Some interactive decision problems emerging in statistics

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Abstract. We consider games which arise when two statisticians must make a decision simultaneously, and the loss function depends on both decisions. We are interested, in particular, in situations when information is detrimental, in a sense to be made precise. We show that in certain problems related to Bayesian testing and prediction the phenomenon of information rejection occurs for certain values of the parameters involved.
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1 Introduction

In this note we expand on previous work by the same authors [4] concerning the possibility that two interacting statisticians might prefer to refuse free information. This phenomenon of *information rejection* may occur when the loss of a statistician depends not only on his action and on the state of Nature, but also on the decision made by another statistician. We refer to [4] for general considerations on the problem, and also for references on the relation between statistics and game theory.

Some real situations fit into the scheme of interacting statisticians. For example, the so-called “inspection games”, where the statistician of the inspected party is trying to cheat the inspecting colleague (see [1]). Think also of a buyer and a seller simultaneously testing a sample each from a stock of items.

We consider here two examples that were presented in [4], drawn from the theory of Bayesian testing and Bayesian prediction, and we rephrase them in greater generality.

2 Two interacting statisticians and information refusal

We shall consider two examples, relevant in statistics, of games in which information rejection occurs, in the sense specified below. We shall use the terms "player" and "statistician" indifferently.

The games considered are as follows:

1. Nature chooses between the following bimatrices of payoffs:

$$G_A: \quad \begin{array}{c|c|c|} & a_1 & a_0 \\ \hline a_1 & 0,0 & 0,0 \\ \hline a_0 & 0,0 & 1,1 \\ \hline \end{array} \qquad G_B: \quad \begin{array}{c|c|c|} & a_1 & a_0 \\ \hline a_1 & 1,1 & 0,0 \\ \hline a_0 & 0,0 & 0,0 \\ \hline \end{array} \tag{1}$$

   We shall refer to G_A and G_B as *state-games*. The probability with which Nature selects each state-game is not exactly known to the players.
2. The two players have a common prior $\mathbb{P}$ about the behavior of Nature.
3. Each player acquires private information about the choice of Nature. This information will be assumed to be binary. Thus, each player can be of two types, in the sense of Harsanyi (1967/1968). We shall say that player I is of type I_0 or of type I_1 according to whether he has seen, say, a Tail or a Head. Analogously for player II. For $k \in \{I, II\}$, we shall denote by $\mathbb{P}_{I_k}$ the conditional probability given the private information acquired by player I_k. We may think of $\mathbb{P}_{I_k}$ as the updated beliefs of I_k about the realized choice of Nature.
4. A binary public signal is shown to both players.
5. Each player chooses his action.
6. The state-game chosen by Nature is revealed and payoffs are collected accordingly.

Several criteria can be taken into account to select actions; among these, we consider the following:

CRITERION A: Each player chooses a_1 if and only if he thinks that G_B is more likely than G_A, conditionally on all the information available, private and public.

CRITERION B: The same as above, but not taking into account the public signal (namely, conditioning on private information only).

We may say that the phenomenon of *information rejection* occurs when both criteria lead to Nash equilibria and Criterion B is more favorable than Criterion A for at least one player. Recall that a Nash equilibrium is a strategy profile such that no player can profit from unilaterally deviating from his strategy in the profile.

Remark 1. Private information plays a crucial role. In fact, it is proved in [3] that in games with the structure described above, if the players have the same information they want as much information as possible. A more general result, relating positive value of information to uniqueness of Pareto optimal Nash equilibria is given in [2].

Remark 2. Although the game presented here is somehow artificial, it is in a sense the simplest possible example in which information refusal may occur. Private information (which is necessary, as we mentioned above) is binary, the action space is binary, public information is binary, the bimatrices of payoff have only one non-zero entry.

Remark 3. The rationale underlying the examples of information refusal which we are going to show can be phrased as follows. The games are coordination games, and the prior law is such that both players believe that G_B is more likely than G_A. It is known that one observation is not enough to reverse this opinion, but two observations may lead a player to believe that G_A is more likely. Thus, after one private observation is taken, the players may prefer to avoid an additional observation, in order not to run the risk of disrupting the initial coordination.

In order to characterize games as described above, we need to specify:

(a) The prior law $\mathbb{P}$ and the way it relates to the mechanism of choice by Nature
(b) The structure of private information and the way it helps to understand the unknown probability distribution of Nature on the two state-games.
(c) The structure of the public signal.

2.1 First example: hypothesis testing

In this example we want to describe the situation when two (Bayesian) statisticians need to simultaneously test a simple hypothesis vs another simple hypothesis, and their payoff is positive iff both make the correct choice. We may think of G_A (resp.:G_B) as the payoffs when the true hypothesis is the null (resp.: the alternative).

We characterize the game along the lines sketched above.

(a) *Description of the prior.* The prior law $\mathbb{P}$ is a distribution on the parameter space $\Psi := \{\theta_0, \theta_1\}$, with $0 < \theta_0 < \theta_1 < 1$. The value θ_0 corresponds to the null hypothesis, and θ_1 to the alternative.
We denote by π_0 the probability $\mathbb{P}$ that the state-game G_A is selected by Nature, i.e. that the null hypothesis holds true.
(b) *Structure of private information.* Let Θ be a Ψ-valued random variable such that $\Theta = \theta_0$ iff G_A is selected by Nature. Let also $X_{\mathrm{I}}, X_{\mathrm{II}}, Y$ be random variables such that, conditionally on $\Theta = \theta$, they are i.i.d. Bernoulli

with parameter θ, $\forall\theta \in \Psi$ (i.e. $\mathbb{P}(X_I = 1 | \Theta = \theta) = \theta$). It is common knowledge that the value of X_{I} is shown to player I only, and that the value of X_{II} is shown to player II only. Thus, each statistician has a private sample of size one from the population to be tested. Y is the public signal.

(c) *Structure of the public signal.* The value of Y is shown to both players. Thus, an additional sample of size one is observed by both statisticians.

A strategy profile in this game is a string of 8 actions: the first two are the actions taken by I_0 (i.e. Player I with private information $X_I = 0$) if $Y = 0$ and $Y = 1$, respectively, and so on. The following proposition shows that information refusal may occur.

Proposition 1. *Consider the game previously described. If the parameters θ_0, θ_1 and π_0 satisfy*

$$\left(\frac{1-\theta_1}{1-\theta_0}\right)^2 \max\left\{\frac{\theta_1}{1-\theta_0}, 1\right\} \leq \frac{\pi_0}{1-\pi_0} \leq \frac{1-\theta_1}{1-\theta_0} \min\left\{\frac{(\theta_1)^2}{\theta_0(1-\theta_0)}, 1\right\},$$

then:

1. *The following strategy profile*

$$\overset{(\mathrm{I}_0)}{a_0 a_1}\ \overset{(\mathrm{I}_1)}{a_1 a_1}\ \overset{(\mathrm{II}_0)}{a_0 a_1}\ \overset{(\mathrm{II}_1)}{a_1 a_1}\ , \tag{2}$$

is an equilibrium. Each action is the same that a single statistician would have taken if he were to maximize his expected utility based on all available information, namely, if he were to choose his action according to Criterion A.

2. *The following strategy profile*

$$\overset{(\mathrm{I}_0)}{a_1 a_1}\ \overset{(\mathrm{I}_1)}{a_1 a_1}\ \overset{(\mathrm{II}_0)}{a_1 a_1}\ \overset{(\mathrm{II}_1)}{a_1 a_1}\ . \tag{3}$$

is an equilibrium. Each action is the same that a single statistician would have taken if he were to maximize his expected utility based on private information only, namely, if he were to choose his action according to Criterion B.

3. *The payoff for I_0 if (1.2) is played is less than his payoff when (1.3) is played if and only if*

$$\frac{\pi_0}{1-\pi_0} \geq \left(\frac{1-\theta_1}{1-\theta_0}\right)^2 \frac{1}{(1-\theta_0)} \tag{4}$$

4. *The payoff for I_1 if (1.2) is played is always less than his payoff when (1.3) is played.*

Proof. First, we write the expressions for the payoffs:

- The expected payoff for I_0 if (1.2) is played is

$$A(\pi_0,\theta_0,\theta_1) := \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_0, Y=0, X_{II}=0\right) + \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_1, Y=1\right). \quad (5)$$

- If player I_0 deviates from (1.2) and plays $a_1 a_1$ (other moves are clearly not advantageous) his expected payoff is

$$B(\pi_0,\theta_0,\theta_1) := \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_1, Y=0, X_{II}=1\right) + \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_1, Y=1\right). \quad (6)$$

- The expected payoff for I_1 if (1.2) is played is

$$C(\pi_0,\theta_0,\theta_1) := \mathbb{P}_{\mathrm{I}_1}\left(\Theta=\theta_1, Y=0, X_{II}=1\right) + \mathbb{P}_{\mathrm{I}_1}\left(\Theta=\theta_1, Y=1\right). \quad (7)$$

- If player I_1 deviates from (1.2) and plays $a_0 a_1$ (other moves are clearly not advantageous) his expected payoff is

$$D(\pi_0,\theta_0,\theta_1) := \mathbb{P}_{\mathrm{I}_1}\left(\Theta=\theta_0, Y=0, X_{II}=0\right) + \mathbb{P}_{\mathrm{I}_1}\left(\Theta=\theta_1, Y=1\right). \quad (8)$$

- The expected payoff for I_0 if (1.3) is played is

$$E(\pi_0,\theta_0,\theta_1) := \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_1\right) \quad (9)$$

- The expected payoff for I_1 if (1.3) is played is

$$F(\pi_0,\theta_0,\theta_1) := \mathbb{P}_{\mathrm{I}_1}\left(\Theta=\theta_1\right) \quad (10)$$

It is clear that (1.3) is an equilibrium. In order to show that (1.2) is an equilibrium, we need to show that $A-B \geq 0$ and $C-D \geq 0$. In fact,

$$\begin{aligned}
A(\pi_0,\theta_0,\theta_1) - B(\pi_0,\theta_0,\theta_1) &= \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_0 Y=0, X_{II}=0\right) - \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_1 Y=0, X_{II}=1\right)\\
&= \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_0\right)\mathbb{P}_{\mathrm{I}_0}\left(Y=0, X_{II}=0 \mid \Theta=\theta_0\right) - \mathbb{P}_{\mathrm{I}_0}\left(\Theta=\theta_1\right)\mathbb{P}_{\mathrm{I}_0}\left(Y=0, X_{II}=1 \mid \Theta=\theta_1\right)\\
&= \frac{\pi_0\left(1-\theta_0\right)}{\pi_0\left(1-\theta_0\right)+\left(1-\pi_0\right)\left(1-\theta_1\right)}\left(1-\theta_0\right)^2 - \frac{\left(1-\pi_0\right)\left(1-\theta_1\right)}{\pi_0\left(1-\theta_0\right)+\left(1-\pi_0\right)\left(1-\theta_1\right)}\theta_1(1-\theta_1)\\
&\geq 0 \Leftrightarrow \frac{\pi_0}{1-\pi_0} \geq \frac{\theta_1}{1-\theta_0}\left(\frac{1-\theta_1}{1-\theta_0}\right)^2
\end{aligned}$$

and

$$\begin{aligned}
C(\pi_0,\theta_0,\theta_1) - D(\pi_0,\theta_0,\theta_1) &= \mathbb{P}_{\mathrm{I}_1}\left(\Theta=\theta_1 Y=0, X_{II}=1\right) - \mathbb{P}_{\mathrm{I}_1}\left(\Theta=\theta_0 Y=0, X_{II}=0\right)\\
&= \frac{(1-\pi_0)\theta_1}{(1-\pi_0)\theta_1+\pi_0\theta_0}\theta_1(1-\theta_1) - \frac{\pi_0\theta_0}{(1-\pi_0)\theta_1+\pi_0\theta_0}\left(1-\theta_0\right)^2\\
&\geq 0 \Leftrightarrow \frac{\pi_0}{1-\pi_0} \leq \left(\frac{1-\theta_1}{1-\theta_0}\right)\frac{\theta_1^2}{\theta_0(1-\theta_0)}
\end{aligned}$$

Next, we show that the actions in (1.2) (resp.: (1.3)) are those that a single statistician following Criterion A (resp.: Criterion B) would have chosen.

Preliminarily, we observe the following: if $\theta_0 < \theta_1$ and if $Z_1, Z_2, \ldots$ are i.i.d. conditionally on $\Theta = \theta$, for $\theta \in \{\theta_0, \theta_1\}$, with conditional distribution Bernoulli with parameter θ, then $\mathbb{P}(\Theta = \theta_1 | \sum Z_i = z)$ is increasing in z, as it is easy to check.

In view of these considerations, it is clear that we need only to show

$$\mathbb{P}_{\mathrm{I}_0}\left(\Theta = \theta_1 | Y = 0\right) < \frac{1}{2} < \mathbb{P}_{\mathrm{I}_0}\left(\Theta = \theta_1\right) \tag{11}$$

In fact,

$$\begin{aligned}
\mathbb{P}_{\mathrm{I}_0}\left(\Theta = \theta_1 | Y = 0\right) &= \frac{\mathbb{P}\left(\Theta = \theta_1\right)\mathbb{P}\left(X_I = 0, Y = 0 | \Theta = \theta_1\right)}{\mathbb{P}\left(X_I = 0, Y = 0\right)} \\
&= \frac{\left(1-\pi_0\right)\left(1-\theta_1\right)^2}{\left(1-\pi_0\right)\left(1-\theta_1\right)^2 + \pi_0\left(1-\theta_0\right)^2} \\
&< \frac{1}{2} \Leftrightarrow \frac{\pi_0}{1-\pi_0} > \left(\frac{1-\theta_1}{1-\theta_0}\right)^2
\end{aligned}$$

Furthermore,

$$\mathbb{P}_{\mathrm{I}_0}\left(\Theta = \theta_1\right) = \frac{\left(1-\pi_0\right)\left(1-\theta_1\right)}{\left(1-\pi_0\right)\left(1-\theta_1\right) + \pi_0\left(1-\theta_0\right)} > \frac{1}{2} \Leftrightarrow \frac{\pi_0}{1-\pi_0} < \frac{1-\theta_1}{1-\theta_0}.$$

Thus, (1.11) is proved, and the claim follows.

Now, we compare the payoffs of I_0 and I_1 in the two equilibria (1.2) and (1.3). It is easy to check that $C - F < 0$ for all values of the parameters. Hence, if given the choice, player I_1 would choose that the additional information Y not be revealed.

As far as I_0 is concerned, we must compare A and E:

$$\begin{aligned}
&A(\pi_0, \theta_0, \theta_1) - E(\pi_0, \theta_0, \theta_1) = \mathbb{P}_{\mathrm{I}_0}\left(\Theta = \theta_0 Y = 0, X_{II} = 0\right) - \mathbb{P}_{\mathrm{I}_0}\left(\Theta = \theta_1 Y = 0\right) \\
&\quad = \frac{\pi_0(1-\theta_0)}{\pi_0(1-\theta_0) + (1-\pi_0)(1-\theta_1)}\left(1-\theta_0\right)^2 - \frac{(1-\pi_0)(1-\theta_1)}{\pi_0(1-\theta_0) + (1-\pi_0)(1-\theta_1)}\left(1-\theta_1\right) \\
&\quad \geq 0 \Leftrightarrow \frac{\pi_0}{1-\pi_0} \geq \left(\frac{1-\theta_1}{1-\theta_0}\right)^2 \frac{1}{(1-\theta_0)}
\end{aligned}$$

Hence, only for high enough values of π_0 the payoff in the equilibrium emerging when Y is considered is higher. Thus, if given the choice, player I_0 would prefer that information be revealed for certain values of π_0 and would prefer that it be withheld for other values.

2.2 Second example: prediction

We want to describe here a situation in which two statisticians must simultaneously predict correctly a binary outcome in order to guarantee for themselves a positive reward.

The setup can be described as follows. Nature chooses repeatedly a state-game, each time with a probability Θ which is unknown to the players. They must predict which state-game Nature will choose next. Preliminary observations will help the players in assessing the value of Θ.

As we shall see below, we shall include in this example the possibility of partial signaling.

Here is a description of the game.

(a) *Description of the prior.* The prior law $\mathbb{P}$ determines the "a priori" distribution of Θ. We assume that this distribution is a Beta(α, β).

(b) *Structure of private information.* Let Y, X, X_I, X_{II} be exchangeable Bernoulli random variables, i.i.d. conditionally on Θ, with $\mathbb{P}(X = 1|\Theta = \theta) = \theta$. It is common knowledge that the value of X_{I} is shown to player I only, and that the value of X_{II} is shown to player II only. Y is the public signal (see below), and X represents the choice of Nature to be predicted: $X = 1$ if and only if the state-game G_B is selected.

(c) *Structure of the public signal.* A binary signal ξ_p is shown to both players $(0 \leq p \leq 1)$. It is such that, independently of the values of Y and of all random variables involved,

$$\mathbb{P}\left(\xi_p = Y\right) = p = 1 - \mathbb{P}\left(\xi_p = Z\right)$$

where Z is the outcome of a fair coin independent of X; thus, with probability p the r.v. ξ_p yields valuable information, namely Y, and with probability $1 - p$ it gives irrelevant information, namely the outcome of an independent coin toss ; for an example of such a variable, see the Remark below.

We may think of p as the clarity of the signal revealed. For each value of p we have a game, say G_p.

A strategy profile is described by a string of 8 actions. The first two are the actions taken by I_0 when $\xi_p = 0$ and $\xi_p = 1$, respectively, and so forth.

Remark 4. In order to describe the public signalling mechanism, consider first three independent Bernoulli random variables Y, W, Z such that

- $Y, X_{\mathrm{I}}, X_{\mathrm{II}}, X$ are exchangeable;
- W is independent of $X_{\mathrm{I}}, X_{\mathrm{II}}, X$ and $\mathbb{P}\left(W = 1\right) = p$;
- Z is independent of $X_{\mathrm{I}}, X_{\mathrm{II}}, X$ and $\mathbb{P}\left(Z = 1\right) = \frac{1}{2}$.

The Bernoulli random variable ξ_p is described as follows: the coin W is tossed by a referee; if $W = 1$, then the value of Y is revealed, otherwise the fair coin Z is tossed and the result of the toss is revealed. Thus

$$\xi_p = \begin{cases} Y \text{ if } W = 1, \\ Z \text{ if } W = 0, \end{cases} \tag{12}$$

i.e.

$$\{\xi_p = k\} = \{W = 1, Y = k\} \cup \{W = 0, Z = k\}, \qquad k \in \{0, 1\}.$$

This mechanism is common knowledge, but the players don't know the outcome of W. They are only told the value of ξ_p (in addition to their private information). Observe that relevant information is given only when $W = 1$, which happens with probability p. If $p = 1$, then the players have an additional observation (exchangeable with X) before predicting X. If $p = 0$, then the additional observation Y is not available to the players.

Proposition 2. *Consider the games G_p as described above. If the parameters of the prior law of Θ satisfy the relation*

$$2 \leq \beta + 1 < \alpha < \beta + 2,$$

then:

1. *The following strategy profile*

$$\overset{(\mathrm{I}_0)}{a_0 a_1}\ \overset{(\mathrm{I}_1)}{a_1 a_1}\ \overset{(\mathrm{II}_0)}{a_0 a_1}\ \overset{(\mathrm{II}_1)}{a_1 a_1}\ . \tag{13}$$

is an equilibrium. For

$$p > p_0 := \frac{1 - 2\frac{\beta+1}{\alpha+\beta+1}}{1 - 4\frac{\beta+1}{\alpha+\beta+1}\frac{\alpha}{\alpha+\beta+2}},$$

each action is the same that a single statistician would have taken if he were to maximize his expected utility based on private information only, namely, if he were to choose his action according to Criterion A.

2. *The following strategy profile*

$$\overset{(\mathrm{I}_0)}{a_1 a_1}\ \overset{(\mathrm{I}_1)}{a_1 a_1}\ \overset{(\mathrm{II}_0)}{a_1 a_1}\ \overset{(\mathrm{II}_1)}{a_1 a_1}\ . \tag{14}$$

is an equilibrium. Each action is the same that a single statistician would have taken if he were to maximize his expected utility based on private information only, namely, if he were to choose his action according to Criterion B. Furthermore, for $p < p_0$, each action is the same that a single statistician following Criterion A would have taken.

3. For $p > p_0$, Criterion A is Pareto-dominated by Criterion B. Obviously, it leads to the same payoff for other values of p.

Proof (of Proposition 2). First of all, since $\beta + 1 < \alpha$, we have

$$\mathbb{P}_{I_k}(X = 1) = \frac{\alpha + k}{\alpha + \beta + 1} > \frac{1}{2}, \qquad k = 0, 1.$$

Hence, it is obvious that Criterion B leads to (1.14), and this is clearly an equilibrium.

Next, we consider Criterion A. We have

$$\mathbb{P}_{\mathrm{I}_0}(\xi_p = 0) = \mathbb{P}_{\mathrm{I}_0}(\xi_p = Y, Y = 0) + \mathbb{P}_{\mathrm{I}_0}(\xi_p = Z, Z = 0) = p\frac{\beta + 1}{\alpha + \beta + 1} + (1-p)\frac{1}{2},$$

and

$$\mathbb{P}_{\mathrm{I}_0}(X = 0, \xi_p = 0) = p\frac{\beta + 1}{\alpha + \beta + 1}\frac{\beta + 2}{\alpha + \beta + 2} + (1-p)\frac{1}{2}\frac{\beta + 1}{\alpha + \beta + 1}.$$

Hence,

$$\begin{aligned}\mathbb{P}_{\mathrm{I}_0}(X = 0 | \xi_p = 0) &= \frac{\frac{\beta+1}{\alpha+\beta+1}\left[p\frac{\beta+2}{\alpha+\beta+2} + (1-p)\frac{1}{2}\right]}{p\frac{\beta+1}{\alpha+\beta+1} + (1-p)\frac{1}{2}} \\ &> \frac{1}{2} \Leftrightarrow p > \frac{1 - 2\frac{\beta+1}{\alpha+\beta+1}}{1 - 4\frac{\beta+1}{\alpha+\beta+1}\frac{\alpha}{\alpha+\beta+2}} = p_0.\end{aligned}$$

Thus, the action of I_0 if he sees $\xi_p = 0$ is a_0. The other strategies in the profile can be established similarly.

Next, we show that the strategy profile (1.13) yields a Nash equilibrium of G_p, for every $p > p_0$. First, we write the expressions of the relevant payoffs:

- The payoff of I_0 in G_p if (1.13) is played is

$$\begin{aligned}&\mathbb{P}_{\mathrm{I}_0}(\xi_p = 0, X_{\mathrm{II}} = 0, X = 0) + \mathbb{P}_{\mathrm{I}_0}(\xi_p = 1, X = 1) \qquad (15)\\ &= \frac{1-p}{2}\mathbb{P}_{\mathrm{I}_0}(X_{\mathrm{II}} = 0, X = 0) + p\mathbb{P}_{\mathrm{I}_0}(Y = 0, X_{\mathrm{II}} = 0, X = 0) + \mathbb{P}_{\mathrm{I}_0}(\xi_p = 1, X = 1)\end{aligned}$$

- The payoff of I_0 in G_p if he deviates and plays $a_1 a_1$ is

$$\begin{aligned}&\mathbb{P}_{\mathrm{I}_0}(\xi_p = 0, X_{\mathrm{II}} = 1, X = 1) + \mathbb{P}_{\mathrm{I}_0}(\xi_p = 1, X = 1) \qquad (16)\\ &= \frac{1-p}{2}\mathbb{P}_{\mathrm{I}_0}(X_{\mathrm{II}} = 1, X = 1) + p\mathbb{P}_{\mathrm{I}_0}(Y = 0, X_{\mathrm{II}} = 1, X = 1) + \mathbb{P}_{\mathrm{I}_0}(\xi_p = 1, X = 1)\end{aligned}$$

We show now that the difference between (1.13) and (1.14), namely

$$\left(\frac{1-p}{2}\right)\left[\frac{\beta+1}{\alpha+\beta+1}\frac{\beta+2}{\alpha+\beta+2}-\frac{\alpha}{\alpha+\beta+1}\frac{\alpha+1}{\alpha+\beta+2}\right]$$
$$+p\left[\frac{\beta+1}{\alpha+\beta+1}\left(\frac{\beta+2}{\alpha+\beta+2}\frac{\beta+3}{\alpha+\beta+3}-\frac{\alpha}{\alpha+\beta+2}\frac{\alpha+1}{\alpha+\beta+3}\right)\right]$$

is positive iff $p > p_0$. In fact, the above quantity is positive iff

$$p > \frac{\frac{\alpha}{\alpha+\beta+1}\frac{\alpha+1}{\alpha+\beta+2}-\frac{\beta+1}{\alpha+\beta+1}\frac{\beta+2}{\alpha+\beta+2}}{\frac{\alpha}{\alpha+\beta+1}\frac{\alpha+1}{\alpha+\beta+2}-\frac{\beta+1}{\alpha+\beta+1}\frac{\beta+2}{\alpha+\beta+2}+2\frac{\beta+1}{\alpha+\beta+1}\left(\frac{\beta+2}{\alpha+\beta+2}\frac{\beta+3}{\alpha+\beta+3}-\frac{\alpha}{\alpha+\beta+2}\frac{\alpha+1}{\alpha+\beta+3}\right)}.$$

Now, we see after some straightforward calculations that the right hand side equals p_0.

Next, we repeat the same arguments for player I_1. If (1.13) is played, his payoff is

$$\mathbb{P}_{I_1}(\xi_p = 1, X = 1) + \mathbb{P}_{I_1}(\xi_p = 0, X_{II} = 1, X = 1),$$

whereas if he deviates and plays $a_0 a_1$ his payoff becomes

$$\mathbb{P}_{I_1}(\xi_p = 1, X = 1) + \mathbb{P}_{I_1}(\xi_p = 0, X_{II} = 0, X = 0).$$

It is a simple matter to check that for every value of p there is no interest in deviating.

In order to prove the last claim of the Proposition, we first show that the payoff for I_0 if (1.14) is played , namely $\mathbb{P}_{I_0}(X = 1)$, is greater than (1.15). In fact, their difference yields

$$\begin{aligned}\mathbb{P}_{I_0}(\xi_p = 0, X = 1) &- \mathbb{P}_{I_0}(\xi_p = 0, X_{II} = 0, X = 0)\\ &= p\frac{\beta+1}{\prod_{k=1}^{3}(\alpha+\beta+k)}[\alpha(\alpha+\beta+3)-(\beta+2)(\beta+3)]\\ &\quad+\frac{1-p}{2\prod_{k=1}^{2}(\alpha+\beta+k)}[\alpha(\alpha+\beta+2)-(\beta+1)(\beta+2)]\end{aligned}$$

This difference is positive, since

$$\alpha(\alpha+\beta+3)-(\beta+2)(\beta+3) \geq (\beta+1)(2\beta+4)-(\beta+2)(\beta+3) = (\beta+2)(\beta-1) \geq 0,$$

and

$$\alpha(\alpha+\beta+2)-(\beta+1)(\beta+2) \geq (\beta+1)\alpha > 0.$$

It is even simpler to show that the payoff for I_1 is greater in the equilibrium (1.14) than in (1.15) In fact, if (1.14) is played, player I_1 collects a non-zero reward if and only if the event $\{X = 1\}$ occurs, whereas in (1.15) his reward is non-zero iff a proper subset of $\{X = 1\}$ occurs.

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