, the game continues as before, with $n - 1$ instead of $n$ stages and $m - 1$ or $m$ inspections left.

<table>
<thead>
<tr>
<th>Inspectee</th>
<th>legal action $I(n-1,m-1)$</th>
<th>violation $1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>inspection</td>
<td>$I(n-1,m-1)$</td>
<td>$1$</td>
</tr>
<tr>
<td>no inspection</td>
<td>$I(n-1,m)$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Figure 4.1. The Dresher game with at most one intended violation for $n$ periods and $m$ inspections with value $I(n,m)$. The entries denote the payoffs to the inspector.

In this game, it is reasonable to assume a circular structure of the players' preferences. That is, the inspector prefers to use his inspection if and only if the inspectee violates, who in turn prefers to violate if and only if the inspector does not inspect. (Formally, this is an inductive hypothesis, which is true for the simplest case $n = 2$, $m = 1$ where the payoff matrix is known, and which can later be proved inductively.) This means that the game has a unique mixed equilibrium point. Thereby, both players choose both of their actions with positive probability. This requires that their expected payoff for both actions is the same. If the inspectee's probability for inspecting at the first stage is $p$, then the inspectee is indifferent between legal action and violation if and only if

$$p \cdot I(n-1,m-1) + (1-p) \cdot I(n-1,m) = p + (1-p) \cdot (-1).$$

Both sides of this equation denote the game value $I(n,m)$. Solving for $p$ and substituting yields

$$I(n,m) = \frac{I(n-1,m) + I(n-1,m-1)}{I(n-1,m) + 2 - I(n-1,m-1)}. \quad (4.3)$$

With this recurrence equation for $0 < m < n$ and the base cases (4.1), the game value is determined for all parameters. Equations (4.1) and (4.3) have an explicit solution, namely

$$I(n,m) = \left(\binom{n-1}{m} / \sum_{i=1}^{m} \binom{n}{i}\right).$$
The payoffs in Dresher’s model have been generalized by Höpfinger (1971). He assumed that the inspectee’s gain for a successful violation need not equal his loss if he is caught, and also solved the resulting recurrence equation explicitly. Furthermore, zero-sum payoffs are not fully adequate since a caught violation, compared to legal action throughout, is usually undesirable for both players since for the inspector this demonstrates a failure of his surveillance system. A non-zero-sum game that takes account of this is shown in Figure 4.2. There, \( I(n, m) \) and \( V(n, m) \) denote the equilibrium payoff to the inspector respectively inspectee (‘\( V \’ \) indicating a potential violator), and \( a \) and \( b \) are positive parameters denoting the losses to both players for a caught violation, where \( a < 1 \). An uncaught violation yields payoff \(-1\) to the inspector and \( 1 \) to the inspectee, which are also the payoffs \( I(n, 0) \) respectively \( V(n, 0) \) for \( n > 0 \). The legal case has reference payoff \( 0 = I(n, n) = V(n, n) \) to both.

<table>
<thead>
<tr>
<th>Inspectee</th>
<th>legal action</th>
<th>violation</th>
</tr>
</thead>
<tbody>
<tr>
<td>inspection</td>
<td>( V(n - 1, m - 1) )</td>
<td>(-b)</td>
</tr>
<tr>
<td></td>
<td>( I(n - 1, m - 1) )</td>
<td>(-a)</td>
</tr>
<tr>
<td>no inspection</td>
<td>( V(n - 1, m) )</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>( I(n - 1, m) )</td>
<td>(-1)</td>
</tr>
</tbody>
</table>

Figure 4.2. Non-zero-sum game for \( n \) stages and \( m \) inspections, with equilibrium payoff \( I(n, m) \) to the inspector and \( V(n, m) \) to the inspectee. Legal action throughout the game has reference payoff zero to both players. A caught violation yields negative payoffs to both, where \( 0 < a < 1 \) and \( 0 < b \).

These assumptions lead again to a unique mixed equilibrium at the first stage of the game, as well as in all subsequent stages where there is still something to decide, that is, where \( 0 < m < n \). In a similar way as before, the inspectee’s equilibrium payoff is given by

\[
V(n, m) = \frac{b \cdot V(n - 1, m) + V(n - 1, m - 1)}{V(n - 1, m - 1) + b + 1 - V(n - 1, m)}
\]

and explicitly by

\[
V(n, m) = \binom{n - 1}{m}/\sum_{i=0}^{m} \binom{n}{i} b^{n-i}.
\]

(4.4)
The inspector's payoff $I(n, m)$ has an analogous form, explicitly given by

$$I(n, m) = -\binom{n-1}{m} \sum_{i=0}^{m} \binom{n}{i} (-a)^{m-i}.$$  

(4.5)

This has been shown by Avenhaus and von Stengel (1992); this short paper generalizes and simplifies the derivations by Dresher (1962) and Höpfinger (1971).

The same payoffs, although with a different normalization, have already been considered by Maschler (1960). He derived an expression for the payoff to the inspector that is equivalent to (4.5), considering the appropriate linear transformations for the payoffs. Beyond this, Maschler introduced the leadership concept where the inspector announces and commits himself to his strategy in advance. Interestingly, this improves his payoff in the non-zero-sum game. We will discuss this in detail in the next section.

It should be emphasized that the recursive description of the games in Figure 4.1 and 4.2 assumes that all players know about the game state after each stage. This is usually not true for the inspector after a stage where he did not inspect, since then he will not learn about a violation. The constant payoffs after a successful violation indicate that the game terminates. In fact, the game continues, but any further action of the inspector is irrelevant since then the inspectee will only behave legally.

As soon as the players' information is such that the recursive structure is not valid, the problem becomes extremely difficult. Therefore, even natural and simple looking extensions of Dresher's model are still open problems. Consider a situation in which there is more than one intended violation. That is, let $k$ denote the maximum number of violations intended by the inspectee, who can violate at most once per stage. It is easily seen that no recursive structure holds without some further assumptions. Von Stengel (1991) who discussed this game assumed that the inspectee collects one unit for each successful violation, which he does not have to return if he is caught at a later stage. As before, the game terminates if all remaining stages either will or will not be inspected, or if the inspector detects a violation, where he receives the positive payoff $b$. Still, this does not lead to a recursive structure since when the inspector does not inspect, he does not know whether there was a violation (in which case the value of $k$ becomes $k - 1$ and a payoff of 1 was 'credited' to the inspectee) or not (in which case the value of $k$ is unchanged). Von Stengel had the insight that this uncertainty on the part of the inspector 'should not be important'. To prove this formally, he assumed first that even after a stage with no inspection, it becomes common knowledge whether there was a violation or not. This leads to a game with a recursive structure whose value $I(n, m, k)$ is the minmax value of the matrix game in Figure 4.3. As before, $n$ and $m$ denote the number of stages and inspections, respectively. The case $k = 1$ of one intended violation is (for $b = 1$) Dresher's game in Figure 4.1.
<table>
<thead>
<tr>
<th>Inspectee</th>
<th>legal action</th>
<th>violation</th>
</tr>
</thead>
<tbody>
<tr>
<td>inspection</td>
<td>(I(n-1,m-1,k))</td>
<td>(b)</td>
</tr>
<tr>
<td>no inspection</td>
<td>(I(n-1,m,k))</td>
<td>(I(n-1,m,k-1)-1)</td>
</tr>
</tbody>
</table>

Figure 4.3. Zero-sum game with \(n\) stages, \(m\) inspections, and up to \(k\) intended violations. The inspectee collects one unit for each successful violation, which he can keep even if he is caught later and has to pay the amount \(b\).

The game in Figure 4.3 was solved explicitly by von Stengel (1991). For the game value, a rather involved recurrence equation obtained analogously to (4.3) permits the explicit representation

\[
I(n, m, k) = \frac{(n-k)}{(m+1)} - \binom{n}{m+1} \quad \text{where} \quad s(n, m) = \sum_{i=0}^{m} \binom{m}{i} b^{m-i}. \tag{4.6}
\]

The probability \(p\) of inspection at the first stage, determined analogously to (4.2), is thereby given by \(p = s(n-1, m-1)/s(n, m)\) and thus does not depend on \(k\). This means that the inspector need not know the value of \(k\) in order to play optimally. Hence, the intermediate assumption about the knowledge of the inspector after a period without inspection can be removed: The solution of the recursive game is also the solution of the original game, without recursive structure. It is important to note, however, that this special feature of the solution holds only for this particular payoff structure. A detailed analysis and proof of this claim uses the extensive form of these games and has been described by von Stengel (1991).

A closed-form solution of recursive games cannot be expected in most cases, since many recurrence equations do not even have an explicit solution. Even if it exists, it can be very hard to find (its verification is usually straightforward). With computers at hand, a specific model can be studied using the recurrences alone, for instance by computing the game value for various parameter constellations. In that way, the recursive approach can be a quick tool for analysis.

### 4.2. Timeliness Games

In this subsection, we consider another inspection game where the inspector may use \(m\) inspections in order to detect a single violation. This violation is detected at the earliest subsequent inspection, but the time that has elapsed in between is the
payoff to the inspectee that the inspector wants to minimize. The game is zero-sum. If the inspectee can observe the inspections, then the game has a simple solution. The case of unobserved inspections has been solved explicitly by Diamond (1982). This work represents the most involved – yet canonical – analytic solution of an inspection game that we are aware of.

The game has a discrete version where there are \( n \) possible stages for inspecting and violating. This discrete game can be solved with a computer so that one may guess the form of its solution. Analytically, however, it is easier to solve the game in a form where time is continuous.

In that game, the players choose their actions from the unit time interval. The inspectee violates at some time \( s \) in \([0, 1)\). The inspector has \( m \) inspections \((m \geq 1)\) that he uses at times \( t_1, \ldots, t_m \), where his last inspection must take place at the end of the interval, \( t_m = 1 \). He can choose the other inspection times freely, with \( 0 \leq t_1 \leq \cdots \leq t_{m-1} \leq t_m = 1 \). The violation is discovered at the earliest inspection following the violation time \( s \). Then, the time elapsed is the payoff to the inspectee given by the function

\[
V(s, t_1, \ldots, t_{m-1}) = t_k - s \quad \text{for } s \in [t_{k-1}, t_k), \quad k = 1, \ldots, m, \tag{4.7}
\]

where \( t_0 = 0 \) and \( t_m = 1 \). The inspector's payoff is the negative of the payoff to the inspectee.

As an application, Derman (1961) and Diamond (1982) considered reliability theory. An operating unit or some material in storage may fail after some time \( s \), and there is a certain (here linear) cost associated with the time that the faulty unit operates until its failure is detected. Inspections are used to limit that damage. The last inspection at time \( t_m = 1 \) represents the normal replacement of the unit. A minmax analysis leads to the described zero-sum game. A more common approach in reliability theory, which is not our topic, is to assume some knowledge about the distribution of the failure time \( s \).

Another application are interim inspections of direct-use nuclear material (see, for example, Avenhaus, Canty, and von Stengel 1991). There, the certain inspection at time \( t_m = 1 \) represents a regular inspection of a nuclear plant at the end of the year, say, but if nuclear material is diverted, one may wish to discover this not after a year but earlier, which is the purpose of interim inspections.

In this game, there is an obvious deterministic strategy of the inspector where he inspects at times \( t_i = i/m \) for \( i = 1, \ldots, m \). In that way, a violation is discovered at most after time \( 1/m \) has elapsed. This is indeed an optimal strategy for the inspector if the inspectee can observe the inspections. In that case, the inspectee will violate immediately after the \( k \)th inspection, where \( k \) is with equal probability a number between 0 and \( m-1 \). That is, the violation time \( s \) is uniformly chosen from
\{t_0, t_1, \ldots, t_{m-1}\}, so that by (4.7) the expected time after the violation is discovered is \(1/m \cdot (t_m - t_0)\), that is, \(1/m\).

Derman (1961) considered this model of observable inspections where each interim inspection detects a violation only with a certain fixed probability. In addition to the time until detection, there is a cost to be paid for each inspection. Derman described a deterministic minmax schedule for the inspection times where \(t_i\) is not proportional to \(i\) as before with \(t_i = i/m\), but a quadratic function of \(i\), where the inspections accumulate towards the end of the time interval because of their cost. Moreover, Derman assumed that the number \(m\) of inspections can be freely chosen by the inspector, and he determined an optimal maximal choice for \(m\). His analysis, however, seems to be valid even if \(m\) is fixed, provided the inspector can give up some inspections if \(m\) is too large.

If the inspections are unobserved, then the violation time \(s\) and the free inspection times \(t_1, \ldots, t_{m-1}\) are chosen simultaneously. This is an appropriate assumption for inspecting an unreliable unit if the inspections do not interfere with the operation of the system. This game was completely solved by Diamond (1982). We will demonstrate the solution of the first nontrivial case \(m = 2\), and explain the features of the general case.

For \(m = 2\), the inspector has one free inspection at time \(t_1\), for short \(t\), chosen from \([0, 1]\). The inspectee may select his violation at time \(s \in [0, 1]\). This is a zero-sum game over the unit square with payoff \(V(s, t)\) to the inspectee where, by (4.7),

\[
V(s, t) = \begin{cases} 
    t - s & \text{if } s < t, \\
    1 - s & \text{if } s \geq t.
\end{cases}
\]

(4.8)

This describes a kind of duel with time reversed where both players have an incentive to act early but after the other.

It turns out that by (4.8), the inspectee's payoff is too small if he violates too late, so that he will select \(s\) with a certain probability distribution from an interval \([0, b]\) where \(b < 1\). Consequently, the inspector will not inspect later than \(b\). He chooses \(t\) according to a certain density function \(p\) from \([0, b]\), where

\[
\int_0^b p(t) dt = 1.
\]

(4.9)

The expected payoff \(V(s)\) to the inspectee for \(s \in [0, b]\) is then given as follows, taking into account that the inspection may take place before or after the violation:

\[
V(s) = \int_0^s (1 - s) p(t) dt + \int_s^b (t - s) p(t) dt
\]

\[
= \int_0^s p(t) dt + \int_s^b t p(t) dt - s \int_0^b p(t) dt.
\]

(4.10)

If the inspectee randomizes as described, then this payoff must be constant (see Karlin 1959, Lemma 2.2.1, p. 27). That is, its derivative with respect to \(s\), which

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by (4.9) is given by \( p(s) = s \rho(s) - 1 \), should be zero. The density for the inspection time is therefore given by

\[
p(t) = \frac{1}{1 - t},
\]

(4.11)

with \( b \) in (4.9) given by \( b = 1 - 1/e \). The constant expected payoff to the violator is then \( V(s) = 1/e \). For \( s > b \), the inspectee's payoff is \( 1 - s \) which is smaller than \( 1/e \) so that he will indeed not violate that late.

Conversely, the optimal distribution of the violation time has an atom at 0 and a density \( q \) on the remaining interval \( (0, b] \). We consider its distribution function \( Q(s) \) denoting the probability that a violation takes place at time \( s \) or earlier. The mentioned atom is \( Q(0) \), the derivative of \( Q \) is \( q \). The resulting expected cost \( V(t) \) for \( t \in [0, b] \) to the inspector is

\[
V(t) = t \cdot Q(0) + \int_0^t (t - s)q(s)ds + \int_t^b (s - t)q(s)ds \\
= t \cdot Q(0) + t \int_0^t q(s)ds + \int_t^b q(s)ds - \int_0^b s q(s)ds \\
= t \cdot Q(t) + Q(b) - Q(t) - \int_0^b s q(s)ds.
\]

Again, the inspector randomizes only if this is a constant function of \( t \), which means that \( (t - 1) \cdot Q(t) \) is a constant function of \( t \). The requirement \( Q(b) = 1 \) yields

\[
Q(s) = \frac{1}{e} \cdot \frac{1}{1 - s} 
\]

(4.12)

for the violation time \( s \in [0, b] \), and \( Q(s) = 1 \) for \( s > b \). The nonzero atom is given by \( Q(0) = 1/e \). The value of the game is \( 1/e \). Unsurprisingly, this is better for the inspector than the value 1/2 of the game with observable inspections.

The solution the game for \( m > 2 \) is considerably more complex. It exhibits the following properties. As before, the inspectee violates with a certain probability distribution on an interval \([0, b]\), leaving out the interval \((b, 1]\) at the end. The inspections take place in disjoint intervals. That is, the first inspection \( t_1 \) takes place in \([a_0, a_1]\) where \( a_0 = 0 \), the second inspection \( t_2 \) takes place in \([a_1, a_2]\), and so on, and \( t_{m-1} \) is chosen from \([a_{m-2}, a_{m-1}]\) where \( a_{m-1} = b \). The inspection times are random, but it is a randomization over a one parameter family of pure strategies: The inspector can be thought of first using a random variable \( X \), uniformly distributed on the unit interval \([0, 1]\). Then, the \((m - i)\)th inspection time is \( t_{m-i} = h_i(X) \) where \( h_i \) is a function that maps \([0, 1]\) continuously to the interval \([a_{m-i-1}, a_{m-i}]\), assuming

\[
h_i(0) = a_{m-i}, \quad h_i(1) = a_{m-i-1}.
\]

(4.13)

In that way, the inspections are fully correlated and take place in disjoint intervals as described. The functions \( h_1, h_2, \ldots \) are iteratively determined by a simple differential equation derived from the equilibrium conditions.
The optimal violation scheme of the inspectors is given by a distribution function \( Q \), which turns out to be piecewise defined on the \( m - 1 \) time intervals in terms the functions \( h_{m-1} \) and \( h_i \) on the \( i \)th interval. Also, \( Q \) has an atom at the beginning of the game and otherwise it is continuous.

Asymptotically (for large \( m \)), it turns out that the inspection times are rather uniformly distributed in their intervals. Using a generating function for the recursive definition of the functions \( h_i \) and contour integration in the complex plane, Diamond showed that these functions are linear with an error that is exponentially small in \( i \). This implies that the violation strategy is also more or less uniform, except for some of the first and last intervals since \( h_1 \) and \( h_2 \) are not as well linearly approximated as the other functions. Moreover, the value of the game rapidly approaches \( 1/(2m-4) \) from above with growing \( m \). This means that if inspections are unobserved, the average detection time is about half as long as the time \( 1/m \) in the case of observable inspections. This is not surprising: assume that inspector chooses his \( i \)th inspection for \( i = 1, \ldots, m - 1 \) not deterministically at time \( i/m \) but at time \( i/m - x \) with some random time shift \( x \) uniformly chosen from \([0, 1/m] \). The resulting expected detection time is \( 1/2m \) for any violation executed between these inspections, where the gap between any two inspections does not change. This is not quite true, of course, since the gap between \( t_{m-1} \) and the last inspection \( t_m \), which is fixed, does get larger. This ‘end effect’ results in a complication for the last free inspection, then for the one that takes place before it, and so on, but this effect vanishes quickly. For all practical purposes, it may be appropriate to choose the inspections uniformly from evenly spaced intervals.

Based on this complete analytic solution, Diamond (1982) demonstrated computational solution methods for non-linear loss functions as well. Thereby, the loss to the inspector may be any monotonic function of the time that the violation remains undetected. The qualitative properties of the solution remain, in particular the one-parameter randomization in the optimal inspection scheme.
5. Inspector Leadership

The leadership principle says that it can be advantageous in a competitive situation to be the first player to select and stay with a strategy. It was suggested first by von Stackelberg (1934) for pricing policies. Maschler (1966) applied this idea to sequential inspections. The notion of leadership consists actually of two elements: The ability of to announce one's strategy first and make it known to the other player, and the ability of the player to commit himself to playing it. In a sense, it would be more appropriate to use the term commitment power. This concept is particularly suitable for inspection games since an inspector can credibly announce his strategy and stick to it, whereas the inspectee cannot do so if he intends to act illegally. Therefore, it is reasonable to assume that the inspector will take advantage of his leadership role.

In this section, we first define the concept of leadership formally and illustrate it with some examples. Then, we consider leadership for two inspection games: first, the recursive inspection game with non-zero-sum payoffs discussed in Section 4.1, which was introduced by Maschler (1966). Then, we will come full circle and reconsider the inspection game defined in Section 3.1 that extends the inspector's statistical decision problem. As we will see, the effect of leadership is quite substantial: If the inspector cannot announce his strategy, then the inspectee violates with some positive probability in equilibrium. In the same inspection game but with inspector leadership, the inspectee behaves legally in equilibrium.

5.1. Definition and Introductory Examples

A leadership game is to be contrasted with a simultaneous game, which is in fact a standard normal form game. Consider a game in normal form in which the players are I and II with strategy sets $\Delta$ and $\Omega$, respectively. For simplicity, we assume that $\Delta$ is closed under the formation of mixed strategies. In particular, $\Delta$ could be the set of mixed strategies over a finite set. The two players select simultaneously their strategies $\delta$ and $\omega$, respectively, and receive their corresponding payoffs, defined as expected payoffs if $\delta$ originally represents a mixed strategy.

In the leadership version of the game, one of the players, say player I, is called the leader. The other player, thus player II, is called the follower. The leader chooses a strategy $\delta$ in $\Delta$ and makes this choice known to the follower, who then chooses a strategy $\omega$ in $\Omega$ which will typically depend on $\delta$, and is therefore denoted by $\omega(\delta)$. The payoffs to the players are those of the simultaneous game for the pair of strategies $(\delta, \omega(\delta))$. It is important that the strategy $\delta$ is executed without giving the leader an opportunity to reconsider it; this is the commitment power of the leader.

The simplest way to construct the leadership game formally is to start with the simultaneous game in extensive form. Player I moves first, and player II has a
simultaneous game:

leadership game:

Figure 5.1. The simultaneous game and its corresponding leadership game in extensive form. The strategies \( \delta \) of player I include all mixed strategies. In the simultaneous game, the choice \( \omega \) of player II is the same irrespective of \( \delta \). In the leadership game, player I is the leader and moves first, and player II, the follower, may choose his strategy \( \omega(\delta) \) depending on the announced leader strategy \( \delta \). The payoffs in both games, which are not shown, are the same.

single information set since he is not informed about that move, and moves second. In the leadership game, that information set is dissolved into singletons, and the rest of the game, including the payoffs, stays the same, as indicated in Figure 5.1.

The leadership game has perfect information because the follower is precisely informed about the announced strategy \( \delta \). He can therefore choose a best response \( \omega(\delta) \) to \( \delta \). We assume that he always does this, even if \( \delta \) is not a strategy announced by the leader in equilibrium. That is, we only consider subgame perfect equilibria. The game has perfect information, so that the follower may respond deterministically to \( \delta \). Nevertheless, we permit randomized responses as well. Because the game has perfect recall, a behavior strategy suffices, defined by a 'local' probability distribution on the set \( \Omega \) of choices of the follower for each announced strategy \( \delta \).
We will not study leadership games in general, which would be a separate topic of research. Instead, we consider a few simple cases of leadership games. In a zero-sum game which has a value, leadership has no effect. This is the essence of the minmax theorem: Each player can guarantee the value even if he announces his (mixed) strategy to his opponent and commits himself to playing it.

![Game matrix]

Figure 5.2. Game where the unique equilibrium \((T, L)\) of the leadership game is one of several equilibria in the simultaneous game.

In a non-zero-sum game, leadership can sometimes serve as a coordination device and as a method for equilibrium selection. The game in Figure 5.2, for example, has two equilibria in pure strategies. There is no rule to select either equilibrium if the players are in symmetric positions. In the leadership game, if player I is made a leader, he will select \(T\), and player II will follow by choosing \(L\). In fact, \((T, L)\) is the only equilibrium of the leadership game. Player II may even consider it advantageous that player I is a leader and accept the payoff 8 in this equilibrium \((T, L)\) instead of 9 in the other pure strategy equilibrium \((B, R)\) as a price for avoiding the undesirable payoff 0.

However, a simultaneous game and the corresponding leadership game may have different equilibria. The simultaneous game in Figure 5.3 has a unique equilibrium in mixed strategies: Player I plays \(T\) with probability \(1/3\) and \(B\) with probability \(2/3\), and player II plays \(L\) with probability \(1/3\) and \(R\) with probability \(2/3\). The resulting payoffs are \(-\frac{2}{3}, \frac{1}{3}\).

In the corresponding leadership game, player I (the leader) uses the same mixed strategy as in the simultaneous game, and player II gets the same payoff, but as a follower he will act to the advantage of the leader, for the following reason. Consider the possible announced mixed strategies, given by the probability \(p\) that the leader plays \(T\). If player II responds with \(L\) and \(R\), the payoffs to player I are \(-p\) and \(p/2 - 1\), respectively. When player II makes these choices, his own payoffs are \(p\) and \(1 - 2p\), respectively. He therefore chooses \(R\) for \(p < \frac{1}{3}\), \(L\) for \(p > \frac{1}{3}\), and is indifferent for \(p = \frac{1}{3}\). The resulting payoff to the leader as a function of \(p\) is shown
Figure 5.3. A simultaneous game with a unique mixed equilibrium. The leadership game has a unique equilibrium where the leader announces the same mixed strategy, but the follower changes his strategy to the advantage of the leader.

in Figure 5.4. The leader tries to maximize this payoff, which he achieves only if the follower plays $L$. Thus, player I announces any $p$ that is greater than or equal to $\frac{1}{3}$ but as small as possible. This means announcing exactly $\frac{1}{3}$ (as first pointed out by Avenhaus, Okada, and Zamir 1991): If the follower, who is then indifferent, would not choose $L$ with certainty, then it is easy to see that the leader could announce a slightly larger $p$ (thus forcing the follower to play $L$) and improve his payoff, in contradiction to the equilibrium property. Thus, the unique equilibrium of the leadership game is that player I announces $p^* = \frac{1}{3}$ and player II responds with $L$ (and deterministically, as described, for all other announcements of $p$, which are not materialized).

5.2. Announced Inspection Strategy

Inspection problems are a natural case for leadership games since the inspector can make his strategies public and thus become a leader. The inspectee cannot announce that he intends to violate with some positive probability. The example in Figure 5.3 demonstrates the effect of leadership in an inspection game, with the inspector as player I and the inspectee as player II. Namely, this game is a special case of the recursive game in Figure 4.2 for two stages ($n = 2$), where the inspector has one inspection ($m = 1$). Inspecting at the first stage is the strategy $T$, and not inspecting (that is, inspecting at the second stage) is represented by $B$. For the inspectee, $L$ and $R$ refer to legal action and violation at the first stage, respectively. The losses to the players in the case of a caught violation in Figure 4.2 have the special form $a = \frac{1}{2}$ and $\delta = 1$.

In that leadership game, we have shown that the inspector announces his equilibrium strategy of the simultaneous game, but receives a better payoff. By the
utility assumptions about the possible outcomes of an inspection, that better payoff is achieved by legal behavior of the inspectee. It can be shown that this applies not only to the simple game in Figure 5.2, but to the recursive game with general parameters $n$ and $m$ in Figure 4.2. Note, however, that the inspectee behaves legally only in the two-by-two games encountered in the recursive definition: If the inspections are used up ($n > 0$ but $m = 0$), then the inspectee can and will safely violate. He will act legally only until then. In that sense, we have to qualify our claim that the inspectee behaves legally if the inspector may announce his strategy.

The definition of this particular leadership game with general parameters $n$ and $m$ is due to Maschler (1966). However, he did not construct the leadership game from a simultaneous game, and explicitly did not try to determine an equilibrium. Instead, Maschler simply postulated that the inspectee would act benevolently to the advantage of the inspector if he is indifferent, calling this behavior 'pareto-optimal'. The essential idea is there, however, since he argued, as we did with the help of Figure 5.4, that the inspector can announce an inspection probability $p$ that is slightly higher than $p^*$, so that he is on the safe side.

Despite the advantage of leadership, announcing such a precisely calibrated inspection strategy looks risky. Above, the equilibrium strategy $p^* = 1/3$ depends on the payoffs of the inspectee which might not be fully known to the inspector. Therefore, Avenhaus, Okada, and Zamir (1991) considered the leadership game for a simultaneous game with incomplete information. There, the gain to the inspectee
for a successful violation is a payoff in some range with a certain (say, uniform) probability distribution assumed by the inspector. In that leadership game, unlike in Figure 5.4, the inspector maximizes over a continuous payoff curve. He announces an inspection probability that just about forces the inspectee to act legally for any value of the unknown penalty \( b \). This strategy has a higher equilibrium probability \( p^* \) (that is, it is on the safer side) than in the simultaneous game.

The simple game in Figure 5.3 and the argument using Figure 5.4 is prototypical for many leadership games. In the same vein, we consider now the inspection game in Figure 3.1 as a simultaneous game, and construct the corresponding leadership game. Recall that in this game, the inspector has collected some data and, using this data, has to decide whether the inspectee acted illegally or not. He uses a statistical test procedure which is designed to detect an illegal action. Then, he either calls an alarm, rejecting the null hypothesis \( H_0 \) of legal behavior of the inspectee in favor of the alternative hypothesis \( H_1 \) of a violation, or not. The first and second error probabilities \( \alpha \) and \( \beta \) of a false rejection of \( H_0 \) or \( H_1 \), respectively, are the probabilities for a false alarm and an undetected violation.

As shown in Section 3.1, the only choice of the inspector is in effect the value for the false alarm probability \( \alpha \) from \([0, 1]\). The non-detection probability \( \beta \) is then determined by the most powerful statistical test and defines the function \( \beta(\alpha) \), which has the properties (3.7). Thereby, the convexity of \( \beta \) implies that the inspector has no advantage in choosing \( \alpha \) randomly since this will not improve his non-detection probability. Hence, for constructing the leadership game as above in Section 5.1, \( \alpha \) can be announced as a deterministic choice. The possible actions of the inspectee are legal behavior \( H_0 \) and illegal behavior \( H_1 \). According to the payoff functions in (3.2) and the definition of \( \beta(\alpha) \), this defines the game shown in Figure 5.5.

<table>
<thead>
<tr>
<th>( \alpha \in [0, 1] )</th>
<th>legal action ( H_0 )</th>
<th>violation ( H_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\alpha )</td>
<td>( -b + (1 + b)\beta(\alpha) )</td>
<td></td>
</tr>
<tr>
<td>( -a - (1 - a)\beta(\alpha) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.5. The game of Figure 3.1 with payoffs (3.2) depending on the false alarm probability \( \alpha \). The non-detection probability \( \beta(\alpha) \) for the best test has the properties (3.7).

In an equilibrium of the simultaneous game, the inspector chooses \( \alpha^* \) as defined by (3.5), that is, as the solution of \( -h \alpha^* = -b + (1 + b)\beta(\alpha^*) \). Furthermore, the inspectee violates with positive probability \( q^* \) according to (3.4). This is different in the corresponding leadership game. In that game, the response of the inspectee is
unique for any announcement of $\alpha$ except for $\alpha^*$. Namely, the inspectee will violate ($H_1$) if $\alpha < \alpha^*$, and act legally ($H_0$) if $\alpha > \alpha^*$. The inspector's payoffs for these responses are shown in Figure 5.6. It follows easily that the only equilibrium of the leadership game is that in which the inspector announces $\alpha^*$, and the inspectee acts legally.

![Figure 5.6](image)

Figure 5.6. The inspector's payoff as a function of $\alpha$ for the best responses of the inspectee. For $\alpha < \alpha^*$ the inspectee behaves illegally, hence the payoff is given by the line $H_1$. For $\alpha > \alpha^*$ the inspectee behaves legally and the payoff is given by the line $H_0$. In the simultaneous game, the inspectee plays $H_1$ with probability $q^*$ according to (3.9), so that the inspector's payoff, shown by the thin line, has its maximum for $\alpha^*$; this is his equilibrium payoff, indicated by $\circ$. The inspector's equilibrium payoff in the leadership game is marked by $\bullet$.

Summarizing, we observe that in the simultaneous game, the inspectee violates with positive probability, whereas he acts legally in the leadership game. The optimal false alarm probability $\alpha^*$ of the inspector stays the same. His payoff is larger in the leadership game.
6. Conclusions

One of our claims is that inspection games constitute a real field of applications of game theory. Is this justified? Have these games actually been used? Did game theory have an impact on more than technical issues?

The decision to implement a particular verification procedure is usually not based on a game theoretical analysis. Practical questions overwhelm, and the allocation of inspection effort to various sites, for example, is usually based on rules of thumb. Most IAEA procedures are of this kind. However, in retrospect, they can often be justified by game theoretic means. We mentioned that the subdivision of a plant into small areas with intermediate balances has no effect on the overall detection probability if the violator acts strategically – such a physical separation may only improve localizing a diversion of material. With all caution with respect to the impact of a theoretical analysis, this observation may have influenced the design of some nuclear processing plants.

Another question concerns the proper scope of a game theoretic model. For example, the course of second-level actions – after the inspector has raised an alarm – is often determined politically. In an inspection game, the effect of a detected violation is usually modeled by an unfavorable payoff to the inspector. The particular magnitude of this penalty, as well as the inspector’s utility for a successful violation, is usually not known to the analyst. This is often used as an argument against game theory. As a counterargument, the signs of these payoffs often suffice, as we have illustrated with a specific model in Section 3.1. Then, the part of the game where a violation is to be discovered can be reduced to a zero-sum game with the detection probability as payoff, as first proposed by Bierlein (1963).

We believe that inspection models should be of this kind, where the merit of game theory as a technical tool becomes clearly visible. ‘Political’ parameters, like the choice of a false alarm probability, are exogeneous to the model. Higher-level models describing the decisions of the states, like whether to cheat or not, and in the extreme whether ‘to go to war’ or not, should in our view not be blended with an inspection game. They are likely to be simplistic and will invalidate the analysis.

Which direction will or should research on inspection games take in the foreseeable future? We saw that the more interesting concrete inspection games were stimulated by problems raised by practitioners. In that sense we expect further progress again from the application side, in particular in the area of environmental control where a fruitful interaction between theorists and environmental experts is still missing. Indeed there are open questions which shall be illustrated with the material accountancy example discussed in the Section 3.2.

We have shown that intermediate inventories should be ignored in equilibrium. This means, however, that a decision about legal or illegal behavior can be made only at the end of the reference time. This may be too long if detection time becomes
a criterion. If one models this appropriately, one enters the area of sequential games and sequential statistics. In a way similar to the result in Section 3.1 and under reasonable assumptions one can show that then the probability of detection and the false alarm probability are replaced by the average sum lengths under $H_1$ and $H_0$, respectively, that is, the expected times until an alarm is raised (see Avenhaus and Okada 1992). However, they need not exist and, more importantly, there is no equivalent to the Neyman Pearson Lemma which gives us a constructive advice for the best sequential test. Thus, in general, sequential statistical inspection games represent a wide field for future research.

From a theoretical point of view, the leadership principle as applied to inspection games deserves more attention. In the case of sequential games, is legal behavior again an equilibrium strategy of the inspectors? How does the leadership principle work if more complicated payoff structures have to be considered? Does the inspector in general improve his equilibrium payoff by credibly announcing his inspection strategy?

In this contribution, it has been attempted to demonstrate that inspection games represent an area of game theory where real applications do exist, and where they are necessary tools for handling practical problems. Therefore, the major effort in the future, also in the interest of sound theoretical development, should be spent in deepening these applications, and in trying to convince practitioners of the usefulness of appropriate game theoretic models.
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