

Formulation of Bayesian Analysis for Games with Incomplete Information

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1. Introduction

In analyzing a game with incomplete information, i.e. games in which players are uncertain about all the parameters defining the strategy spaces and the payoff functions, one is led naturally to handle “an infinite hierarchy of beliefs” for each player: His beliefs (i.e. subjective probabilities) on the parameters of the games, his beliefs on the beliefs of the other players on the parameters of the games, his beliefs on the other players’ beliefs on his own beliefs on the parameters of the games, his beliefs on the other players’ beliefs on his own beliefs on their beliefs on the parameters of the games, etc. . . .

In an attempt to overcome the difficulty of having to work with infinite sequences of mutual beliefs. *Harsanyi* [1967 – 1968] introduced the concept of *type* which proved to be very useful in making games with incomplete information much more manageable. *Harsanyi*’s idea was to summarize all parameters and beliefs concerning a certain player, by one vector which he calls the *attribute* vector. In his words [see *Harsanyi*, 1967, p. 171]: “. . . we can regard the vector c_i as representing certain physical, social, and psychological *attributes* of player *i* himself in that it summarizes some crucial parameters of player *i*’s own payoff function U_i as well as the main parameters of his beliefs about his social and physical environment . . . the rules of the game as such allow any given player *i* to belong to any one of a number of possible *types*, correspond-

ing to the alternative values of his attribute vector c_i could take . . . Each player is assumed to know his own actual type but to be in general ignorant about the other players' actual types."

Can this idea be formalized mathematically? In other words: Starting from a set S of all possible values of the parameters of the game can one identify a mathematically well defined set \mathcal{V} of the "states of the world" in which every point contains all characteristics, beliefs and mutual beliefs of all players?

If yes, would any infinite hierarchy of beliefs lead to some point in \mathcal{V} ? This is exactly the construction we do in Section 2 of this paper. The space \mathcal{V} defined there is what we call "*the universal beliefs space generated by S* " and it includes, roughly speaking, all possible states of the world arising from S . Furthermore, there is a well defined space \mathcal{T} , called the space of all possible types of a player in such a game, such that \mathcal{V} and \mathcal{T} satisfy (up to some appropriate homeomorphism) the following two relations:

(i) $\mathcal{V} = S \times [\mathcal{T}]^n$; (ii) $\mathcal{T} =$ the set of all probability distributions on $(S \times [\mathcal{T}]^{n-1})$. The first equality says that a state of the world $y \in \mathcal{V}$ consists of a state of nature $s \in S$ and an n -tuple of types, one for each player. The second relation says that a type of a player is just a joint probability distribution on S and types of the other $(n - 1)$ players. This is exactly the formalization of the notion of 'type' as used by Harsanyi.

Typically in an actual situation many of the points in \mathcal{V} will be considered impossible by all of the players. In other words what is then relevant is only some subset of \mathcal{V} . (This is for instance the case if all players know one parameter in S but are uncertain about the others). This leads to the notion of what we call *beliefs subspaces* of \mathcal{V} .

It turns out, as it can be easily seen, that even if we start with a set S which is finite, both \mathcal{V} and most of its beliefs subspaces will be sets of high cardinality. On the other hand, most of the work on games with incomplete information assume *finitely* many possible states of the world. In Section 3 we provide some justification for this by proving that any beliefs subspace of \mathcal{V} can be "approximated" by a *finite* beliefs subspace which is arbitrarily close to it in the Hausdorff distance between closed sets.

In Section 4 we consider the concept of *consistency*, also discussed by Harsanyi and later by *Aumann/Maschler*. Generally speaking, a state of the world represents a consistent situation if there is a probability distribution on all the states of the world such that the beliefs of each player equal the conditional probability distribution given his private information. We define this concept formally and prove that it in fact captures the intuitive meaning of consistency. We then show that the consistency of an actual situation is common knowledge, in the sense that each player, based on his own information only, can test the hypothesis that the state of the world is consistent, if yes to compute the consistent set of states to which it belongs and compute the global probability distribution on \mathcal{V} corresponding to the consistent situation he is in. In such a test each player has (subjective) probability 0 of committing any error in his conclusion.

Finally, in Section 5, we define a game in strategic form determined by the beliefs space (or subspace). This will be typically a game "with vector payoffs", but the Nash equilibria are well defined. For a consistent beliefs subset, the Nash equilibria will be the same as those of a certain extensive form game in which the state of the world is chosen according to the (uniquely determined) probability distribution, and each player is informed on what is his own type. This is Harsanyi's theorem [Harsanyi, 1967, part II, p. 321] which is in the background of most models of games with incomplete information.

It should be pointed out that works in the same direction were done by Böge *et al.* who, being interested mainly in the equilibrium points of games with incomplete information, incorporated the strategy choices of the players as part of the space of parameters on which the infinite hierarchy of beliefs is built.

2. The Universal Beliefs Space \mathcal{V}

The main objective of this section is to prove Theorem 2.9 which establishes the existence of a space of infinite hierarchy of beliefs. We consider a situation of incomplete information involving a set of players $I = \{1, \dots, n\}$, the members of which are uncertain about the parameters of the game they are playing which may be any element of some set S (we may think of a point of S as a full listing of the strategy spaces and the payoff functions). We shall refer to S as the *parameter-space*.

Assumption: S is a compact space.

Remark: To see that this assumption is not too restrictive, let us see how, in a typical and rather general model of incomplete information, the space S will in fact be compact: Observe that S has most generally to include all the parameters of the game including the parameters of the utility functions of the player. So let S_0 be the set of possible values of all the parameters of the game. Clearly S_0 may be assumed (by enlarging it if necessary) to be compact. For each player i let A^i be his action set (enlarged so as to become independent of $S \in S_0$). The set of outcomes can

then be identified with the set $C = S_0 \times \prod_{i=1}^n A^i$ and is compact if A^i are compact. The

Von-Neumann Morgenstern utility function of player i is a (continuous) real map $u_i: C \rightarrow \mathbb{R}$, which we may want to assume to be bounded (for instance by applying the Von Neumann-Morgenstern theory to all countable lotteries, in order to avoid the St. Petersburg paradox). Hence we may take $u_i: C \rightarrow [0, 1]$ and the set of all possible games is then $S = S_0 \times [[0, 1]^C]^n$ which is compact. A special case is of course that in which S_0 and A^i are finite then S will be in addition metrisable.

For any compact space X , $\Pi(X)$ will denote the compact space of probability measures on X , endowed with the weak* topology. [It is clearly closed in the set of all measures of norm ≤ 1 since the function 1 is continuous-, and the set is by Riesz theorem the unit ball of the dual of $C(X)$, hence weak*-compact by Alaoglu's theorem.]

Definition 2.1: A coherent beliefs hierarchy of level K ($K = 1, 2, \dots$) is a sequence (C_0, C_1, \dots, C_K) where:

- 1) C_0 is a compact subset of S and for $k = 1, \dots, K$, C_k is a compact subset of $C_{k-1} \times [\Pi(C_{k-1})]^n$ (as topological spaces). (We denote by ρ_{k-1} and t^i the projections of C_k on C_{k-1} and the i -th copy of $\Pi(C_{k-1})$ respectively.)

$$\rho_{k-1}(C_k) = C_{k-1}; k = 1, \dots, K. \tag{2}$$

$$\forall c_k \in C_k \text{ let } c_{k-1} = \rho_{k-1}(c_k), \text{ then } \forall i: \tag{3}$$

H1) the marginal distribution of $t^i(c_k)$ on C_{k-2} is $t^i(c_{k-1})$;

H2) the marginal distribution of $t^i(c_k)$ on the i -th copy of $\Pi(C_{k-2})$ is the unit mass at $t^i(c_{k-1}) = t^i(\rho_{k-1}(c_k))$.

$$\forall i, \forall t \in t^i(C_k); k = 1, \dots, K, t \{ \rho_{k-1} [(t^i)^{-1}(t)] \} = 1. \tag{4}$$

We interpret C_k as a set of beliefs up to level k and thus a point in C_k consists of hierarchy of beliefs up to level $(k - 1)$ (i.e. a point in C_{k-1}) and for each player i a probability distribution t_k^i on hierarchies of beliefs up to level $(k - 1)$ (i.e.

$t_k^i \in \Pi(C_{k-1})$). Condition H1 says that player i 's k -level beliefs coincide with his $(k - 1)$ level beliefs in whatever concerns hierarchies up to level $(k - 2)$. Condition H2 says that player i knows his own previous order beliefs.

In the next definition we formalize the properties of the space of states of the world C we would like to obtain: Any point $c \in C$ determines uniquely a set of parameters $s \in S$ and the type t^i of each player. The type t^i is a probability distribution on C which is coherent in the sense that each player knows his own type. In other words if $t^i \in \Pi(C)$ is a certain type of player i , then in all points in the support of t^i ($\text{Supp}(t^i)$) player i is of type t^i . This motivates the following.

Definition 2.2: An S -based abstract beliefs space (BL -space) is an $(n + 3)$ tuple $(C, S, f, (t^i)_{i=1}^n)$ where C is a compact set, S is some compact space, f is a continuous mapping $f: C \rightarrow S$ and $t^i, i = 1, \dots, n$, are continuous mappings $t^i: C \rightarrow \Pi(C)$ (with respect to the weak* topology) satisfying:

$$(*) \quad \tilde{c} \in C \text{ and } \tilde{c} \in \text{Supp}(t^i(c)) \Rightarrow t^i(\tilde{c}) = t^i(c).$$

When no confusion may result we shall denote the BL -space simply by C .

The space C is a space in which each point $c \in C$ contains a full description not only of the state of nature $s \in S$ but also of all beliefs, beliefs on beliefs etc. on S . In fact if we interpret t^i as player i 's (subjective) probability distribution on C , then

combined with f it defines a probability distribution on S , which is the first level beliefs of player i . But t^i also defines a probability distribution on $(t^j)_{j \neq i}$, and hence on the first level beliefs of the other players. This may be called the second level beliefs of player i . Proceeding inductively we find that with each $c \in C$ is associated one infinite hierarchy of beliefs for each player. The condition (*) is a consistency condition which says basically that a player i assigns positive probability (in the discrete case) only to points of C in which he has the same beliefs. In other words he is certain of his own beliefs.

Let us write now formally the above mentioned observation:

Given S we define the spaces X_k, T_k , by

$$X_0 = S$$

$$T_k = \Pi(X_{k-1})$$

$$X_k = X_{k-1} \times [T_k]^n = S \times \prod_{l=1}^k [T_l]^n; \quad k = 1, 2, \dots$$

Define also $X = S \times \prod_{l=1}^{\infty} [T_l]^n$, which is a well defined compact space when so is S .

Note that X is generated by S and whenever we want to specify the generating space we shall write $X(S)$. We shall denote a typical point in $X(S)$ as

$$x = (s, t_1^1, \dots, t_1^n, \dots, t_k^1, \dots, t_k^n, \dots), \text{ where for each } i \text{ and each } k \quad t_k^i \in T_k^i = \Pi(X_{k-1}).$$

If $\varphi: C \rightarrow \tilde{C}$ is a continuous mapping between two compact spaces C and \tilde{C} , we denote by $\hat{\varphi}$ the mapping $\Pi(C) \rightarrow \Pi(\tilde{C})$ canonically induced by φ , namely the mapping $\hat{\varphi}: \Pi(C) \rightarrow \Pi(\tilde{C})$ which maps $\mu \in \Pi(C)$ to $\hat{\mu} \in \Pi(\tilde{C})$ such that for any continuous function f on \tilde{C} , $\int_{\tilde{C}} f(\tilde{c}) d\hat{\mu} = \int_C (f \circ \varphi)(c) d\mu$.

To any S -based abstract BL -space $(C, S, f, (t^i)_{i=1}^n)$ we define now a certain natural continuous mapping $h: C \rightarrow X(S)$. This will be done by defining for each

$k = 0, \dots, 2, \dots$ a mapping $h_k: C \rightarrow X_k$ such that

$$k \leq l \Rightarrow \rho_k(h_l(c)) = h_k(c) \quad \forall c \in C,$$

in other words, $h_k(c)$ is the projection of $h(c)$ on X_k .

The mappings h_k are defined inductively as follows: $h_0(c) = f(c)$. Assume $h_k: C \rightarrow X_k$ is defined then we want to define $h_{k+1}: C \rightarrow X_{k+1}$. Take any $c \in C$ and let $h_k(c) = (s, t_1^1, \dots, t_1^n, \dots, t_k^1, \dots, t_k^n) \in X_k$ then $h_{k+1}(c) = (s, t_1^1, \dots, t_1^n, \dots, t_k^1, \dots, t_k^n, t_{k+1}^1, \dots, t_{k+1}^n) \in X_{k+1}$ where $\forall i$,

$t_{k+1}^i = \hat{h}_k \circ t^i: C \rightarrow \Pi(X_k) = T_{k+1}$, and \hat{h}_k is the mapping $\hat{h}_k: \Pi(C) \rightarrow \Pi(X_k)$ canonically induced by h_k .

It follows that the so defined $h: C \rightarrow X(S)$ is continuous. Let $H = h(C) \subseteq X(S)$. When we want to emphasize the underlying S we shall write $H(S)$. By construction, the image $h(c)$ contains all possible information concerning S and beliefs on S . Therefore it is intuitively pretty clear that $h(c) \neq h(c')$ for $c \neq c'$ unless c and c' are identical in whatever concerns S and differ only by something which is redundant to S and to the beliefs structure on S .

To define this notion of nonredundancy more formally, given an BL -space $(C, S, f, (t^i)_{i=1}^n)$ let F be the smallest σ -field (of subsets of C) for which f is measurable and $\forall i, (t^i(c))(B)$ is measurable $\forall B \in F$.

Definition 2.4: A BL -space $(C, S, f, (t^i)_{i=1}^n)$ is said to satisfy the *non-redundancy condition* (NR-condition) if the σ -field F separates each two distinct points in C .

By our previous discussion we thus have:

Proposition 2.5: If an S -based abstract BL -space $(C, S, f, (t^i)_{i=1}^n)$ satisfies the NR-condition then the mapping $h: C \rightarrow H$ is also one to one hence it is an isomorphism.

In dealing with BL -spaces we would like to consider homeomorphisms between BL -spaces which (in addition to their topological properties) will also preserve the beliefs structure. These mappings will be called *BL-morphisms* and we proceed now to define them formally.

Definition 2.6: A *beliefs morphism* (BL -morphism) from a BL -space $(C, S, f, (t^i)_{i=1}^n)$ to a BL -space $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ is a pair (φ, φ') where φ' is a continuous mapping of C onto \tilde{C} and φ is a continuous mapping of S onto \tilde{S} such that for each $i, i = 1, \dots, n$, the following diagram comutes:

$$\begin{array}{ccc}
 S & \xrightarrow{\varphi} & \tilde{S} \\
 f \uparrow & & \uparrow \tilde{f} \\
 C & \xrightarrow{\varphi'} & \tilde{C} \\
 t^i \downarrow & & \downarrow \tilde{t}^i \\
 \Pi(C) & \xrightarrow{\hat{\varphi}'} & \Pi(\tilde{C})
 \end{array}$$

where $\hat{\varphi}'$ is the mapping $\hat{\varphi}': \Pi(C) \rightarrow \Pi(\tilde{C})$ canonically induced by φ' .

Definition 2.7: A BL-morphism (φ, φ') from $(C, S, f, (t^i)_{i=1}^n)$ to $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ is called a BL-isomorphism if the inverse mappings φ^{-1} and $(\varphi')^{-1}$ exist and $(\varphi^{-1}, (\varphi')^{-1})$ is a BL-morphism from $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ to $(C, S, f, (t^i)_{i=1}^n)$. The two BL-spaces are said to be BL-isomorphic.

Some thought on the diagram of Definition 2.6 leads us to the observation that if (φ, φ') is a BL-morphism from C to \tilde{C} then there is actually one essential mapping and not two since φ' seems to be determined by φ via the above diagram. This is in fact true provided \tilde{C} satisfies the NR-condition:

Lemma 2.8: If (φ, φ') is a BL-morphism from $(C, S, f, (t^i)_{i=1}^n)$ to $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ and if the latter satisfies the NR-condition, then φ' is uniquely determined by φ .

Proof: Using our notation $h: C \rightarrow X(S)$ and $\tilde{h}: \tilde{C} \rightarrow X(\tilde{S})$ we denote by $h \circ \varphi: C \rightarrow X(\varphi(S)) \subseteq X(\tilde{S})$ the mapping which maps $c \in C$ to $h(c)$ in which the underlying S is replaced by $\varphi(S)$. The fact that the diagram of Definition 2.6 commutes implies that $\forall c \in C$ we have $\tilde{h}(\varphi'(c)) = (h \circ \varphi)(c) \in X(\tilde{S})$. Since \tilde{C} satisfies the NR-condition \tilde{h} is one to one (by Proposition 2.5) and hence invertible. Therefore:

$$\varphi'(c) = \tilde{h}^{-1}(h \circ \varphi)(c).$$

In words, the idea of the proof is that φ combined with the diagram determines for each $c \in C$ uniquely the infinite hierarchy $\tilde{h}(c')$ associated with $c' = \varphi'(c)$, and hence it determines uniquely c' itself since \tilde{C} satisfies the NR-condition.

Remark: In view of Lemma 2.8 we shall shorten our notation and terminology and speak of BL-morphism φ from BL-space C to BL-space \tilde{C} . This is the BL-morphism induced by the mapping $\varphi: S \rightarrow \tilde{S}$.

We are now ready to state the main theorem of this section.

Theorem 2.9: For any compact S and positive integer n there are spaces \mathcal{Y} and \mathcal{T} such that:

- 1) $\mathcal{Y} = S \times [\mathbb{T}]^n$
 - 2) $\mathcal{T} = \Pi(S \times [\mathbb{T}]^{n-1})$
- } up to BL-morphisms.
- 3) There are compact spaces $\{Y_k\}_{k=0}^\infty$ s.t. $\forall k (Y_0, Y_1, \dots, Y_k)$ is a coherent beliefs hierarchy and \mathcal{Y} is the projective limit $\{Y_k\}_{k=0}^\infty$ (with respect to the natural projection $\rho_{k-1}: Y_k \rightarrow Y_{k-1}$. We denote by ρ_k also the projection of \mathcal{Y} on Y_k).
 - 4) \mathcal{Y} is an S -based BL-space (with the projections $f: \mathcal{Y} \rightarrow S$ and $t^i: \mathcal{Y} \rightarrow \mathbb{T}^i$).
 - 5) Any S -based abstract BL-space, which satisfies the NR-condition, is canonically BL-homeomorphic to a compact subset of \mathcal{Y} (which will be called a BL-subspace of \mathcal{Y}).

- 6) For any coherent beliefs hierarchy (C_0, C_1, \dots, C_K) there is a BL-subspace C of \mathcal{Y} s.t. $\rho_k(C) = C_k, k = 0, \dots, K$.
- 7) Any $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{T}}$ which satisfy 1) and 2) or 4) and 5) can be mapped continuously onto \mathcal{Y} and \mathcal{T} respectively. This map induces a BL-homeomorphism between \mathcal{Y} and a BL-subspace of $\tilde{\mathcal{Y}}$. Any $\tilde{\mathcal{Y}}$ which satisfies 3) and 6) can be BL-morphically mapped onto \mathcal{Y} .

\mathcal{Y} will be called the Universal BL-space generated by S (and n) and \mathcal{T} will be called the Universal type space generated by S (and n).

Proof: We shall prove the theorem by constructing the sequence $\{Y_k\}_{k=0}^\infty$ in (3) and define \mathcal{Y} as its projective limit and \mathcal{T}^i as the projection of \mathcal{Y} on player i 's coordinates. Then we shall prove that these \mathcal{Y} and \mathcal{T} satisfy the required properties.

Construction of \mathcal{Y}

Define the sequence of spaces $\{Y_k\}_{k=0}^\infty$ as follows:

$$Y_0 = S \text{ and for } k = 1, 2, \dots$$

- (2.1) $Y_k = \{y_k \in Y_{k-1} \times [\Pi(Y_{k-1})]^n \mid (a) \forall i \text{ the marginal distribution of } t^i(y_k) \text{ on } Y_{k-2} \text{ is } t^i(y_{k-1}) \text{ and (b) the marginal distribution of } t^i(y_k) \text{ on the } i\text{-th copy of } \Pi(Y_{k-2}) \text{ is the unit mass at } t^i(y_{k-1})\}$.

As we have already noted if X is compact, then $\Pi(X)$ is also compact. Note also that the conditions (a) and (b) in the definition of Y_k are closed conditions. It follows that if Y_{k-1} is compact, then Y_k is also compact. Since $Y_0 = S$ is compact, it follows inductively that Y_k is compact $\forall k$. Let \mathcal{Y} be the projective limit of $\{Y_k\}_{k=0}^\infty$ with respect to the natural projections $\rho_{k-1}: Y_k \rightarrow Y_{k-1}$. \mathcal{Y} is a well defined compact set.

Now by definition of Y_k we have that $\forall k, (Y_0, \dots, Y_k)$ satisfy automatically all properties of a coherent beliefs hierarchy (Definition 2.1) except for condition (2), namely that $\rho_k(Y_{k+1}) = Y_k, k = 0, 1, \dots$. This we prove now:

Proposition 2.10: $\rho_k(Y_{k+1}) = Y_k, k = 0, 1, 2, \dots$

This proposition has the following immediate corollaries.

Corollary 2.11:

- i) $\forall k, (Y_0, Y_1, \dots, Y_k)$ is a coherent beliefs hierarchy;
 (ii) $\forall k, \rho_k(\mathcal{Y}) = Y_k$, in particular $\mathcal{Y} \neq \emptyset$.

The proof of Proposition 2.10 will follow from the following.

Lemma 2.12: Let A and B be compact sets, D a compact subset of $A \times B$ and $q \in \Pi(A)$. A necessary and sufficient condition for the existence of $p \in \Pi(D)$ whose marginal distribution on A is q , is that $q(D_A) = 1$, where D_A is the projection of D on A .

Proof: Since $D \subset D_A \times B$, the necessity is obvious. To prove the sufficiency assume $q(D_A) = 1$. Define $L_q(f) = \int f dq$. L_q is a linear functional defined on $C(D_A)$, (the linear space of continuous real functions on D_A). If we consider a function on D_A as a function on D , by the natural definition $F(a, b) = f(a) \forall (a, b) \in D$, and write $L_q(F) = \int F dq$, L_q is then a linear functional defined on a linear subspace of $C(D)$. This is clearly a positive functional with $\|L_q\| = 1$. By Hahn-Banach extension theorem L_q can be extended to a positive linear functional L of norm 1 on $C(D)$. Finally by Riesz representation theorem there is a probability measure $p \in \Pi(D)$, s.t. $L(f) = \int_D f dp \forall f \in C(D)$. This p is the required extension of q .

Proof of Proposition 2.10: We prove the proposition inductively on k . It holds for $k = 0$ since $Y_0 = S$ and $Y_1 = S \times [\Pi(S)]^n$, thus $\rho_0(Y_1) = Y_0$. Assume that $\rho_{k-1}(Y_k) = Y_{k-1}$ and let us prove that $\rho_k(Y_{k+1}) = Y_k$. In other words we have to show that any point $y \in Y_k$ can be extended to a point $(y, t_{k+1}^1, \dots, t_{k+1}^n) \in Y_{k+1}$. So we have to establish the existence of an n -tuple $t_{k+1}^1, \dots, t_{k+1}^n$ of probability distributions $t_{k+1}^i \in \Pi(Y_k)$ satisfying conditions (a) and (b) in the definition of Y_k , namely that the marginal distribution on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$ is $t_k^i \times \delta_{t^i(y)}$, where $\delta_{t^i(y)}$ is the element $\Pi(Y_{k-1})$ which assigns mass 1 to $t^i(y)$. We have thus to show that each of these marginals can be extended to a probability distribution t_{k+1}^i on $Y_{k-1} \times [\Pi(Y_{k-1})]_1 \times \dots \times [\Pi(Y_{k-1})]_n$ supported by its subset Y_k i.e. $t_{k+1}^i(Y_k) = 1$. Using Lemma 2.12 it remains to prove that

$$\text{Supp}(t^i(y)) \times \{t^i(y)\} = \text{Supp}[t^i(y) \times \delta_{t^i(y)}] \subseteq \text{projection of } Y_k \text{ on } Y_{k-1} \times [\Pi(Y_{k-1})]_i.$$

So let $(\tilde{y}_{k-1}, t^i(y)) \in \text{Supp } t^i(y) \times \{t^i(y)\}$ i.e. $\tilde{y}_{k-1} \in \text{Supp } t^i(y) \subseteq Y_{k-1}$. Since $t^i(y)$ assigns probability 1 to $t^i(\rho_{k-1}(y))$ it follows that $t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))$. Since by induction hypothesis $\rho_{k-1}(Y_k) = Y_{k-1}$, there is an extension $(\tilde{y}_{k-1}, \tilde{t}_k^1, \dots, \tilde{t}_k^n) \in Y_k$. We claim that if in this point we replace \tilde{t}_k^i by $t^i(y)$ we obtain a point which is also in Y_k , proving that $(\tilde{y}_{k-1}, t^i(y))$ is in the projection of Y_k on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$ and thus completing the proof.

