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Imperfect Inspection Games Over Time

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Abstract. We consider an inspection game played on a continuous finite time interval. The inspector wishes to detect a violation as soon as possible after it has been made by the operator. The loss to the inspector is assumed to be linear in the duration of the time elapsed between the violation and its detection. This paper is mostly an extension of Diamond's models for a single inspection, which includes the uncertainty aspect, by relaxing the assumption that the inspection is perfect. Here the inspection is imperfect; it has a Type One Error which means that the inspector may call a false alarm (with probability α), and a Type Two Error which means that the inspection may fail to detect (with probability β) a violation which did occur. In addition we will assume that the inspection is silent, i.e., the operator is unaware of the inspection when it takes place, unless the inspector calls a false alarm.

1. Introduction

Environmental damage is a problem causing increasing concern in the modern technological world. In any discussion relating to the damage done to the environment by an industrial or other activity, time is of essence. Specifically, when pollution is the cause of the damage, the harmful effect to the environment becomes more serious, when the duration of the pollution becomes longer, and in extreme cases it may become irreversible. Unfortunately in most cases the agency which is responsible for the source of pollution measures its activities in monitory terms, where polluting is cheaper than taking costly measures to avoid it. Since the environmental damage does not have an immediate influence on the agency's budget, the effects are not necessarily a part of its cost–benefit equation. To illustrate this, let us consider the problem that the management of a plant, or a city municipality, faces when the question of waste disposal arises. A simple solution is, for example, to let sewage flow into the rivers and then into the sea. The longer the sewage runs into water resources the lower the cost, when compared with better (more expensive) means of waste disposal. In order to prevent such a situation, the government – the Department of the Environment – has to check the quality of the water and to enforce actions that will protect the environment.

We will formulate this type of situation in the following manner: The enforcing authority will be called the "inspector" and the manager of a plant the "operator". The operator can violate proper waste disposal procedures by polluting water resources or the air surrounding his plant, in which case he causes damages. The inspector checks for violation once in a fixed interval of time. The inspection test is imperfect, for several reasons, such as measurement errors and environmental conditions (e.g., temperature and season of the year). When the operator acts illegally, causing damage to the environment, we say that the operator is guilty of causing a violation. According to the customary statistical terminology the inspector is facing the following problem of hypotheses testing:

H_0: the operator acted legally.

H_1: the operator acted illegally.

This is not a standard statistical problem, because the inspector does not face "natural uncertainties". H_0 and H_1 are not "states of nature", but they are determined by "strategic choices" of the operator whether to act legally or not. This is not known a priori to the inspector. Therefore the tools of game theory are more appropriate here. The inspector can make two possible errors:

- A "Type One Error" implies that the inspector may declare the operator to be guilty of a violation, when in fact he is not. This will also be called a "false alarm".
- A "Type Two Error" means that the inspector may assume that the operator acted legally, while in fact he is guilty of a violation. This will also be called an "oversight".

We will assume that the inspection is silent:

A *silent* inspection: the event that an inspection took place is not known to the operator unless the inspector calls an alarm.

Structure of the game

We consider a zero sum game with two players: the operator (denoted as O) and the inspector (denoted as I). The game is played over an interval of time $[0, D]$, which without loss of generality we assume to be $[0, 1]$.

Strategies

- The operator strategy is determination of the violation time T, which is a random variable in $[0, 1]$ (that is a mixed strategy).
- The inspector is allowed to have one inspection, hence her strategy is the inspection time S, which is a random variable in $[0, 1]$ (again, a mixed strategy). The inspection system detects a violation which occurred before the inspection (i.e., $S \geqslant T$) with probability $1 - \beta$. With probability α a false alarm is called, i.e., the inspector calls an alarm although no violation took place before the inspection ($S < T$).

This type of inspection problem is an extension of what occurs usually in reliability theory, the detection of a system failure. The damage that is caused by a late detection is a function of the time, which elapsed between the failure and its detection. More inspections performed may lead to an early detection of a violation, but inspections are costly. In such models one usually assumes that the failure distribution is known, hence the problem is an optimization problem for the inspector. Derman [5] considered a unit in an operating system that deteriorates with time and is no longer functioning after a random period. The distribution failure is not known. To detect the failure of the unit, one has to inspect it from time to time. Since the inspections are costly, and the length of the time where the unit is not functioning involves estimable cost, the objective is to find a schedule for the inspections, which minimize the cost. Diamond [6] studied a game where the operating system (from Derman's model) is replaced by a strategic agent – the operator, who has his own strategy, which is not known to the inspector. The game is played over continuous time. In this game the payoffs are functions of the time interval between violation and its detection. This is a zero sum game where the inspector tries to minimize this interval while the operator maximize it. Diamond assumed that the inspection is perfect, i.e. if a violation was made earlier it would be detected certainly by the inspection. Diamond finds the minimax solution of this game which involves randomized inspection strategies for the inspector.

The assumption of perfect inspection will be relaxed in this paper. In our game the inspector is allowed to conduct only one inspection in order to detect a violation by the operator. We assume that the detection system is not perfect, i.e., there are positive probabilities of making Type One and Type Two Errors.

1.1. The payoffs

The payoffs are time dependent, since early detection is better for the inspector and worse to the operator. Illustrated by our previous example, the earlier the inspector detects a violation, the lower will be the environmental damage, decreasing the inspector's losses. The operator is expected to lose from such an early detection, since he will not be able to use the cheaper means of waste disposal, and will thus be forced to use more costly actions.

The assumption of this model are:

(1) The probabilities α and β are fixed and given as part of the data of the model.

(2) The inspection is not observed by the operator unless the inspector calls an alarm, i.e., the inspection is *silent*.

(3) Both players can distinguish between a justified alarm and a false alarm after it has been called (for example by a further detailed investigation).

(4) The payoffs are linear in the duration of the time elapsed between the violation and its detection (by appropriate choice of the utility we assume that the payoff is equal to the time).

(5) At time $t = 1$ there is a perfect inspection which ends the game.

Let $S \in [0, 1]$ be the time of inspection and $(t, \tilde{t})$ the time of violation, i.e., $t \in [0, 1]$ is the time of violation as long as no false alarm is called, and $\tilde{t} : [0, 1] \to [0, 1]$ s.t. $\tilde{t}(s) \geqslant s\ \forall s \in [0, 1]$ is the revised violation contingent on observing a false alarm call at time $s < t$.

The operator payoff function $u(s, (t, \tilde{t}))$ is defined by:

$$\begin{aligned} u\big(s, (t, \tilde{t})\big) = {} & \big[\alpha\big(1 - \tilde{t}(s)\big) + (1 - \alpha)(1 - t)\big] J(s < t) \\ & + \big[(1 - \beta)(s - t) + \beta(1 - t)\big] J(s \geqslant t), \end{aligned} \tag{1}$$

where $J(A)$ is the indicator function of the event A.

The operator strategy $(t, \tilde{t})$ can be written as $(t, s+\mu)$ where $t \in [0,1]$ and $\tilde{t} = s+\mu$ and $0 \leqslant \mu \leqslant 1-s$. It is straightforward that the strategy (t, s) weakly dominates any other strategy $(t, \tilde{t})$, since it yields the same payoff against any strategy $s \geqslant t$; on the other hand against any strategy $s < t$ the former yields

$$(1-\alpha)(1-t) + \alpha\bigl(1-\tilde{t}(s)\bigr),$$

where the latter yields

$$(1-\alpha)(1-t) + \alpha(1-s)$$

which is greater or equal to the former. According to the above one can rewrite the operator payoff function:

$$\begin{aligned} u(S,T) = {} & \bigl[\alpha(1-S) + (1-\alpha)(1-T)\bigr]J(S<T) \\ & + \bigl[(1-\beta)(S-T) + \beta(1-T)\bigr]J(S \geqslant T), \end{aligned} \tag{2}$$

where T is the time of violation and S is the time of inspection, which are independent random variables in the interval [0, 1], reflecting the fact that the players choose their strategies privately and execute them independently (clearly, the payoff is a random variable as the players play mixed strategies).

The payoff function of the operator is minus the payoff function of the inspector (zero sum game).

We shall analyze the game by studying its Nash equilibria: a Nash equilibrium point of the game, as we shall formally define later on, is a pair of strategies (one for each player), which are best reply to each other.

In a two-person zero-sum game, a Nash equilibrium point coincides with a pair of minimax strategies.

1.2. Equilibrium

We first note that there is no equilibrium in which one of the players uses a "pure strategy", i.e., with either S or T constant. To see this let us assume that in equilibrium the operator violates at time t with probability 1 (which is a pure strategy). The inspector's best reply is to inspect immediately after time t, guaranteeing the minimal loss possible $\beta(1-t)$. But this is not an equilibrium since the operator will do better by violating a short while after that time, etc. The same kind of argument applies when the inspector inspects with probability 1 at time s. Hence the strategies of both players are mixed strategies, i.e., the inspector chooses the time of inspection S according to a cumulative distribution function H, and the operator violates at time T according to a cumulative distribution function G, if the inspector had not called an alarm until time T. If she did, the operator violates immediately after the alarm. Therefore the operator will act at time $\min(A, T)$, where A is the time when the inspector calls a false alarm. We restrict our attention to strategies in which the continuous parts of the distributions G and H have densities which we denote by $h(s)$ and $g(t)$ respectively (according to common practice s, t denote the realization of the random variables S and T respectively). The strategy space of both players is thus $\Delta[0, 1]$, the set of probability distributions on $[0, 1]$ (with densities on the continuous parts), which we shall denote shortly by Δ ($H \in \Delta$ and $G \in \Delta$).

Using the above notation, and denoting the operator and the inspector expected payoffs by

$$O(H, G) := E_{H,G}\big[u(S, T)\big] \quad \text{and} \quad I(H, G) := -E_{H,G}\big[u(S, T)\big]$$

respectively, one can formulate the definition of the Nash equilibrium as follows: (H, G) is in equilibrium if

$$\begin{aligned} I(H, G) &\geqslant I(H', G) \quad \forall H' \in \Delta, \\ O(H, G) &\geqslant O(H, G') \quad \forall G' \in \Delta. \end{aligned}$$

That is, the strategy of each player is the best response to the strategy of the other.

For mathematical convenience, we will use later on, S or H to describe the inspector strategy and T or G the operator strategy.

Theorem 1. If (S, T) is an equilibrium, then

- The support of both S and T is in the interval $[0, b]$, with $b < 1$.
- The random variable S has no atoms in $[0, b]$, and T has no atoms in $(0, b]$.

The proof is given in the appendix.

According to theorem 1 none of the players will act beyond a certain time b which is strictly less than 1. Intuitively one may expect that, when the game approaches its end, the probability that a diversion has occurred increases and with it the damage to the inspector. On the other hand the loss that is caused to the inspector by calling a false alarm decreases. By taking these two considerations into account the best that the inspector can do, is to inspect before the end. Given that the inspector does not check after time b, the operator's best reply is not to violate latter than b, since otherwise he could increase his payoff by shifting the probability of diversion in $(b, 1]$ to diversion immediately after b.

According to theorem 1, we look for an equilibrium (S, T) with respective distributions H and G having densities h and g respectively in $(0, b]$, where T may have an atom at $t = 0$, $G(0)$. The expected payoffs are then given by:

- For the operator

$$\begin{aligned} O(H, t) = &\int_0^t \alpha(1 - s)h(s)\,\mathrm{d}s + \int_0^t (1 - \alpha)(1 - t)h(s)\,\mathrm{d}s \\ &+ \int_t^b (1 - \beta)(s - t)h(s)\,\mathrm{d}s + \int_t^b \beta(1 - t)h(s)\,\mathrm{d}s. \end{aligned} \tag{3}$$

In equilibrium each t in the support of G has to be a best reply to H, i.e., it maximizes $O(H, t)$. This function cannot have (in equilibrium) a global maximum at $t \geqslant 0$, since it would yield an atom for G in the interval $(0, b)$, in contradiction to theorem 1, hence the maximum of $O(H, t)$ is local. By equating the first derivative of $O(H, t)$ with respect to t, to zero, we obtain:

$$\frac{\mathrm{d}O(H,t)}{\mathrm{d}t} = \alpha(1-t)h(t) + (1-\alpha)(1-t)h(t) - (1-\alpha)\int_0^t h(s)\,\mathrm{d}s$$
$$-(1-\beta)\int_t^b h(s)\,\mathrm{d}s - \beta(1-t)h(t) - \beta\int_t^b h(s)\,\mathrm{d}s = 0$$

which is

$$(1-\beta)(1-t)h(t) + \alpha\int_0^t h(s)\,\mathrm{d}s - 1 = 0 \tag{4}$$

since by theorem 1, H has no atom in $[0, b]$. This differential equation is solved by the following distribution function:

$$H(t) = \frac{1}{\alpha} + k(1-t)^{\alpha/(1-\beta)}.$$

The condition $H(0) = 0$ (H has no atom in point zero by theorem 1) implies $k = -1/\alpha$. Consequently the inspector's strategy is the distribution function $H(s)$ given by:

$$H(s) = \frac{1}{\alpha} - \frac{1}{\alpha}(1-s)^{\alpha/(1-\beta)}. \tag{5}$$

The point b satisfies $H(b) = 1$. Hence from equation (5):

$$b = 1 - (1-\alpha)^{(1-\beta)/\alpha}. \tag{6}$$

• For the inspector, by theorem 1, her expected loss function when the operator uses the strategy G is:

$$I(s,G) = sG(0)(1-\beta) + G(0)\beta$$
$$+\int_0^s (1-\beta)(s-t)g(t)\,\mathrm{d}t + \int_0^s \beta(1-t)g(t)\,\mathrm{d}t$$
$$+\int_s^b \alpha(1-s)g(t)\,\mathrm{d}t + \int_s^b (1-\alpha)(1-t)g(t)\,\mathrm{d}t. \tag{7}$$

Setting the derivative with respect to s to zero we have:

$$\frac{\mathrm{d}I(s,G)}{\mathrm{d}s} = G(0)(1-\beta) + (1-\beta)\int_0^s g(t)\,\mathrm{d}t + \beta(1-s)g(s)$$
$$-\alpha(1-s)g(s) - \alpha\int_s^b g(t)\,\mathrm{d}t - (1-\alpha)(1-s)g(s) = 0,$$

which is simplified to:

$$\frac{1-\beta+\alpha}{\alpha}G(s) - \frac{1-\beta}{\alpha}(1-s)g(s) - 1 = 0. \tag{8}$$

This differential equation has the same form as equation (4), but with different coefficients. Using $G(b) = 1$ one obtains, for the operator's distribution function $G(s)$:

$$G(s) = \frac{\alpha}{1-\beta+\alpha} + \frac{(1-\alpha)^{(1-\beta+\alpha)/\alpha}(1-\beta)}{1-\beta+\alpha}(1-s)^{-(1-\beta+\alpha)/(1-\beta)}$$

and since by equation (6)

$$(1-\alpha)^{(1-\beta+\alpha)/\alpha} = (1-\alpha)(1-b)$$

we conclude that the distribution G is given by:

$$G(t) = \frac{\alpha}{1-\beta+\alpha} + \frac{(1-\alpha)(1-\beta)(1-b)}{1-\beta+\alpha}(1-t)^{-(1-\beta+\alpha)/(1-\beta)}. \tag{9}$$

Substituting into equation (9) the point $t = 0$ leads to:

$$G(0) = \frac{\alpha + (1-\alpha)(1-\beta)(1-b)}{1-\beta+\alpha} > 0 \tag{10}$$

which shows that in equilibrium the operator's strategy has an atom at point zero.

Note. Since G and H were uniquely determined, it follows from theorem 1, that the above equilibrium is unique (among the strategies with atoms and continuous part with density).

Proposition 1. The value of the game equals $G(0)$. That is the atom of the operator's violation strategy is also the value of the game.

Proof. The value of the game, can be calculated via the operator's expected payoff (equation (3)). The right-hand side of this equation is equal to the value of the game for any $t \in [0, b]$, hence for $t = 0$ we get:

$$\text{Value} = (1-\beta)\int_0^b sh(s)\,\mathrm{d}s + \beta, \tag{11}$$

where $h(s)$ is the density function of the inspection time in equilibrium.

$$h(s) = \frac{1}{1-\beta}(1-s)^{\alpha/(1-\beta)-1}. \tag{12}$$

Thus the expected inspection time is:

$$\begin{aligned}\int_0^b sh(s)\,\mathrm{d}s &= -(1-s)^{\alpha/(1-\beta)}\left(\frac{1-\beta+\alpha s}{\alpha(1-\beta+\alpha)}\right)\Bigg|_0^b \\ &= \frac{1-\beta-(1-b)^{\alpha/(1-\beta)}(1-\beta+\alpha b)}{\alpha(1-\beta+\alpha)}.\end{aligned}$$

Recall that $b = 1 - (1-\alpha)^{(1-\beta)/\alpha}$, according to equation (11) the value is:

$$\text{Value} = \frac{\alpha}{1-\beta+\alpha} + \frac{(1-\alpha)^{(1-\beta+\alpha)/\alpha}(1-\beta)}{1-\beta+\alpha} = G(0). \qquad \square \tag{13}$$

To check the special case of perfect inspection we set $\beta = 0$ in (9), (12) and (13) and take the limits as $\alpha \to 0$. We obtain

$$h(s) = \frac{1}{1-s}, \qquad G(t) = \frac{1}{e}\frac{1}{1-t}$$

in accordance with Diamond's results [6]. Furthermore, by equation (10) we get that $1 - b = 1/e = G(0)$. However this does not hold when $\alpha > 0$ and $\beta > 0$.

2. Summary

Inspection problems cover a large spectrum of real life situations. This paper emphasizes two main aspects of such problems (uncertainty of the inspection system and the time element), while presenting a methodological approach to their analysis: extension of the classical statistical approach.

2.1. *Main aspects of inspection procedure*

The two main aspects that are emphasized in this paper are uncertainty of the inspection system and the time element.

- The inspector's uncertainty of the operator's actions, at the time of the inspection, due to imperfect inspection procedures (e.g., measurement errors and environmental conditions). The imperfection of the inspection system is captured by the probabilities of type one errors and type two errors.
- The time element: early detection of a violation is vital for the inspector. The damage increases with the duration of time that the air or water resources are polluted.

2.2. *Extension of the classical statistical approach*

An important methodological aspect presented in this paper suggests that using game theoretical analysis for the study of inspection problems may be viewed as an extension of the classical statistical approach.

Both the statistical decision problem and the game have the same structure (from the point of view of the inspector).

Based on some random observation (inaccurate measure), the statistician (the inspector) has to choose one of two actions:

(1) Reject H_0 (call an alarm: the operator behaved illegally).

(2) Do not reject H_0 (do not call an alarm).

The game theoretical framework takes into account the possibility that the state of nature (H_0 or H_1) as well as the distribution of the random observation under H_1 are not fixed (though they may be unknown to the statistician) as in the statistical setup. In the game theoretical framework, however, they are strategic choices of another decision maker – the operator.

Appendix

Proof of Theorem 1. The operator payoff function $u(S, T)$ is:

$$\begin{aligned} u(S,T) = {} & \big[\alpha(1-S) + (1-\alpha)(1-T)\big]J(S < T) \\ & + \big[(1-\beta)(S-T) + \beta(1-T)\big]J(S \geqslant T). \end{aligned} \tag{14}$$

Note, that if the inspector and the operator act at the same time $s = t$, the inspector detects this violation immediately with probability $1-\beta$.

If (S, T) is an equilibrium, then the value of the game is $Eu(S, T)$. Since it is a zero-sum game, the value is equal to the operator's expected payoff and to the inspector expected loss.

There are four stages in the proof, each of which is formulated as a proposition. The first proves that the support of S is in the interval $[0, b]$, for some $b < 1$. The second proves that the support of T is in the interval $[0, b]$, for the same b. The third proposition shows that S has no atom in the support. The last proposition completes the proof of theorem 1, by showing that T has no atom in the half open interval $(0, b]$.

Lemma 1. If (S, T) is an equilibrium, then

$$P(T = 1) < \frac{1-\beta}{1-\beta+\alpha}.$$

Proof. Let $p = P(T = 1)$. The expected loss of the inspector, when inspecting at time $0 \leqslant s < 1$, against this strategy T of the operator is:

$$\begin{aligned} Eu(s,T) &= E\big\{\big[(1-\beta)(s-T) + \beta(1-T)\big]J(T \leqslant s)\big\} \\ &\quad + E\big\{\big[\alpha(1-s) + (1-\alpha)(1-T)\big]J(T > s)\big\} \\ &= E(1-T) - (1-s)(1-\beta)P(T \leqslant s) + \alpha E\big\{(T-s)J(T > s)\big\} \end{aligned}$$

which can be rewritten as:

$$\begin{aligned} Eu(s,T) = {} & E(1-T) + (1-s)\big[\alpha p - (1-\beta)(1-p)\big] \\ & + \alpha E\big\{(T-s)J(s < T < 1)\big\} + (1-s)(1-\beta)P(s < T < 1). \end{aligned} \tag{15}$$

For $s = 1$ the expected loss is:

$$Eu(1, T) = E(1-T).$$

Note that

$$\alpha E\{(T-s)J(s<T<1)\}+(1-s)(1-\beta)P(s<T<1)\geqslant 0$$

is nonnegative and decreasing in s. Now, we want to prove that $\alpha p-(1-\beta)(1-p)<0$.

(1) Assume that $\alpha p-(1-\beta)(1-p)>0$ (in particular $p>0$), then $Eu(s,T)>Eu(1,T)$ for any $0\leqslant s<1$. Hence, the best that the inspector can do is to inspect at time $s=1$, i.e., the inspector *should not inspect at all.* If this is the case T cannot be a best reply to this inspection strategy, since the operator can increase his payoff by replacing the atom at $T=1$ by an atom at $T=0$, a contradiction to $p>0$.

(2) Assume now that $\alpha p-(1-\beta)(1-p)=0$, we will show that there is no such equilibrium.

(a) If $P(s<T<1)=0$ for all $s\in[0,1)$, then the operator's strategy T consists of two atoms, at $t=0$ and $t=1$, since $P(T=1)=p=(1-\beta)/(1-\beta+\alpha)$, i.e., $0<p<1$. His expected payoff for a violation at time $t=0$ is

$$Eu(S,0)=E\big[(1-\beta)S+\beta\big]=(1-\beta)E(S)+\beta$$

and for violation at $t=1$ is

$$Eu(S,1)=E\big[\alpha(1-S)\big]=\alpha E(1-S).$$

Since the operator is indifferent of violating at time $t=0$ and $t=1$, the expected payoffs are equal, which yield:

$$E(S)=\frac{\alpha-\beta}{1-\beta+\alpha}.$$

If $\alpha<\beta$, this cannot be satisfied and hence there can be no such equilibrium.

For $\alpha\geqslant\beta$, it will be shown that the operator can use a different strategy than the one with atoms at $t=0$ and $t=1$, which will give him a higher expected payoff. First note that if $S\in[0,1]$ is a random variable and $E(S)=\mu$, then there exist t $(0<t<1)$ that satisfies $P(S\geqslant t)\leqslant\mu$. In fact otherwise we would have:

$$\mu=E(S)=\int_0^1 P(S\geqslant t)\,\mathrm{d}t>\mu.$$

In our case $\mu=E(S)=(\alpha-\beta)/(1-\beta+\alpha)$, therefore there exist $t^*<1$ such that

$$P(S\geqslant t^*)\leqslant\frac{\alpha-\beta}{1-\beta+\alpha}.$$

If the operator violates at time t^*, then his expected payoff is (equation (14))

$$\begin{aligned}Eu(S,t^*)=&\,\alpha E(1-S)+(1-\alpha)(1-t^*)P(S<t^*)\\&+E\big\{\big[(1-t^*)-(1-S)(1-\beta+\alpha)\big]J(S\geqslant t^*)\big\}.\end{aligned}$$

Substituting $(1 - S)$ by $(1 - t^*)$ in the second line of the above equation

$$\begin{aligned} Eu(S, t^*) &\geqslant \alpha E(1 - S) + (1 - t^*)\big[1 - \alpha - (1 - \beta)P(S \geqslant t^*)\big] \\ &\geqslant \alpha E(1 - S) + (1 - t^*)\left[1 - \alpha - (1 - \beta)\frac{\alpha - \beta}{1 - \beta + \alpha}\right]. \end{aligned}$$

Since $(\alpha < 1)$

$$\begin{aligned} 1 - \alpha - (1 - \beta)\frac{\alpha - \beta}{1 - \beta + \alpha} &= \frac{1 - \alpha - (\beta - \alpha)^2}{(1 - \beta + \alpha)} \\ &\geqslant \frac{1 - \alpha - (1 - \alpha)^2}{1 - \beta + \alpha} > 0 \end{aligned}$$

we obtain that $Eu(S, t^*) > Eu(S, 1)$, i.e., the T with two atoms is not in equilibrium, since it is not a best reply to S.

(b) if $P(s < T < 1) > 0$ for all $0 \leqslant s < 1$, then from equation (15) $Eu(s, T) > Eu(1, T)$ for all $0 \leqslant s < 1$, hence by the above argument (item (1)) we end with a contradiction that T has an atom at $t = 1$.

(c) If there exist $0 \leqslant \hat{s} = \sup\{s \mid P(s < T < 1) > 0\} < 1$, then any $s < \hat{s}$ is not in the support of S, since in equation (15)

$$\alpha E\big\{(T - s)J(s < T < 1)\big\} + (1 - s)(1 - \beta)P(s < T < 1) > 0$$

and it is decreasing in s. It implies that $Eu(s, T) > Eu(1, T)$ for all $s < \hat{s}$. Inspecting at $s = 1$ is better for the inspector than inspecting at $s < \hat{s}$. Consequently the support of S is $(\hat{s}, 1]$. Against this strategy of the inspector, the best that the operator can do is to shift all the weight of the support in the interval $[0, \hat{s}]$, to the atom at $t = 0$ and increase his expected payoff. This contradicts the definition of $\hat{s}$.

Consequently

$$\alpha p - (1 - \beta)(1 - p) < 0$$

or

$$p = P(T = 1) < \frac{1 - \beta}{1 - \beta + \alpha}. \qquad \square$$

Proposition 2. For every equilibrium (S, T), $1 \notin \mathrm{Supp}(S)$.

Proof. Assume that $1 \in \mathrm{Supp}(S)$. When the inspector inspects at time $s = 1$, the expected loss for the inspector, according to equation (14), is given by:

$$Eu(1, T) = E\big[(1 - \beta)(1 - T) + \beta(1 - T)\big] = 1 - E(T).$$

When the inspector inspects at $s = 1 - \varepsilon$, her expected loss is

$$\begin{aligned} Eu(1 - \varepsilon, T) &= E\big\{\big[(1 - \beta)(1 - \varepsilon - T) + \beta(1 - T)\big]J(T \leqslant 1 - \varepsilon)\big\} \\ &\quad + E\big\{\big[\alpha\varepsilon + (1 - \alpha)(1 - T)\big]J(T > 1 - \varepsilon)\big\} \\ &= \big(1 - E(T)\big) + \Lambda, \end{aligned}$$

where

$$\begin{aligned}\Lambda &= -\varepsilon(1-\beta)P(T \leqslant 1-\varepsilon) + \alpha E\{[\varepsilon-(1-T)]J(T > 1-\varepsilon)\}\\ &\leqslant -\varepsilon[(1-\beta)-(1-\beta+\alpha)P(T > 1-\varepsilon)].\end{aligned}$$

Since $P(T = 1) < (1-\beta)/(1-\beta+\alpha)$ (lemma 1), then for small enough $\varepsilon > 0$, the term Λ is negative. It indicates that the inspector will lose less if he inspects at $s = 1-\varepsilon$. This leeds to a contradiction to the assumption that in equilibrium $1 \in \mathrm{Supp}(S)$. □

Denote by b, the upper bound of the support of S, since $1 \notin \mathrm{Supp}(S)$. Therefor the support of S is in the interval $[0, b]$ where $b < 1$. The inspector will not inspect beyond time b.

Proposition 3. For every equilibrium (S, T), $\mathrm{Supp}(T) \subset [0, b]$.

It is seen intuitively clear that any violation at time $t > b$ is dominated by violating at time $t = b$, since there is certainly no inspection after time b.

Proof. The expected payoff of the operator against S, when violating at $t = b$, is:

$$Eu(S, b) = E[\alpha(1-S) + (1-\alpha)(1-b)].$$

If he violates at $t = b + \varepsilon$, where $0 < \varepsilon < 1 - b$ his expected payoff is:

$$Eu(S, b+\varepsilon) = E[\alpha(1-S) + (1-\alpha)(1-b) - (1-\alpha)\varepsilon]$$

which is smaller then $Eu(S, b)]$ since $(1-\alpha)\varepsilon > 0$ for sufficiently small ε. Therefore if (S, T) is in equilibrium, we conclude that $\mathrm{Supp}(T) \subset [0, b]$. □

Proposition 4. If (S, T) is an equilibrium, then S has no atom in $[0, b]$.

Proof. The proof consist of three lemmas. The first (lemma 2) shows that S has no atom at $s = 0$. The second (lemma 3) states that if S has an atom at $s_0 \in (0, b]$ in equilibrium, then there is an interval before s_0 where the operator will not violate, and in this case the inspector's strategy cannot have an atom at $s_0 \in (0, b]$ (lemma 4). Consequently there is no atom for S in the interval $[0, b]$.

Lemma 2. If (S, T) is an equilibrium then S does not have an atom at $s = 0$.

Proof. Assume that S has an atom at $s = 0$. If the operator violates at $t = 0$, his expected payoff against S (by equation (14)), is:

$$Eu(S, 0) = (1-\beta)E(S) + \beta.$$

His expected payoff, if he violates at time $t = \delta > 0$ is:

$$\begin{aligned}Eu(S, \delta) &= E\{[\alpha(1-S) + (1-\alpha)(1-\delta)]J(S < \delta)\}\\ &\quad + E\{[(1-\beta)(S-\delta) + \beta(1-\delta)]J(S \geqslant \delta)\}.\end{aligned}$$

The difference between $E[u(S,\delta)]$ and $E[u(S,0)]$ is:

$$\begin{aligned}
&Eu(S,\delta) - Eu(S,0)\\
&\quad = E\{[\alpha(1-S) + (1-\alpha)(1-\delta) - (1-\beta)S - \beta]J(S<\delta)\}\\
&\qquad + E\{[(1-\beta)(S-\delta) + \beta(1-\delta) - (1-\beta)S - \beta]J(S\geqslant\delta)\}\\
&\quad = [(1-\beta) - \delta(1-\alpha)]P(S=0)\\
&\qquad + E\{[(1-\beta) - (1-\beta+\alpha)S - \delta(1-\alpha)]J(0<S<\delta)\} - \delta P[S\geqslant\delta].
\end{aligned}$$

Since $P(S=0)>0$, the above expression is positive for δ small enough. The operator prefers, in this case, to violate at time $t=\delta>0$ rather than at $t=0$. Hence, T has no atom at $t=0$.

Against this strategy of the operator, the expected loss for an inspection at time $s=0$ is given by (equation (14)):

$$Eu(0,T) = E\{[\alpha + (1-\alpha)(1-T)]J(T>0)\} = \alpha + (1-\alpha)(1-E(T)).$$

The expected loss for an inspection at time $s=\varepsilon$ is:

$$\begin{aligned}
Eu(\varepsilon,T) &= E\{[(1-\beta)(\varepsilon-T) + \beta(1-T)]J(T\leqslant\varepsilon)\}\\
&\quad + E\{[\alpha(1-\varepsilon) + (1-\alpha)(1-T)]J(T>\varepsilon)\}\\
&= \alpha + (1-\alpha)(1-E(T)) - \varepsilon\alpha P(T>\varepsilon)\\
&\quad - (1-\beta)(1-\varepsilon)P(T\leqslant\varepsilon) - \alpha E[TJ(T\leqslant\varepsilon)]\\
&< \alpha + (1-\alpha)(1-E(T)).
\end{aligned}$$

For any $\varepsilon>0$, the expected loss for the inspector is larger if he inspects at time $s=0$, than if he does so at time $s=\varepsilon$. This contradicts the assumption that S has an atom at $s=0$. □

Lemma 3. If (S,T) is an equilibrium and S has an atom at $s_0\in(0,b]$, then there exists an $\varepsilon>0$, such that

$$(s_0-\varepsilon, s_0)\cap \mathrm{Supp}(T) = \emptyset.$$

Proof. Assume that S has an atom at s_0 $(0<s_0\leqslant b)$. We will show that the expected payoff of the operator is smaller if he violates before and close enough to time s_0, than if he violates immediately after s_0. If the operator violates at $t=s_0-\varepsilon$, his expected payoff according to equation (14) is:

$$\begin{aligned}
Eu(S,s_0-\varepsilon) &= E\{[\alpha(1-S) + (1-\alpha)(1-s_0+\varepsilon)]J(S<s_0-\varepsilon)\}\\
&\quad + E\{[(1-\beta)(S-s_0+\varepsilon) + \beta(1-s_0+\varepsilon)]J(S\geqslant s_0-\varepsilon)\}.
\end{aligned}$$

If the operator violates immediately after s_0 his expected payoff is:

$$\begin{aligned}
Eu(S,s_0) &= E\{[\alpha(1-S) + (1-\alpha)(1-s_0)]J(S\leqslant s_0)\}\\
&\quad + E\{[(1-\beta)(S-s_0) + \beta(1-s_0)]J(S>s_0)\}.
\end{aligned}$$

The difference between $Eu(S, s_0)$ and $Eu(S, s_0 - \varepsilon)$ is:

$$\begin{aligned}
&Eu(S, s_0) - Eu(S, s_0 - \varepsilon)\\
&\quad = -\varepsilon\big[(1-\alpha)P(S < s_0 - \varepsilon) + P(S \geqslant s_0 - \varepsilon)\big]\\
&\qquad + E\big\{\big[(1-\beta)(1-s_0) + (1-\beta+\alpha)(s_0 - S)\big]J(s_0 - \varepsilon \leqslant S < s_0)\big\}\\
&\qquad + (1-\beta)(1-s_0)P(S = s_0).
\end{aligned}$$

Since $P(S = s_0) > 0$, the above difference is positive for small enough $\varepsilon > 0$. The operator expected payoff is larger if he violates at $t = s_0$ than for a violation at $s_0 - \varepsilon \leqslant t < s_0$, i.e., the operator does not violate in the interval $(s_0 - \varepsilon, s_0)$. □

Lemma 4. If (S, T) is an equilibrium and $(s_0 - \varepsilon, s_0] \cap \mathrm{Supp}(T) = \emptyset$, then S cannot have an atom at s_0.

Proof. Let T be a violation strategy such that $(s_0 - \varepsilon, s_0] \cap \mathrm{Supp}(T) = \emptyset$.

The expected loss for an inspection at time $s = s_0 - \varepsilon$, against strategy T of the operator is (equation (14)):

$$\begin{aligned}
Eu(s_0 - \varepsilon, T) = {}& E\big\{\big[(1-\beta)(s_0 - \varepsilon - T) + \beta(1-T)\big]J(T \leqslant s_0 - \varepsilon)\big\}\\
&+ E\big\{\big[\alpha(1 - s_0 + \varepsilon) + (1-\alpha)(1-T)\big]J(T > s_0 - \varepsilon)\big\}
\end{aligned}$$

and at inspection time $s = s_0$,

$$\begin{aligned}
Eu(s_0, T) = {}& E\big\{\big[(1-\beta)(s_0 - T) + \beta(1-T)\big]J(T \leqslant s_0)\big\}\\
&+ E\big\{\big[\alpha(1-s_0) + (1-\alpha)(1-T)\big]J(T > s_0)\big\}.
\end{aligned}$$

Noting that

$$J(T > s_0 - \varepsilon) = J(T > s_0)$$

and

$$J(T \leqslant s_0 - \varepsilon) = J(T \leqslant s_0)$$

it is seen that

$$Eu(s_0 - \varepsilon, T) - Eu(s_0, T) = \varepsilon\big[\alpha - (1-\beta+\alpha)P(T \leqslant s_0)\big].$$

Hence the condition which makes the inspector choose to inspect at time $s_0 - \varepsilon$ rather than at time s_0 is given by:

$$P(T \leqslant s_0) > \frac{\alpha}{1-\beta+\alpha}. \tag{16}$$

If the inspection is at $s = s_0 + \delta$, the inspector's expected loss against the strategy T of the operator is:

$$\begin{aligned}
Eu(s_0 + \delta, T) = {}& E\big\{\big[(1-\beta)(s_0 + \delta - T) + \beta(1-T)\big]J(T \leqslant s_0 + \delta)\big\}\\
&+ E\big\{\big[\alpha(1 - s_0 - \delta) + (1-\alpha)(1-T)\big]J(T > s_0 + \delta)\big\}.
\end{aligned}$$

Noting that

$$J(T \leqslant s_0 + \delta) = J(T \leqslant s_0) + J(s_0 < T \leqslant s_0 + \delta).$$

The difference between the expected loss at time $s = s_0 + \delta$ and time $s = s_0$ is given by:

$$\begin{aligned} &Eu(s_0 + \delta, T) - Eu(s_0, T) = \delta(1 - \beta)P(T \leqslant s_0) \\ &\qquad + E\{[(1 - \beta)(s_0 + \delta - 1) + \alpha(s_0 - T)]J(s_0 < T \leqslant s_0 + \delta)\} - \delta\alpha P(T > s_0 + \delta) \\ &\quad = \delta[-\alpha + (1 - \beta + \alpha)P(T \leqslant s_0)] \\ &\qquad + E\{[(1 - \beta)(s_0 + \delta - 1) + \alpha(s_0 + \delta - T)]J(s_0 < T \leqslant s_0 + \delta)\}. \end{aligned}$$

In this case the strategy S with an atom at time s_0 is dominated by other strategies if:

$$P(T \leqslant s_0) < \frac{\alpha}{1 - \beta + \alpha} \tag{17}$$

since, for δ small enough, the difference is negative.

It is also true for

$$P(T \leqslant s_0) = \frac{\alpha}{1 - \beta + \alpha},$$

which is proved below:

(1) If $P(s_0 < T \leqslant s_0 + \delta) > 0$, then for $\delta > 0$ small enough

$$E\{[(1 - \beta)(s_0 + \delta - 1) + \alpha(s_0 + \delta - T)]J(s_0 < T \leqslant s_0 + \delta)\} < 0.$$

This shows that the operator prefers to make his violation at time $s_0 + \delta$ rather than at time s_0, since the difference between the expected loss for the inspector is negative. This conclusion is in contradiction to the assumption that there is an atom for S at time $s = s_0$.

(2) If $P(s_0 < T \leqslant s_0 + \delta) = 0$, the difference is zero, i.e., the inspector expected loss is the same for any $s \in [s_0 - \varepsilon, s_0 + \delta]$, where

$$\delta = \max_{0 \leqslant \gamma < 1 - s_0} \{\gamma \mid P(s_0 < T \leqslant s_0 + \gamma) = 0\}.$$

We now distinguish between two possibilities:

(a) If $\delta = 1 - s_0$, no violation occurs after time $t = s_0 - \varepsilon$. It is clear that in this case a better reply of the inspector is to shift the atom from $s = s_0$ to $s = s_0 - \varepsilon$, which contradicts the assumption that S has an atom at s_0.

(b) If $\delta < 1 - s_0$, the inspector is indifferent to inspecting at any point in the interval $[s_0 - \varepsilon, s_0 + \delta]$. Denote $s_0 + \delta$ by s_0' and $0 \leqslant \delta' < 1 - s_0'$. Note that in this case

$$P(s_0' < T \leqslant s_0' + \delta') > 0.$$

By using an identical argument of item (1), for small enough δ', the inspector prefers to inspect at time $s_0' + \delta'$ rather than at $s_0 - \varepsilon \leqslant s \leqslant s_0 + \delta$ and in specific at time s_0, which contradicts the assumption that S has an atom at $s = s_0$.

In view of the above we conclude that S with an atom at s_0 is dominated by some other strategy if

$$P(T \leqslant s_0 - \varepsilon) \leqslant \frac{\alpha}{1 - \beta + \alpha}. \tag{18}$$

Thus each of the two conditions (16) and (18) is incompatible with S having an atom at s_0. Since, obviously, one these two inequalities always holds by lemma 3, there is no atom in the inspector's strategy S at time s_0. □

By means of the three lemmas 2, 3 and 4 we have seen that S has no atom in the interval $[0, b]$, which completes the proof of proposition 4. □

Proposition 5. If (S, T) is an equilibrium, then T has no atom in $(0, b]$.

Proof. We will prove first that if T has an atom at $t_0 \in (0, b]$, then the inspector will not inspect in an interval time $(t_0 - \varepsilon, t_0)$ for some $\varepsilon > 0$ (lemma 5). Given this strategy of the inspector, the best strategy of the operator cannot have an atom at t_0 in contradiction to the assumption (lemma 6). □

Lemma 5. If (S, T) is an equilibrium and T has an atom at $t_0 \in (0, b]$, there exist $\varepsilon > 0$ such that $(t_0 - \varepsilon) \cap \mathrm{Supp}(S) = \emptyset$.

Proof. Assume that there is an atom at t_0. We will show that the expected loss of the inspector is greater if he inspects before and near enough to time t_0, than if he does so immediately after t_0.

If the inspector inspects at $s = t_0 - \varepsilon$, the expected loss, according to equation (14) is:

$$\begin{aligned} Eu(t_0 - \varepsilon, T) = {} & E\{[(1-\beta)(t_0 - \varepsilon - T) + \beta(1-T)]J(T \leqslant t_0 - \varepsilon)\} \\ & + E\{[\alpha(1 - t_0 + \varepsilon) + (1-\alpha)(1-T)]J(T > t_0 - \varepsilon)\}. \end{aligned}$$

If the inspector inspects immediately after t_0, her expected loss is given by:

$$\begin{aligned} Eu(t_0, T) = {} & E\{[(1-\beta)(t_0 - T) + \beta(1-T)]J(T \leqslant t_0)\} \\ & + E\{[\alpha(1 - t_0) + (1-\alpha)(1-T)]J(T > t_0)\}. \end{aligned}$$

Then

$$\begin{aligned} Eu(t_0, T) - Eu(t_0 - \varepsilon, T) = {} & (1-\beta)\varepsilon P(T \leqslant t_0 - \varepsilon) \\ & + E\{[(1-\beta)(t_0 - 1) - \alpha(T - t_0 + \varepsilon)]J(t_0 - \varepsilon < T < t_0)\} \\ & - (1-\beta)(1-t_0)P(T = t_0) - \alpha\varepsilon P(T \geqslant t_0) \\ = {} & \varepsilon[(1-\beta)P(T \leqslant t_0 - \varepsilon) - \alpha P(T \geqslant t_0)] \\ & - E\{[(1-\beta)(1-t_0) + \alpha(T - t_0 + \varepsilon)]J(t_0 - \varepsilon < T < t_0)\} \\ & - (1-\beta)(1-t_0)P(T = t_0). \end{aligned}$$

For small enough $\varepsilon > 0$, the above difference is negative, since $P(T = t_0) > 0$. This shows that if T has an atom at t_0, there exist an interval before t_0 in which the inspector will not inspect. □

Lemma 6. If (S, T) is an equilibrium and $(t_0 - \varepsilon) \cap \mathrm{Supp}(S) = \emptyset$, the best reply of the operator cannot have an atom at t_0.

Proof. If the operator violates at t_0 against the inspector's strategy S, his expected payoff is (according to equation (14)):

$$\begin{aligned} Eu(S, t_0) = {} & E\big\{\big[\alpha(1 - S) + (1 - \alpha)(1 - t_0)\big]J(S < t_0)\big\} \\ & + E\big\{\big[(1 - \beta)(S - t_0) + \beta(1 - t_0)\big]J(S \geqslant t_0)\big\}. \end{aligned}$$

If the operator violates at $t_0 - \varepsilon$ against the inspector strategy S, his expected payoff is:

$$\begin{aligned} Eu(S, t_0 - \varepsilon) = {} & E\big\{\big[\alpha(1 - S) + (1 - \alpha)(1 - t_0) + (1 - \alpha)\varepsilon\big]J(S < t_0 - \varepsilon)\big\} \\ & + E\big\{\big[(1 - \beta)(S - t_0) + \beta(1 - t_0) + \varepsilon\big]J(S \geqslant t_0 - \varepsilon)\big\}. \end{aligned}$$

Noting that

$$J(S < t_0) = J(S < t_0 - \varepsilon)$$

and

$$J(S \geqslant t_0) = J(S \geqslant t_0 - \varepsilon)$$

the difference between $Eu(S, t_0 - \varepsilon)$ and $Eu(S, t_0)$ is given by:

$$E\big[u(S, t_0 - \varepsilon)\big] - E\big[u(S, t_0)\big] = (1 - \alpha)\varepsilon P(S < t_0 - \varepsilon) + \varepsilon P(S \geqslant t_0) > 0.$$

At point $t_0 - \varepsilon$ the expected payoff of the operator is larger than if he violates at time t_0. Therefore there cannot be an atom for T at t_0 which contradicts the assumption. □

Consequently there is no atom of T in the half open interval $(0, b]$. This lemma concludes the proof of the proposition and by that the proof of the theorem. □

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