On a Repeated Game Without a Recursive Structure

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Abstract: The solution is given here for the infinitely repeated two-person zero-sum games of incomplete information characterized by $2 \times 2$ games, with information matrices $(\mathbf{A}, \mathbf{B})$ for the first game and $(\mathbf{C}, \mathbf{D})$ for the second game.

1. Introduction

Two main classes of repeated two-person zero-sum games with incomplete information are solved up to now:
- Games in which the information matrices matrices may depend on the player but not on the state of nature [Mertens and Zamir, Mertens].
- Games in which the information matrices do not depend on the players, may depend on the state of nature, with the additional assumption that each player recalls all prior moves [Kohlb erg and Zamir, Kohlb erg].

It seems that without those assumptions one loses the recursive structure that made those cases tractable.

Here an example is solved of a game not fulfilling those assumptions. It was mentioned as an open problem some six years ago [Zamir]:

There are two possible states of nature and accordingly two payoff matrices,

$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ q & r \end{pmatrix}$, the actual payoff matrix (i.e. the actual state of nature) is chosen once and for all by the referee (with probability $p$ for matrix $A$), and told to neither player. There are in addition two information matrices $H^A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$ and $H^B = \begin{pmatrix} e & f \\ q & r \end{pmatrix}$, $a$ and $b$ being two different letters. After each stage, if $T$ is the true payoff matrix $(A$ or $B)$, and the players $I$ and II played their pure strategies $i$ and $j$ respectively, the referee transfers $t_{ij}$ from player II's account to player I's and tell both players the letter $t_{ij}$. The players get no statement on their accounts before the end of the game. It is crucial here that the moves $i$ and $j$ are not stated explicitly by the referee. However, each player recalls his own move ($i$ or $j$) and all his own previous moves in addition to the information statements $t_{ij}$ made by the referee up to that stage.

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Notice that as soon as the letter e is announced by the referee, the true matrix is revealed to both players.

The payoff in the infinitely repeated game is thought of as being the limit $\lim_{n \to \infty} E \left( \frac{1}{n} \sum_{k=1}^{n} t_{kk} \right)$, but is not defined due to the possible non-existence of the limit. Nevertheless we will show that Min Max (and dually Max Min) of the infinite game exists in a well defined (and rather strong) sense:

Player II has an infinite game strategy that guarantees even in all sufficiently large finite games $E \left( \frac{1}{n} \sum_{k=1}^{n} t_{kk} \right) < \text{Min Max} + e$; conversely, for every infinite game strategy $\tau$ of player II, player I has an infinite game strategy $\sigma$ such that $\liminf_{n \to \infty} E \left( \frac{1}{n} \sum_{k=1}^{n} t_{kk} \right) > \text{Min Max} - e$.

For a proof of this result let us introduce a few conventions: We may obviously subtract from the matrices $A$ and $B$ their values $\nu (A)$ and $\nu (B)$ respectively, which will subtract from all payoffs the constant $p \nu (A) + (1 - p) \nu (B)$. Hence we may assume without loss of generality that $\nu (A) = \nu (B) = 0$. We may multiply $A$ by $p$ and $B$ by $(1 - p)$, and consider the payoff to be the sum of the payoffs that would be obtained if $A$ was the true matrix and if $B$ was the true matrix. We will do this in order to simplify slightly notations. Finally $x'$ will always stand for $1 - x$.

1. We define the following auxiliary game $\tilde{T}$:

$\begin{align*}
\tilde{T} & = \begin{bmatrix}
L & R \\
\tilde{T} & \tilde{B}
\end{bmatrix} \\
(1 - e) T & = \begin{bmatrix}
\tilde{B} & \tilde{A}
\end{bmatrix} \\
(1 - e) B & = \begin{bmatrix}
\tilde{B} & \tilde{A}
\end{bmatrix}
\end{align*}$

Here $\tilde{T}$ (resp. $\tilde{R}, \tilde{F}, \tilde{B}$) stands for the strategy (of player II) of playing always Left (resp. Right, Top, Bottom); $(1 - e) T$ (resp. $(1 - e) B$) stands for the strategy of playing at every stage independently with probability $(1 - e)$ Top (resp. $B$) and with probability $e$ Bottom (resp. $T$). $(\beta, \beta')$ stands for strategy of playing at each stage and independently with probability $\beta$ Left and with probability $\beta'$ Right. Finally $T_1$ (resp. $B_1$) stands for a strategy consisting of playing once $T$ (resp. $B$) and all other times $B$ (resp. $T$). The entries $\tilde{T}$ can be easily obtained as asymptotic payoffs corresponding to those strategies, using our previous conventions (and thinking of $\beta$ as strictly between 0 and 1).
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Denote by \( \overline{v} \) the value of \( \bar{\Gamma} \). If we denote by Max Min \( \bar{\Gamma} \) and Min Max \( \bar{\Gamma} \) the max Min and Min Max value of our original game in the strong sense that we described we shall prove that Min Max \( \bar{\Gamma} = \overline{v} \) and that it may be different from Max Min \( \bar{\Gamma} \). To make these statements rigorous we need still two more definitions:

2. Let us define:

\[
\underbar{v}_\infty = \inf \{ U | \exists N \exists \tau \text{ s.t. } \forall n \geq N \forall \sigma_n, \rho_n (\sigma_n, \tau) < U \}
\]

\[
\overbar{v}_\infty = \sup \{ U | \forall \tau \exists \sigma \text{ s.t. } \lim_{n \to \infty} \inf \rho_n (\sigma, \tau) \geq U \}
\]

where \( \sigma \) (resp. \( \tau \)) stands for a strategy of player I (resp. II) in the infinite game while \( \sigma_n \) (resp. \( \tau_n \)) stands for a strategy of player I (resp. II) in a game consisting of \( n \) stages only, \( \rho_n \) (\( \sigma, \tau \)) is the expected average payoff per stage in the first \( n \) stages, given \( \sigma, \tau \) and \( \rho \); i.e., \( \rho_n (\sigma, \tau) = E \sum_{t=0}^{n-1} \rho (T_{k+1}, k_t \sigma, k_t \tau) \), where \( T = (t_N) \) is the true payoff matrix chosen by the referee at the beginning of the game.

Loosely speaking, \( \overbar{v}_\infty \) is the lowest value of \( \lim sup \rho_n \) that player II can guarantee in the infinite game while \( \underbar{v}_\infty \) is the highest value of \( \lim inf \rho_n \) that cannot be guaranteed by player II. Clearly \( \overbar{v}_\infty \leq \underbar{v}_\infty \). In the next section we will prove that \( \overline{v} = \overbar{v}_\infty - \underbar{v}_\infty \), which establishes that \( \overline{v} \) is Min Max \( \bar{\Gamma} \) in the above explained sense.

3. Proofs

For a strategy of player I in \( \bar{\Gamma} \) let \( (\alpha, \alpha') \) be the probability distribution induced on \( (1 - \varepsilon) T \) and \( (1 - \varepsilon) B \).

**Lemma 1.** For any \( \alpha \) corresponding to any undominated optimal strategy of Player I in \( \bar{\Gamma} \) that uses \((1 - \varepsilon) T \) or \((1 - \varepsilon) B \) with a positive probability, one of the following holds:

\[
\alpha b_{12} > 0 \quad \text{and} \quad \alpha' b_{21} > 0 \quad (3.1)
\]

\[
\alpha b_{11} > 0 \quad \text{and} \quad \alpha' b_{22} > 0 \quad (3.2)
\]

**Proof.** Assume that for some optimal \( \alpha \) neither of (3.1) and (3.2) holds, so for instance \( \alpha b_{12} < 0 \) (the case \( \alpha' b_{21} < 0 \) is completely symmetric). Since \( \alpha b_{11} < 0 \) and \( \nu (B) = 0 \), we have \( b_{21} > 0 \) and thus also \( \alpha b_{12} < 0 \) (since 3.1 does not hold). Since \( \alpha a_{12} < 0 \) and \( \nu (A) = 0 \), we have \( a_{22} > 0 \). But this implies that the strategy \((1 - \varepsilon) B \) of player I strictly dominates his strategy \((1 - \varepsilon) T \) in \( \bar{\Gamma} \), and thus that \( \alpha = 0 \) which contradicts the assumption \( \alpha b_{11} < 0 \).
Theorem 1. \( v \gg v \).

Proof. Consider an arbitrary strategy \( \tau \) of player II and an arbitrary \( e \) (\( 0 < e < 1 \)). \( \tau \) may be considered as a probability measure \( P \) on the space \( \Omega \) of all sequences of \( L \) (left) and \( R \) (right) with the understanding that as soon as the true matrix is revealed, player II switches to his optimal strategy in that matrix.

Let \( p_1 = P(L), p_2 = P(R), p_3 = P(\Omega \setminus (L \cup R)) = 1 - p_1 - p_2 \). Let \( \Omega_{\omega} \) denote the subset of \( \Omega \) consisting of sequences with infinitely many \( L \) and infinitely many \( R \). Let \( L_{\omega} \) denote the subset of \( \Omega \) with a finite non-zero number of \( L \) in the sequence, and similarly \( R_{\omega} \) is the subset of those sequences with a finite non-zero number of \( R \).

We shall refer to these finitely many \( L \) or \( R \) as the exceptional moves.

Define \( N_2 \) by:

\[
\text{Prob. [player II has not played all his exceptional moves before } N_1 \setminus (L_{\omega} \cup R_{\omega}) < e \text{ and } N_2 \text{ by:}
\]

\[
\text{Prob. [number of } L \text{ and } R \text{ in the interval }]_{N_1, N_2} \text{ are both at least}
\]

\[
\ln \frac{e}{1 - \varepsilon} > 1 - \varepsilon
\]

with the understanding that whenever the conditioning set has zero probability, the corresponding integer takes its least possible value (1 or \( N_1 + 1 \)).

It follows from the definitions that even if player I plays \( (1 - e) \) in \( J_{N_1, N_2} \) matrix \( A \) (if it is the true matrix) will be revealed with probability greater than \( 1 - 2e \), given \( \Omega_{\omega} \), and also that:

\[
\text{Prob. [Both } L \text{ and } R \text{ appear before } N_1 \setminus (L_{\omega} \cup R_{\omega}) < e \text{]
}

Let \( (q_1, q_2, q_3, q_4, q_4, q_4, q_4, q_4) \) be an undominated optimal strategy of player I in \( \pi \). For any \( k > N_1 \), let \( q_k \) be the following strategy of player I:

- with probability \( q_1 \), play \( L \);
- with probability \( q_2 \), play \( R \);
- with probability \( q_3 \), choose \( H \) with probability \( \alpha \) and \( S \) with probability \( \alpha' \) and play:
  - if (3.1) holds: if \( H : \; F \) up to \( N_1 \) and \( (1 - e) T \) after \( N_1 \),
  - if \( S : \; F \) up to \( N_1 \) and \( (1 - e) T + B \) after \( N_1 \),
- if (3.2) holds: if \( H : \; F \) up to \( N_1 \) and \( (1 - e) T + B \) after \( N_1 \),
- with probability \( q_4 \), play a strategy \( B_1 \), with the time of playing \( B \) chosen independently of all other choices and uniformly in \([1, k]\);
- with probability \( q_4 \), play a strategy \( T_1 \), with the time of playing \( T \) chosen independently of all other choices and uniformly in \([1, k]\).

We have for all \( n > k \):

\[
1/n \ll N_1/n \ll N_1/n \ll k/n; 1/n \ll k/n, 1/k \ll e.
\]

(3.3)

Let \( f_n = (1/n) \cdot (\text{number of } L \text{ up to time } n) \).
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Let \( M = \max \{ \max_{ij} a_{ij} - \min_{ij} a_{ij}; \max_{ij} b_{ij} - \min_{ij} b_{ij} \} \),

and let 0(e) stand for any quantity x such that |x| ≤ Me

similarly 0(1/k) stands for any y such that |y| ≤ M/k, etc.

Denote by \( \rho_n (\sigma_k, \tau) \) the average payoff per stage resulting from strategies \( \sigma_k \) and \( \tau \),

we have that \( \rho_n (\sigma_k, \tau) \) is the expectation of:

\[
\begin{align*}
q_1 [ & p_1(b_{11} + 0(1/n)) + p_2(a_{12} + b_{12}) + p_3(\gamma a_{b_{11}} + \gamma a_{b_{12}}) + 0(N_1/n) + 0(e)] \\
+ & q_2 [p_1(a_{21} + b_{21}) + p_2(a_{22} + 0(1/n)) + p_3(\gamma a_{a_{21}} + \gamma a_{a_{22}}) + 0(N_1/n) + 0(e)] \\
+ & q_3 [p_1(\gamma a_{a_{b_{11}}}) + \gamma a_{a_{b_{12}}}) + 0(N_1/n) + 20 + (0(1/n)) \text{ if (3.2) holds} + \\
+ & q_4 [\gamma a_{a_{b_{11}}} + \gamma a_{a_{b_{12}}}) + 0(N_1/n) + 20 + 0(e) + 0(1/n)) \text{ if (3.2) holds}], \\
+ & P([\Omega_n] + 0(e) + 0(N_1/n))
\end{align*}
\]

if (3.1) holds:

\[
P([L_1]) (\omega a_{11} + 30 + 0(N_1/n)) + P([R_1]) (\omega a_{b_{11}} + 30 + 0(N_1/n))
\]

if (3.2) holds:

\[
P([L_1]) (\omega a_{11} + 30 + 0(N_1/n) + 0(1/n)) + P([R_1]) (\omega a_{b_{11}} + 30 + 0(N_1/n) + 0(1/n))
\]

\[
+ q_4 [p_1(b_{11} + 0(1/n)) + p_2(a_{12} + 0(k/n)) + p_3(\gamma a_{b_{11}} + \gamma a_{b_{12}}) + 0(1/k) + 0(e) + 0(k/n)])
\]

\[
+ q_3 [p_1(b_{21} + 0(k/n)) + p_2(a_{22} + 0(1/n)) + p_3(\gamma a_{b_{21}} + \gamma a_{b_{22}}) + 0(1/k) + 0(e) + 0(k/n)]
\]

Using relations (3.3) and Lemma 1 we get that for all \( n > k \):

\[
\rho_n (\sigma_k, \tau) \geq E(H(k, n, \tau, \omega)) - 4Me - 0(k/n),
\]

where

\[
H(k, n, \tau, \omega) = \begin{bmatrix}
q_1 & b_{11} & a_{12} + b_{12} & b_{11} & b_{12} \\
q_2 & a_{21} + b_{11} & a_{22} & a_{21} & a_{22} \\
q_3 & b_{11} & a_{12} & 0 & 0 & \gamma a_{b_{11}} + \gamma a_{b_{12}} \\
q_4 & b_{21} & a_{22} & 0 & 0 & \gamma a_{a_{b_{11}}} + \gamma a_{a_{b_{12}}} \\
q_5 & b_{21} & a_{22} & \gamma a_{a_{b_{11}}} + \gamma a_{a_{b_{12}}} \\
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
\end{bmatrix}
\]
Denote $E(H(k,n,\tau,\omega))$ by $\phi(f_k,f_n)$. The function $\phi : L_\infty \times L_\infty \rightarrow \mathbb{R}$ is weakly continuous and affine in each variable separately on $L_\infty$ endowed with the weak topology $\sigma(L_\infty,L_1)$. 

Let $C$ be the closed convex hull of $\{f_i \mid i > N_1\}$ in $L_\infty$, and consider $\phi$ on $C \times C$. We have $\phi(f,f) \geq \sup \{f \in C, \inf f_k = f, H(k,n,\tau,\omega) \geq \gamma\}$ holds for each value of $\omega$ since $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ is an optimal strategy for player I in $\Gamma$. In addition $C$ is compact and convex for $\sigma(L_\infty,L_1)$ and is affine and continuous in each variable separately on $C$. It follows that $\phi$ has a saddle point, hence:

$$\exists f \in C \ \forall f \in C : \phi(g,f) \geq \inf \sup \phi(u,f) \geq \inf \phi(f,f) = \bar{u}.$$ 

Now $g$ is also in the closure of the convex hull of $\{f_i \mid i > N_1\}$ when $L_\infty$ is endowed with the Mackey topology $\tau(L_\infty,L_1)$ due to the convexity of the set $-$ this is a well-known result that follows from the Hahn-Banach theorem. Since on bounded sets of $L_\infty$ the Mackey topology $\tau(L_\infty,L_1)$ coincides with the topology of convergence in probability, it follows that there exist $\lambda_i (1 \leq i \leq l, \lambda_i \geq 0, \sum_{i=1}^l \lambda_i = 1)$ and $k_i (1 \leq i \leq l, k_i > N_1)$ such that:

$$P(\sum_{i=1}^l \lambda_i f_{k_i} - g \geq \epsilon) < \epsilon.$$ 

Let now $\sigma_{\epsilon,n}$ be the strategy of player I consisting of choosing at the start of the game a number $i (1 \leq i \leq l)$ with probability $\lambda_{i}$ and thereafter using his strategy $\sigma_{k_i}$. Let also $K_M = \max \{K_i \mid 1 \leq i \leq l\}$ then we have:

$$\rho_n (\sigma_{\epsilon,n},\tau) \geq \phi(g,f_n) - 6\epsilon - 0 (K_M/n) \text{ for all } n > K_M.$$ 

Thus: $\forall \tau$, strategy of player II, $\forall \epsilon, 0 < \epsilon \leq 1$, $\exists \sigma_{\epsilon,n}$, strategy of player I, such that:

$$\lim_{n \rightarrow \infty} \inf_{\tau} \rho_n (\sigma_{\epsilon,n},\tau) \geq \inf_{\tau} \phi(g,f) - 6\epsilon \geq \bar{u} - 6\epsilon.$$ 

This completes the proof of Theorem 1.

**Lemma 2.** Player II has an optimal strategy in $\Gamma$ using only a single value of $\beta$.

**Proof.** A priori player II's optimal strategy in $\Gamma$ consists of a probability vector $(p_1, p_2, p_3)$ together with a probability measure $\mu$ on $[0,1]$ to choose $\beta$. We want to show that player II has an optimal strategy in which $\mu$'s support is a single point in $[0,1]$.

1. If $b_{11} > b_{12}$ and $a_{11} > a_{12}$ the result follows from the convexity in $\beta$ of the payoff function.
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L 2) Otherwise we have either $b_{11} < b_{12}$ or $a_{22} < a_{21}$, by symmetry we may assume that $a_{22} < a_{21}$. Since $\nu(A) = 0$ it follows that $a_{22} \leq 0$.

L 2.1) If in addition $b_{11} < b_{12}$, the payoff function is concave in $\beta$ and thus $\nu$ is dominated by the probability on $(0, 1)$ that has the same mean. So without loss of generality we may assume that in this case $\mu(D, I) = 0$. We get thus for $\Gamma = 6 \times 4$ matrix with $\bar{L}, \bar{R}, \bar{b} = 1$ and $\beta = 0$ as pure strategies for player II. The other strategies are eliminated by dominance. In addition $\nu(B) = 0$ implies $b_{11} \leq 0$, and thus we conclude that rows $B_{1}$, and $T_{1}$ are dominated by $(1 - \epsilon T$ and $(1 - \epsilon) B$ respectively.

Let $R = \begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad C = \begin{pmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{pmatrix}$

$L 2.1.1$ If $r \leq 0$ and if we denote by $(\beta, \beta')$, the relative weights of the columns $\beta = 1$ and $\beta = 0$, then there exists an optimal $\beta$ for which $\beta' b_{11} + \beta b_{12} \leq 0$ and $\beta' a_{21} + \beta b_{22} \leq 0$ (if $c < 0$, the required $\beta$ is the relative weight of the last two columns in the equalizing strategy of player II, if $c > 0$ the value of the game is 0 and an optimal strategy of player II is $(0, 0, \beta, \beta')$ where $\beta$ is optimal in $R$ and hence satisfies the required inequalities). It follows that if in that optimal strategy, player II would replace the columns $\beta = 1$ and $\beta = 0$ by i.i.d. $(\beta, \beta')$, rows $B_{1}$ and $T_{1}$ would still be dominated by $(1 - \epsilon) T$ and $(1 - \epsilon) B$ respectively and hence player II has in this case an optimal strategy using a single $\beta$.

$L 2.1.2$ If $r > 0$, the optimal mixture of the columns $\beta = 1$ and $\beta = 0$ is $(\beta, \beta')$. $\beta$ being optimal in $R$ and hence $\beta' b_{11} + \beta b_{12} > 0$ and $\beta' a_{21} + \beta b_{22} > 0$. It follows again that replacing the last two columns by i.i.d. $(\beta', \beta')$, rows $B_{1}$ and $T_{1}$ remain dominated, this time by $T$ and $\bar{B}$ respectively, providing again a single $\beta$ optimal strategy for player II.

L 2.2) We are thus left with the case:

$a_{22} < a_{21}, \quad a_{22} < 0, \quad b_{12} < b_{11}, \quad b_{12} < 0$.

Consider player II's optimal strategy in the game $\bar{\Gamma}$ without the rows $B_{1}$ and $T_{1}$; it obviously implies $\beta = 0$. For this $\beta, B_{1}$, is dominated by $(1 - \epsilon) T$ and $T_{1}$ by $(1 - \epsilon) B$ and thus this single $\beta$ strategy is also optimal in $\bar{\Gamma}$ This completes the proof of Lemma 2.

Notice that the strategies $\beta = 1$ and $\beta = 0$ in $\bar{\Gamma}$ should be interpreted as playing i.i.d. $(1 - \epsilon, \epsilon)$ and $(\epsilon, 1 - \epsilon)$ respectively. Thus in the single $\beta$ optimal strategy for player II established in Lemma 2 we may assume $0 < \beta < 1$. 

Theorem 2. \( \varphi_n \leq \nu. \)

Proof. We will show that whenever player II plays in \( \Gamma \) one of his strategies \( r \) in \( \Gamma, \)
consisting of a mixture of \( \bar{L}, \bar{R} \) and one \( (\beta, \beta') \) with \( 0 < \beta < 1 \) — any pure strategy
of player I yields in \( \Gamma_n \) a payoff dominated up to terms \( 0 (1/n) \) by a convex combination
of rows of \( \Gamma. \) Since by Lemma 2 player II can guarantee \( r \) — up to \( r \) — by such
mixtures \( r \) against rows of \( \Gamma, \) the result will then follow.

If the pure strategy of player I is \( \bar{L} \) or \( \bar{R} \) then it is already a row of \( \Gamma. \) Take any other
pure strategy that begins say with \( T \) (for strategies starting with \( \bar{R} \) the discussion is completely
dual). Let \( \omega_1 = 1 \) if \( T \) occurs at time \( 1 \) in the strategy and \( \omega_1 = 0 \)
otherwise. Let \( f_n = \frac{1}{n} \sum_{i=1}^{n} \omega_i, \) and \( \omega_{n+1} \) be the first zero in the sequence \( (\omega_i). \) Let
\( y = t/n; \) we have \( 1/n \leq y \leq f_n \leq (n - 1)/n. \)

Let \( D = \beta b_{11} + \beta' b_{21}, \) \( G = \beta b_{21} + \beta' b_{22}, \)
\( X = \frac{1}{n} \left[ \omega_1 (1 - \beta \omega_1) + \ldots + \omega_n (1 - \beta \omega_n) \right] \)
\( Y = \frac{1}{n} \left[ \omega_1 (1 - \beta' \omega_1) + \ldots + \omega_n (1 - \beta' \omega_n) \right] \).

We have \( f_n \beta^m \leq X \leq f_n \beta^m y \) and
\( y + (\gamma_n - y) \beta^m \leq Y \leq y + (\gamma_n - y) \beta, \) neglecting terms \( 0 (1/n). \)

The strategy of player I obtains, up to \( 0 (1/n); \) against \( \bar{L}, f, b_{11} + f_n b_{21}; \) against
\( \bar{R} ; y b_{12} + f_n a_{12} + f_n a_{22} \) and against \( (\beta, \beta') : GX + DY. \) Majoring the last term
according to the sign of \( G \) and \( D \) we obtain (writing \( f \) for \( f_n) : \)

\[
\begin{array}{ccc}
\text{against} & \bar{L} & \bar{R} \\
\text{a payoff} \leq & f b_{11} + f' b_{21} & y b_{12} + f a_{12} + f' a_{22} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{against} & (\beta, \beta') \text{ with} & G \geq 0, D \geq 0 & G < 0, D < 0 & G > 0, D < 0 & G < 0, D > 0 \\
\text{a payoff} \leq & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') \\
\end{array}
\]

Since all terms are convex in \( y, \) we may replace \( y \) by its extreme values \( 1/n \) and \( f. \)
Neglecting terms \( 0 (1/n) \) one gets thus:

\[
\begin{array}{c|ccc|ccc}
\chi = 1/n & f b_{11} - f b_{21} & f a_{12} - f a_{22} & G \geq 0, D \geq 0 & G < 0, D > 0 & G > 0, D < 0 & G < 0, D < 0 \\
\chi = f & f b_{11} - f b_{21} & f a_{12} - f a_{22} & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c}
\chi = 1/n & f b_{11} - f b_{21} & f a_{12} - f a_{22} & G \geq 0, D \geq 0 & G < 0, D > 0 & G > 0, D < 0 & G < 0, D < 0 \\
\chi = f & f b_{11} - f b_{21} & f a_{12} - f a_{22} & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') & (y' \delta + D y' \delta') \\
\end{array}
\]
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$f^{m'}$ and $f^{m''}$ are convex. When their coefficients are negative let us majorize them by zero. All functions get then linear or convex in $f$, so we may replace $f$ by its extreme values $1/n$ and $(1 - 1/n)$. Neglecting terms $O(1/n)$ one obtains:

<table>
<thead>
<tr>
<th>$f = 1/n$</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$0$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f = m/n$</td>
<td>$b_1$</td>
<td>$a_{11}$</td>
<td>$G b$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$m' = m/n$</td>
<td>$b_{11}$</td>
<td>$a_{11} + b_{11}$</td>
<td>$D$</td>
<td>$D$</td>
<td>$D$</td>
</tr>
</tbody>
</table>

We conclude that player I's strategy is dominated by the mixture of three similar strategies with $(y = 1/n, f = 1 - 1/n), (y = f = 1/n)$ and $(y = f = 1 - 1/n)$, the weights being $f', f''$ and $y$ respectively. But this mixture is dominated by the convex combination with the same weights of the following rows of $f':$

<table>
<thead>
<tr>
<th>Case</th>
<th>$G &gt; 0, D &gt; 0$</th>
<th>$G &lt; 0, D &gt; 0$</th>
<th>$G &gt; 0, D &lt; 0$</th>
<th>$G &lt; 0, D &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f' - y$</td>
<td>$B_1$</td>
<td>$B_1$</td>
<td>$(1 - e)T$</td>
<td>$(1 - e)T$</td>
</tr>
<tr>
<td>$f''$</td>
<td>$T_1$</td>
<td>$(1 - e)B$</td>
<td>$T_1$</td>
<td>$(1 - e)B$</td>
</tr>
</tbody>
</table>
| $y$ | $T$ | $T$ | $T$ | $T$f

This completes the proof of Theorem 2.

4. Conclusions

(i) $\tilde{w}_m = \tilde{w}_m = \tilde{v} = \min \max \Gamma$

(ii) Player II has an "e-MinMax" strategy of the type: With probability $p_1$ play always $L$, with probability $p_2$ play always $R$ and with probability $1 - p_1 - p_2$ play always i.i.d. with probability $\beta), L$ and with probability $\beta, R$.

(iii) This strategy also guarantees that in any finite sufficiently long game the payoff is less than $\tilde{v} + e$.

(iv) Dual results hold for player I.

(v) Analysis of the game $\Gamma'$ and its' dual $\Gamma$ shows that the only cases where there is no value (i.e. $\tilde{v} > y$) are: $(c \lor r < 0$ and $a_{21} > 0$ and either $a_{12} \wedge b_{21} > 0$ or $a_{12} (a_{21} - a_{22}) + b_{21} a_{22} < 0$) and its symmetries obtained by either permuting the games

\[(a_{ij} \leftrightarrow b_{ij}'$ where $1' = 2$ and $2' = 1$) or permuting the players $(a_{ij} \leftrightarrow -a_{ij}; b_{ij} \leftrightarrow -b_{ij})$ or both.
An example of a game without value is the following:

\[ A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \]

Optimal strategies in \( \Gamma \) are: For player I; \((1/4, 1/4, 1/4, 1/4, 0, 0)\) and for player II; \((1/4, 1/4, 1/2, 1/2, 1/2)\) giving \( \gamma = -1/2 \). Optimal strategies in \( \Gamma^l \) are: For player I; \((1/6, 1/6, 2/3, 1/2, 1/2)\) and for player II; \((1/6, 1/6, 0, 0, 1/3, 1/3)\) giving \( \gamma = -2/3 \).

References


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