INCOMPLETE INFORMATION GAMES
AND THE NORMAL DISTRIBUTION

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Incomplete Information Games and the Normal Distribution

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Abstract

We consider a repeated two-person zero-sum game in which the payoffs in the stage game are given by a $2 \times 2$ matrix. This is chosen (once) by chance, at the beginning of the game, to be either $G^1$ or $G^2$, with probabilities $p$ and $1 - p$ respectively. The maximizer is informed of the actual payoff matrix chosen but the minimizer is not. Denote by $v_n(p)$ the value of the $n$-times repeated game (with the payoff function defined as the average payoff per stage), and by $v_\infty(p)$ the value of the infinitely repeated game. It is proved that

$$v_n(p) = v_\infty(p) + K(p)\frac{\phi(p)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

where $\phi(p)$ is an appropriately scaled normal distribution density function evaluated at its $p$-quantile, and the coefficient $K(p)$ is either 0 or the absolute value of a linear function in $p$. 
1 Introduction

In this paper we treat a two-person 0-sum repeated game with incomplete information on one side (the informed player is the maximizer-player 1) in which there are two states of nature, corresponding to payoff matrices:

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \]

occurring with probabilities \( p \) and \( p' = 1 - p \) respectively. The information structure is standard, i.e. Only the moves of the players are announced after each stage.

Letting \( u(p) = \text{val}(pA + p'B) \) and for all \( n \), \( v_n(p) \) be the value of such a game with \( n \) repetitions, we know that \( \lim_{n \to \infty} v_n(p) = \text{Cav} u(p) \) (see Aumann and Maschler (1995)). We also know that \( \left| v_n(p) - \text{Cav} u(p) \right| \) which we call the error term, is bounded in order of magnitude by \( 1/\sqrt{n} \) and that this speed of convergence can in fact be reached (Zamir (1971)).

The purpose of this work is:

(i) To give the general expression for the coefficient of \( 1/\sqrt{n} \) in terms of the matrices \( A \) and \( B \). It turns out that this coefficient is always a multiple of what we shall call the Normal Function \( \phi(p) \), which is the standard normal distribution evaluated at its \( p \)-quantile (see Mertens and Zamir (1976)):

\[ \phi(p) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}p^2} \quad \text{where} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{p} e^{-\frac{1}{2}x^2} dx = p \]

(ii) To prove that the next term after \( 1/\sqrt{n} \) in the expansion of \( v_n(p) \) as a function of \( n \) is bounded by \( O(1/n^{2/3}) \).

To state our result more precisely we shall use the following notation:

\( (t_A, t'_A) \) and \( (t_B, t'_B) \) will denote optimal strategies for player II in games \( A \) and \( B \) respectively (except if one matrix, say \( A \), is constant in which case \( (t_A, t'_A) \) is an optimal strategy in the other matrix, \( B \) ). \( t_p \) will denote an optimal strategy of player II in the game \( pA + p'B \).

\[ \delta(A, B) = \begin{cases} 1 & \text{if player I has a strategy } (s, s') \text{ which is optimal in } pA + p'B \text{ for all } p \in [0,1] \\ 0 & \text{Otherwise.} \end{cases} \]

\[ \Delta_A = (a_{11} + a_{22} - a_{12} - a_{21}) \quad \text{and} \quad \Delta_B = (b_{11} + b_{22} - b_{12} - b_{21}). \]

Our result can be stated now as follows:

**Theorem 1.1** In the general \( 2 \times 2 \) games with payoff matrices \( A \) and \( B \); for all \( n \) and for all \( p \in [0,1] \),

\[ \left| v_n(p) - \text{Cav} u(p) - K(A, B, p)\phi(q(p))(1/\sqrt{n}) \right| \leq \frac{C}{n^{2/3}} \quad \quad (1) \]

where \( C > 0 \), \( q(p) = q \) is defined by \( t_p = qt_A + q't_B \) and

\[ K(A, B, p) = \delta(A, B) \left| t_A - t_B \right| \sqrt{ss'} \left| p\Delta_A + p'\Delta_B \right|. \]
Lemma 1.2 The statement is unambiguous, in the sense that $K(A, B, p)$ is always uniquely defined, and that if it is nonzero, $q(p)$ is always uniquely defined, with $0 \leq q(p) \leq 1$.

Proof Observe that if in a nonconstant $2 \times 2$ game player I has a strictly mixed optimal strategy, then player II has a unique optimal strategy. In other words if, for some choice of $s, \delta(A, B)\sqrt{s^2} | p\Delta_A + p'\Delta_B |$ is not identically zero, we know that one of the matrices, say $A$, is not constant, so $t_A$ is uniquely defined - and thus $t_B$ also, both in the case of constant and non constant $B$. It remains thus only to show that $K(A, B, p)$ does not depend on the choice of $s$. Assume that, for some choice of $s, K(A, B, p)$ is not identically zero: by the above, $t_A$ and $t_B$ are uniquely defined with $t_A \neq t_B$ - hence both matrices $A$ and $B$ are non constant. On the other hand, if player I has several optimal strategies in the $2 \times 2$ game $pA + p'B$, then one of its columns must be constant; this being so for all $p$, the constant column can be chosen independently of $p$ - e.g. the first. By subtracting this constant from the payoffs, we can assume that the first column payoffs in both $A$ and $B$ are zero and thus

$$pA + p'B = \begin{pmatrix} 0 & pa_{12} + p'b_{12} \\ 0 & pa_{22} + p'b_{22} \end{pmatrix}$$

The fact that player I has several strategies in this game which are optimal for all $p \in [0, 1]$ implies that either

$$pa_{12} + p'b_{12} = pa_{22} + p'b_{22} \leq 0 \quad \forall p \in [0, 1],$$

which is impossible since it implies $\Delta_A = \Delta_B = 0$, or the second column of both $A$ and $B$ contain a positive entry. But then the left column is player II's optimal strategy in both $A$ and $B$ contradicting $t_A \neq t_B$.

This proves that $K(A, B, p)$ is always uniquely defined, independently of the choice of $s$, $t_A$ and $t_B$, and that, whenever it is not identically zero, $s, t_A, t_B$ and $t_p$ are uniquely defined and satisfy $t_A \neq t_B$, so $q(p)$ also is uniquely defined. The proof of the Theorem will show that in fact $0 \leq q(p) \leq 1$.

Remarks (1) Both $K(A, B, p)$ and $q(p)$ are, as they should be, invariant under addition of a constant to all payoffs in one of the matrices.

(2) The theorem states that the error term is of the order $1/\sqrt{n}$, i.e. $K(A, B, p) > 0$, if and only if player I has the same strictly mixed optimal strategy in $pA + p'B$ for all $p \in [0, 1]$ while player II's optimal strategies in $A$ and in $B$ are different. In such a case the coefficient of $1/\sqrt{n}$ is a multiple of the normal function $\phi$ (with some transformation of the variable).

(3) In the example considered in our earlier paper (Mertens and Zamir (1976)) we have $\delta(A, B) = 1, \ s = 1/2, \ t_A = 1/4, \ t_B = 1/2, \Delta_A = \Delta_B = 8$ and hence $K(A, B, p) = 1$. Also, $t_p = (p + 2p')/4 = pt_A + p't_B$ implying that $q(p) = p$ for all $p$ and therefore the coefficient of $1/\sqrt{n}$ is $\phi(p)$.

2 Games with highest order error term

As a first step in proving the general result we find a class of games in which the error term is of the order $1/\sqrt{n}$ with Normal Distribution coefficient. As it is suggested by the definition of $K(A, B, p)$, these games will be characterized by the following strategic property:
• Player I has a strictly mixed strategy \((s, s')\) which is optimal in both \(A\) and \(B\).

• Player II has no such uniformly optimal strategy i.e. \(t_A \neq t_B\).

Theorem 2.1 If \(A\) and \(B\) are of the form

\[
A = \begin{pmatrix}
\theta a & -\theta a' \\
-\theta a & \theta a'
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
\theta b & -\theta b' \\
-\theta b & \theta b'
\end{pmatrix}
\]

where \(a, b\) and \(\theta\) are in \([0, 1]\), then (1) holds. More precisely there exists \(C > 0\) such that for all \(n\), and for all \(p \in [0, 1]\),

\[
|v_n(p) - \frac{1}{\sqrt{n}}|a - b|\sqrt{\theta \theta'}\phi(p)| \leq C \frac{\ln n}{n} \quad (2)
\]

Remark  In these games \(u(p) = 0, \delta(A, B) = 1, s = \theta, t_A = a', t_B = b', \Delta A = \Delta B = 1\) and so \(K(A, B, p) = |a - b|\sqrt{\theta \theta'}\phi(p).\) Also, the optimal strategy of player II in \(pA + p'B\) is \(t_p = pa' + pb' = pt_A + pt_B\) implying \(g(p) = p\) and hence (2) implies (1) in this special case since \((\ln n)/n \leq 1/n^{2/3}\) (for sufficiently large \(n\)).

Proof  If \(a = b\), we have a game with complete information with \(v_n(p) = 0\) for all \(n\) and all \(p\) and so (2) holds trivially with \(C = 0\). So assume, w.l.o.g. that \(a > b\) (for \(a < b\), interchange \(A\) with \(B\), and \(p\) with \(p'\)). If \(\theta = 0\) or \(\theta' = 0\), player II can guarantee 0 at each stage by playing always Right (if \(\theta = 0\)) or always Left (if \(\theta' = 0\)) and again (2) holds with \(C = 0\). Therefore we shall assume from now on also that \(\theta \theta' > 0\).

The recursive formula for \(v_n\) (see MSZ, [1994]) is;

\[
v_{n+1}(p) = \frac{1}{n+1} \max_{0 \leq s \leq 1, 0 \leq t \leq 1} \left\{ H(s, t) + n \left[ p(T)v_n \left( \frac{ps}{p(T)} \right) + p(B)v_n \left( \frac{ps'}{p(B)} \right) \right] \right\}
\]

where \(p(T) = ps + pt, p(B) = 1 - p(T) = ps' + st'\) and

\[
H(s, t) = \min \left[ pa(\theta's - \theta's') + pb(\theta't - \theta't'), \ pa'(-\theta's + \theta's') + pb'(-\theta't + \theta't') \right]
\]

Define the constant \(d = p(a - a') + p'(b - b')\) and note that if \(a \neq b\) then \(|d| < 1\). Then we introduce two new variables \(\sigma\) and \(\delta\):

\(\sigma = ps + pt\) (the average mixed strategy of player I in the first stage),

\(\delta = pp'(s - t)\) (a multiple of the difference of the two strategies),

and let \(\psi_n(p) = \sqrt{n}v_n(p).\) The recursive formula then becomes (recalling that \(|a - b| = a - b|):\)

\[
\psi_{n+1}(p) = \frac{1}{\sqrt{n+1}} \max_{(\sigma, \delta) \in S(p)} \left\{ \tilde{H}(\sigma, \delta) + \sqrt{n}[(\sigma'\psi_n(p - \delta/\sigma') + \sigma\psi_n(p + \delta/\sigma)] \right\} \quad (3)
\]

where

\[
\tilde{H}(\sigma, \delta) = \frac{1}{2}[(\sigma - \theta)d - |\sigma - \theta|] + (a - b)\delta
\]

\(S(p) = \{(\sigma, \delta) | 0 \leq \sigma \leq 1, -pp' \leq \delta \leq pp', \delta \leq ps' + pt, \delta \leq ps + pt\}.
\)
When $\sigma = 0$ or $\sigma' = 0$ we define:

$$\sigma' \psi_n(p - \delta/\sigma') + \sigma \psi_n(p + \delta/\sigma) = \psi_n(p).$$

To derive the desired result from this recursive formula we need the following property of the normal function $\phi$. The proof, which is rather technical is given in the appendix at the end of the paper.

**Lemma 2.2** For any $\sigma$ s.t. $0 < \sigma < 1$, there exists a constant $C > 0$ s.t. for all $n$ and for all $p \in [0, 1]$

\[
|\phi(p) - \frac{1}{\sqrt{n} + 1} \max_{\xi \in D(p)}[\xi + \sqrt{n} \phi_\sigma(p, \xi)]| \leq \frac{C}{n^{1/2}}
\]

where

$$\phi_\sigma(p, \xi) = \sigma\phi(p - \sigma\xi) + \sigma\phi(p + \sigma\xi)$$

and

$$D(p) = \left\{ \xi | 0 \leq \xi \leq \sqrt{\frac{\sigma}{\sigma'}} p \land \sqrt{\frac{\sigma'}{\sigma}} p' \right\}.$$

With Lemma 2.2 at hand the proof of Theorem 2.1 proceeds as follows:

**Lemma 2.3** There exists $\alpha > 0$ such that for all $n$ for all $p \in [0, 1]$

$$\sqrt{n} \psi_n(p) - |a - b| \sqrt{\theta' \phi(p)} \geq -\frac{\alpha \ln n}{\sqrt{n}}.$$ 

**Proof** Recalling the recursive formula (3) for $\psi_n(p) = \sqrt{n} \psi_n(p)$, we define a sequence of functions $\{\tilde{\psi}_n\}_{n=0}^{\infty}$ by $\tilde{\psi}_0 \equiv 0$ and:

$$\tilde{\psi}_{n+1}(p) = \frac{1}{n + 1} \max_{0 \leq \xi \leq \sigma \theta \wedge \sigma' \theta'} \left\{ |a - b| \sqrt{n} \left[ \theta' \tilde{\psi}_n(p - \delta/\theta') + \theta \tilde{\psi}_n(p + \delta/\theta) \right] \right\}.$$

Compared with (3) these functions are what player I can guarantee when he restricts himself to $\sigma = \theta$, and hence clearly $\sqrt{n} \psi_n(p) = \tilde{\psi}_n(p)$ for all $n$ and for all $p$. It suffices then to prove the statement of the lemma for $\psi_n(p)$ i.e.,

$$\tilde{\psi}_n(p) - |a - b| \sqrt{\theta' \phi(p)} \geq -\frac{\alpha \ln n}{\sqrt{n}}.$$ 

We shall show that by proving inductively that for $n = 1, 2, \ldots$,

$$\tilde{\psi}_n(p) \geq |a - b| \sqrt{\theta' \phi(p)} - \frac{1}{\sqrt{n}} \left( \frac{1}{2} + 2C \sum_{i=1}^{n-1} \frac{1}{i} \right),$$

where $C > 0$ is a constant satisfying Lemma 2.2. This will yield the required result since

$$\sum_{i=1}^{n-1} \frac{1}{i} = O(\ln n).$$
For \( n = 1 \) inequality (5) is true since \( \tilde{\psi}_1(p) \geq 0 \geq |a - b| \sqrt{\theta'}(\phi(p) - 1/2) \), for all \( p \in [0, 1] \). Assume it is true for \( n \) then (changing variable to \( \xi = \delta/\sqrt{\theta'} \)):

\[
\tilde{\psi}_{n+1}(p) \geq \frac{|a - b| \sqrt{\theta'}}{\sqrt{n} + 1} \left[ H(p, \theta) - \left( \frac{1}{2} + 2C \sum_{i=1}^{n-1} \frac{1}{i} \right) \right],
\]

where

\[
H(p, \theta) = \max_{\xi \in D(p)} \left\{ \xi + (\theta' \phi(p - \sqrt{\theta}/\xi) + \theta \phi(p + \sqrt{\theta}/\xi) \right\},
\]

and

\[
D(p) = \left\{ \xi \mid 0 \leq \xi \leq \sqrt{\theta'} p \land \sqrt{\theta'} p \right\}.
\]

Using Lemma 2.2, (noticing that \( C/n \sqrt{n} \leq 2C/n \sqrt{n+1} \)), we obtain:

\[
\tilde{\psi}_{n+1}(p) \geq |a - b| \sqrt{\theta'} \left[ \phi(p) - \left( \frac{2C}{n \sqrt{n} + 1} + \frac{1}{\sqrt{n} + 1} \left( \frac{1}{2} + 2C \sum_{i=1}^{n-1} \frac{1}{i} \right) \right) \right]
\geq |a - b| \sqrt{\theta'} \left[ \phi(p) - \frac{1}{\sqrt{n} + 1} \left( \frac{1}{2} + 2C \sum_{i=1}^{n-1} \frac{1}{i} \right) \right],
\]

completing the proof of Lemma 2.3.

\[ \star \]

**Lemma 2.4** There exists \( \beta \geq 0 \) such that for all \( n \) for all \( p \in [0, 1] \):

\[ \sqrt{n} \psi_n(p) - |a - b| \sqrt{\theta'} \phi(p) \leq -\frac{\beta \ln n}{\sqrt{n}}. \]

**Proof** The proof consist of a combination of two arguments.

- In the recursive formula (3), if we maximize first over \( \delta \) we obtain for the right hand side approximately:

\[
\psi_n(p) + \max_{\sigma} \left\{ \frac{1}{2}[(\sigma - \theta)d - |\sigma - \theta]| + \sigma \sigma' f_{\sigma}(p) O \left( \frac{1}{\sqrt{n}} \right) \right\},
\]

where \( f_{\sigma}(p) \) is a uniformly bounded function of \( p \) and \( \sigma \). For \( n \) sufficiently large this attains its maximum at \( \sigma = \theta \) (since \( |d| < 1 \)). Therefore, one should expect that the maximum over \( \sigma \) in (3) is reached at \( \sigma = \theta \) from some \( n_0 \) on.

- By taking \( \sigma = \theta \) in (3) we can use repeatedly the right part of (4) to prove that

\[
\psi_n(p) \leq |a - b| \sqrt{\theta'} \left( \phi(p) + \frac{K}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{1}{i} \right),
\]

similar to what we did in proving Lemma 2.3.
However, the two arguments are interlinked in such a way that it is very difficult to use Lemma 2.2 as it is although we have to repeat many elements of its proof. Therefore we shall sketch the proof leaving the details, most of which appear in a very similar form in the proof of Lemma 2.2 (see the proof of (iii) in the Appendix).

- If for some $n$

$$\psi_n(p) \leq |a - b| \sqrt{\theta^2 \left( \phi(p) + \frac{\tilde{K}}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{1}{i} \right)} ,$$

then by (3):

$$\psi_{n+1}(p) \leq \frac{1}{\sqrt{n + 1}} \max_{\sigma} \left\{ \frac{1}{2} |(\sigma - \theta)d - | \sigma - \theta | | + |a - b| \max_{\delta} \left\{ \sqrt{\theta^2} \sqrt{n} \left( \phi(p - \delta/\sigma') + \phi(p + \delta/\sigma') \right) \right\} + \frac{\tilde{K}}{\sqrt{n + 1}} \sum_{i=1}^{n-1} \frac{1}{i} .$$

- If in the second maximum we replace $\sigma'\phi(p - \delta/\sigma') + \sigma\phi(p + \delta/\sigma)$ by its second order approximation: $\phi(p) - \delta^2/(2\sigma\sigma'\phi(p))$, the maximum is attained at:

$$\sigma_0 = \frac{\sigma\sigma'\phi(p)}{\sqrt{\theta} \sqrt{n}}$$

and it equals to:

$$\sqrt{\theta^2} \sqrt{n} \phi(p) + \frac{\sigma\sigma'\phi(p)}{2\sqrt{\theta} \sqrt{n}} .$$

- By repeating almost exactly the proof of (iii) of Lemma 2.2 (in the Appendix) we show that the original maximum over $\delta$ is less or equal to the maximum of the approximation plus $\tilde{K}/n$, for some $\tilde{K} > 0$.

- Thus:

$$\psi_{n+1}(p) \leq \frac{1}{\sqrt{n + 1}} \max_{\sigma} \left\{ \frac{1}{2} |(\sigma - \theta)d - | \sigma - \theta | | + |a - b| \left( \sqrt{\theta^2} \sqrt{n} \phi(p) + \frac{\sigma\sigma'\phi(p)}{2\sqrt{\theta} \sqrt{n}} \right) \right\} + \frac{\tilde{K}}{\sqrt{n + 1}} \sum_{i=1}^{n-1} \frac{1}{i} + \frac{\tilde{K}}{\sqrt{n + 1} 2n} .$$

- Choose $n_0$ such that for $n > n_0$ and for $p \in [0, 1]$, the maximum is obtained at $\sigma = \theta$. For such $n$:

$$\psi_{n+1}(p) \leq |a - b| \sqrt{\theta^2} \phi(p) + \left( \frac{\sqrt{n + 1/2}}{\sqrt{n + 1}} \right) \phi(p) + \frac{\tilde{K}}{\sqrt{n + 1}} \left( \sum_{i=1}^{n-1} \frac{1}{i} + \frac{1}{2n} \right) .$$

- Since if $n_0$ is large enough: $(\sqrt{n + 1/2} - \sqrt{n + 1}) \phi(p) \leq \tilde{K}/2$ (assuming $\tilde{K} \geq 1$), we finally get:

$$\psi_n(p) \leq |a - b| \sqrt{\theta^2} \left( \phi(p) + \frac{\tilde{K}}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{1}{i} \right) .$$
for all \( n \geq n_0 \). So if we enlarge \( \tilde{K} \), if necessary, to make this inequality hold for
\( n \leq n_0 \), we just showed that by induction it holds for every \( n \), completing the proof of
Lemma 2.4 and of Theorem 2.1.

We want now to extend the result of Theorem 2.1 to a larger class of games. As a first
step we consider the class generated from the game of Theorem 2.1 by two simple operations
which preserve the strategic aspects of the game: adding a constant to or multiplying by
positive constant all payoffs of one matrix.

**Proposition 2.5** (i) Adding a constant to all payoffs of one matrix, adds the same linear
function of \( p \) to all value functions of the game, i.e., If \( \tilde{A} = A + c \) and \( \tilde{B} = B + d \) then
for all \( n \) and for all \( p \in [0, 1] \),

\[
\tilde{u}(p) = u(p) + cp + dp' \\
\Cav \tilde{u}(p) = \Cav u(p) + cp + dp' \\
\tilde{v}_n(p) = v_n(p) + cp + dp'
\]

(ii) For any two constants \( c > 0 \) and \( d > 0 \), If \( \tilde{A} = cA \) and \( \tilde{B} = dB \) then:

\[
\tilde{u}(p) = (pc + p'd)u \left( \frac{pc}{pc + p'd} \right) \\
\Cav \tilde{u}(p) = (pc + p'd)\Cav u \left( \frac{pc}{pc + p'd} \right) \\
\tilde{v}_n(p) = (pc + p'd)v_n \left( \frac{pc}{pc + p'd} \right)
\]

for all \( n \) and for all \( p \in [0, 1] \).

**Proof.** Part (i) is straightforward. To prove (ii) observe that if \( (\sigma, \tau) \) is a pair of strategies
for the two players (in the nonrevealing game \( \Delta(p) \) or in \( \Gamma_n(p) \) or in \( \Gamma_\infty(p) \)) and if we denote
by \( H(\sigma, \tau; p) \) and \( \tilde{H}(\sigma, \tau; p) \) the expected payoffs in the \( A, B \) game and in the \( \tilde{A}, \tilde{B} \) game,
respectively, then:

\[
\tilde{H}(\sigma, \tau; p) = pE_\sigma(\text{payoffs} \mid \tilde{A}) + p'E_\tau(\text{payoffs} \mid \tilde{B}) \\
= pcE_\sigma(\text{payoffs} \mid A) + p'dE_\tau(\text{payoffs} \mid B) \\
= (pc + p'd) \left[ \frac{pc}{pc + p'd} \right] E_\tau(\text{payoffs} \mid A) + \frac{p'd}{pc + p'd} E_\tau(\text{payoffs} \mid B) \\
= (pc + p'd)H(\sigma, \tau; \frac{pc}{pc + p'd})
\]

As a particular case of (ii), when \( c = d \), we have the obvious fact that multiplying all
payoffs by the same positive constant multiply all value functions by the same constant. We
shall refer to the transformation of type (i) and (ii) as standard transformations. By
Proposition 2.5 a transformation of type (i) leaves \( \sqrt{n}(v_n - \Cav u)(p) \) unchanged while a
transformation (ii) transforms it to \( \sqrt{n}(pc + p'd)(v - \Cav u) \left( \frac{pc}{pc + p'd} \right) \).
Proposition 2.6 If a game satisfies the conclusion of the Main Theorem, then any game obtainable from it by standard transformations also satisfies this conclusion.

Proof It is clear that (1) is invariant under (i) since it leaves \( v_n - \text{Cav } u \) as well as \( K(A, B, p) \) unchanged. Let us prove that (1) is invariant under transformations of type (ii): Assume that \( A, B \) are the matrices of a game satisfying (1) and let \( c > 0 \), \( d > 0 \), \( \tilde{A} = cA \), and \( \tilde{B} = dB \). Labeling with all the functions of the transformed game we have to show that for some \( \tilde{C} > 0 \):

\[
\left| \tilde{v}_n(p) - \text{Cav } \tilde{u}(p) - K(\tilde{A}, \tilde{B}, p)\phi(\tilde{q}(p)) \frac{1}{\sqrt{n}} \right| \leq \tilde{C} \frac{\ln n}{n},
\]

By Proposition 2.5 we have:

\[
\tilde{v}_n(p) - \text{Cav } \tilde{u}(p) = \frac{(pc + p'd)(v_n - \text{Cav } u)}{pc + p'd}.
\]

Also clearly \( t_A = t_A, t_B = t_B \) and \( \delta A, A = \delta(A, B) \). Therefore if \( K(A, B, p) = 0 \) then also \( K(\tilde{A}, \tilde{B}, p) = 0 \) and (6) follows from (1) with \( \tilde{C} = C(pc + p'd) \). If \( \delta(A, B) = 1 \) then (clearly \( \tilde{s} = s \)):

\[
K(\tilde{A}, \tilde{B}, p) = \left| t_A - t_B \right| \sqrt{ss'} | p\Delta_A - p'\Delta_B |
= \left| t_A - t_B \right| \sqrt{ss'} | pc\Delta_A - p'd\Delta_B |
\]

Also, by definition \( \tilde{q}(p) = \tilde{q} \) such that \( \tilde{t}_p = \tilde{q}t_A + \tilde{q}'t_B \), where \( \tilde{t}_p \) is player II's optimal strategy in \( p\tilde{A} - p'\tilde{B} = pcA + p'dB \). Clearly \( \tilde{t}_p \) is also the optimal strategy in \( \frac{pc}{pc+p'd} A + \frac{p'd}{pc+p'd} B \) which is, by definition, \( t_p \) for \( \tilde{p} = \frac{pc}{pc+p'd} p \). It follows that \( t_p = \tilde{q}t_A + \tilde{q}'t_B \) and therefore \( \tilde{q}(p) = q(\tilde{p}) \). Substituting this in (6) we get:

\[
\left| \tilde{v}_n(p) - \text{Cav } \tilde{u}(p) - K(\tilde{A}, \tilde{B}, p)\phi(\tilde{q}(p)) \frac{1}{\sqrt{n}} \right|
= \left| (pc + p'd)v_n(\tilde{p}) - \text{Cav } u(\tilde{p}) - | t_A - t_B | \sqrt{ss'} | pc\Delta_A + p'd\Delta_B | \phi(q(\tilde{p})) \frac{1}{\sqrt{n}} \right|
= \left| pc + p'd \right| v_n(\tilde{p}) - \text{Cav } u(\tilde{p}) - | t_A - t_B | \sqrt{ss'} | \tilde{p}\Delta_A + \tilde{p}'\Delta_B | \phi(q(\tilde{p})) \frac{1}{\sqrt{n}} \right|
= \left| pc + p'd \right| v_n(\tilde{p}) - \text{Cav } u(\tilde{p}) - K(A, B, \tilde{p})\phi(q(\tilde{p})) \frac{1}{\sqrt{n}} \leq (pc + p'd)C \frac{\ln n}{\sqrt{n}},
\]

where \( C \) is the constant satisfying (1) for the \( A, B \) game.

From Theorem 2.1 and Proposition 2.6 we have:

Corollary 2.7 All games where player I has a completely mixed strategy which is optimal in both \( A \) and \( B \) satisfy the conclusion of the Main Theorem.

Proof All such games are obtainable by standard transformation from the games in Theorem 2.1.
3 Partitioning \([0, 1]\) into subintervals

In proving the main theorem we partition \([0, 1]\) into subintervals according to the character of the function \(u(p)\) as follows.

**Proposition 3.1** For any game, the interval \([0, 1]\) can be divided into finitely many subintervals of the following types:

(i) **Intervals in which** \(\text{Cav } u(p) = v_1(p)\).

(ii) **Intervals on the interior of which** \(u(p) = \text{Cav } u(p) < v_1(p)\) and \(u(p)\) is linear.

(iii) **Intervals in which** \(u(p) = \text{Cav } u(p)\) is strictly concave.

(iv) **Intervals** \([a, b]\) in which \(u(a) = \text{Cav } u(a), u(b) = \text{Cav } u(b), u(p) < \text{Cav } u(p) < v_1(p)\) in the open interval \((a, b)\).

One can further assume that, on the interior of each interval, all order relations between elements of \(\Delta(p)\) remain the same, and \(v_1(p)\) is linear.

**Proof** It is clear that the above list exhausts all cases regarding the linearity of \(u(p)\) and its relationship to \(\text{Cav } u(p)\) and \(v_1(p)\). There are finitely many subintervals since all functions involved are piecewise rational and \(v_1(p)\) is piecewise linear.

**Remark** An interval of type (iv) can be divided into two subintervals \([a, c]\) and \([c, b]\) so that \(u(p) = \text{Cav } u(p)\) holds at one end of each interval \((a \text{ and } b)\) and strict inequality holds in the semi-open intervals \((a, c]\) and \([c, b)\). These two subtypes of intervals can be obtained from each other by interchanging the names of the matrices \(A\) and \(B\). Hence, it is enough to consider only one subtype and replace type (iv) by

(iv') **Intervals** \([a, b]\) in which \(u(a) = \text{Cav } u(a), u(p) < \text{Cav } u(p) < v_1(p)\) in the semi-open interval \((a, b]\).

**Corollary 3.2** (i) In each subinterval, \(u(p)\) and \(\text{Cav } u(p)\) are rational (i.e. equal to a rational fraction whose denominator does not vanish in the subinterval). (ii) In each subinterval, \(u(p)\) is either linear, or has its second derivative bounded away from zero.

**Proof** Since the order relations between elements of \(\Delta(p)\) completely determine the 'formula' for the value of \(u(p)\), and this is either a saddle point, hence linear in \(p\), or completely mixed, hence a rational fraction with numerator of degree \(\leq 2\) and denominator of degree \(\leq 1\). So anyway, after dividing out common factors, it is such a rational fraction, where the denominator, if of degree \(1\), does not divide the numerator. Since \(u(p)\) is bounded, this implies that the denominator does not vanish in the subinterval. This proves (i) for \(u(p)\) and for \(\text{Cav } u(p)\), since it is either linear or equal to \(u(p)\).

To prove (ii), observe that if the degree of the denominator zero, \(u(p)\) is a parabola, hence \(u''(p)\) is constant, while if this degree is 1, then division shows that \(u(p)\) differs by a linear function from \(A/(p - p_0)\), with \(A \neq 0\) and \(p_0 \in [0, 1]\) outside the subinterval and so \(u''(p) = 2A/(p - p_0)^3\) is bounded away from zero.
Reduction of the case of type (ii) subintervals

In this subsection we eliminate those sub cases of the type (ii) subintervals which are covered by other cases and determine the remaining sub cases to be treated. More precisely we prove:

Proposition 3.3 Except for the “normal distribution case” (thus: for games which are not covered by Corollary 2.7), intervals $[a, b]$ of type (ii) occur only for games (up to symmetries):

$$A = \begin{pmatrix} \ 0 & 0- \\ + & 0- \end{pmatrix} \quad B = \begin{pmatrix} \ 0 & + \\ - & - \end{pmatrix}$$

with

$$a_{21}b_{22} - a_{12}b_{21} \leq 0.$$

In this case, $b \leq \frac{b_{22}}{b_{12} - a_{12}},$ with TL as saddle point for $\Delta(p)$.

Proof \quad $u(p) = \text{Cav} u(p)$ is linear in the interval $[a, b]$ with $0 \leq a < b \leq 1$. By subtracting appropriate constants from all elements of $A$ and from all elements of $B$ we may assume that $u(p) = \text{Cav} u(p) = 0$ on $[a, b]$ and, since $\text{Cav} u(p) = 0$ on $[a, b]$, $u(p) \leq 0$ everywhere on $[0, 1]$. The first condition is:

$$\forall p \in [a, b], \quad V = \begin{pmatrix} p \alpha_{11} + p' \beta_{11} & p \alpha_{12} + p' \beta_{12} \\ p \alpha_{21} + p' \beta_{21} & p \alpha_{22} + p' \beta_{22} \end{pmatrix} = 0.$$

By the further clause in Proposition 3.1, we have to consider the following two cases:

Case 1 \quad $u(p) = 0$ corresponds to a constant saddle point - say, by permutation of strategies, TL. This case is then reduced as follows:

Proposition 3.4 In Case 1, the payoff matrices have the following configuration:

$$A = \begin{pmatrix} \ 0 & - \\ + & 0- \end{pmatrix} \quad B = \begin{pmatrix} \ 0 & + \\ - & - \end{pmatrix}$$

Proof \quad Clearly, $a_{11} = b_{11} = 0$.

- If both $a_{21} \leq 0$ and $b_{21} \leq 0$ then by playing Left player II can guarantee 0 in $u_1(p)$. So this is a type (i) interval and the error term is 0.

- So at least one of $a_{21}$ and $b_{21}$ is strictly positive. Assume without loss of generality that $a_{21} > 0$. Then necessarily $a_{12} \leq 0$ and $a_{22} \leq 0$ (since otherwise $u(A) > 0$).

- It must be that $b_{21} < 0$ since otherwise $p \alpha_{21} + p' \beta_{21} > 0$ for all interior $p$ and so 0 could not be a saddle point in a subinterval.
- Eliminating cases corresponding to type (i) intervals i.e. cases in which \( v_1(p) = u(p) = 0 \) in the interval \([a, b]\), we impose now the condition that there exists \( p_0 \in [0, 1] \) for which \( v_1(p_0) > 0 \):

\[
\text{Val.} \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22} \\
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix} > 0,
\]

which implies that at least one of the \( 2 \times 2 \) sub matrices has a positive value. Since both \( A \) and \( B \) have non positive value, one of the following inequalities must hold:

1. \( \text{Val.} \begin{pmatrix}
  a_{11} & a_{12} \\
  b_{11} & b_{12}
\end{pmatrix} > 0 \)

2. \( \text{Val.} \begin{pmatrix}
  a_{11} & a_{12} \\
  b_{21} & b_{22}
\end{pmatrix} > 0 \)

3. \( \text{Val.} \begin{pmatrix}
  a_{21} & a_{22} \\
  b_{11} & b_{12}
\end{pmatrix} > 0 \)

4. \( \text{Val.} \begin{pmatrix}
  a_{21} & a_{22} \\
  b_{21} & b_{22}
\end{pmatrix} > 0 \)

Cases (1) and (4) are excluded since they imply \( u(p_0) > 0 \) for some \( p_0 \in [0, 1] \). Case (2) is also excluded since \( a_{11} = 0 \) and \( b_{21} < 0 \). Hence Case (3) holds and since \( a_{22} \leq 0 \) this implies \( b_{12} > 0 \).

We conclude that the only configuration left to be considered is that stated in the proposition.

We proceed to study further consequences of the case under consideration:

\[
\text{Val.} \begin{pmatrix}
  0 & \Delta_{12}(p) \\
  \Delta_{21}(p) & \Delta_{22}(p)
\end{pmatrix} = \text{Val.} \begin{pmatrix}
  0 & pa_{12} + p'b_{12} \\
  pa_{21} + p'b_{21} & pa_{22} + p'b_{22}
\end{pmatrix} \leq 0,
\]

holds for all \( p \in [0, 1] \) and the (TL) entry 0 is a saddle point for \( p \in [a, b] \). By the last proposition \( \Delta_{21}(p) \) is increasing in \( p \) and \( \Delta_{12}(p) \) is decreasing in \( p \). Thus if 0 is a saddle point at some \( p > 0 \), it is also a saddle point in the whole interval \([0, p]\); that is, \( a = 0 \).

If \( p_0 \) satisfies \( \Delta_{21}(p_0) = 0 \) (i.e. \( p_0 = -b_{21}/(a_{21} - b_{21}) \)) then \( \Delta_{12}(p_0) \leq 0 \) (otherwise we would have \( u(p_0 + e) > 0 \) for sufficiently small \( e \)). This is equivalent to:

\[
a_{21}b_{12} - a_{12}b_{21} \leq 0
\]

Finally \( b \), the right end of the subinterval, must satisfy

\[
a_{12}p + b_{12}p' \geq 0 \quad \text{i.e.} \quad b \leq \frac{b_{12}}{b_{12} - a_{12}}.
\]

This concludes the proof of Proposition 3.1 in Case 1.

Case 2 \( u(p) = 0 \) corresponds to a completely mixed solution, everywhere in the interior of \([a, b]\).
The value of a completely mixed 2 x 2 matrix is proportional to its determinant. Thus if
its value is 0 so must be its determinant and this, in turn, means the matrix $ij$-th element
is of the form $x_iy_j$.

So consider the case where $\Delta(p)$ has a completely mixed solution with value 0 in an
interval $[a, b]$. This implies

$$(a_{11}p + b_{11}p')(a_{22}p + b_{22}p') - (a_{12}p + b_{12}p')(a_{21}p + b_{21}p') = 0, \quad (8)$$

for $p \in [a, b]$. This implies that the coefficients of $p^2, (p')^2$ and $pp'$ are all zero. These
coefficients are $\det(A), \det(B)$ and $(a_{11}b_{22} + a_{22}b_{11}) - (a_{12}b_{21} + a_{21}b_{12})$ respectively. Thus we have:

- $\det(A) = 0$ and therefore $A = \begin{pmatrix} \alpha_1 \alpha_2 \\ \bar{\alpha}_1 \bar{\alpha}_2 \end{pmatrix}$

- $\det(B) = 0$ and therefore $B = \begin{pmatrix} \beta_1 \beta_2 \\ \bar{\beta}_1 \bar{\beta}_2 \end{pmatrix}$

- $(a_{11}b_{22} + a_{22}b_{11}) - (a_{12}b_{21} + a_{21}b_{12}) = 0$ and therefore

$$\alpha_1 \alpha_2 \bar{\beta}_1 \bar{\beta}_2 + \bar{\alpha}_1 \bar{\alpha}_2 \beta_1 \beta_2 - \alpha_1 \bar{\alpha}_2 \beta_1 \beta_2 - \bar{\alpha}_1 \alpha_2 \beta_1 \bar{\beta}_2 = 0.$$

The last equality is equivalent to $(\alpha_1 \beta_1 - \bar{\alpha}_1 \bar{\beta}_1)(\alpha_2 \beta_2 - \bar{\alpha}_2 \bar{\beta}_2) = 0$ and therefore

- Either $\det \begin{pmatrix} \alpha_1 & \bar{\alpha}_1 \\ \beta_1 & \bar{\beta}_1 \end{pmatrix} = 0$ which implies \{ $\alpha_1 = xy \bar{\alpha}_1 = xy$  \\
  $\beta_1 = \bar{xy} \bar{\beta}_1 = \bar{xy}$  

- Or $\det \begin{pmatrix} \alpha_2 & \bar{\alpha}_2 \\ \beta_2 & \bar{\beta}_2 \end{pmatrix} = 0$ which implies \{ $\alpha_2 = xy \bar{\alpha}_2 = xy$  \\
  $\beta_2 = \bar{xy} \bar{\beta}_2 = \bar{xy}$  

There are therefore two cases to consider:

- $A = \begin{pmatrix} xy & x\bar{\alpha}_2 \\ x\bar{\alpha}_1 & xy \bar{\alpha}_2 \end{pmatrix}$ and $B = \begin{pmatrix} \bar{xy} & \bar{x}\bar{\beta}_2 \\ \bar{x}\beta_1 & xy \bar{\beta}_2 \end{pmatrix}$

- $A = \begin{pmatrix} xy & x\bar{\alpha}_1 \\ x\bar{\alpha}_2 & xy \bar{\alpha}_1 \end{pmatrix}$ and $B = \begin{pmatrix} \bar{xy} & \bar{x}\beta_1 \\ \bar{x}\bar{\beta}_2 & xy \bar{\beta}_1 \end{pmatrix}$

In the first case let $\alpha = x\alpha_2$, $\alpha = x\bar{\alpha}_2$, $\beta = \bar{x}\beta_1$, $\beta = \bar{x}\bar{\beta}_1$, $\theta = y$ and $\bar{\theta} = \bar{y}$. Similarly in
the second case let $\alpha = x\alpha_1$, $\alpha = x\bar{\alpha}_1$, $\beta = \bar{x}\beta_1$, $\beta = \bar{x}\bar{\beta}_1$, $\tau = y$ and $\bar{\tau} = \bar{y}$. The two cases
to be considered are then rewritten as:

(*) $A = \begin{pmatrix} \theta & \bar{\theta} \\ \bar{\theta} & \theta \end{pmatrix}$ and $B = \begin{pmatrix} \theta \beta & \theta \bar{\beta} \\ \bar{\theta} \bar{\beta} & \theta \bar{\beta} \end{pmatrix}$

(**) $A = \begin{pmatrix} \alpha \tau & \alpha \bar{\tau} \\ \bar{\alpha} \tau & \bar{\alpha} \bar{\tau} \end{pmatrix}$ and $B = \begin{pmatrix} \beta \tau & \beta \bar{\tau} \\ \bar{\beta} \tau & \bar{\beta} \bar{\tau} \end{pmatrix}$

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We claim that (**) is impossible. In fact, in this case the game

\[
\Delta(p) = \left( \begin{array}{c}
\tau(\alpha p + \beta p') \\
\tau(\alpha p + \beta p')
\end{array} \right)
\left( \begin{array}{c}
\tau(\alpha p + \beta p') \\
\tau(\alpha p + \beta p')
\end{array} \right)
\]

being completely mixed, satisfies, for all \( p \in [a, b] \),

\[
\begin{align*}
\tau(\alpha p + \beta p') &\geq 0 \\
\tau(\alpha p + \beta p') &\leq 0
\end{align*}
\]

or the opposite inequalities, with strict inequalities in the interior of \([a, b]\). But then player II guarantees zero in \( \Gamma_1(p) \) by the mixed strategy \( \left( \frac{|p|}{|p|+|p'|}, \frac{|p|}{|p|+|p'|} \right) \).

In the remaining case (**), the game

\[
\Delta(p) = \left( \begin{array}{c}
\theta(\alpha p + \beta p') \\
\theta(\alpha p + \beta p')
\end{array} \right)
\left( \begin{array}{c}
\theta(\alpha p + \beta p') \\
\theta(\alpha p + \beta p')
\end{array} \right)
\]

satisfies, as above, for all \( p \in [a, b] \) (permuting columns if necessary),

\[
\begin{align*}
\theta(\alpha p + \beta p') &\geq 0 \\
\theta(\alpha p + \beta p') &\leq 0
\end{align*}
\]

with \( \theta \bar{\theta} < 0 \).

Changing, if necessary, the signs of \( \alpha, \bar{\alpha}, \beta \) and \( \bar{\beta} \) and multiplying all these constants by \( |\theta| + |\bar{\theta}| \), we may assume that \( 0 < \theta < 1 \) and \( \bar{\theta} = -\theta' \). Inequalities (10) are then written as

\[
\begin{align*}
\alpha p + \beta p' &\geq 0 \\
\bar{\alpha} p + \bar{\beta} p' &\leq 0
\end{align*}
\]

for \( p \in [a, b] \),

with strict inequalities for \( p \) in the interior \((a, b)\).

Recall now that the conditions \( u(p) \leq 0 \) for all \( p \) and \( v_1(p_0) > 0 \) for some \( p_0 \in [0, 1] \) imply

\[
\text{Val.} \left( \begin{array}{cc}
a_{11} & a_{12} \\
b_{11} & b_{12}
\end{array} \right) > 0 \quad \text{or} \quad \text{Val.} \left( \begin{array}{cc}
a_{21} & a_{22} \\
b_{21} & b_{22}
\end{array} \right) > 0,
\]

which in this case is

\[
\text{Val.} \left( \begin{array}{cc}
\theta \alpha & \theta \bar{\alpha} \\
\theta \beta & \theta \bar{\beta}
\end{array} \right) > 0 \quad \text{or} \quad \text{Val.} \left( \begin{array}{cc}
\bar{\theta} \alpha & \bar{\theta} \bar{\alpha} \\
\bar{\theta} \beta & \bar{\theta} \bar{\beta}
\end{array} \right) > 0.
\]

Since \( \theta \bar{\theta} < 0 \), this is equivalent to:

\[
\text{Val.} \left( \begin{array}{cc}
\alpha & \bar{\alpha} \\
-\beta & -\beta
\end{array} \right) > 0 \quad \text{or} \quad \text{Val.} \left( \begin{array}{cc}
-\alpha & -\bar{\alpha} \\
\beta & \beta
\end{array} \right) > 0. \quad (12)
\]

Since \( u(p) \leq 0 \) for all \( p \in [0, 1] \) the inequality

\[
(\alpha p + \beta p')(\bar{\alpha} p + \bar{\beta} p') \leq 0,
\]

must hold for all $p$ since otherwise $\Delta(p)$ would have a positive row and hence a positive value. For the product of the two linear forms to be nonpositive for all $p$ they must either be of different sign on the whole interval $[0, 1]$ or they must cross 0 at the same value of $p$. The second possibility is equivalent - up to permutation of columns - to $\bar{\alpha} = -\lambda \alpha$ and $\bar{\beta} = -\lambda \beta$ with $\lambda \geq 0$, which is in contradiction with inequalities (12) (since both matrices would have a value zero.) Thus it must be that either

$$a \rho + b \beta' \geq 0 \geq \bar{a} \rho + \bar{\beta} \beta' \forall \rho \in [0, 1]$$

or

$$a \rho + b \beta' \leq 0 \leq \bar{a} \rho + \bar{\beta} \beta' \forall \rho \in [0, 1].$$

The second case is excluded by inequality (11), hence

$$\alpha \geq 0 \ ; \ \beta \geq 0 \ ; \ \bar{\alpha} \leq 0 \ ; \ \bar{\beta} \leq 0.$$  

Inequalities (12) become now

$$\alpha \beta \neq \bar{\alpha} \beta,$$  

hence in particular $\alpha - \bar{\alpha} > 0$ and $\beta - \bar{\beta} > 0$. Dividing thus the elements of $A$ by $\alpha - \bar{\alpha}$ and the elements of $B$ by $\beta - \bar{\beta}$ and letting $\bar{a} = \alpha / (\alpha - \bar{\alpha}), \bar{b} = \beta / (\beta - \bar{\beta})$, we obtain after this transformation (of type (ii) in Proposition 2.5):

$$A = \left( \begin{array}{cc} \theta' a & -\theta' a' \\ -\theta a & \theta a' \end{array} \right) \quad \text{and} \quad B = \left( \begin{array}{cc} \theta' b & -\theta' b' \\ -\theta b & \theta b' \end{array} \right)$$

with $0 < \theta < 1$, $0 \leq a \leq 1$, $0 \leq b \leq 1$ and, by (13), $a \neq b$. This is the case considered in Theorem 2.1. Hence Case (*) is covered by Corollary 2.7.

This concludes the proof of Proposition 3.3.

4 Aggregation of bounds on subintervals

In this section we prove that error term estimations can be made separately on subintervals of $[0, 1]$ for each of the types listed in Proposition 3.1. Although in this paper we are concerned with $2 \times 2$ payoff matrices, the following Proposition is stated and proved for any game with two states of nature and arbitrary strategy sets for the two players.

If $(x, y)$ denote the mixed strategies used by player I in $A$ and $B$, at the first stage, we denote by $\bar{p}$ the (random) conditional probability of $A$ given the first move $i$ of player I (and given $p$) that is, $\bar{p} = P(A \mid i)$. The first stage expected payoff is denoted by $h(x, y; j)$, where $j \in J$ is the move of player II in the first stage, and the expectation with respect to the probability measure determined by $(x, y)$ (and $p$) is denoted by $E_{x, p}$.

**Proposition 4.1** Let $\{\delta_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive real numbers. Assume that the interval $[0, 1]$ is partitioned into finitely many proper subintervals, such that for every subinterval, say $[a, b]$, there exists a sequence of concave functions $W_n(p)$ on $[a, b]$ such that for all $n \geq n_0$: For all $p \in [a, b], \ldots$
(1) \[ W_{n+1}(p) + C(\delta_{n+1} - \delta_n) \geq \max_{\{x,y\}\in\mathcal{D}([0,1])} \min_{\mathcal{J} \in \mathcal{J}} \{ h(x,y; j) + E_{x,y} W_n(\beta) \}, \]

(2) \[ n v_\infty(p) \leq W_n(p) \leq n v_\infty(p) + C \delta_n. \]

At every common boundary point \( p_0 \) between two subintervals,

(3) \[ W'_n(p_0^+) - W'_n(p_0^-) \leq C \delta_n. \]

(with the obvious notation for the right and left derivatives at \( p_0 \).)

With \( \Delta_n = W_n(p_0^+) - W_n(p_0^-) \) (with \( \Delta_{-1} = 0 \) then,

(4) \[ \sum_{i=1}^{n} | \Delta_i - \Delta_{i-1} | \leq C \delta_n. \]

Then, for some \( C > 0, \)

\[ v_n(p) \leq v_\infty(p) + C \frac{\delta_n}{n} \quad \forall p \in [0,1] \text{ and } \forall n \geq 1. \]

Further, condition (4) is satisfied in particular if

- either \( | \Delta_n | \leq C(\delta_n - \delta_{n-1}) ; \forall n \geq n_0 \)
- or \( \Delta_n \) is eventually monotonic in \( n \).

Proof of the 'Further' clause.

- The first condition yields \( | \Delta_t - \Delta_{t-1} | \leq | \Delta_t | + | \Delta_{t-1} | \leq C(\delta_t - \delta_{t-2}) \forall t > n_0 \), hence

\[
\sum_{t=1}^{n} | \Delta_t - \Delta_{t-1} | \leq C(\delta_n + \delta_{n-1} - 2\delta_{n_0}) + \sum_{t=1}^{n_0} | \Delta_t - \Delta_{t-1} | \leq 2C\delta_n + K \leq \left(2C + \frac{K}{\delta_1}\right) \delta_n
\]

- The second condition yields, if \( \Delta_t \) is monotonic (say increasing) for \( t \geq t_0 \), that

\[
\sum_{t=1}^{n} | \Delta_t - \Delta_{t-1} | \leq L + | \Delta_n - \Delta_{t_0} | \leq L' + | \Delta_n |
\]

For some constant \( L' \) (which can be selected independently of the boundary point \( p_0 \) since there are only finitely many of them).

\[ ^1 \text{It is in this form that the Proposition is mostly applied, since for all functional forms of } W_n \text{ typically used, like polynomials in fractional powers of } n \text{ and of } \ln n, \text{ the presence of highest order terms assures that the differences } \Delta_n \text{ will be eventually monotonic in } n. \]
Since by condition (2) of the Proposition we have \( |\Delta_n| \leq C \delta_n \), it follows that

\[
\sum_{i=1}^{n} |\Delta_i - \Delta_{i-1}| \leq C' \delta_n, \quad \text{with} \quad C' = C + \frac{C}{\delta_1}.
\]

The rest of the proof of the Proposition consists of several steps.

**Step 0:** Reduction to \( n_0 = 0 \).

Since the numbers \( \delta_n \) \( (1 \leq n \leq n_0) \) are positive, all inequalities, in the Proposition, except (1) become valid for all \( n \) by increasing sufficiently the constant \( C \). To be able to use the same argument for (1), replace first \( \delta_n \) by \( \delta'_n = \frac{\Delta_n + \Delta_{n-1}}{\Delta_n + \Delta_{n-1}} \delta_n \) for \( n \leq n_0 \). All assumptions are still valid with the new sequence \( \delta'_n \), which is now in addition strictly increasing for \( n \leq n_0 \). Thus, the previous argument applies now also for inequality (1).

**Step 1:** Making the functions \( W_n \) continuous.

For a boundary point \( p_0 \), choose minimal non-decreasing sequences \( y_n \geq 0 \) and \( z_n \geq 0 \) such that \( y_n - z_n = \Delta_n \) for all \( n \). Thus,

\[
y_n = \Delta_n^+ + \alpha_n, \quad \text{and} \quad z_n = \Delta_n^- + \alpha_n, \quad \text{with} \quad \alpha_n = \alpha_{n-1} + \max[\Delta_{n-1}^+ - \Delta_n^+, \Delta_{n-1}^- - \Delta_n^-].
\]

Letting \( s_n = y_n + z_n = |\Delta_n| + 2\alpha_n \), we get

\[
s_n = y_n + z_n
\]

\[
= |\Delta_n| + 2\alpha_{n-1} + \max[|\Delta_{n-1}| + |\Delta_{n-1}| - |\Delta_n| - |\Delta_n| - |\Delta_{n-1}| - |\Delta_{n-1}| + |\Delta_n|
\]

\[
= |\Delta_{n-1}| + 2\alpha_{n-1} + \max[\Delta_{n-1} - \Delta_n, \Delta_n - \Delta_{n-1}]
\]

\[
= s_{n-1} + |\Delta_n - \Delta_{n-1}|
\]

Hence (letting \( \Delta_0 = 0 \), since there is no loss of generality in assuming \( W_0 \equiv 0 \))

\[
s_n = \sum_{i=1}^{n} |\Delta_i - \Delta_{i-1}| \leq C \delta_n.
\]

The above procedure defines sequences \( \{y_n(p_0)\} \) and \( \{z_n(p_0)\} \) for every boundary point \( p_0 \), which are bounded by \( C \delta_n \). Let also \( y_n(0) = z_n(0) = 0 \).

Now, for every subinterval \([a, b]\), replace the corresponding sequence of functions \( W_n \) by \( W_n^*(p) = W_n(p) + A_n a + B_n \) where \( A_n a + B_n = z_n(a) \) and \( A_n b + B_n = y_n(b) \) that is,

\[
A_n = \frac{y_n(b) - z_n(a)}{b - a} \quad \text{and} \quad B_n = \frac{b z_n(a) - ay_n(b)}{b - a}.
\]

Then we have:

(i) The functions \( W_n^* \) are continuous on \([0, 1]\) since

\[
W_n^*(p_0) = W_n(p_0^+) + y_n(p_0) = W_n(p_0^+) + z_n(p_0) = W_n^*(p_0).\]
(ii) The functions $A_n p + B_n$ are non-decreasing (in $n$) on $[a, b]$, by linearity, being so at both $a$ and $b$.

(iii) By (ii) and the linearity of $A_n p + B_n$, the functions $W_n^*$ also satisfy condition (1) of the Proposition.

(iv) Since $y_n + z_n \leq C\delta_n$, the functions $W_n^*$ also satisfy condition (2) of the Proposition (with $C' = 2C$).

(v) Our formula for $A_n$ and the bounds on $y_n$ and $z_n$ yield that

$$|W_n^*(a+) - W_n^*(a+)| \leq \frac{C\delta_n}{b-a}$$

and

$$|W_n^*(b-) - W_n^*(b-)| \leq \frac{C\delta_n}{b-a}.$$

Hence, for every common boundary point $p_0$, condition (3) of the Proposition yields that

$$W_n^*(p_0+) - W_n^*(p_0-) \leq C'\delta_n,$$

with $C' = C \left(1 + \frac{1}{\ell}\right)$,

where $\ell$ is the length of the shortest subinterval.

Thus, we have obtained a new sequence of functions $\{W_n\}$ (the $\{W_n^*\}$), which are furthermore continuous on $[0, 1]$, and satisfy conditions (1), (2) and (3) of the Proposition with a modified constant $C$.

**Step 2: Making the functions $W_n$ concave.**

By Step 1 we assume now that the functions $W_n$ are continuous on $[0, 1]$, and satisfy conditions (1), (2) and (3) of the Proposition. For an appropriate constant $D \geq 0$, let

$$\bar{W}_n = W_n + D_n \sum_{p_0} \left(\frac{p}{p_0} \wedge \frac{p'}{p_0'}\right),$$

with

$$\frac{D_n}{p_0} + W_n'(p_0-) \geq -\frac{D_n}{p_0} + W_n'(p_0+) + D$$

i.e.

$$D_n \geq p_0 p'_0 (D + W_n'(p_0^+) - W_n'(p_0^-)).$$

Thus, using condition (3) of the Proposition, we may choose

$$D_n = \frac{1}{4} (D + C\delta_n).$$

Now, the functions $\bar{W}_n$ are still continuous on $[0, 1]$ and for every boundary point $p_0$,

$$\bar{W}_n'(p_0-) \geq \bar{W}_n'(p_0+) + D. \tag{14}$$

In particular, since $\bar{W}_n$ is obtained from $W_n$ by adding linear functions on every interval $[a, b]$, the concavity is preserved on each subinterval. Since the concavity of $\bar{W}_n$ at all boundary points $p_0$ is guaranteed by inequality (14) we conclude that the functions $\bar{W}_n$ are concave.
on \([0, 1]\). Further, since \(D_n\) is non-decreasing, the same argument as in Step 1 shows that
the functions \(\bar{W}_n\) still satisfy condition (1) of the Proposition. Next, if \(N\) is the number of
common boundary points \(p_0\), we have:

\[
\bar{W}_n(p) \leq \frac{N}{4} (D + C \delta_n) + W_n(p) \leq \frac{N}{4} (D + C \delta_n) + C \delta_n + n_v(p).
\]

So, \(\bar{W}_n\) also satisfy condition (2) of the Proposition with \(\bar{C} = C + \frac{N}{4} (C + \frac{D}{\delta_n})\).

Thus, we can assume now that the functions \(\bar{W}_n\) are concave on \([0, 1]\), satisfy conditions
(1) and (2) of the Proposition and that

\[
\bar{W}_n(p_0- - W_n(p_0+) \geq D \text{ for every boundary point } p_0.
\]

If we replace \(W_n\) by \(\bar{W}_n = W_n + \bar{C} \delta_n\), these properties are preserved. Condition (1)
implies now that \(\bar{W}_n\) satisfy the recursive formula, when the posterior probabilities \(\bar{p}\) are
restricted to remain in the same subinterval as the prior \(p\):

\[
\bar{W}_{n+1}(p) \geq \max_{\{x,y\} \in [a,b]} \min_{j \in J} \left\{ h(x, y; j) + E_{x,y} \bar{W}_n(\bar{p}) \right\}.
\] (15)

**Step 3: Obtaining the recursive formula.**

We prove now that for an appropriate choice of the constant \(D\), the recursive formula
without restrictions on \(\bar{p}\) holds for \(\bar{W}_n\) that is:

\[
\bar{W}_{n+1}(p) \geq \max_{x, y} \min_{\substack{j \in J \\text{opt} \in S_j}} \left\{ h(x, y; j) + E_{x,y} \bar{W}_n(\bar{p}) \right\}.
\] (16)

For \(p\) in a given subinterval \([a, b]\), we will express the fact that any first stage strategy
\((x, y)\) of player I in the recursive formula (16) yields less than some modified strategy which
yields \(\bar{p} \leq b\). Further, in this modification, all values of the posterior probability \(\bar{p}\) either
remain the same or move towards \(b\), so that if before the modification all values satisfied
\(\bar{p} \geq a\), they are still so after the modification. Hence, making first this modification to have
\(\bar{p} \leq b\) and then a similar one to get \(\bar{p} \geq a\) will result in \(\bar{p} \in [a, b]\).

Order the set \(J\) of pure moves of player I such that \(i \leq i' \iff \pi_i \leq \pi_{i'}\), where \(\pi_i\) denotes
the corresponding value of \(\bar{p}\) i.e. \(\pi_i = P_{x_i} \{ A \mid i \}\). Let \(\sigma_i = px_i + py_i\) and \(e_i = (\sum_{i \geq i_0} \sigma_i \pi_i) / (\sum_{i \geq i_0} \sigma_i)\). Choose \(i_0\) such that \(e_{i_0} \leq b < e_{i_0+1}\),
and \(\theta \in [0, 1]\) such that

\[
\frac{\theta \sigma_i \pi_{i_0} + \sum_{i > i_0} \sigma_i \pi_i}{\theta \sigma_i + \sum_{i > i_0} \sigma_i} = b.
\]

Define now the modification of \((x, y)\) to \((\hat{x}, \hat{y})\) as follows:

- For \(i < i_0\), let \(\hat{x}_i = x_i\) and \(\hat{y}_i = y_i\) (hence \(\hat{\sigma}_i = \sigma_i\) and \(\hat{\pi}_i = \pi_i\)).

- For \(i > i_0\) let

\[
\hat{x}_i = \frac{\theta x_{i_0} + \sum_{i > i_0} x_i}{\theta \sigma_{i_0} + \sum_{i > i_0} \sigma_i}, \quad \hat{y}_i = \frac{\theta y_{i_0} + \sum_{i > i_0} y_i}{\theta \sigma_{i_0} + \sum_{i > i_0} \sigma_i}.
\]
From condition (2) of the Proposition we finally conclude that
\[ v_n(p) \leq \frac{1}{n} \tilde{W}_n(p) \leq v_{\infty}(p) + \tilde{C} \delta_n, \]
completing the proof of Proposition 4.1.

**Corollary 4.2** The following conditions are sufficient for the conclusion of the Proposition to hold: (1) and (2) as above and:

(3') \( |D_{p_0}[W_n - n\text{Cav } u]| \leq C\delta_n \), where \( D_{p_0} \) stands for any one sided derivative at \( p_0 \).

(4') \( W_n(p_0) = n\text{Cav } u(p_0) \).

**Proof** Condition (4') implies that \( W_n(p) \) are continuous and hence condition (4) is trivially satisfied (since \( \Delta_n = 0 \)). By condition (3'),
\[ W'_n(p_0+) \leq n v'_{\infty}(p_0+) + C\delta_n \]
and
\[ W'_n(p_0-) \geq n v'_{\infty}(p_0-) - C\delta_n \]
Hence
\[ W'_n(p_0+) - W'_n(p_0-) \leq 2C\delta_n + n(v'_{\infty}(p_0+) - v'_{\infty}(p_0-)) \leq 2C\delta_n, \]
since \( v_{\infty}(p) \) is concave, and condition (3) is also satisfied.

**5 Solutions for the various types of subintervals**

In this section we provide a sequence of functions \( W_n \) and a non-decreasing sequence \( \delta_n \) satisfying the conditions of Corollary 4.2 for the types of intervals specified in Proposition 3.1.

**5.1 Case (i): Intervals in which \( \text{Cav } u(p) = v_1(p) \) is linear**

In this case the functions \( W_n(p) = nv(p) = nv_1(p) \) satisfy conditions of Corollary 4.2 with \( \delta_n = 0 \). In fact conditions (2), (3') and (4') are obviously satisfied. To see that (1) is also satisfied note that since \( W_n(p) \) is linear in the interval \([a, b]\) under consideration, we have
\[ \hat{p} \in [a, b] \Rightarrow E_{x, y} W_n(\hat{p}) = W_n(p). \]
Also, by the definition of \( v_1(p) \),
\[ \max_{(x, y)} \min_{j \in J} h(x, y; j) = v_1(p). \]
Hence
\[ W_{n+1}(p) = (n + 1)v_1(p) \]
\[ = \max_{(x, y)} \min_{j \in J} h(x, y; j) + \max_{(x, y)} \max_{\{E_{x, y} W_n(\hat{p})\}} \]
\[ \geq \max_{\{E_{x, y} \in [a, b]\}} \min_{j \in J} \{h(x, y; j) + E_{x, y} W_n(\hat{p})\}. \]
5.2 Case (ii): The remaining sub case

In view of Proposition 3.3, the only case remaining to be considered in the type (ii) intervals \( u(p) = \text{Cav} u(p) < u_1(p) \) and \( u(p) \) is linear), is the case (up to symmetries) in which the matrices are of the form:

\[
A = \begin{pmatrix}
0 & a_{12} \\
\frac{a_{21}}{a_{22}} & a_{22}
\end{pmatrix},
\quad B = \begin{pmatrix}
0 & b_{12} \\
\frac{b_{21}}{b_{22}} & b_{22}
\end{pmatrix},
\]

where \( b_{21} < 0 < b_{12}, \ a_{12} \leq 0, \ a_{21} b_{12} \leq a_{12} b_{21} \) and the interval \([a, b]\) under consideration satisfies \( b \leq \pi = b_{12}/(b_{12} - a_{12}) \). Note that in this case \( u(p) = \text{Cav} u(p) = 0 \) on the interval \([0, \pi]\).

**Proposition 5.1** \( W_n(p) = 0, \ \forall n, \) satisfy the conditions of Corollary 4.2 on \([0, \pi]\), with \( \delta_n = 0 \) for all \( n \).

**Proof** Since conditions (2), (3') and (4') are trivially satisfied, we have only to show that the zero functions satisfy the recursive formula (1) restricted to \([0, \pi]\).

Let \((1 - x, x)\) and \((1 - y, y)\) be the first stage mixed strategies of player I in games \( A \) and \( B \) respectively. If \( p_2 \) is the conditional probability of \( A \) given that the second row is played, we have:

\[
p_2 \leq \pi \Rightarrow \frac{px}{px + p'y} \leq \frac{b_{12}}{b_{12} - a_{12}} \Rightarrow px \frac{a_{12}}{b_{12}} + p'y \geq 0.
\]

Next, since \( b_{12} > 0 \), we have \( a_{21} \leq a_{12} b_{21}/b_{12} \) and so

\[
px a_{21} + p'y b_{21} \leq px \frac{a_{12} b_{21}}{b_{12}} + p'y b_{21} = b_{21} \left( px \frac{a_{12}}{b_{12}} + p'y \right) \leq 0,
\]

since \( b_{21} \leq 0 \) and \( px \frac{a_{12}}{b_{12}} + p'y \geq 0 \). This means that the left strategy of player II yield a payoff \( px a_{21} + p'y b_{21} \leq 0 \), hence the right hand side of (1) is non positive and therefore the recursive inequality is satisfied as claimed.

\[
\]

5.3 Case (iii): Intervals in which \( u(p) = \text{Cav} u(p) \)

is strictly concave

Denote by \( \sigma \) and \( \tau \) one stage mixed strategies of players I and II respectively and by \( h(\sigma, \tau) \) the corresponding first stage payoff.

**Proposition 5.2** In intervals on which \( \text{Cav} u(p) = u(p) \) is strictly concave, the functions \( W_n(p) = nu(p) \) satisfy all the conditions of Corollary 4.2 with \( \delta_n = \ln n \) and an appropriate constant \( C \).
**Proof**  Condition (2), (3') and (4') are trivially satisfied. To prove (1) note first that since u is strictly concave on the closed interval [a, b], there are constants c > 0 and K > 0 such that \( u''(p) \leq -c \) for all p in [a, b] and

\[
\min_{\sigma, \tau} Eh(\sigma, \tau) \leq u(p) + KE\|\tilde{p} - p\| \leq \text{Cav} u(p) + KE\|\tilde{p} - p\|.
\]

Subtract \( W_n(p) \) from both sides of (1) and rewrite it as

\[
W_{n+1}(p) - W_n(p) + C(\delta_{n+1} - \delta_n) \geq \max_{z,y \in [a,b]} \min_j \{ h(x, y; j) \} + E_{x,y} W_n(\tilde{p}) - W_n(p). \tag{17}
\]

By the definition of \( K \) it is enough to prove

\[
W_{n+1}(p) - W_n(p) C(\delta_{n+1} - \delta_n) \geq \text{Cav} u(p) + H_n(p), \tag{18}
\]

where

\[
H_n(p) = \max_{a \leq p_r \leq p, \ p \leq p_t \leq b} \left\{ 2K \left( \frac{p_r - p)(p - p_t)}{p_r - p} + \frac{p_r - p}{p_r - p} W_n(p_t) + \frac{p - p_t}{p_r - p_t} W_n(p_r) - W_n(p) \right) \right\}.
\]

Since \( W_{n+1}(p) - W_n(p) = u(p) = \text{Cav} u(p) \) and \( \delta_{n+1} - \delta_n = \ln \frac{1 + n}{n} \geq 1/(2n) \), to prove condition (1) it suffices to prove that for some constant \( M > 0 \), the inequality \( H_n(p) \leq M/n \) holds for all \( n \) and all \( p \) in [a, b].

Let \( \xi = p_r - p \), \( \eta = p - p_t \) and rewrite \( H_n(p) \) as:

\[
H_n(p) = \max_{0 \leq \xi \leq b - p, \ 0 \leq \eta \leq p - a} \left\{ 2K \left( \frac{\xi \eta}{\xi + \eta} + \frac{\xi \eta^2}{\xi + \eta} W''(\tilde{p}_t) + \frac{\xi^2 \eta}{\xi + \eta} W''(\tilde{p}_r) \right) \right\},
\]

for some \( \tilde{p}_t \) and \( \tilde{p}_r \) satisfying: \( p - \eta \leq \tilde{p}_t \leq p \) and \( p \leq \tilde{p}_r \leq p + \xi \).

Now \( W''(\tilde{p}_t) = nu''(\tilde{p}_t) \leq -nc \) hence,

\[
H_n(p) \leq \max_{\xi, \eta} \left\{ 2K \left( \frac{\xi \eta}{\xi + \eta} - nc \xi \eta \right) \right\}.
\]

For given \( \xi + \eta \), the bracket is maximized at \( \xi = \eta \) or at \( \xi \eta = 0 \) hence

\[
H_n(p) \leq \max_{\xi} \left\{ K\xi - nc \xi^2 \right\} = \frac{K^2}{2nc}.
\]

Taking \( M = K/(2c) \), this completes the proof of Proposition 5.2.
5.4 Case (iv') : Intervals in which \( u(a) = \text{Cav} u(a) \) and \( u(p) < \text{Cav} u(p) \) in \((a, b]\).

To complete the proof of our Main Theorem, there remains thus only to exhibit a sequence of functions \( W_n \) and constants \( \delta_n \leq n^{-1/2} \) satisfying the conditions of Corollary 4.2 for each of the types of intervals \([a, b]\) in which \( u(p) < \text{Cav} u(p) \) in the semi-open interval \( a < p \leq b \), and \( u(a) = \text{Cav} u(a) \). Adding constants to the two games, we may assume that \( \text{Cav} u = 0 \) on the interval \([a, b]\). As it turns out, the result (i.e. the best bound for the error term) is different for \( a = 0 \) (or \( a = 1 \)) and for \( 0 < a < 1 \). Accordingly we split the treatment of this case into two parts.

The case \( a = 0 \) (or \( a = 1 \)).

Consider an interval \([0, b]\) on which \( \text{Cav} u(p) = 0 \), \( u(0) = 0 \) and \( u(p) < 0 \) for \( 0 < p \leq b \).

**Proposition 5.3** For such an interval, there is \( C > 0 \) such that the functions

\[
W_n(p) = C \left( \ln(1 + np) + \sum_{i=1}^{n} \frac{1}{i^2} \right),
\]

satisfy the recursive condition (1) of Proposition 4.1 with \( \delta_n = 0 \).

**Proof** By Corollary 3.2, the highest possible degree of tangency between \( u(p) \) and (the linear) \( \text{Cav} u(p) \) (at \( p = 0 \)) is 2. It follows that there is \( \theta > 0 \) such that

\[
u(p) \leq -\theta p^2, \; \forall p \in [0, b]. \tag{19}\]

The posterior probability \( \tilde{p} \), as player I has only 2 moves, has two values which we denote as \( p_L \) and \( p_R \), with \( p_L \leq p \leq p_R \) and the corresponding probabilities (when \( p_L < p_R \)) are \((p_R - p)/(p_R - p_L)\) and \((p - p_L)/(p_R - p_L)\) respectively, thus

\[
E_{x, y} W_n(p) = \frac{p_R - p}{p_R - p_L} W_n(p_L) + \frac{p - p_L}{p_R - p_L} W_n(p_R),
\]

where the right hand side is defined as \( W_n(p) \) if \( p_L = p \). Recall that the excess of the first stage payoff over \( u(p) \) is bounded by

\[
\min_j h(x, y; j) \leq u(p) + 2a \frac{(p_R - p)(p - p_L)}{p_R - p_L},
\]

where the quotient at the right hand side is defined as 0 if \( p_L = p_R \). The surplus over \( W_n(p) \) of the right hand side of the recursive formula is thus less or equal to

\[
\max_{0 \leq p_L \leq p} \left\{ 2a \frac{(p_R - p)(p - p_L)}{p_R - p_L} + \frac{p_R - p}{p_R - p_L} W_n(p_L) + \frac{p - p_L}{p_R - p_L} W_n(p_R) - W_n(p) \right\}. \tag{20}\]

Subtracting \( W_n(p) \) from both sides of the recursive formula in condition (1), it is thus enough to prove the existence of \( C \) such that

\[
W_{n+1}(p) - W_n(p) \geq -\theta p^2 + H_n(p), \tag{21}\]
where the function $H_n(p)$ is given by (20). Letting $\xi = p_r - p$ and $\eta = p - p_l$, we rewrite $H_n(p)$ as

$$H_n(p) = \max_{0 \leq \xi \leq b - p, 0 \leq \eta \leq p - a} \left\{ 2\alpha \frac{\xi \eta}{\xi + \eta} + \frac{\xi}{\xi + \eta} (W_n(p - \eta) - W_n(p)) + \frac{\eta}{\xi + \eta} (W_n(p + \xi) - W_n(p)) \right\}.$$  

For the proposed $W_n$ this is

$$H_n(p) = \max_{0 \leq \xi \leq b - p, 0 \leq \eta \leq p} \left\{ 2\alpha \frac{\xi \eta}{\xi + \eta} + C \frac{\xi}{\xi + \eta} \ln \left( 1 - \frac{\eta}{p + \frac{1}{n}} \right) + C \frac{\eta}{\xi + \eta} \ln \left( 1 + \frac{\xi}{p + \frac{1}{n}} \right) \right\}. \tag{22}$$

Letting $x = \xi/(p + 1/n)$ and $y = \eta/(p + 1/n)$ we get

$$H_n(p) = \max_{0 \leq x \leq \frac{b - p}{p + 1/n}, 0 \leq y \leq \frac{p}{p + 1/n}} C \left\{ \frac{2\alpha}{C} \left( \frac{1}{n} + \frac{x}{y} \right) \ln \left( 1 - \frac{y}{x + y} \right) + \frac{x}{x + y} \ln(1 - y) + \frac{y}{x + y} \ln(1 + x) \right\}. \tag{24}$$

We need now the following property of the ln function whose proof is given in the Appendix.

**Lemma 5.4** There exists $\beta > 0$ such that for all $c > 0$,

$$\max_{0 \leq x} \left\{ c \frac{x}{x + y} \ln(1 - y) + \frac{x}{x + y} \ln(1 + x) \right\} \leq \beta c^2 \quad \text{for all } 0 \leq y \leq 1.$$

**Remark** Note that our interest in this bound is for low values of $c$. In fact, for large $c$ this is a rather poor bound: since the left hand side is the maximum of linear functions of $c$ with slope $\leq 1$, it follows that for $c \geq 1/(2\beta)$, the maximum is at most $c - 1/(4\beta)$ (the value on the highest line of slope $1$ below the parabola $y = \beta c^2$). Hence, the bound in the Lemma can be improved to $c^2/(c + \epsilon)$ for some positive constant $\epsilon$.

Applying this Lemma in (24) we have

$$H_n(p) \leq \frac{4\alpha^2 \beta}{C} \left( \frac{1}{n} + \frac{1}{n} \right)^2 \tag{25}$$

Inserting $W_n(p)$ in the left hand side of (21) and using this bound for $H_n(p)$, it is enough to show the existence of $C > 0$ such that:

$$C \ln \left( 1 + \frac{p}{1 + np} \right) + \frac{C}{(n + 1)^2} \geq -\theta p^2 + \frac{4\alpha^2 \beta}{C} \left( \frac{1}{n} + \frac{1}{n} \right)^2. \tag{26}$$

In fact,
• Choosing \( C \geq 4\alpha^2\beta/\theta \), it is enough to guarantee
\[
C \ln \left( 1 + \frac{p}{1 + np} \right) + \frac{C}{(n + 1)^2} \geq \frac{8\alpha^2\beta p}{Cn} + \frac{4\alpha^2\beta}{Cn^2}
\]

• Choosing \( C \geq 4\alpha\sqrt{\beta} \), it is enough to guarantee (since \( 4/(n + 1)^2 \geq 1/n^2 \))
\[
C \ln \left( 1 + \frac{p}{1 + np} \right) \geq \frac{8\alpha^2\beta p}{Cn}
\]

• Since \( \ln(1 + x) \geq x - x^2/2 \) it is enough to show
\[
\frac{Cp}{1 + np} \geq \frac{8\alpha^2\beta p}{Cn} + \frac{Cp^2}{2(1 + np)^2}
\]

• Since \( p/(1 + np) < 1 \), half of the left hand side is greater than the second term in the right hand side therefore it is enough to guarantee
\[
\frac{Cp}{2(1 + np)} \geq \frac{8\alpha^2\beta p}{Cn},
\]
which is satisfied if \( C^2 \geq 32\alpha^2\beta \) (since \((1 + np)/n \leq 2\)).

• The proof is thus completed by choosing \( C = \max\{4\alpha^2\beta/\theta, \sqrt{32\alpha^2\beta}\} \).

\[\Box\]

**Corollary 5.5** If \( u(p) < \text{Cav} \ u(p) \) in the open interval \((0, 1)\) then, the order of the error term is bounded by \( O(\ln n/n) \).

**Proof** Apply Proposition 4.1 for the case in which there are only two subintervals; \([0, b]\) and \([b, 1]\), both of the type treated in Proposition 5.3. The functions
\[
\tilde{W}_n(p) = C \left[ \ln(1 + np) + \ln(1 + n(1 - p)) + \sum_{i=1}^{n} \frac{1}{i^2} \right],
\]
satisfy all the conditions of Proposition 4.1 the conclusion of which is then:
\[
v_n(p) \leq \text{Cav} \ u(p) + \frac{C}{n} \left( \ln(1 + np) + \sum_{i=1}^{n} \frac{1}{n^2} \right).
\]

\[\Box\]

In fact it can be further shown that the order is \( O(1/n) \) if there is no tangency at \( p = 0 \). However, if \( \text{Cav} \ u(p) \) is tangent to \( u(p) \) at \( p = 0 \) or at \( p = 1 \), this is then the best bound (see Mertens-Sorin-Zamir Exercise 10f, page 305).
The case $0 < a < 1$.

Note first that a result similar to Proposition 5.3 can be proved also for $a > 0$ namely that the functions

$$W_n(p) = C \left( \ln \left( 1 + n(p - a) \right) + \sum_{i=1}^{n} \frac{1}{i^2} \right),$$

satisfy the recursive formula in condition (1) (and also condition (1) of Proposition 4.1.) However, this cannot be used to prove that the error term is of the order of $\ln n/n$ since to do that, we have to apply Proposition 4.1 with $\delta_n = \ln n/n$. But then, condition (3) of the proposition does not hold since $W'_n(a) - W'_n(a-)$ is of the order of $n$.

Consider an interval $[a, b]$ on which $\text{Cav} u(p) = 0$, $u(a) = 0$ and $u(p) < 0$ for $a < p < b$. The analogue of Proposition 5.3 is now,

**Proposition 5.6** For such an interval, there is $K > 0$ such that the functions

$$W_n(p) = K \left[ \ln \left( 1 + n^{1/3}(p - a) \right) - \frac{p - a}{b - a} \ln \left( 1 + n^{1/3}(b - a) \right) \right] + \nu(x)_n(p)$$

satisfy the conditions of Corollary 4.2 with $\delta_n = n^{1/3}$.

**Proof** Conditions (2) and (4') are clearly satisfied. For condition (3') we have

$$\left| W'_n(b) - \nu'(\rho)_n(b) \right| \leq K \ln(1 + n) \quad \left| W'_n(a) - \nu'(\rho)_n(a) \right| \leq Kn^{1/3},$$

so the condition is satisfied.

In proving condition (1) observe that the condition is invariant to an addition of a linear function of $p$ on $[a, b]$. Also, since $\nu(x)_n(p) = 0$ on the interval $[a, b]$, it follows that it suffice to prove condition (1) for $\tilde{W}_n(p) = K \ln \left( 1 + n^{1/3}(p - a) \right)$; Let $x = p - a$ and $F_n(x) = \ln(1 + n^{1/3}x)$. Proceeding as in the proof of the previous Proposition, using the tangency condition and subtracting $W_n(p)$ from both sides of the recursion condition (1) this becomes:

$$\tilde{W}_{n+1}(p) - \tilde{W}_n(p) + C \left[ (n + 1)^{1/3} - n^{1/3} \right] \geq -\theta x^2 + H_n(p),$$

where

$$\frac{1}{K} H_n(p) = \max_{0 \leq \xi \leq b - p} \left\{ \frac{2}{K} \frac{\xi \eta}{\xi + \eta} + \frac{\xi}{\xi + \eta} \left( F_n(x - \eta) - F_n(x) \right) + \frac{\eta}{\xi + \eta} \left( F_n(x + \xi) - F_n(x) \right) \right\}$$

(27)

Since $\tilde{W}_n(p)$ is increasing in $n$, it is enough to prove the existence of sufficiently large $C$ such that

$$-\theta x^2 + H_n(p) \leq C \left[ (n + 1)^{1/3} - n^{1/3} \right],$$

or, equivalently, we have to prove that $n^{2/3}[H_n(p) - \theta x^2]$ is bounded from above. To bound this expression we again make use of Lemma 5.4 with the variables

$$y = \frac{\eta}{x + n^{-1/3}} \quad \text{and} \quad z = \frac{\xi}{x + n^{-1/3}}.$$
Note that

\[ 0 \leq y \leq \frac{x}{x + n^{-1/3}} < 1 \text{ and } 0 \leq x \leq \frac{b - p}{x + n^{-1/3}}. \]

By Lemma 5.4 we then obtain

\[ \frac{1}{K} \mathcal{H}_n(p) \leq \beta c^2 \text{ with } c = \frac{2\alpha}{K}(x + n^{-1/3}). \]

So, we have to show that

\[ n^{2/3} \left[ \frac{4\alpha^2 \beta}{K} \left( x + n^{-1/3} \right)^2 - \theta x^3 \right] \]

is bounded above.

Using the fact \((a + b)^2 \leq 2(a^2 + b^2)\), it is enough to show that

\[ n^{2/3} \left[ \frac{8\alpha^2 \beta}{K} \left( x^2 + n^{-2/3} \right) - \theta x^3 \right] \]

is bounded above.

This is guaranteed if we choose the constant \(K\) to satisfy \(K \geq 8\alpha^2 \beta / \theta\).

This concludes the proof of the Main Theorem. It turns out that the bound in Proposition 5.6 is tight; Example 6.1 in section 6.3 provides a game with an error term of order \(O(n^{-2/3})\).

6 Appendix

6.1 Proof of Lemma 2.2

Proof Recall that:

\[ \phi(p) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}p^2} \text{ where } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, dx = 1. \]

The following is obtained by straightforward differentiation:

\[ \phi'(p) = -xp; \quad x_p' = 1/\phi(p); \quad \phi''(p) = -1/\phi(p) = -x_p' \]

\[ \phi^{(3)}(p) = -x_p/\phi^3(p); \quad \phi^{(4)}(p) = -(1 + 2x_p^2)/\phi^5(p) \]

We first expand \( \phi(p, \xi) = \sigma\phi(p - \sqrt{\xi} \xi) + \sigma\phi(p + \sqrt{\xi} \xi) \):

\[ \phi(p, \xi) = \phi(p) - \frac{\xi^2}{2\phi(p)} + \frac{\xi^3}{\sigma \sqrt{\sigma^2 \phi^2(p)}} \left[ \frac{\sigma^2 1 + 2x_p^2 - \sqrt{\sigma/\sigma^2} \xi \eta}{\sigma^2 \phi^3(p - \sqrt{\sigma/\sigma^2} \xi \eta)} + \frac{(\sigma^2 1 + 2x_p^2 - \sqrt{\sigma/\sigma^2} \xi \eta)}{\sigma \phi^3(p - \sqrt{\sigma/\sigma^2} \xi \eta)} \right] \]

for some \(0 \leq \eta \leq 1\).
If we consider only the first two terms, let \( \hat{\phi}_n(p, \xi) = \phi(p) - \xi^2/(2\phi(p)) \) and take this instead of \( \phi_n \) in the maximand in (4) we have (relaxing the constraints on \( \xi \)),

\[
\frac{1}{\sqrt{n + 1}} \max_{\xi \geq 0} \left\{ \xi + \sqrt{n}(\phi(p) - \frac{\xi^2}{2\phi(p)}) \right\} = \frac{1}{\sqrt{n + 1}} \left\{ \xi + \sqrt{n}(\phi(p) - \frac{\xi^2}{2\phi(p)}) \right\}_{\xi = \hat{\phi}_n(p, \xi)} = \frac{\sqrt{n + 1}/(2\sqrt{n})}{\sqrt{n + 1}} \phi(p)
\]

Since \( 1 \leq \frac{\sqrt{n + 1}/(2\sqrt{n})}{\sqrt{n + 1}} \leq 1 + \frac{1}{n^2} \), we have that for all \( p \in [0, 1] \) and for all \( n \):

\[
\left| \frac{1}{\sqrt{n + 1}} \max_{\xi \geq 0} \left\{ \xi + \sqrt{n}\hat{\phi}(p, \xi) \right\} - \phi(p) \right| \leq \frac{1}{n^2}
\]  

(28)

From this we intend to prove the lemma by proving the following three statements:

(i) There exist \( n_0 \) and a constant \( C_1 > 0 \) such that for each \( n \geq n_0 \) there exists \( p_n < 1 \) s.t. for \( p_n \leq p \leq p'_n \) we have \( \xi_0 = \phi(p)/\sqrt{n} \in D(p) \) i.e.

\[
\phi(p)/\sqrt{n} \leq \sqrt{\sigma^2 - \sigma^2 p} \quad \text{for } n \geq n_0, \quad p_n \leq p \leq p'_n,
\]

(29)

while for \( p < p_n \) or \( p > p'_n \) we have \( \sqrt{n/(n + 1)}\phi(p) \geq \phi(p) - C_1/n^2 \).

(ii) There exists \( \hat{C} > 0 \) s.t. for all \( n \geq n_0 \) and for all \( p_n \leq p \leq p'_n \),

\[
\sqrt{n/n + 1}(\xi_0 - \hat{\phi}_n(p, \xi_0)) \geq -\hat{C}/n\sqrt{n}
\]

(30)

(iii) There exists \( \hat{K} > 0 \) s.t. for all \( n \) and \( p \in [0, 1] \),

\[
\frac{1}{\sqrt{n + 1}} \max_{\xi \in D(p)} \left\{ \xi + \sqrt{n}\hat{\phi}_n(p, \xi) \right\} \leq \frac{1}{\sqrt{n + 1}} \max_{\xi \geq 0} \left\{ \xi + \sqrt{n}\hat{\phi}_n(p, \xi) \right\} + \frac{\hat{K}}{n\sqrt{n}}
\]

(31)

The proof of the Lemma is then obtained as follows: Inequality (28), (i) and (ii) imply that for \( n \geq n_0 \),

\[
\frac{1}{\sqrt{n + 1}} \max_{\xi \in D(p)} \left\{ \xi + \sqrt{n}\hat{\phi}_n(p, \xi) \right\} \geq \frac{1}{\sqrt{n + 1}} \left\{ \xi_0 + \sqrt{n}\hat{\phi}_n(p, \xi_0) \right\} \geq \phi(p) - \frac{\hat{C}}{n\sqrt{n}} - \frac{1}{n^2}
\]

if \( p_n \leq p \leq p'_n \) and

\[
\frac{1}{\sqrt{n + 1}} \max_{\xi \in D(p)} \left\{ \xi + \sqrt{n}\phi(p, \xi) \right\} \geq \frac{1}{\sqrt{n + 1}} \left\{ 0 + \sqrt{n}\phi(p, 0) \right\} \geq \frac{\sqrt{n}}{\sqrt{n + 1}} \phi(p) \geq \phi(p) - \frac{C_1}{n^2}
\]

if \( p < p_n \) or \( p > p'_n \). On the other hand, by inequality (28) and (iii) we have that for \( n \geq n_0 \)

\[
\frac{1}{\sqrt{n + 1}} \max_{\xi \in D(p)} \left\{ \xi + \sqrt{n}\phi(p, \xi) \right\} \leq \frac{1}{\sqrt{n + 1}} \max_{\xi \geq 0} \left\{ \xi + \sqrt{n}\phi(p, \xi) \right\} + \frac{\hat{K}}{n\sqrt{n}} \leq \phi(p) + \frac{\hat{K}}{n\sqrt{n}} \frac{1}{n^2}.
\]

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Enlarging the constants $\tilde{C}$ and $\tilde{K}$ to incorporate the terms $1/n^2$ and $C_1/n^2$ and to make these inequalities hold for $n \leq n_0$, we obtain the inequality (4).

It remains thus to prove the above three statements.

**Proof of (1)** For $n = 1, 2, \ldots$ define $p_n$ by:

\[ e^{-\frac{1}{2}x_p^2} = \frac{1}{n\sigma'} \text{ and } p_n \leq \frac{1}{2} \]

then

\[ p_n \leq p \leq p_n' \iff x_p^2 \leq x_{p_n}^2 \iff e^{-\frac{1}{2}x_p^2} \geq e^{-\frac{1}{2}x_{p_n}^2} = \frac{1}{n\sigma'} \]

We prove first that $p_n \leq p \leq p_n'$ implies $\phi(p)/\sqrt{n} \in D(p)$. That is we prove that $e^{-\frac{1}{2}x_p^2} \geq \frac{1}{n\sigma'}$ implies:

\[ \frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2}x_p^2} \leq \min \left\{ \sqrt{\frac{\sigma'}{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_p} e^{-\frac{1}{2}x^2} dx, \sqrt{\frac{\sigma}{\sigma'}} \cdot \frac{1}{\sqrt{2\pi}} \int_{x_p}^{\infty} e^{-\frac{1}{2}x^2} dx \right\} \]

Since $\sqrt{\sigma\sigma'} \leq \min(\sqrt{\sigma'}/\sigma, \sqrt{\sigma}/\sigma')$, it is enough to prove that $e^{-\frac{1}{2}x_p^2} \geq \frac{1}{n\sigma'}$ implies:

\[ \frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2}x_p^2} \leq \sqrt{\frac{\sigma'}{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-|x_p|} e^{-\frac{1}{2}x^2} dx \]

We may thus assume that $x_p \leq 0$, i.e. $p \leq 1/2$. Letting $y = x_p \leq 0$ we therefore want to prove that for $n$ large enough $e^{-\frac{1}{2}y^2} \geq \frac{1}{n\sigma'}$ implies

\[ \frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2}y^2} \leq \sqrt{\frac{\sigma'}{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}x^2} dx \]

(32)

Choose $n_1$ such that $n \geq n_1$ implies that (32) holds for $-1 \leq y \leq 0$. Note that the left hand side of (32) is concave on $-1 \leq y \leq 0$ while the right hand side is convex, therefore it is enough to choose $n_1$ big enough so as to make the left hand side at $y = -1$ and its slope there smaller than the right hand side at $y = -1$ and its slope there respectively (see Figure 1).

---

**Figure 1**

![Figure 1](image-url)
For $y \leq -1$ note that $\int_{-\infty}^{y} e^{-\frac{1}{2}x^2} \, dx \geq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$ therefore in that case it is enough to show that: $e^{-\frac{1}{2}y^2} \geq \frac{1}{n\sigma^2}$ implies $\frac{\sqrt{\sigma^2}}{2y} \geq \frac{1}{\sqrt{n}}$, for $n$ sufficiently large. In fact choose $n_2$ to satisfy $n_2 \geq \frac{1}{\sigma^2}$ and $\frac{\sqrt{\sigma^2}}{2\sqrt{2}\ln(\sigma^2 n_2)} > \frac{1}{\sqrt{n_2}}$, then for $n \geq n_2$ we have:

$$e^{-\frac{1}{2}y^2} \geq \frac{1}{n\sigma^2} \implies |y| = -y \leq \sqrt{2\ln(\sigma^2 n)}$$

$$\implies \frac{\sqrt{\sigma^2}}{2y} \geq \frac{\sqrt{\sigma^2}}{2\sqrt{2}\ln(\sigma^2 n)} > \frac{1}{\sqrt{n}}.$$ 

The required $n_0 = n_0 = \max(n_1, n_2)$ which will satisfy (29) for the above definition of $p_n$.

To finish the proof of (i) note that $p < p_n$ or $p > p'_n$ imply $e^{-\frac{1}{2}y^2} < \frac{1}{n\sigma^2}$ and therefore $\phi(p) \leq \frac{1}{n\sigma^2\sqrt{2\pi}}$ and so:

$$\phi(p) - \sqrt{n + 1} \phi(p) \leq \frac{1}{\sigma^2\sqrt{2\pi}} \frac{1}{n(n + 1)} < C_1 \frac{1}{n^2}$$

for an appropriate $C_1 > 0$.

**Proof of (ii)** Using the expansion of $\phi_n(p, \xi)$:

$$\sqrt{n + 1} (\phi(p, \xi_0) - \phi(p, \xi_0)) = \sqrt{n + 1} (A(p, \sigma) \xi_0^2 - B(p, \sigma) \xi_0^4),$$

where

$$A(p, \sigma) = \frac{(\sigma' - \sigma) x_p}{6\sqrt{\sigma^2 \phi^2(p)}}$$

$$B(p, \sigma) = \frac{1}{24} \left[ \frac{\sigma^2 (1 + 2x_p^2)}{\sigma' \phi^3(p - \sqrt{\eta^2 / \sigma'^2} \xi_0)} + \frac{\sigma'^2 (1 + 2x_p^2)}{\sigma' \phi^3(p + \sqrt{\sigma' / \sigma^2} \xi_0)} \right],$$

where $0 \leq \eta \leq 1$.

For the first term note that $|\phi(p)x_p|$ is bounded on [0, 1] so:

$$\sqrt{n + 1} \xi_0^2 (\sigma' - \sigma) x_p = \sqrt{n + 1} \left( \frac{\phi(p)}{\sqrt{n}} \right)^3 (\sigma' - \sigma) x_p$$

$$\geq \frac{1}{6\sqrt{\sigma^2}} \frac{|\phi(p)| x_p (\sigma' - \sigma)}{\sqrt{n}} \geq -\frac{C_1}{n\sqrt{n}},$$

for some $C_1 > 0$. It remains to show that for $n$ sufficiently large the second term is uniformly bounded by $-C_2/(n\sqrt{n})$ for some $C_2 > 0$. We shall show that each of the two parts of that term is bounded that way. Consider the first which we denote by $B_1$. Let $\delta = \eta \sqrt{\sigma'/\sigma^2}(\phi(p)/\sqrt{n})$ (hence $0 \leq \delta \leq \sqrt{\sigma'/\sigma^2}(\phi(p)/\sqrt{n})$). Using the definition of $\phi(p)$ and the inequality $1 + 2x^2 \leq 8e^{x^2/4}$ we get, for $p \leq 1/2$ (the case $p \geq 1/2$ is treated similarly):

$$B_1 \leq \frac{1}{3\sqrt{2\pi}} \frac{\sigma^2}{\sigma} \sqrt{\frac{n}{n + 1}} \cdot \frac{1}{n^2} \exp\left(\frac{7}{4} x_p^2 - 2x_p^2\right).$$
We shall show that there exist $\hat{Q} > 0$ such that $x_p - x_{p-\delta} \leq \hat{Q}/\sqrt{n}$ holds for sufficiently large $n$ and for all $p_n \leq p \leq p'$. Since $x_p - x_{p-\delta}$ increases in $\delta$ it is enough to show that for $
abla_0 = \sqrt{\sigma/\sigma'} \phi(p)/\sqrt{n}$. Letting $\Delta = x_p - x_{p-\delta}$ and $y = x_p \leq 0$, we want to show that for $n$ sufficiently large if:

$$\sqrt{\frac{\sigma}{\sigma'} \phi(p)} = \frac{1}{\sqrt{n}} \sqrt{\frac{\sigma}{\sigma'}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= \delta_0 = p - (p - \delta_0) = \int_{y-\Delta}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx,$$

then $\Delta \leq \hat{Q}/\sqrt{n}$.

Consider the graph of the normal density $(1/\sqrt{2\pi}) e^{-\frac{1}{2}y^2}$ at $y = -1$ (Figure 2):

![Figure 2](image)

The triangular area formed by the tangent at $y = -1$ is $(1/2)\phi(-1) = 1/2\sqrt{2\pi e}$. By choosing $n$ large enough to have $\sqrt{\sigma/\sigma'} \phi(p)/\sqrt{n} < 1/2\sqrt{2\pi e}$, we know then that $\sqrt{\sigma/\sigma'} \phi(p)/\sqrt{n} = \int_{y-\Delta}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$ and $-1 \leq y \leq 0$ imply that $\Delta < 1$ and hence:

$$\int_{y-\Delta}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \geq \int_{-1-\Delta}^{-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$\geq \phi(-1) \left(1 - \frac{\Delta}{2}\right) \frac{\Delta}{2\sqrt{2\pi e}},$$

thus:

$$\frac{1}{\sqrt{n}} \sqrt{\frac{\sigma}{\sigma'}} \frac{1}{\sqrt{2\pi}} \geq \sqrt{\frac{\sigma}{\sigma'} \phi(p)/\sqrt{n}} = \int_{y-\Delta}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \geq \frac{\Delta}{2\sqrt{2\pi e}},$$

implying $\Delta \leq 2\sqrt{2e\sigma/\sigma'}/\sqrt{n}$.

For $y \leq -1$ we do the same type of evaluation, using again the normal density function for a general value $y \leq -1$ (see Figure 3):

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The triangular area bounded by the tangent at \( y = \frac{\phi(p)}{2|w|} \). Also \( p \geq p_n \) implies \( |y| = |x_p| \leq \sqrt{2 \log(\sigma/\sigma_n)} \leq \frac{1}{2} \sqrt{n} \sqrt{\sigma/\sigma_1} \), for \( n \) sufficiently large. Hence the triangular area is greater or equal to \( \sqrt{\sigma/\sigma_1} \phi(p) = \int_{y-\Delta}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \). This implies that \( \Delta < 1/|y| \) and so the integral can be bounded by the area of the trapezoid:
\[
\sqrt{\sigma/\sigma_1} \phi(p) = \int_{y-\Delta}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \geq \phi(p)(2 - |y|/\Delta) \frac{\Delta}{2},
\]
implying that \( \Delta < \frac{Q}{\sqrt{n}} \) for \( p_n \leq p \leq 1/2 \) and \( n \) sufficiently large.

Now, since \( x_{p-\delta} \leq x_p \leq 0 \), we have:
\[
x_{p-\delta}^2 = (x_p - (x_p - x_{p-\delta}))^2 \leq (x_p - \frac{Q}{\sqrt{n}})^2 = x_p^2 - \frac{2Qx_p}{\sqrt{n}} + \frac{Q^2}{n},
\]
and hence (since \( x_p \leq 0 \)):
\[
\frac{7}{4}x_{p-\delta}^2 - 2x_p^2 = -\frac{1}{4}x_p^2 - \frac{7Qx_p}{2\sqrt{n}} + \frac{7Q^2}{4n} \leq \frac{1}{4}x_p^2 - \frac{7Q}{2}x_p + \frac{7Q^2}{4} \leq Q,
\]
where \( Q > 0 \) is the maximum of the parabola (in \( x_p \)) at the right hand side.

We conclude that there exists \( n_1 \) and \( C_1 > 0 \) such that for \( n \geq n_1 \), \( B_1 \leq C_1/n^2 \) for all \( p_n \leq p \leq p_1 \).

The second term in the expansion (34), the one with \( \xi_k \), is treated in the same way by evaluating \( x_{p+\sqrt{\sigma/\sigma_1}\phi(p)/\sqrt{n}} - x_p \). Noticing that for \( p \leq 1/2 \), \( x_{p+\delta} - x_p \leq x_{p-\delta} - x_p \), the required bound is obtained from the bound for \( A_1 \) (for different \( \delta \)). This completes the proof of (ii).

Proof of (iii) Since \( \phi'(0+) = +\infty \) and \( \phi'(1-) = -\infty \), \( \max_{\xi \in \mathbb{D}(p)} \{ \xi + \sqrt{n}\phi_s(p, \xi) \} \) is not attained at the boundaries and is thus an interior local maximum say at \( \xi_0 \). Equating the first derivative to 0 we have:
\[
1 + \frac{\sqrt{n}}{\sigma_1} \phi'(p - \sqrt{\sigma/\sigma_1\xi_0}) + \frac{\sqrt{\sigma_1}}{\sigma_1} \phi'(p + \sqrt{\sigma/\sigma_1\xi_0}) = 0.
\]
Since $\phi'(p) = -x_p$, using the mean value theorem we get:

$$\frac{1}{\sqrt{\sigma' \sigma n}} = x_{p+\sqrt{\sigma'/\sigma} \xi_0} - x_{p-\sqrt{\sigma'/\sigma} \xi_0} = \left(\sqrt{\sigma'/\sigma} + \sqrt{\sigma'/\sigma}\right) \xi_0 x_p'$$

where $p - \sqrt{\sigma'/\sigma} \xi_0 \leq \tilde{p} \leq p + \sqrt{\sigma'/\sigma} \xi_0$. So (since $x_p' = 1/\phi(p)$) we have $\xi_0 = \phi(\tilde{p})/\sqrt{n}$.

Now:

$$\frac{\phi(\tilde{p})}{\phi(p)} = \exp \left\{ \frac{1}{2} \left( x_p^2 - x_{\tilde{p}}^2 \right) \right\} = \exp \left\{ \frac{1}{2} (x_p + x_{\tilde{p}})(x_p - x_{\tilde{p}}) \right\}.$$ 

Since $x_p$ is increasing in $p$ we have:

$$x_p - x_{\tilde{p}} \leq x_{p+\sqrt{\sigma'/\sigma} \xi_0} - x_{p-\sqrt{\sigma'/\sigma} \xi_0} = \frac{1}{\sqrt{\sigma' \sigma n}},$$

and hence $x_p + x_{\tilde{p}} \leq 2 |x_p| + \frac{1}{\sqrt{\sigma' \sigma n}}$, thus:

$$\frac{\phi(\tilde{p})}{\phi(p)} \leq \exp \left\{ \frac{1}{2} \left( 2 |x_p| + \frac{1}{\sqrt{\sigma' \sigma n}} \right) \right\} \leq L \exp \left( \frac{|x_p|}{\sqrt{\sigma' \sigma n}} \right),$$

for some $L > 0$ (independent of $p$). In particular we have,

$$\xi_0 \leq \frac{L}{\sqrt{n}} \phi(p) \exp \left( \frac{|x_p|}{\sqrt{\sigma' \sigma n}} \right).$$

Coming back to the proof of (iii) we have:

$$\max_{\xi \in D(p)} \left\{ \xi + \sqrt{n} \phi_\sigma(p, \xi) \right\} = \sqrt{n+1} \max_{\xi \geq 0} \left\{ \xi + \sqrt{n+1} \phi_\sigma(p, \xi) \right\} + \frac{x_p}{\sqrt{n+1}} \left( \phi_\sigma(p, \xi_0) - \tilde{\phi}_\sigma(p, \xi_0) \right) \leq \sqrt{n+1} \max_{\xi \geq 0} \left\{ \xi + \sqrt{n+1} \phi_\sigma(p, \xi) \right\} + \frac{x_p}{\sqrt{n+1}} \left( \phi_\sigma(p, \xi_0) - \tilde{\phi}_\sigma(p, \xi_0) \right).$$

It remains therefore to bound the last term by $\tilde{K}/(n\sqrt{n})$. In fact, since the last term in this difference, the one involving $\xi_0$, is negative (that is $-B(p, \sigma)\xi_0^4$, see (34)),

$$\sqrt{\frac{n}{n+1}} \left( \phi_\sigma(p, \xi_0) - \tilde{\phi}_\sigma(p, \xi_0) \right) \leq \frac{\xi_0^3 (\sigma' - \sigma) x_p}{6 \sqrt{\sigma' \sigma} \phi^2(p)} \leq \frac{L^3 (\sigma' - \sigma) (\phi(p) x_p^2)}{6 \sqrt{\sigma' \sigma n} \sqrt{n}} \exp \left( \frac{3 |x_p|}{\sqrt{\sigma' \sigma n}} \right) \leq \frac{1}{n} \frac{L^3 (\sigma' - \sigma)}{\sqrt{\sigma' \sigma}} x_p \exp \left( \frac{-1}{2} x_p^2 + \frac{3 |x_p|}{\sqrt{\sigma' \sigma n}} \right).$$

Since $x_p \exp \left( -\frac{1}{2} x_p^2 + 3 |x_p| / \sqrt{\sigma' \sigma} \right)$ is uniformly bounded for $-\infty \leq x_p \leq \infty$ (and hence for $0 \leq p \leq 1$), the last term is bounded by $\tilde{K}/n\sqrt{n}$ for some $\tilde{K} > 0$, completing the proof.
of (iii) and the proof of Lemma 2.2.

Remark Note that in the proof of Lemma 2.2 both constants C and K involve \( (\sigma' - \sigma) \). In fact when \( \sigma = \sigma' = 1/2 \), sharper bounds can be obtained namely, in (4) we can replace \( n^{\sqrt{n}} \) by \( n^2 \) on both sides of the inequality. This follows easily from two general lemmas about \( \phi(p) \) proved in an earlier paper (Mertens and Zamir (1977) Lemma 4.4 and 4.5. See also Mertens Sorin Zamir (1991) Lemma V.4.9 and V.4.10).

6.2 Proof of Lemma 5.5

Note first that by the concavity of the ln function:

\[
\frac{x}{x+y} \ln(1 - y) + \frac{y}{x+y} \ln(1 + x) \leq \ln 1 = 0.
\]

So denoting by \( F_c(x,y) \) the function under maximization we have:

\[
\max_{0 \leq x} F_c(x,y) \leq \max_{0 \leq x} \frac{cy}{x+y} = c.
\]

Hence the statement of the proposition certainly holds for all \( c \geq c_0 > 0 \), for any fixed \( c_0 \). It is therefore enough to prove it for c sufficiently small (which is where the inequality is of interest).

First let us check that for each fixed c, the inequality \( F_c(x,y) \leq c \) holds near the boundaries of the maximization region.

- In the neighborhood of \( x = 0 \) and \( y = 0 \), \( F_c(x,y) \approx 0 \).
- In the neighborhood of \( y = 1 \) and \( x \to + \infty \), \( F_c(x,y) \to - \infty \).
- In the neighborhood of \( x = + \infty \), \( F_c(x,y) \leq y + \ln(1 - y) \leq c^2 \).

It remains to bound \( F_c(x,y) \) on the interior local maxima:

\[
\frac{\partial F_c}{\partial x} = 0 \iff cy + \frac{y+x}{1+x} + \ln \left( \frac{1-y}{1+x} \right) = 0.
\]

let \( z = \frac{y+x}{1+x} \) and \( \psi(z) = -z - \ln(1-z) \) then \( 0 < z < 1 \) and since \( 0 < y < 1 \) we also have \( \psi(z) \leq c \) and:

\[
\psi(z) = -z - \ln(1-z) = \frac{z^2}{2} + \frac{z^3}{3} + \ldots \geq \frac{z^2}{2}.
\]

So at the maximum \( z \leq \sqrt{2c} \) and hence, \( \frac{y+x}{1+x} = 1 - z \leq 1 - \sqrt{2c} \) implying (for c sufficiently small), \( y \leq 2\sqrt{c} \) and \( x \leq 2\sqrt{c} \). Thus we can use Taylor expansions of \( F_c(x,y) \) up to terms in \( x^4 \) and \( y^4 \) since the remainder will be small with respect to \( c^2 \). Replacing in \( F_c(x,y) \) the term \( \ln(1+x) \) by \( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \), and \( \ln(1-y) \) by \( -y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} \), we obtain:

\[
F_c(x,y) = c \frac{xy}{x+y} - \frac{xy}{2} + \frac{x^2y - xy^2}{3} - \frac{x^3y - x^2y^2 + xy^3}{4}.
\]
The condition \( \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} = 0 \) yields,

\[
\frac{x^2 + y^2}{(x + y)^2} - \frac{x + y}{2} + \frac{x^2 - y^2}{3} - \frac{1}{4}(x^3 + x^2y + xy^2 + y^3) = 0.
\]

Thus

\[
\frac{x^2 + y^2}{(x + y)^3} = \frac{1}{2} - \frac{x - y}{3} + \frac{x^2 + y^2}{4} \geq \frac{1}{3},
\]

from which we deduce: \((x + y)^3 \leq 3c(x^2 + y^2) \leq 3c(x + y)^2\) and hence \(x + y \leq 3c\) (at the maximum).

We conclude that up to an error smaller than \(c^2\) we have:

\[
\max_{0 \leq x, 0 \leq y \leq 1} F_c(x, y) \leq \frac{c}{x + y} \leq \frac{c}{x + y} \left( \frac{x + y}{2} \right)^2 = \frac{c}{4}(x + y) \leq \frac{3}{4}c^2.
\]

6.3 A game with an error term of order \(O(n^{-2/3})\).

Our main theorem states that for games with two states, the highest order of error term, \((n^{-1/3})\), has always a normal distribution coefficient. When this coefficient vanishes, the error term is bounded by \(n^{-2/3}\). This bound for the second highest term is obtained at the neighborhood of an interior tangency point of Cav \(u\) and \(u\). If the degree of tangency is \(k\) then the bound for the error term is \(n^{-k/(2k-1)}\) which is \(n^{-2/3}\) for \(k = 2\), the highest degree of tangency when the payoff matrices are \(2 \times 2\), (see the proof of Proposition 5.3 and the remark following it.)

In this section we give an example of a game in which the error term is of the order \(O(n^{-2/3})\). The purpose of the example is not only to prove that this bound is attained but also to give an idea of the rather nontrivial and delicate arguments and techniques involved in determining the error term of a given game.

**Example 6.1** Consider the game with two states in which the payoff matrices are:

\[
A = \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}
\]

The nonrevealing game is

\[
\Delta(p) = \begin{pmatrix} 2p - p' & -p' \\ -p & -p + 2p' \end{pmatrix},
\]

and its value is

\[
u(p) = \begin{cases} -p & \text{for } 0 \leq p \leq \frac{1}{3}; a \text{ saddle point at Bottom-Left} \\ -(p - p')^2 & \text{for } \frac{1}{3} \leq p \leq \frac{3}{4}; \text{ a mixed game} \\ -p' & \text{for } \frac{3}{4} \leq p \leq 1; \text{ a saddle point at Top-Right} \end{cases}
\]

Thus Cav \(u \equiv 0\) and it is tangent to \(u(p)\) at \(p = 1/2\) (see Figure 4).
If we denote \( V_n(p) = n v_n(p) \), the value of the \( n \)-stage game with cumulative payoff, then it is readily verified that the recursive formula for \( V_n(p) \) is:

\[
V_{n+1}(p) = \max_{0 \leq s \leq 1, 0 \leq t \leq 1} \{ \min(2ps - ps' - p't, -ps' - p't + 2p't') + P(T) V_n(p_T) + P(B) V_n(p_B) \} ,
\]

where \( V_0 \equiv 0 \) and

\[
P(T) = ps + p't \quad P(B) = ps' + p't' \quad p_T = \frac{ps}{ps + p't} \quad p_B = \frac{ps'}{ps' + p't'} .
\]

It will be more convenient to work with the variables \( x = ps \) and \( y = p't' \) and with the homogeneous functions of two variables \( W_n(x, y) = (x + y)V_n(x) \) (in particular \( W_n(p, p') = V_n(p) \)). We have \( W_0 \equiv 0 \) and the recursive formula for \( n = 1, 2, \ldots \) is:

\[
W_{n+1}(p, p') = -p - p' + \max_{0 \leq x \leq p, 0 \leq y \leq p'} \{ x + y + \min(x, y) + W_n(p - x, y) + W_n(x, p' - y) \} ,
\]

Our result in this section is:

**Proposition 6.2** There exist a constant \( C > 0 \) such that

\[
W_n(p, p') \geq C \sqrt{n} \min(p, p') \quad \text{and hence} \quad v_n(p) \geq C n^{-2/3} \min(p, p') .
\]

**Proof** By appropriate choice of \( C \) this can certainly made true for any bounded number of initial values of \( n \) (note for instance that \( W_1(p) = v_1(p) = 2 \min(p, p') \)). By the concavity of \( W_n \) it is sufficient to prove that

\[
\liminf_{n \to \infty} \left[ \frac{W_n \left( \frac{1}{2} \right)}{\sqrt[3]{n}} \right] > 0 .
\]
Considering then the game $\Gamma_n(1/2)$ for $n \geq 1$, let $\delta_n = n^{-1/3}$ and $\epsilon_n = \delta_n^2 = n^{-2/3}$ and let player $I$ play the following strategy $\sigma$ in $\Gamma_n(p)$. At a stage in which the posterior probability of $A$ is $p$ then:

- If $\frac{1}{2} - \delta_n \leq p \leq \frac{1}{2} + \delta_n$, use $x = y = \frac{1}{2} + \epsilon_n$.
- If $\frac{1}{2} \leq p \wedge p' \leq \frac{1}{2} - \delta_n$, use $x = y = \frac{1}{2} [p \lor p']$.
- If $0 \leq p \leq \frac{1}{2}$, use $x = p$, $y = p' - p$.
- If $0 \leq p' \leq \frac{1}{2}$, use $x = p - p'$, $y = p'$.

**Remark** Note that the posterior probability, $p$ (computable by both players), is the conditional probability of $A$ given the strategy of player $I$ and his past moves up to stage $t - 1$ hence the above definition of the strategy $\sigma$ is an implicit one since $p$ depends on $\sigma$. However there is no problem in deducing from this an explicit recursive definition of $\sigma$ (which will of course be much more awkward to write down.)

Using the strategy $\sigma$, the payoff of stage $t$ (which is $3x + y - 1$) is at least $\min(p, p', 4\epsilon_n)$. Since the expected number of times in which $0 < p < 4\epsilon_n$ is bounded and small, we may assume that the total payoff is at least $4\epsilon_n T$, where $T$ is the number of stages before $p$ is absorbed in $\{0, 1\}$.

So, it is enough to show that $E(T) \geq cn$ for some $c > 0$ or that $P(T \geq cn) \geq \beta$ for some $\alpha > 0$, $\beta > 0$.

Given $p$, denote the possible values of $p_{x+y}$ by $p' = (p-x)/(p-x+y)$ and $p" = x/(x+p'-y)$. These values have the probabilities $(p-x+y)$ and $(x+p'-y)$ respectively.

Remark now that as soon as $p$ is outside the interval $[\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n]$, say in $(0, \frac{1}{2} - \delta_n)$ (we adopt the usual notation of [ and ( for closed and open ends of an interval), then $p' = 1/2$. If $p \leq 1/3$ then $p' = 0$. If $\frac{1}{2} \leq p \leq \frac{1}{2} - \delta_n$ then the probability of $p'$ (which is $1/2$) is at least $1/2$. If $p \leq 1/4$ then the probability of $p'$ (which is $0$) is at least $1/2$.

Thus at every stage (in which $p$ is outside the interval $[\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n]$), the probability of hitting 0 or 1/2 is at least 1/2, so the expected time of hitting the set $\{0, 1/2\}$ is less or equal to 2 (the distribution of this hitting time is bounded by a geometric distribution with $q = 1/2$.)

When $p$ hits the interval $[0, \frac{1}{2} - \delta_n]$ coming from $[\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n]$, one has

$$p \geq \frac{\frac{1}{2} - \delta_n - (\frac{1}{2} + \epsilon_n)}{\frac{1}{2} - \delta_n} = \frac{1 - 4(\epsilon_n + \delta_n)}{2(1 - 2\delta_n)}.$$

Starting with $p$, the posterior probabilities will first hit 1/2 with probability $\pi$, or 0 with probability $(1-\pi)$. Since the posterior probabilities form a martingale, we have $p = \pi(1/2) + (1-\pi)0$ and thus:

$$\pi \geq \frac{1 - 4(\epsilon_n + \delta_n)}{1 - 2\delta_n} \geq 1 - 3\delta_n,$$

for $n$ sufficiently large.

So every time the martingale restarts from 1/2, there is a probability of at least $1 - 3\delta_n$ to return to 1/2 in an expected time of at most 2 after leaving $[\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n]$, independently of the length of time required to leave that interval.

Thus there is a positive probability, bounded away from zero (for all $n \geq n_0$), of returning at least $1/\delta_n = n^{1/3}$ times to 1/2, independently of the length of the cycles between successive visits.
Now, starting at $p_0 = 1/2$, the variable

$$M_t = \sqrt{\frac{(p'_t - p_t)^2 + 4\epsilon_n}{(1 + 4\epsilon_n)^t}}$$

form a martingale under the strategy $\sigma$, as long as $p_{t-1} \in [\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n]$. This is easily verified directly from the definition of $\sigma$. So if we denote by $T$ the first hitting time of the complement of the set $[\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n]$, we have:

$$2\sqrt{\epsilon_n} = M_0 = E(M_T) \geq E \sqrt{\frac{(2\delta_n)^2 + 4\epsilon_n}{(1 + 4\epsilon_n)^T}} ,$$

thus

$$E \left( \left( \frac{1}{\sqrt{1 + 4\epsilon_n}} \right)^T \right) \leq \frac{1}{\sqrt{\epsilon_n + \delta_n^2}} = \frac{1}{\sqrt{2}} .$$

For any (positive large number) $L$, denoting by $\mathbf{1}$ the indicator function, it follows that

$$E \left[ \left( \frac{1}{\sqrt{1 + 4\epsilon_n}} \right)^L \mathbf{1}(T \leq L) \right] \leq \frac{1}{\sqrt{2}} ,$$

i.e. (using $1 + x \leq e^x$),

$$P(T \leq L) \leq \left( \frac{\sqrt{1 + 4\epsilon_n}}{\sqrt{2}} \right)^L \leq \exp \left( 2L\epsilon_n - \frac{\ln 2}{2} \right) .$$

Choosing $L = 1/8\epsilon_n = n^{2/3}/8$ we get

$$P \left( T \leq \frac{n^{2/3}}{8} \right) \leq \exp \left( \frac{1}{4} - \frac{\ln 2}{2} \right) = \sqrt{\frac{e}{4}} < 0.9 .$$

Thus $P \left( T \geq \frac{n^{2/3}}{8} \right) > 0.1$ and therefore, using the Central Limit Theorem for Binomial random variables, the sum of $n^{1/8}$ copies of such a $T$ will be at least $0.1n$ with probability bonded away from 0. This completes the proof of the proposition.

Combined with the upper bound of $n^{-2/3}$ for the case of an interior point strategy (established in Proposition 5.6), we conclude that $v_n$ in this game (which is also the error term since $\text{Cav} u \equiv 0$) is of the order $n^{-2/3}$.
7 References


