CHAPTER 17

Formulation of Bayesian Analysis for Games with Incomplete Information

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1. Introduction

In analyzing a game with incomplete information, i.e. games in which players are uncertain about all the parameters defining the strategy spaces and the payoff functions, one is led naturally to handle "an infinite hierarchy of beliefs" for each player: His beliefs (i.e. subjective probabilities) on the parameters of the games, his beliefs on the beliefs of the other players on the parameters of the games, his beliefs on the other players' beliefs on his own beliefs on the parameters of the games, his beliefs on the other players' beliefs on his own beliefs on their beliefs on the parameters of the games, etc. . .

In an attempt to overcome the difficulty of having to work with infinite sequences of mutual beliefs. *Harsanyi* [1967 – 1968] introduced the concept of *type* which proved to be very useful in making games with incomplete information much more manageable. Harsanyi's idea was to summarize all parameters and beliefs concerning a certain player, by one vector which he calls the *attribute* vector. In his words [see *Harsanyi,* 1967, p. 171]: ". . . we can regard the vector c_i as representing certain physical, social, and psychological *attributes* of player *i himself* in that it summarizes some crucial parameters of player *i*'s own payoff function U_i as well as the main parameters of his beliefs about his social and physical environment . . . the rules of the game as such allow any given player *i* to belong to any one of a number of possible *types*, correspond-

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ing to the alternative values of his attribute vector c_i could take . . . Each player is assumed to know his own actual type but to be in general ignorant about the other players' actual types."

Can this idea be formalized mathematically? In other words: Starting from a set S of all possible values of the parameters of the game can one identify a mathematically well defined set Y of the "states of the world" in which every point contains all characteristics, beliefs and mutual beliefs of all players?

If yes, would any infinite hierarchy of beliefs lead to some point in Y? This is exactly the construction we do in Section 2 of this paper. The space Y defined there is what we call *"the universal beliefs space generated by S"* and it includes, roughly speaking, all possible states of the world arising from S. Furthermore, there is a well defined space T, called the space of all possible types of a player in such a game, such that Y and T satisfy (up to some appropriate homeomorphism) the following two relations:

(i) $Y = S \times [T]^n$; (ii) $T =$ the set of all probability distributions on $(S \times [T]^{n-1})$. The first equality says that a state of the world $y \in Y$ consists of a state of nature $s \in S$ and an n-tuple of types, one for each player. The second relation says that a type of a player is just a joint probability distribution on S and types of the other $(n-1)$ players. This is exactly the formalization of the notion of 'type' as used by Harsanyi.

Typically in an actual situation many of the points in Y will be considered impossible by all of the players. In other words what is then relevant is only some subset of Y. (This is for instance the case if all players know one parameter in S but are uncertain about the others). This leads to the notion of what we call *beliefs subspaces* of Y.

It turns out, as it can be easily seen, that even if we start with a set S which is finite, both Y and most of its beliefs subspaces will be sets of high cardinality. On the other hand, most of the work on games with incomplete information assume *finitely* many possible states of the world. In Section 3 we provide some justification for this by proving that any beliefs subspace of Y can be "approximated" by a *finite* beliefs subspace which is arbitrarily close to it in the Hausdorff distance between closed sets.

In Section 4 we consider the concept of *consistency,* also discussed by Harsanyi and later by *Aumann/Maschler*. Generally speaking, a state of the world represents a consistent situation if there is a probability distribution on all the states of the world such that the beliefs of each player equal the conditional probability distribution given his private information. We define this concept formally and prove that it in fact captures the intuitive meaning of consistency. We then show that the consistency of an actual situation is common knowledge, in the sense that each player, based on his own information only, can test the hypothesis that the state of the world is consistent, if yes to compute the consistent set of states to which it belongs and compute the global probability distribution on Y corresponding to the consistent situation he is in. In such a test each player has (subjective) probability 0 of comitting any error in his conclusion.

Finally, in Section 5, we define a game in strategic form determined by the beliefs space (or subspace). This will be typically a game "with vector payoffs", but the Nash equilibria are well defined. For a consistent beliefs subset, the Nash equilibria will be the same as those of a certain extensive form game in which the state of the world is chosen according to the (uniquely determined) probability distribution, and each player is informed on what is his own type. This is Harsanyi's theorem [*Harsanyi*, 1967, part II, p. 321] which is in the background of most models of games with incomplete information.

It should be pointed out that works in the same direction were done by *Böge et al.* who, being interested mainly in the equilibrium points of games with incomplete information, incorporated the strategy choices of the players as part of the space of parameters on which the infinite hierarchy of beliefs is built.

2. The Universal Beliefs Space Y

The main objective of this section is to prove Theorem 2.9 which establishes the existence of a space of infinite hierarchy of beliefs. We consider a situation of incomplete information involving a set of players $I = \{1, \ldots, n\}$, the members of which are uncertain about the parameters of the game they are playing which may be any element of some set S (we may think of a point of S as a full listing of the strategy spaces and the payoff functions). We shall refer to S as the *parameter-space.*

Assumption: S is a compact space.

Remark: To see that this assumption is not too restrictive, let us see how, in a typical and rather general model if incomplete information , the space S will in fact be compact: Observe that S has most generally to include all the parameters of the game including the parameters of the utility functions of the player. So let S_0 be the set of possible values of all the parameters of the game. Clearly S_0 may be assumed (by enlarging it if necessary) to be compact. For each player i let A^i be his action set (enlarged so as to become independent of $S \in S_0$). The set of outcomes can then be identified with the set $C = S_0 \times \mathop{\mathsf{X}}_{i=1}^{n} A^i$ and is compact if A^i are compact. The Von-Neumann Morgenstern utility function of player i is a (continuous) real map $u_i\colon C \to R$, which we may want to assume to be bounded (for instance by applying the Von Neumann-Morgenstern theory to all countable lotteries, in order to avoid the St. Petersburg paradox). Hence we may take $u_i\colon C \to [0, 1]$ and the set of all possible games is then $S = S_0 \times [[0, 1]^C]^n$ which is compact. A special case is of course that in which S_0 and A^i are finite then S will be in addition metrisable.

For any compact space X, $\Pi\,(X)$ will denote the compact space of probability measures on X, endowed with the weak* topology. [It is clearly closed in the set of all measures of norm $\leqslant 1$ since the function 1 is continuous-, and the set is by Riesz theorem the unit ball of the dual of $C\,(X)$, hence weak*-compact by Alaoglu's theorem.]

Definition 2.1: A coherent beliefs hierarchy of level K ($K = 1, 2, \ldots$) is a sequence $(C_0, C_1, \ldots, C_K)$ where:

1) C_0 is a compact subset of S and for $k = 1, \ldots, K$, C_k is a compact subset of $C_{k-1} \times [\Pi(C_{k-1})]^n$ (as topological spaces). (We denote by ρ_{k-1} and t^i the projections of C_k on C_{k-1} and the i-th copy of $\Pi(C_{k-1})$ respectively.)

$$\rho_{k-1}(C_k) = C_{k-1}; k = 1, \ldots, K. \tag{2}$$

$$\forall c_k \in C_k \text{ let } c_{k-1} = \rho_{k-1}(c_k), \text{ then } \forall i: \tag{3}$$

H1) the marginal distribution of $t^i(c_k)$ on C_{k-2} is $t^i(c_{k-1})$;

H2) the marginal distribution of $t^i(c_k)$ on the i-th copy of $\Pi(C_{k-2})$ is the unit mass at $t^i(c_{k-1}) = t^i(\rho_{k-1}(c_k))$.

$$\forall i, \forall t \in t^i(C_k); k = 1, \ldots, K, t\,\{\rho_{k-1}[(t^i)^{-1}(t)]\} = 1. \tag{4}$$

We interpret C_k as a set of beliefs up to level k and thus a point in C_k consists of hierarchy of beliefs up to level $(k-1)$ (i.e. a point in C_{k-1}) and for each player i a probability distribution t_k^i on hierarchies of beliefs up to level $(k-1)$ (i.e. $t_k^i \in \Pi(C_{k-1})$). Condition H1 says that player i's k-level beliefs coincide with his $(k-1)$ level beliefs in whatever concerns hierarchies up to level $(k-2)$. Condition H2 says that player i knows his own previous order beliefs.

In the next definition we formalize the properties of the space of states of the world C we would like to obtain: Any point $c \in C$ determines uniquely a set of parameters $s \in S$ and the type t^i of each player. The type t^i is a probability distribution on C which is coherent in the sense that each player knows his own type. In other words if $t^i \in \Pi(C)$ is a certain type of player i, then in all points in the support of t^i (Supp (t^i)) player i is of type t^i. This motivates the following.

Definition 2.2: An S-based abstract beliefs space (BL-space) is an $(n+3)$ tuple $(C, S, f, (t^i)_{i=1}^n)$ where C is a compact set, S is some compact space, f is a continuous mapping $f: C \to S$ and t^i, $i = 1, \ldots, n$, are continuous mappings $t^i: C \to \Pi(C)$ (with respect to the weak* topology) satisfying:

(*) $\tilde{c} \in C$ and $\tilde{c} \in \text{Sup}(t^i(c)) \Rightarrow t^i(\tilde{c}) = t^i(c)$.

When no confusion may result we shall denote the BL-space simply by C.

The space C is a space in which each point $c \in C$ contains a full description not only of the state of nature $s \in S$ but also of all beliefs, beliefs on beliefs etc. on S. In fact if we interpret t^i as player i's (subjective) probability distribution on C, then combined with f it defines a probability distribution on S, which is the first level beliefs of player i. But t^i also defines a probability distribution on $(t^j)_{j \neq i}$, and hence on the first level beliefs of the other players. This may be called the second level beliefs of player i. Proceeding inductively we find that with each $c \in C$ is associated one infinite hierarchy of beliefs for each player. The condition (*) is a consistency condition which says basically that a player i assigns positive probability (in the discrete case) only to points of C in which he has the same beliefs. In other words he is certain of his own beliefs.

Let us write now formally the above mentioned observation:

Given S we define the spaces X_k, T_k, by

$$X_0 = S$$

$$T_k = \Pi\,(X_{k-1})$$

$$X_k = X_{k-1} \times [T_k]^n = S \times \mathop{\times}_{l=1}^{k} [T_l]^n; \quad k = 1, 2, \ldots$$

Define also $X = S \times \mathop{\times}_{l=1}^{\infty} [T_l]^n$, which is a well defined compact space when so is S.

Note that X is generated by S and whenever we want to specify the generating space we shall write $X\,(S)$. We shall denote a typical point in $X\,(S)$ as $x = (s, t_1^1, \ldots, t_1^n, \ldots, t_k^1, \ldots, t_k^n, \ldots)$, where for each i and each k $t_k^i \in T_k^i =$ $= \Pi\,(X_{k-1})$.

If $\varphi\colon C \to \tilde{C}$ is a continuous mapping between two compact spaces C and $\tilde{C}$, we denote by $\hat{\varphi}$ the mapping $\Pi\,(C) \to \Pi\,(\tilde{C})$ canonically induced by φ, namely the mapping $\hat{\varphi}\colon \Pi\,(C) \to \Pi\,(\tilde{C})$ which maps $\mu \in \Pi\,(C)$ to $\hat{\mu} \in \Pi\,(\tilde{C})$ such that for any continuous function f on $\tilde{C}$, $\int_{\tilde{C}} f\,(\tilde{c})\,d\hat{\mu} = \int_{C} (f \circ \varphi)\,(c)\,d\mu$.

To any S-based abstract BL-space $(C, S, f, (t^i)_{i=1}^n)$ we define now a certain natural continuous mapping $h\colon C \to X\,(S)$. This will be done by defining for each $k = 0, ., 2, \ldots$ a mapping $h_k\colon C \to X_k$ such that

$$k \leqslant l \Rightarrow \rho_k\,(h_l\,(c)) = h_k\,(c) \qquad \forall\, c \in C,$$

in other words, $h_k\,(c)$ is the projection of $h\,(c)$ on X_k.

The mappings h_k are defined inductively as follows: $h_0\,(c) = f\,(c)$. Assume $h_k\colon C \to X_k$ is defined then we want to define $h_{k+1}\colon C \to X_{k+1}$. Take any $c \in C$ and let $h_k\,(c) = (s, t_1^1, \ldots, t_1^n, \ldots, t_k^1, \ldots, t_k^n) \in X_k$ then $h_{k+1}\,(c) =$ $= (s, t_1^1, \ldots, t_1^n, \ldots, t_k^1, \ldots, t_k^n, t_{k+1}^1, \ldots, t_{k+1}^n) \in X_{k+1}$ where $\forall\, i$,

$t^i_{k+1} = \hat{h}_k \circ t^i : C \to \Pi(X_k) = T_{k+1}$, and $\hat{h}_k$ is the mapping $\hat{h}_k : \Pi(C) \to \Pi(X_k)$ canonically induced by h_k.

It follows that the so defined $h: C \to X(S)$ is continuous. Let $H = h(C) \subseteq X(S)$. When we want to emphasize the underlying S we shall write $H(S)$. By construction, the image $h(c)$ contains all possible information concerning S and beliefs on S. Therefore it is intuitively pretty clear that $h(c) \neq h(c')$ for $c \neq c'$ unless c and c' are identical in whatever concerns S and differ only by something which is redundant to S and to the beliefs structure on S.

To define this notion of nonredundancy more formally, given an BL-space $(C, S, f, (t^i)^n_{i=1})$ let $\mathcal{F}$ be the smallest σ-field (of subsets of C) for which f is measurable and $\forall i, (t^i(c))(B)$ is measurable $\forall B \in \mathcal{F}$.

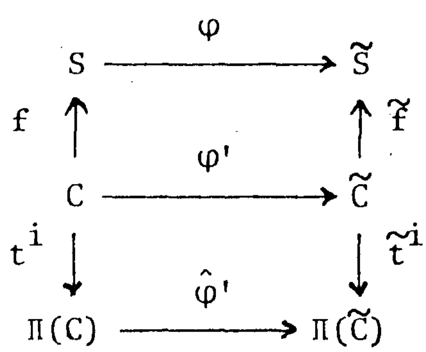
Definition 2.4: A BL-space $(C, S, f, (t^i)^n_{i=1})$ is said to satisfy the *non-redundancy* condition (NR-condition) if the σ-field $\mathcal{F}$ separates each two distinct points in C.

By our previous discussion we thus have:

Proposition 2.5: If an S-based abstract BL-space $(C, S, f, (t^i)^n_{i=1})$ satisfies the NR-condition then the mapping $h: C \to H$ is also one to one hence it is an isomorphism.

In dealing with BL-spaces we would like to consider homeomorphisms between BL-spaces which (in addition to their topological properties) will also preserve the beliefs structure. These mappings will be called *BL-morphisms* and we proceed now to define them formally.

Definition 2.6: A beliefs morphism (BL-morphism) from a BL-space $(C, S, f, (t^i)^n_{i=1})$ to a BL-space $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)^n_{i=1})$ is a pair (φ, φ') where φ' is a continuous mapping of C onto $\tilde{C}$ and φ is a continuous mapping of S onto $\tilde{S}$ such that for each i, $i = 1, \ldots, n$, the following diagram comutes:



where $\hat{\varphi}'$ is the mapping $\hat{\varphi}' : \Pi(C) \to \Pi(\tilde{C})$ canonically induced by φ'.

Definition 2.7: A BL-morphism (φ, φ') from $(C, S, f, (t^i)_{i=1}^n)$ to $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ is called a BL-isomorphism if the inverse mappings φ^{-1} and $(\varphi')^{-1}$ exist and $(\varphi^{-1}, (\varphi')^{-1})$ is a BL-morphism from $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ to $(C, S, f, (t^i)_{i=1}^n)$. The two BL-spaces are said to be *BL-isomorphic.*

Some thought on the diagramm of Definition 2.6 leads us to the observation that if (φ, φ') is a BL-morphism from C to $\tilde{C}$ then there is actually one essential mapping and not two since φ' seems to be determined by φ via the above diagram. This is in fact true provided $\tilde{C}$ satisfies the NR-condition:

Lemma 2.8: If (φ, φ') is a BL-morphism from $(C, S, f, (t^i)_{i=1}^n)$ to $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ and if the latter satisfies the NR-condition, then φ' is uniquely determined by φ.

Proof: Using our notation $h: C \to X(S)$ and $\tilde{h}: \tilde{C} \to X(\tilde{S})$ we denote by $h \circ \varphi: C \to X(\varphi(S)) \subseteq X(\tilde{S})$ the mapping which maps $c \in C$ to $h(c)$ in which the underlying S is replaced by $\varphi(S)$. The fact that the diagram of Definition 2.6 computes implies that $\forall\, c \in C$ we have $\tilde{h}(\varphi'(c)) = (h \circ \varphi)(c) \in X(\tilde{S})$. Since $\tilde{C}$ satisfies the NR-condition $\tilde{h}$ is one to one (by Proposition 2.5) and hence invertible. Therefore:

$$\varphi'(c) = \tilde{h}^{-1}(h \circ \varphi)(c).$$

In words, the idea of the proof is that φ combined with the diagram determines for each $c \in C$ uniquely the infinite hierarchy $\tilde{h}(c')$ associated with $c' = \varphi'(c)$, and hence it determines uniquely c' itself since $\tilde{C}$ satisfies the NR-condition.

Remark: In view of Lemma 2.8 we shall shorten our notation and terminology and speak of BL-morphism φ from BL-space C to BL-space $\tilde{C}$. This is the BL-morphism induced by the mapping $\varphi: S \to \tilde{S}$.

We are now ready to state the main theorem of this section.

Theorem 2.9: For any compact S and positive integer n there are spaces $\mathcal{Y}$ and $\mathcal{T}$ such that:

1) $\mathcal{Y} = S \times [\mathcal{T}]^n$
2) $\mathcal{T} = \Pi(S \times [\mathcal{T}]^{n-1})$
$\Big\}$ *up to BL-morphisms.*

3) There are compact spaces $\{Y_k\}_{k=0}^{\infty}$ s.t. $\forall\, k$ $(Y_0, Y_1, \ldots, Y_k)$ *is a coherent beliefs hierarchy and* $\mathcal{Y}$ *is the projective limit* $\{Y_k\}_{k=0}^{\infty}$ *(with respect to the natural projection* $\rho_{k-1}: Y_k \to Y_{k-1}$. *We denote by* ρ_k *also the projection of* $\mathcal{Y}$ *on* Y_k*).*

4) $\mathcal{Y}$ *is an S-based BL-space (with the projections* $f: \mathcal{Y} \to S$ *and* $t^i: \mathcal{Y} \to \mathcal{T}^i$*).*

5) Any S-based abstract BL-space, which satisfies the NR-condition, is canonically BL-homeomorphic to a compact subset of $\mathcal{Y}$ *(which will be called a BL-subspace of* $\mathcal{Y}$*).*

6) For any coherent beliefs hierarchy $(C_0, C_1, \ldots, C_K)$ *there is a BL-subscpace C of* $\mathcal{Y}$ *s.t.* $\rho_k(C) = C_k$, $k = 0, \ldots, K$.

7) Any $\tilde{\mathcal{Y}}$ *and* $\tilde{\mathcal{T}}$ *which satisfy 1) and 2) or 4) and 5) can be mapped continuously onto* $\mathcal{Y}$ *and* $\mathcal{T}$ *respectively. This map induces a BL-homeomorphism between* $\mathcal{Y}$ *and a BL-subspace of* $\tilde{\mathcal{Y}}$. *Any* $\tilde{\mathcal{Y}}$ *which satisfies 3) and 6) can be BL-morphically mapped onto* $\mathcal{Y}$.

$\mathcal{Y}$ *will be called the Universal BL-space generated by S (and n) and* $\mathcal{T}$ *will be called the Universal type space generated by S (and n).*

Proof: We shall prove the theorem by constructing the sequence $\{Y_k\}_{k=0}^{\infty}$ in (3) and define $\mathcal{Y}$ as its projective limit and $\mathcal{T}^i$ as the projection of $\mathcal{Y}$ on player *i*'s coordinates. Then we shall prove that these $\mathcal{Y}$ and $\mathcal{T}$ satisfy the required properties.

Construction of $\mathcal{Y}$

Define the sequence of spaces $\{Y_k\}_{k=0}^{\infty}$ as follows:

$Y_0 = S$ and for $k = 1, 2, \ldots$

(2.1) $Y_k = \{y_k \in Y_{k-1} \times [\Pi(Y_{k-1})]^n \mid (a)\ \forall i$ the marginal distribution of $t^i(y_k)$ on Y_{k-2} is $t^i(y_{k-1})$ and (b) the marginal distribution of $t^i(y_k)$ on the i-th copy of $\Pi(Y_{k-2})$ is the unit mass at $t^i(y_{k-1})\}$.

As we have already noted if X is compact, then $\Pi(X)$ is also compact. Note also that the conditions (a) and (b) in the definition of Y_k are closed conditions. It follows that if Y_{k-1} is compact, then Y_k is also compact. Since $Y_0 = S$ is compact, it follows inductively that Y_k is compact $\forall k$. Let $\mathcal{Y}$ be the projective limit of $\{Y_k\}_{k=0}^{\infty}$ with respect to the natural projections $\rho_{k-1}: Y_k \to Y_{k-1}$. $\mathcal{Y}$ is a well defined compact set.

Now by definition of Y_k we have that $\forall k$, $(Y_0, \ldots, Y_k)$ satisfy automatically all properties of a coherent beliefs hierarchy (Definition 2.1) except for condition (2), namely that $\rho_k(Y_{k+1}) = Y_k$, $k = 0, 1, \ldots$. This we prove now:

Proposition 2.10: $\rho_k(Y_{k+1}) = Y_k$, $k = 0, 1, 2, \ldots$

This proposition has the following immediate corollaries.

Corollary 2.11:

i) $\forall k, (Y_0, Y_1, \ldots, Y_k)$ *is a coherent beliefs hierarchy;*

(ii) $\forall k, \rho_k(\mathcal{Y}) = Y_k$, *in particular* $\mathcal{Y} \neq \emptyset$.

The proof of Proposition 2.10 will follow from the following.

Lemma 2.12: Let A and B be compact sets, D a compact subset of $A \times B$ and $q \in \Pi(A)$. A necessary and sufficient condition for the existence of $p \in \Pi(D)$ whose marginal distribution on A is q, is that $q(D_A) = 1$, where D_A is the projection of D on A.

Proof: Since $D \subset D_A \times B$, the necessity is obvious. To prove the sufficiency assume $q(D_A) = 1$. Define $L_q(f) = \int f dq$. L_q is a linear functional defined on $C(D_A)$, (the linear space of continuous real functions on D_A). If we consider a function on D_A as a function on D, by the natural definition $F(a, b) = f(a) \ \forall (a, b) \in D$, and write $L_q(F) = \int F dq$, Lq is then a linear functional defined on a linear subspace of $C(D)$. This is clearly a positive functional with $\| L_q \| = 1$. By Hahn-Banach extension theorem L_q can be extended to a positive linear functional L of norm 1 on $C(D)$. Finally by Riesz representation theorem there is a probability measure $p \in \Pi(D)$, s.t. $L(f) = \int_D fdp \ \forall f \in C(D)$. This p is the required extension of q.

Proof of Proposition 2.10: We prove the proposition inductively on k. It holds for $k = 0$ since $Y_0 = S$ and $Y_1 = S \times [\Pi(S)]^n$, thus $\rho_0(Y_1) = Y_0$. Assume that $\rho_{k-1}(Y_k) = Y_{k-1}$ and let us prove that $\rho_k(Y_{k+1}) = Y_k$. In other words we have to show that any point $y \in Y_k$ can be extended to a point $(y, t^1_{k+1}, \ldots, t^n_{k+1}) \in Y_{k+1}$. So we have to establish the existence of an n-tuple $t^1_{k+1}, \ldots, t^n_{k+1}$ of probability distributions $t^i_{k+1} \in \Pi(Y_k)$ satisfying conditions (a) and (b) in the definition of Y_k, namely that the marginal disitribution on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$ is $t^i_k \times \delta_{t^i(y)}$, where $\delta_{t^i(y)}$ is the element $\Pi(Y_{k-1})$ which assigns mass 1 to $t^i(y)$. We have thus to show that each of these marginals can be extended to a probability distribution t^i_{k+1} on $Y_{k-1} \times [\Pi(Y_{k-1})]_1 \times \ldots \times [\Pi(Y_{k-1})]_n$ supported by its subset Y_k i.e. $t^i_{k+1}(Y_k) = 1$. Using Lemma 2.12 it remains to prove that

$$\text{Sup}(t^i(y)) \times \{t^i(y)\} = \text{Supp}[t^i(y) \times \delta_{t^i(y)}] \subseteq \text{projection of } Y_k \text{ on } Y_{k-1} \times [\Pi(Y_{k-1})]_i.$$

So let $(\tilde{y}_{k-1}, t^i(y)) \in \text{Supp}\, t^i(y) \times \{t^i(y)\}$ i.e. $\tilde{y}_{k-1} \in \text{Supp}\, t^i(y) \subseteq Y_{k-1}$. Since $t^i(y)$ assigns probability 1 to $t^i(\rho_{k-1}(y))$ it follows that $t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))$. Since by induction hypothesis $\rho_{k-1}(Y_k) = Y_{k-1}$, there is an extension $(\tilde{y}_{k-1}, \tilde{t}^1_k, \ldots, \tilde{t}^n_k) \in Y_k$. We claim that if in this point we replace $\tilde{t}^i_k$ by $t^i(y)$ we obtain a point which is also in Y_k, proving that $(\tilde{y}_{k-1}, t^i(y))$ is in the projection of Y_k on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$ and thus completing the proof.

To see that $(\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, t^i(y), \ldots, \tilde{t}_k^n) \in Y_k$, note that all conditions concerning $\tilde{t}_k^j$, $j \neq i$ are satisfied since $(\tilde{y}_{k-1}, \tilde{t}_k, \ldots, \tilde{t}_k^n) \in Y_k$. The conditions concerning $t^i(y)$ are satisfied since these are the conditions required for $y \in Y_k$ (recalling that $t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))$). This completes the proof of Proposition 2.10.

Remark: Note that when $y \in Y_k$ is such that all distributions t_j^i are of finite support, the extension of y to a point in Y_{k+1} is straightforward and an extension, also with finite support, can be pointed at explicitely.

For any $y = (y_0, y_1, \ldots) \in \mathcal{Y}$ and for each $i \in N$, consider the sequence of probabilities $t^i(y_1), t^i(y_2), \ldots$ on $Y_0, Y_1, Y_2, \ldots$ respectively. By the definition of $\{Y_k\}_{k=0}^{\infty}$, this sequence satisfies that $\forall k$, the marginal of $t^i(y_{k+1})$ on Y_{k-1} is $t^i(y_k)$. Since also $\rho_k(\mathcal{Y}) = Y_k$ $\forall k$, it follows that for any continuous real function f_K on $\mathcal{Y}$ which depends only on K coordinates, the sequence of integrals $(\int f_K \, d\, t^i(k))_{k=1}^{\infty}$ is well defined and constant for $k \geqslant K+1$. Therefore the sequence $(t^i(y_k))_{k=1}^{\infty}$ defines a linear positive functional of norm 1 on the space of all such functions f_K and hence on the closure of this space which is the space of all continuous functions on $\mathcal{Y}$. By Riesz representation theorem there is a uniquely determined probability measure in $\Pi(\mathcal{Y})$ which represents this linear functional.

Definition 2.13:

(i) For each $y \in \mathcal{Y}$ and $\forall i \in N$, define by $t^i(y)$ the probability distribution on $\mathcal{Y}$ determined by y in the above described way

(ii) Let $\mathcal{T}^i = t^i(\mathcal{Y}) \subseteq \Pi(\mathcal{Y})$.

Remark: Note that the mappings t^i are continuous.

Clearly all $\mathcal{T}^i$ are copies of the same space which we denote by $\mathcal{T}$.

The spaces $\mathcal{Y}$ and $\mathcal{T}$ are respectively the universal beliefs space and the universal type space generated by S (and n), and the rest of this section is devoted to prove that these $\mathcal{Y}$ and $\mathcal{T}$ in fact satisfy the properties claimed in Theorem 2.9. So far we have that 3) is satisfied by construction.

Property 1: $\mathcal{Y} = S \times [\mathop{\times}_{i=1}^{n} \mathcal{T}^i]$ *(homeomorphically)*.

Proof: First let us establish a one to one mapping between the two sets. Each $y \in \mathcal{Y}$ determines uniquely some $s \in S$ (namely $s = \rho_0(y)$), also by definition of $\mathcal{T}$, y determines uniquely $t^i(y) \in \mathcal{T}^i$ $\forall i$. This establishes a mapping f: $\mathcal{Y} \to S \times [\mathop{\times}_{i=1}^{n} \mathcal{T}^i]$.

On the other hand by its definition $\mathcal{Y}$ can be represented as:

$$\mathcal{Y} = \{y_0, (t^1(y_k))_{k=1}^{\infty}, \ldots, (t^n(y_k))_{k=1}^{\infty} \mid y_0 \in S \;\forall k \;\forall i$$

$$t^i(y_k) \in \Pi(Y_{k-1}) \text{ and conditions a) and b) of formula (2.1)}$$

$$\text{are satisfied}\}.$$

But for certain i the conditions $t^i(y_k) \in \Pi(y_{k-1})$, a) and b) $\forall k$ are conditions only on the sequence $(t^i(y_k))_{k=1}^{\infty}$, which are satisfied by the sequence $(t_k^i)_{k=1}^{\infty}$ on $\{Y_k\}_{k=0}^{\infty}$ derived from any $t^i \in \mathcal{T}^i$. Thus any point in $S \times [\overset{n}{\underset{i=1}{\times}} \mathcal{T}^i]$ determines uniquely a sequence $(y_0, y_1, \ldots)$ corresponding to some $y \in \mathcal{Y}$. So we have a mapping $g: S \times [\overset{n}{\underset{i=1}{\times}} \mathcal{T}^i] \to \mathcal{Y}$ which is easily verified to be the inverse of f.

Now note that by Stone-Weierstrass theorem, any continuous function on $\mathcal{Y}$ can be approximated by continuous functions on Y_k. This implies that the mappings $t^i: \mathcal{Y} \to \Pi(\mathcal{Y})$ are continuous and hence $\mathcal{T}^i$ is compact $\forall i$ (since $\mathcal{Y}$ is compact). Also clearly the projection $\rho_0: \mathcal{Y} \to S$ is continuous. So the mapping $\rho_0 \overset{n}{\underset{i=1}{\times}} t^i: \mathcal{Y} \to S \times$ $\times\, (\overset{n}{\underset{i=1}{\times}} \mathcal{T}^i)$ is one to one and continuous, and therefore it is a homeomorphism since $\mathcal{Y}$ is compact and $S \times (\overset{n}{\underset{i=1}{\times}} \mathcal{T}^i)$ is a Hausdorff space.

The following lemma establishes an important property of the mappings t^i which will be needed for the rest of the proof.

Lemma 2.14: $\forall i \;\forall y \in \mathcal{Y}$ *if* $\tilde{y} \in \operatorname{Supp}(t^i(y))$, *then* $t^i(\tilde{y}) = t^i(y)$.

Proof: Let $(t_1^i, t_2^i, \ldots)$ and $(\tilde{t}_1^i, \tilde{t}_2^i, \ldots)$ be the sequences of marginal distributions of $t^i(y)$ and $t^i(\tilde{y})$ respectively on $Y_0, Y_1, \ldots \tilde{y} \in \operatorname{Supp}(t^i(y))$ implies that $\forall k$ the support of the marginal distribution of $t_t^i \in \Pi(Y_{k-1})$ on $\overset{k-1}{\underset{l=0}{\times}} [\Pi(Y_l)]_i$ contains $(\tilde{t}_1^i, \ldots, \tilde{t}_{k-1}^i)$. But since $y \in \mathcal{Y}$ it follows by using repeatedly properties a) and b) of (2.1) that the marginal distribution of t_k^i on $\overset{k-1}{\underset{l=0}{\times}} [\Pi(Y_l)]_i$ assigns probability 1 to $(t_1^i, \ldots, t_{k-1}^i)$. Therefore $(\tilde{t}_1^i, \ldots, \tilde{t}_{k-1}^i) = (t_1^i, \ldots, t_{k-1}^i)$ $\forall k$ and thus $t^i(\tilde{y}) = t^i(y)$.

As an immediate consequence of Lemma 2.14, the continuity of t^i and of the projection $\mathcal{Y} \to S$, we have:

Property 4: $\mathcal{Y}$ is an S-based abstract BL-Space.

Property 2: $\mathcal{T} = \Pi(S \times [\mathcal{T}]^{n-1})$ *(homeomorphically).*

Proof: We shall prove that $\forall i$, T^i is homeomorphic to $\Pi(S \times (\underset{j \neq i}{\times} T^j))$. Each $t^i \in T^i$ is an element in $\Pi(Y)$, hence in $\Pi(S \times (\overset{n}{\underset{j=1}{\times}} T^j))$ (by Property 1). But by Lemma 2.14, $(s, \tilde{t}^1, \ldots, \tilde{t}^n) \in \text{Supp}(t^i) \Rightarrow \tilde{t}^i = t^i$. Therefore there is natural mapping f^i of T^i to $\Pi(S \times (\underset{j \neq i}{\times} T^j))$ which maps each $t^i \in T^i$ to its marginal on $S \times (\underset{j \neq i}{\times} T^j)$.

We want to show now that this f^i is homeomorphism: T^i being compact and $S \times (\underset{j \neq i}{\times} T^j)$ being Hausdorff, it is sufficient to prove that f^i is one to one and onto. For this we shall exhibit the inverse mapping of f^i: Given $\mu \in \Pi(S \times (\underset{j \neq i}{\times} T^j))$ we want to show the existence of $y \in Y$ s.t. the marginal of $t^i(y)$ on T^i is a unit mass at $t^i(y)$ and on $S \times (\underset{j \neq i}{\times} T^j)$ is μ. By Property 1) it is enough to define a sequence $(t_1^i, t_2^i, \ldots)$ of marginal distributions on $Y_0, Y_1, \ldots$, respectively which will satisfy conditions a) and b) of (2.1) $\forall k$ and which define an element of $\Pi(Y)$ having the correct marginal distributions.

For each $k \geqslant 1$, let μ_k be the marginal distribution of μ on $(S \times \overset{k-1}{\underset{l=0}{\times}} \underset{j \neq i}{\times} [\Pi(Y_l)]_j)$ (that is the factor space of Y_k which does not involve coordinate i). Let $t_1^i = \rho_0(\mu) =$ the marginal distribution of μ on S and define inductively

$t_k^i \in \Pi(Y_{k-1})$ by: $t_k^i = \mu_k \times \delta_{(t_1^i, \ldots, t_{k-1}^i)}$, $k \geqslant 2$. It follows readily from the construction that $(t_k^i)_{k=1}^\infty$ has the required properties. This completes the proof of Property 2.

Definition 2.15: A closed subset $\tilde{C}$ of Y which satisfies

$$\forall y \in \tilde{C}, \ \forall i \in I, \ t^i(y) \text{ is supported by } \tilde{C} \tag{2.2}$$

will be called *beliefs closed (BL-*closed*)* or a *beliefs subspace (BL-*subspace*)* of Y.

Property 5: The S-based abstract BL-spaces are the BL-closed subspaces of Y (by BL-morphism), under the non-redundancy condition (Definition 2.4).

Proof: Let C be an S-based abstract BL-subspace. We shall define a mapping y: $C \to Y$ by defined for each $c \in C$ an $y(c) = (s, (t_k^1)_{k=1}^\infty, \ldots, (t_k^n)_{k=1}^\infty)$ where $s \in S$ and each of the sequences $(t_k^i)_{k=1}^\infty$ is a sequence of distributions on $(Y_k)_{k=0}^\infty$ respectively, satisfying conditions a), b) (of 2.1) for all $k \geqslant 1$: Remark that any point $(s, (t_k^1)_{k=1}^K, \ldots, (t_k^n)_{k=1}^K)$ which satisfy a) and b) $\forall i$ and $\forall k$ determines uniquely a

point in Y_k, therefore by defining $y_K(c) = (s, (t_k^1)_{k=1}^K, \ldots, (t_k^n)_{k=1}^K)$ $\forall c \in C$ we are defining a mapping $y_K : C \to Y_K$ and hence an induced mapping $\hat{y}_K : \Pi(C) \to \Pi(Y_K)$. We construct these mappings inductively on K: $\forall c \in C$ let $y_0(c) = f(c) \in S$ and for $K = 1, 2, \ldots$, define $\forall i$, $t_K^i = \hat{y}_{K-1} \circ t^i : C \to \Pi(Y_{K-1})$, where $\hat{y}_{k-1}$ is the mapping $\hat{y}_{k-1} : \Pi(C) \to \Pi(Y_{k-1})$ canonically induced by y_{k-1}.
Using condition (*) of Definition (2.2), it follows that $\forall c \in C$ the above defined $y(c)$ in fact satisfies the required condition and hence corresponds to a point in Y. Furthermore, since f and t^i are continuous, it follows inductively that y_K $\forall K$ are continuous mappings and hence the defined $y: C \to Y$ is continuous. If we denote $y(C) = \tilde{C} \subseteq Y$ then it is clear from the cobstruction that $\tilde{C}$ satisfies (2.2) i.e. it is a BL-subspace of Y. At this point we have to notice the following proposition whose proof follows readily from the definitions:

Proposition 2.16: Any BL-closed subset of Y is an S-based abstract BL-space (with respect to the projections of $Y = S \times (\times_{i=1}^{n} T^i)$ on its factor spaces).

Using the terminology of Lemma 2.8 and the remark that follows it, the mapping $y : C \to \tilde{C}$ we constructed is the BL-morphism from C to Y induced by the identity on S (since Y clearly satisfies the NR-condition). Using the same notation the above constructed y is clearly invertible, and hence BL-isomorphism between $H = h(C)$ (the space of infinite hierarchies generated by C) and $\tilde{C}$. Therefore if C satisfies the NR-condition we use Proposition 2.5 to deduce that $y: C \to \tilde{C}$ is a BL-isomorphism. This concludes the proof of property 5.

Property 6: For any coherent beliefs hierarchy $(C_0, C_1, \ldots, C_K)$ there is a BL-subspace C of Y s.t. $\rho_k(C) = C_k$, $k = 0, \ldots, K$.

Proof: By condition (4) of Definition 2.1, $\forall t_K^i \in t^i(C_K)$:

$$\text{Supp}\,(t_K^i \times \delta_{t_K^i}) \subseteq \text{Projection of } C_K \text{ on } C_{K-1} \times [\Pi(C_{K-1})]_i.$$

It follows (for instance by Lemma 2.12) that there is an extension of t_K^i to a probability distribution $\tilde{t}^i$ on $C_K \subseteq C_{K-1} \times [\Pi(C_{K-1})]_1 \times \ldots \times [\Pi(C_{K-1})]_n$. Take all possible such extensions for each $t_K^i \in t^i(C_K)$, $\forall i$, to define C_{K+1}. Prove that $C_0, \ldots, C_K, C_{K+1}$ is a coherent beliefs hierarchy of level $K + 1$, and proceed inductively as in the construction of Y to construct a limiting $C \subseteq Y$ which be the required BL-subspace.

Property 7: The minimality properties of Y *and* T.

- If $\tilde{Y}$ and $\tilde{T}$ satisfy 1) and 2), then $\tilde{Y}$ is an S-based abstract BL-space therefore by the proof of 5), it can be mapped BL-morphically onto some BL-subspace $\tilde{C}$ of Y. By 1), $\rho_0(\tilde{C}) = S$ and inductively (using 1) and 2)) $\rho_k(\tilde{C}) = Y_k$ $\forall k$ hence $\tilde{C} = Y$. The mapping from $\tilde{T}$ onto T is induced accordingly.
- Assume that $\tilde{Y}$ and $\tilde{T}$ satisfies 4) and 5). By 5) since the Y we constructed satisfy the NR-conditions, there is a compact $\tilde{C} \subseteq \tilde{Y}$ and a BL-morphism φ: $Y \to \tilde{C}$ which induces the identity on S. On the other hand, by 4) $\tilde{Y}$ is an S-based BL-subspace, it follows from the proof of 5) that there is a BL-morphism ψ from $\tilde{Y}$ to a BL-subspace of Y which also induces the identity on S, therefore the composed BL-morphisms $\psi \circ \varphi$: $Y \to Y$ must be the identity and hence $\psi = \varphi^{-1}$ and Y is BL-isomorphic to the BL-subspace $\tilde{C}$ of $\tilde{Y}$. The mapping of $\tilde{T}$ onto T is induced in the natural way.
- If $\tilde{Y}$ an S-based BL-space which satisfy 6), then since the $(Y_0, Y_1, \ldots)$ we defined is a coherent beliefs hierarchy, there is a BL-subspace $\tilde{C}$ of $\tilde{Y}$, s.t. $\rho_k(\tilde{C}) = Y_k$, $\forall k$, thus $\tilde{C}$ is BL-homeomorphic to the projective limit of $(Y_0, Y_1, \ldots)$ namely Y. By the same argument, $\tilde{Y}$ satisfy 3) and Y satisfy 6) imply that $\tilde{Y}$ is BL-homeomorphic to a BL-subspace of Y. Since the two BL-morphism induce the identity on S we obtain the required result.

This concludes the proof of Theorem 2.9.

Remark 2.17: A very common situation of incomplete information is that in which in addition to incomplete information about S each player has some ***private information*** which *may depend on the state of nature.* For instance if each player know his own utility function. Can such a situation be incorporated in our model? In other words can we construct a BL-subspace in which each player knows his private information and it is a common knowledge that such is the situation? This in fact can be done as follows: Let $h_i: S \to H_i$ be the private information function of player i which assigns to each state $s \in S$ the element $h_i(s)$ of some space H_i. We would like to construct a BL-subspace $C \subseteq Y$, with the property: $\forall i\ \forall y \in C$, the distribution of $h_i \circ \rho_0$ under $t^i(y)$ is a unit mass at $h_i \circ \rho_0(y)$. To do this let $C_0 = S$ and

$$C_1 = \{(s, t_1, \ldots, t_n) \mid s \in S,\ t_i \in \Pi(h_i^{-1}(h_i(s)));\quad i = 1, \ldots, n\}$$

(C_0, C_1) is trivially a beliefs hierarchy which can therefore be closed to a BL-subspace by property 6. This BL-subspace will have the required property.

Remark 2.18: If S is finite or countable or a standard Borel space, all our results are purely measure theoretic: indeed the set of probabilities Π on a standard Borel space S is again a standard Borel space (with σ-field generated by $\{\pi \mid \pi(B) \geqslant \alpha\}$; B is a

Borel set in S and $\alpha \in R$) and this σ-field is the same as the one we derive from the weak* topology.

To carry also the results on BL-subspaces and on abstract BL-spaces to the measure theoretic setup, one has first to rewrite the above proofs for the case where those concepts would be defined as analytic spaces instead of compact spaces. We are quite convinced that except for technical complication, this extension of the theory causes no serious difficulty.

3. Approximation of a *BL*-Subspace by a Finite *BL*-Subspace

In this section we prove the following approximation theorem.

Theorem 3.1: For any closed BL-subspace C of $\mathcal{Y}$ *and any finite open cover* $\mathcal{O}$ *of* $\mathcal{Y}$, *there is a finite BL-subspace* C^* *of* $\mathcal{Y}$ *s.t.*

(i) $C \subseteq \cup \{0 \in \mathcal{O} \mid 0 \cap C^* \neq \emptyset\}$

(ii) $C^* \subseteq \cup \{0 \in \mathcal{O} \mid 0 \cap C \neq \emptyset\}$.

In other words, Theorem 3.1 states that:

The finite BL-subspaces of $\mathcal{Y}$ are dense in the set of all BL-subspaces of $\mathcal{Y}$, in the Hausdorff topology on closed subsets on $\mathcal{Y}$.

To prove this theorem we use the following known result (see e.g. Kelley, General Topology 6.33, p. 199).

Lemma 3.2: Let X be a compact space. For any finite open cover $\mathcal{O}$ *of X there is a neighborhood* $\mathcal{V}$ *of the diagonal in* $X \times X$ *s.t.* $\forall x \in X, \exists\ 0_x \in \mathcal{O}$ *which satisfies:* $(x, y) \in \mathcal{V} \Rightarrow y \in 0_x$.

Remark 3.3: Clearly $\mathcal{V}$ in Lemma 3.2 can be taken to be a basic neighborhood of the diagonal (for instance one which is generated by a finite open cover of X).

Lemma 3.4: For any finite open cover $\mathcal{O}$ *of a compact space X there is a finite open cover* $\mathcal{W}$ *s.t.* $\forall x, y, z \in X$, *if* (x,y) *are* $\mathcal{W}$*-close and* (y, z) *are* $\mathcal{W}$ *close, then* (x, z) *are* $\mathcal{O}$*-close.*

Proof: By Remark 3.3 let $\mathcal{V}$ be a neighorhood of the diagonal $X \times X$ satisfying the conclusion of Lemma 3.2 and which is generated by some finite open cover $\mathcal{W}$ of X. This $\mathcal{W}$ satisfies the required property.

Notation: We shall denote by Ref ($\mathcal{O}$) all such finite open covers $\mathcal{W}$ given by Lemma 3.4.

Lemma 3.5: Let K be a compact space. Given a finite open cover $\mathcal{O}$ *of* $\Pi(K)$, *then there is a finite set of continuous functions* $f_1, \ldots, f_n$ *on K s.t.* $\forall \nu \in \Pi(K), \exists\ 0_\nu \in \mathcal{O}$ *such that* $|\mu(f_j) - \nu(f_j)| \leqslant 1, \forall j = 1, \ldots, n$ *implies* $\mu \in 0_\nu$.

Proof: Using Lemma 3.2 for $X = \Pi(K)$ let $\mathcal{V}$ be a neighborhood of the diagonal in $\Pi(K) \times \Pi(K)$ satisfying the conclusions of the lemma. In view of Remark 3.3, $\mathcal{V}$ can be taken to be a basic open neighborhood of the diagonal, i.e. of the form:

$$\mathcal{V} = \{(\nu, \mu) \mid |\nu(f_j) - \mu(f_j)| \leqslant 1; j = 1, \ldots, n\},$$

where $f_1, \ldots, f_n$ are continuous functions on K. This finite set of functions satisfy the required properties.

Lemma 3.6: Given a finite open cover $\mathcal{O}$ of $\Pi(K)$, then there is a finite open cover $\mathcal{U}$ of K with the property that $\forall \mu \in \Pi(K) \exists 0 \in \mathcal{O}$ s.t. if $S: K \to K$ is a measurable mapping for which $\forall x \in K$, $(x, S(x)) \in U \times U$ for some $U \in \mathcal{U}$, then $((\mu, S(\mu)) \in 0 \times 0$.

Proof: Let $f_1, \ldots, f_n$ be the continuous functions determined by Lemma 3.5 for the finite cover $\mathcal{O}$. Let $\mathcal{U}$ be a finite open cover of K s.t.:

$$(x, y) \in U \times U \text{ for some } U \in \mathcal{U} \text{ implies } |f_j(x) - f_j(y)| \leqslant 1,\ j = 1, \ldots, n.$$

We claim that this finite open cover $\mathcal{U}$ is the required one. In fact, let $\mu \in \Pi(K)$, let $0 \in \mathcal{O}$ be the open set containing μ and satisfying the conclusion of Lemma 3.5 and take such a measurable mapping φ, then $\forall j = 1, \ldots, n$:

$$|\mu(f_j) - \varphi(\mu)(f_j)| = |\mu(f_j) - \mu(f_j \circ \varphi)| \leqslant \mu(\operatorname*{Max}_{x \in K} |f_j(x) - f_j(\varphi(x))|)$$

But $\forall x \in K$, $(x, \varphi(x)) \in U \times U$ for some $U \in \mathcal{U}$ and hence $|f_j(x) - f_j(\varphi(x))| \leqslant 1, \forall j$. It follows that $|\mu(f_j) - \varphi(\mu)(f_j)| \leqslant 1$ for $j = 1, \ldots, n$ which imply by the definition of f_j that $(\mu, \varphi(\mu)) \in 0 \times 0$.

Lemma 3.7: Let $X = \mathop{\times}\limits_{i=1}^{n} X_i$ where $\forall i$, X_i is a compact space. For any finite open cover $\mathcal{O}$ of X there are finite open covers $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of $X_1, \ldots, X_n$ respectively s.t. $\mathcal{V} = \mathop{\times}\limits_{i=1}^{n} \mathcal{V}_i$ is an open cover of X which is finer than $\mathcal{O}$.

Proof: Let $\mathcal{O} = \{0_1, \ldots, 0_n\}$ and let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a rectangular open cover which is finer than $\mathcal{O}$. As usual denote by ρ_i the projection $X \to X_i$ and $\forall x_i \in X_i$ let $V_{x_i} = \cap \{\rho_i(U_j) \mid x_i \in \rho_i(U_j)\}$. Then take the finite covers $\mathcal{V}_i = \{V_{x_i} \mid x_i \in X_i\}$.

Notation: We shall denote by $\mathcal{RP}(\mathcal{O}, X_1, \ldots, X_n)$ the set of all such product covers refining $\mathcal{O}$, provided by Lemma 3.7.

Having done these preparations we proceed now to prove the main result of this section, Theorem 3.1.

By Theorem 1.3 we write $Y = \underset{i=0}{\overset{n}{\times}} T^i$, where $T^0 = S$ and for $i = 1, \ldots, n$ $T^i = t^i(Y)$ is the type set of player i.

Consider $A = \{O^\alpha\}$, the increasing net of all finite open covers of Y with the partial order: $O^\alpha \leqslant O^\beta$ iff O^α refines O^β. When no confusion may result we will denote the elements of A by $\alpha, \beta, \ldots$, instead $O^\alpha, O^\beta, \ldots$ Accordingly we will write $\alpha \leqslant \beta$ instead of $O^\alpha \leqslant O^\beta$.

Let C be a closed BL-subspace of Y. $\forall\, \alpha \in A$ let $(O_0, \ldots, O_n) \in RP(O^\alpha, T^0, \ldots, T^n)$ and $\forall\, i$ let $P^i = (P^i_1, \ldots, P^i_n)$ be a measurable partition of T^i s.t. $P \in P^i \Rightarrow \exists\, 0 \in O_i, P \subseteq 0$. Such $(n+1)$tuple of partitions $P = (P^0, \ldots, P^n)$ will be *finer than the open cover* α.

$\forall\, i, i = 0, 1, \ldots, n, \forall\, j, j = 1, \ldots, n_i$, let t^i_j be any fixed point in $P^i_j \cap \rho^i(C)$ if this intersection is non empty and any point in P^i_j otherwise. $\forall\, i$ let $X^i = \{t^i_j\}$ and let $X = \underset{i=0}{\overset{n}{\times}} X^i$.

Define the mapping $\varphi_0 : Y \to Y$ by

$$\varphi_0(t^0, \ldots, t^n) = (\bar{t}^0, \ldots, \bar{t}^n)$$

where $\forall\, i\, \exists\, j$: $t^i \in P^i_j$ and $\bar{t}^i = t^i_j$.

Clearly $\rho_0(Y) = X \subseteq Y$. Remark also that $\bar{t}^i$ depends only on t^i, therefore φ_0 defines also uniquely mappings $T^i \to T^i$ which will all be denoted by φ_0 to avoid additional notation.

For $i \geqslant 1$ and for each $t^i \in X^i$, define the following probability distribution $P^i_{t^i}$ on X by $P^i_{t^i}(x) = t^i(\varphi_0^{-1}(x))$. Remark that $P^i_{t^i}(t^0_{j_0}, \ldots, t^n_{j_n}) > 0 \Rightarrow t^i_{j_i} = t^i$.

If we denote $\forall\, i$ by P^i the mapping $t^i \to P^i_{t^i}$ from X^i to $\Pi(X)$, then by our definition $(X, P^1, \ldots, P^n)$ is some S-based abstract BL-space and so is also $(\tilde{X}, P^1, \ldots, P^n)$ where $\tilde{X} = \varphi_0(C)$. By Proposition 3.5, it is homeomorphic to some (finite) BL- subspace of Y which we will denote by $\tilde{C}_\alpha$. Since $(X, P^1, \ldots, P^n)$ is determined solely by φ_0, we have a mapping $\psi_{\varphi_0} : X \to Y$ such that:

$$\tilde{C}_\alpha = \psi_{\varphi_0}(\tilde{X}) = (\psi_{\varphi_0} \circ \varphi_0)(C).$$

Proposition 3.8: $\tilde{C}_\alpha$ converges to C (in the Hausdorff topology on closed subsets of Y).

Proof: Considering the mapping $\varphi_\alpha = \psi_{\varphi_0} \circ \varphi_0 : C \to \tilde{C}_\alpha$, note that φ_α is not determined uniquely by α but also by the special choice of the finite measurable partitions $P^0, \ldots, P^n$ and by the special choice of the points $\{t_j^i\}$. So let ϕ_α be the set of all such mappings φ_α i.e.,

$$\phi_\alpha = \{\varphi \mid \text{There is a partition } P = (P^0, \ldots, P^n) \text{ finer than } \alpha \text{ and a choice of } \{t_j^i\} \text{ that yield } \varphi\}.$$

It is sufficient to prove that ϕ_α converges uniformly to the identity mapping on C, i.e.

$$\forall \alpha \in A \; \exists \beta \in A \text{ such that } \forall \varphi \in \phi_\beta \; \forall x \in C, \varphi(x) \text{ is } \mathcal{O}^\alpha\text{-close to } x.$$

For the next argument we recall the definition of Y as the (projective) limit of Y_k and write a generic point in Y as $y = (s, t_1, \ldots, t_k, \ldots)$ where $s \in S$ and $\forall k$, $t_k = (t_k^1, \ldots, t_k^n)$, $t_k^i \in \Pi(Y_{k-1}) \; \forall i$. We shall refer to t_k as the k-th coordinate of y (s being the 0-coordinate) and $\forall k \geqslant 0$ define:

$$A_k = \{\alpha \in A \mid \forall 0 \in \mathcal{O}^\alpha, 0 \text{ is defined in terms of the first } k \text{ coordinates}).$$

Since any cover $\mathcal{O}^\alpha$ is refined by some cover $\tilde{\mathcal{O}}^\alpha$ involving only a finite number of coordinates, it is sufficient to prove that:

(*) $\quad \forall \alpha \in A_k, \exists \beta \in A : \forall \gamma \geqslant \beta, \forall \varphi \in \phi_\gamma, \forall x \in C; \varphi(x) \text{ is } \mathcal{O}^\alpha\text{-close to } x.$

We shall prove (*) by induction on k:

For $k = 0$ the statement is obvious from our definitions, taking $\beta = \alpha$. Assume that (*) is true for k and let us prove it for $k + 1$: Let $\alpha \in A_{k+1}$ and $\forall i \geqslant 1$ let V_i be a finite open cover of T_{k+1}^i and let V_0 be a finite open cover of Y_k such that

$$V_0 \times \mathop{\mathsf{X}}_{i=1}^{n} V_i \text{ is finer than } \mathcal{O}^\alpha.$$

$\forall i$, let $\tilde{V}_i \in \mathcal{R}\mathrm{ef}(V_i)$ and $\forall i \geqslant 1$, let W_i be a finite open cover of Y_k such that for any measurable W_i-shift ψ of Y (i.e., $\forall y \in Y_k, (y, \psi(y)) \in W \times W$ for some $W \in W_i$) and $\forall t^i \in T^i$, $\psi(t^i)$ is V_i-close to t^i (see Lemma 4.2). Let $\bar{V}_0$ be any common refinement of $(\tilde{V}_0, W_1, \ldots, W_n)$ and let $\tilde{V} = \bar{V}_0 \times \mathop{\mathsf{X}}_{i=1}^{n} \tilde{V}_i$.

Finally, if we denote by $\beta_{\bar{V}_0}$ the $\beta \in A$ satisfying (*) for $\bar{V}_0 \in A_k$ (by induction hypothesis), the required β which corresponds to the given $\alpha \in A_{k+1}$ is $\beta = \max(\beta_{\bar{V}_0}, \tilde{V})$. Let us prove that this β in fact satisfies the property stated in (*). For $x \in Y_{k+1}$, it will be convenient to use the notation $t^i(x)$ for $\rho^i(x)$.

Let $\gamma \geqslant \beta$ and let $\varphi \in \phi_\gamma$, $x \in C$, we have to show that $\varphi(x)$ is $\mathcal{O}^\alpha$-close to x.

By definition of $\beta_{\bar{V}_0}$ we have that $\varphi(x)$ is $\bar{V}_0$-close to x, therefore there remains to show that $\forall\, i$, $t^i(\varphi(x))$ is V_i-close to $t^i(x)$.

Since $\gamma \geqslant \beta \geqslant \widetilde{V}$ we know that if $\varphi = \psi_{\varphi_0} \circ \varphi_0$, then $\varphi_0(x)$ is $\widetilde{V}$-close to x. Thus $\forall\, t^i \in T^i$, $\varphi_0(t^i)$ is $\widetilde{V}_i$-close to t^i. Extend φ (defined on C) to φ: $\mathcal{Y} \to \mathcal{Y}$ by defining $\varphi(x) = x$ for $x \notin C$. We have then that $\varphi(x)$ is $\bar{V}_0$-close to x $\forall x \in \mathcal{Y}$ and hence $\forall\, i \geqslant 1$, $\forall\, t^i \in T^i$, $t^i \circ \varphi^{-1}$ is $\widetilde{V}_i$-close to t^i (see definition of W_i). Thus $\forall\, t^i \in \rho^i(C)$, t^i and $t^i \circ \varphi^{-1}$ are two probability distributions on C and on $\widetilde{C}_\alpha$ respectively which are $\widetilde{V}_i$-close. In particular for $t^i = t^i_j \in \rho^i(C)$, $t^i_j \circ \varphi^{-1} = P^i_{t^i_j}$ is $\widetilde{V}$-close to t^i_j. Therefore $\forall\, x \in C$, $t^i(\varphi_0(x)) \circ \varphi^{-1} = P^i_{t^i(\varphi_0(x))}$ is a probability distribution on $\widetilde{C}_\alpha$ which is $\widetilde{V}_i$-close to $t^i(\varphi_0(x))$ which is on the other hand $\widetilde{V}_i$-close to $t^i(x)$ (on C; since $\varphi_0(x)$ is V-close to x $\forall\, x \in C$).

Since by definition of $\widetilde{C}_\alpha$ $P_{t^i(\varphi_0(x))} = t^i(\varphi(x))$ and since $\widetilde{V}_i \in \mathrm{Ref}(V_i)$ we conclude that $t^i(\varphi(x))$ is V_i-close to $t^i(x)$, completing the proof of Theorem 3.1.

4. Consistency

Summing up the structure developed so far: We started from a compact set S of possible games and we constructed from it the universal BL-space $\mathcal{Y}$ generated by S. This may be thought of as the space of "states of the world" in the sense that each point $y \in \mathcal{Y}$ defines completely all levels of beliefs and mutual beliefs for all the players. At each state $y \in \mathcal{Y}$, player i certainly knows his own (subjective) probability (distribution $t^i(y)$) on $\mathcal{Y}$. We shall also denote this distribution by P^i_y.

Nothing was said so far as to *what is* the actual state of the world? According to what procedure is it determined? What are the relations, if any, between the beliefs of the different players? Following Harsanyi we ask: Are there situations in which the subjective beliefs of the players, namely P^i_y, are equal to the conditional probabilities, given each player's private information, derived from some "prior" probability distribution P on $\mathcal{Y}$? Can one characterize those points in y for which this is in fact the case? In this section, we answer these questions and in the next section we discuss their game theoretical relevance.

Let Y be a closed BL-subspace of $\mathcal{Y}$.

Definition 4.1: A probability distribution $P \in \Pi(Y)$ is said to be *consistent* if:

$$P = \int_Y P^i_y \, dP \qquad \forall\, i,\ i = 1, \ldots, n. \tag{4.1}$$