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The game for the speed of convergence in repeated games of incomplete information

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Abstract. We consider an infinitely repeated two-person zero-sum game with incomplete information on one side, in which the maximizer is the (more) informed player. Such games have value $v_\infty(p)$ for all $0 \le p \le 1$. The informed player can guarantee that all along the game the average payoff per stage will be greater than or equal to $v_\infty(p)$ (and will converge from above to $v_\infty(p)$ if the minimizer plays optimally). Thus there is a conflict of interest between the two players as to the speed of convergence of the average payoffs-to the value $v_\infty(p)$. In the context of such repeated games, we define a game for the speed of convergence, denoted $SG_\infty(p)$, and a value for this game. We prove that the value exists for games with the highest error term, i.e., games in which $v_n(p) - v_\infty(p)$ is of the order of magnitude of $\frac{1}{\sqrt{n}}$. In that case the value of $SG_\infty(p)$ is of the order of magnitude of $\frac{1}{\sqrt{n}}$. We then show a class of games for which the value does not exist. Given any infinite martingale $\mathfrak{X}^\infty = \{X_k\}_{k=1}^\infty$, one defines for each $n : V_n(\mathfrak{X}^\infty) := E\sum_{k=1}^n |X_{k+1} - X_k|$. For our first result we prove that for a uniformly bounded, infinite martingale $\mathfrak{X}^\infty$, $V_n(\mathfrak{X}^\infty)$ can be of the order of magnitude of $n^{1/2-\varepsilon}$, for arbitrarily small $\varepsilon > 0$.

Key words: Repeated Games, Incomplete Information, Variation of Bounded martingales.

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1. Introduction

In this paper we treat a two-person zero-sum infinitely repeated game with incomplete information on one side.[1] Let A_1, A_2 be 2×2 matrices, each

[1] For background see e.g. p. 175 of [1] and p. 116 of [8].

corresponding to the payoff of a two-person zero-sum game, with elements a_{ij}^k, where $k \in \{1,2\}$ represents the number of the matrix and $i \in I = \{T, B\}$ and $j \in J = \{L, R\}$ are the pure strategies of PI (the maximizer) and PII (the minimizer), respectively. For each p, $0 \leq p \leq 1$, we consider the n-stage repeated game $G_n(p)$, defined as follows:

- At stage 0, chance chooses $k = 1$ with probability p, and $k = 2$ with probability $p' = 1 - p$. Both players know p, but (only) PI is informed about the chosen value of k.
- At stage 1, PI, knowing k, chooses $i_1 \in I$, PII chooses $j_1 \in J$, and (i_1, j_1) is publicly announced.
- At stage m, $m = 2, 3..$ PI, knowing k and $(i_1, j_1) \cdots (i_{m-1}, j_{m-1})$, chooses $i_m \in I$. PII, knowing $(i_1, j_1) \cdots (i_{m-1}, j_{m-1})$, chooses $j_m \in J$ and (i_m, j_m) is announced.
- After stage n, PI receives from PII the amount:

$$\frac{1}{n}\sum_{m=1}^{n} a_{i_m j_m}^k.$$

Define $v_n(p) = val\, G_n(p)$ (the minmax value of the game $G_n(p)$).

The strategies in $G_n(p)$: Denote by h_m, the random variable that represents the history of announcements up to stage m: $(i_1, j_1) \cdots (i_{m-1}, j_{m-1})$, and by $H_m = (I \times J)^{m-1}$ the set of all m-stage histories. ($H_1 = \emptyset$.)

A (behavioral) strategy for PI is $\sigma_n = (\sigma_n^1, \sigma_n^2)$, where for each $k \in \{1, 2\}$: $\sigma_n^k = (s_1^k, \ldots s_n^k)$, and for all m, $1 \leq m \leq n$, s_m^k is a function from H_m into the set of probability distributions on I. A (behavioral) strategy for PII is $\tau_n = (t_1 \cdots t_n)$, where for all m, $1 \leq m \leq n$, t_m is a function from H_m into the set of probability distributions on J.

Remark 1.1. *The difference in the structure of the strategies of the two players is due to the fact that only PI knows the chosen value of k, and therefore can play differently in each of the two matrices.*

We now define the infinitely repeated game $G_\infty(p)$, as follows:

A strategy for PI in $G_\infty(p)$ is $\sigma = (\sigma^1, \sigma^2)$, where for all $k \in \{1, 2\}$, σ^k is an infinite sequence $\{s_n^k : n \geq 1\}$, and each s_n^k is a function from H_n into the set of probability distributions on I. A strategy for PII in $G_\infty(p)$ is τ, where $\tau = \{t_n : n \geq 1\}$, and t_n is a function from H_n into the set of probability distributions on J.

For any pair of strategies, σ, τ, let $\gamma_n(\sigma, \tau)$ be the average expected payoff for the n first stages in $G_\infty(p)$ (or in any $G_l(p)$ $l \geq n$), when σ and τ are played. That is:

$$\gamma_n(\sigma, \tau) = E_{p,\sigma,\tau}\left(\frac{1}{n}\sum_{m=1}^{n} a_{i_m j_m}^k\right).$$

($E_{p,\sigma,\tau}$ is the expectation with respect to the probability measure on H_{n+1} induced by p, σ, τ.) From now on, we use σ and τ to denote strategies of PI and PII, respectively, in the game $G_\infty(p)$.

Definition 1.2. *(see e.g. p. 187 in [1].)*

- *We say that PI can guarantee $f(p)$ in $G_\infty(p)$ if for any $\varepsilon > 0$ there is σ_ε and N_ε, such that:*

$$\gamma_n(\sigma_\varepsilon, \tau) - f(p) \geq -\varepsilon \quad \forall n > N_\varepsilon, \forall \tau.$$

- *We say that PII can guarantee $g(p)$ in $G_\infty(p)$ if for any $\varepsilon > 0$ there is τ_ε and N_ε, such that:*

$$\gamma_n(\sigma, \tau_\varepsilon) - g(p) \leq \varepsilon \quad \forall n > N_\varepsilon, \forall \sigma.$$

- *We say that $G_\infty(p)$ has a value $v_\infty(p)$ if both players can guarantee $v_\infty(p)$.*

An alternative definition for the value of an infinitely repeated game would be the limit of the values of the n-stage games $G_n(p)$, namely: $\lim_{n\to\infty} v_n(p)$, if this limit exists. (For more details see Zamir [10]).

Denote by $D(p)$ the one-stage zero-sum game with payoff matrix of $pA^1 + p'A^2$.[2] $D(p)$ can be interpreted as the one-stage game in which *both* players are not informed of the matrix chosen. Let $u(p) = val\, D(p)$, and $Cav\, u(p)$ be the smallest concave function that is greater than or equal to $u(p)$ on $[0, 1]$.

Theorem (Aumann and Maschler [1]):[3] $v_\infty(p)$ and $\lim_{n\to\infty} v_n(p)$ both exist, and:

$$v_\infty(p) = \lim_{n\to\infty} v_n(p) = Cav\, u(p).$$

In other words, in this model as long as we are only interested in the asymptotic properties of $G_n(p)$, both concepts – 'the value of the limit-game', and the 'limit of the values' (of the finite games) – lead to the same result.

2. Motivation

We interpret $G_\infty(p)$ as a model for *finite* long games in which the number of stages n is not known in advance. In such situations, the players have to play "uniformly well" in n, since any stage n can turn out to be the end of the game after which the payment procedure takes place. Aumann and Maschler constructed a strategy σ^* in $G_\infty(p)$ (the splitting strategy, see e.g. p. 126 of [8],) such that

$$\gamma_n(\sigma^*, \tau) \geq Cav\, u(p), \quad \forall n, \forall \tau. \tag{1}$$

Similarly, they constructed an optimal strategy τ^* for PII (the Blackwell approachability strategy), such that there is a $c > 0$ satisfying that:

$$\gamma_n(\sigma, \tau^*) \leq Cav\, u(p) + \frac{c}{\sqrt{n}}, \quad \forall n, \forall \sigma. \tag{2}$$

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[2] see e.g., Definition 3.10 p. 123 of [8].

[3] Aumann and Maschler proposition A, p. 187 and Theorem C.

Property (1) is the key property of our model, on the basis of which we construct the game for the speed of convergence. By this result of Aumann and Maschler, PI can guarantee at least $Cav\, u(p)$ for *all n*. Thus playing optimally for him means guaranteeing the *slowest* possible speed of convergence *from above* to $Cav\, u(p)$. Note that monotonicity of the sequence of payoffs is not a requirement (nor can it be guaranteed by PI, since it depends also on PII's actions). The fact that PI should be interested in a slow speed of convergence follows from Property (1) which states that PI can guarantee that the payoffs will converge to $v_\infty(p)$ from *above*. Similarly, for PII playing optimally means guaranteeing the *fastest* speed possible. This leads naturally to the definition of the game for the speed of convergence, which we will denote $SG_\infty(p)$. The two main applications of $SG_\infty(p)$ are:

- To define formally the level at which PI plays uniformly well in all finite games. The fact that PI does not know n and therefore cannot in general achieve $v_n(p)$, causes him a loss. A strategy which guarantees just $Cav\, u(p)$ yields a loss of $(v_n(p) - Cav\, u(p))$ (compared to the situation in which PI knows n). This difference can be of the order of magnitude of $\frac{1}{\sqrt{n}}$ (Section 3). We will show that PI can reduce his loss to the order of $\frac{1}{\sqrt{n}}\left(1-\frac{1}{n^\varepsilon}\right)$ $\forall n$ for arbitrarily small $\varepsilon > 0$.
- Our game refines the notion of optimality (for PI) for the strategies in $G_\infty(p)$: Any strategy σ satisfying (1) is optimal in $G_\infty(p)$. We shall parameterize this set of optimal strategies by the speed of convergence of $\inf_\tau \gamma_n(\sigma,\tau)$ to $Cav\, u(p)$. An optimal strategy for which $(\inf_\tau \gamma_n(\sigma,\tau) - Cav\, u(p))$ is of the same order of magnitude as $f(n)$ will be denoted by σ_f. This defines a natural "preference" of PI: If f is of a greater order of magnitude than g (i.e., $\frac{g(n)}{f(n)} \to 0$), then $\sigma_g \prec_{PI} \sigma_f$.

Although this paper deals mostly with repeated games of incomplete information on one side, it is clear from this discussion that $SG_\infty(p)$ is of interest in any infinite zero-sum game with the following properties: The infinite game has a value $v_\infty(p)$, and one of the players has the advantage of being able to guarantee $v_\infty(p)$ uniformly in n. (e.g., PII has a strategy τ, satisfying that: $\gamma_n(\sigma,\tau) \le v_\infty(p)\ \forall n, \forall \sigma$).

3. Definitions and preliminary results

For all x denote: $x' = 1 - x$.

$e_n(p) := v_n(p) - Cav\, u(p)$ $n \ge 1$ is called the *nth error term* of the game. The nth error term is the extra gain that PI can guarantee (over $Cav\, u(p)$) in $G_n(p)$. The sequence $e(p) = \{e_n(p)\}_{n=1}^\infty$ is called the *error term* of the game. In analogy with the nth error term defined in $G_n(p)$, given σ in $G_\infty(p)$, we look at

$$\inf_\tau \gamma_n(\sigma,\tau) - Cav\, u(p).$$

This is the extra gain that PI can guarantee by using σ if the game ends after n stages (and similarly for PII: $\sup_\sigma \gamma_n(\sigma,\tau) - Cav\, u(p)$).

The following proposition states the relationship between $v_n(p)$ and its analogs in $G_\infty(p)$.

Proposition 3.1. *For all σ', τ', strategies in $G_\infty(p)$ and for all n:*

$$\inf_\tau \gamma_n(\sigma', \tau) \le v_n(p) \le \sup_\sigma \gamma_n(\sigma, \tau').$$

This proposition expresses the intuition that both PI and PII have more freedom in their choices of strategies in $G_n(p)$ than in $G_\infty(p)$, since a strategy in $G_n(p)$ can depend on n, and a strategy in $G_\infty(p)$ (where the end of the game is not known to the players) cannot depend on n. In other words, any strategy available to any of the players in $G_\infty(p)$ is available to them also in $G_n(p)$ (as its n-truncation); hence they can not "do better" in $G_\infty(p)$ than in $G_n(p)$.

Proof: For any pair of strategies σ and τ of PI and PII in $G_\infty(p)$, we denote by σ_n, τ_n the n-truncation of σ, τ, respectively. Then for all σ', τ' and for all n:

- $\inf_\tau \gamma_n(\sigma', \tau) = \inf_{\tau_n} \gamma_n(\sigma', \tau_n) \le v_n(p)$.
- $\sup_\sigma \gamma_n(\sigma, \tau') = \sup_{\sigma_n} \gamma_n(\sigma_n, \tau') \ge v_n(p)$. □

We are now in a position to define the game $SG_\infty(p)$ and its value:

- The payoff matrices and the strategy spaces for each player are the same as in $G_\infty(p)$.
- The payoff function in this game is not a real number but rather the infinite sequence of expected payoffs in the nth stage game generated by σ and τ, namely: $\{\gamma_n(\sigma, \tau)\}_{n=1}^\infty$. Since the value of $SG_\infty(p)$ that we will define next involves only the *order of magnitude* of sequences, comparing different strategies will be done only by comparing the order of magnitude of their respective payoff sequences.

Definition 3.2.

- *Two sequences f and g of non-negative numbers are said to be of the same order of magnitude if there are constants $c_1, c_2 > 0$, and N, such that*

  $$c_2 g(n) \le f(n) \le c_1 g(n), \quad \forall n \ge N.$$

  This will be denoted by $f = O^*(g)$, or $g = O^*(f)$.
- *If there is a $c > 0$, and N, such that $cg(n) \le f(n)$, $\forall n \ge N$, we write that $g \le O^*(f)$.*

Definition 3.3.

- *We say that PI can guarantee a rate of convergence of $f \ge 0$ in $SG_\infty(p)$ if $\exists c > 0$ such that for all $\varepsilon > 0$ $\exists \sigma_\varepsilon(f)$:*

  $$\gamma_n(\sigma_\varepsilon(f), \tau) - v_\infty(p) \ge cf(n)n^{-\varepsilon}, \quad \forall \tau, n.$$

- *We say that PII can guarantee a rate of convergence of $g \ge 0$ in $SG_\infty(p)$ if $\exists b > 0$ such that for all $\varepsilon > 0$ $\exists \tau_\varepsilon(g)$:*

  $$\gamma_n(\sigma, \tau_\varepsilon(g)) - v_\infty(p) \le bg(n)n^{\varepsilon}, \quad \forall \sigma, n.$$

- *We say that $SG_\infty(p)$ has a value $v_s(p)$ if both PI and PII can guarantee $v_s(p)$ in $SG_\infty(p)$.*

Remark 3.4. *Note that if $g = O^*(f)$, then if PI (or PII) can guarantee f, then he can also guarantee g. In other words, if $v_s(p)$ exists, then it represents not a unique function, but an equivalence class of functions, where the equivalence relation is defined in Definition 3.2.*

Remark 3.5. • *The reason we define asymptotic guaranteeing is that PI can never guarantee a speed as slow as $\frac{1}{\sqrt{n}}$, (as we will soon show). However, he can guarantee $\frac{1}{n^{1/2+\varepsilon}}$ for arbitrarily small ε.*

- *The choice of n^ε for the definition of the asymptotic convergence is somewhat arbitrary and one could think of other, more general definitions. However, this is rather natural in the context of repeated games of incomplete information, where the upper bounds of the speed of convergence are known to be $\frac{1}{\sqrt{n}}$ (see Section 3) and $\frac{1}{n^{2/3}}$ (see [6]).*

Note that for $SG_\infty(0)$ and $SG_\infty(1)$, there is always a value, and this value is $\mathbf{0}$ (where $\mathbf{0}$ is the constant zero sequence), since this is a game with complete information.

Proposition 3.6. *If there is a value $v_s(p)$ for $SG_\infty(p)$, then it satisfies*

$$v_s(p) = O^*(e(p)).$$

(Recall that $e(p) = \{e_n(p)\}_{n\geq 1} = \{v_n(p) - v_\infty(p)\}_{n\geq 1}$ is the error term of the game.)

Proof: From Proposition 3.1 we get that for any σ', τ':

$$\inf_\tau \gamma_n(\sigma', \tau) - v_\infty(p) \leq e_n(p) \leq \sup_\sigma \gamma_n(\sigma, \tau') - v_\infty(p), \quad \forall n.$$

Denote $v_s(p, n)$ as the nth element of the sequence $v_s(p)$.

If there is a value $v_s(p)$ for $SG_\infty(p)$, then by Definition 3.3 there are $c, b > 0$ such that for all $\varepsilon > 0$ there exist $\sigma_\varepsilon = \sigma_\varepsilon(v_s(p))$, $\tau_\varepsilon = \tau_\varepsilon(v_s(p))$ satisfying that

$$cn^{-\varepsilon} v_s(p,n) \leq \inf_\tau \gamma_n(\sigma_\varepsilon, \tau) - v_\infty(p) \leq e_n(p) \leq \sup_\sigma \gamma_n(\sigma, \tau_\varepsilon) - v_\infty(p)$$

$$\leq bn^\varepsilon v_s(p,n), \quad \forall n.$$

Letting ε go to 0, we get that $cv_s(p,n) \leq e_n(p) \leq bv_s(p,n)$; hence by Definition 3.2: $v_s(p) = O^*(e(p))$. □

Remark 3.7. *For any sequence $\{a_n\}$, we abbreviate $O^*(\{a_n\})$ as $O^*(a_n)$.*

Our main result is that although $v_s(p)$ (unlike $e(p)$) does not always exist, it does exist for the special class of games in which $e(p) = O^*\left(\frac{1}{\sqrt{n}}\right)$ $\forall p \in (0,1)$, and hence equals $O^*\left(\frac{1}{\sqrt{n}}\right)$ by Proposition 3.6.

The games for which $e(p) = O^*\left(\frac{1}{\sqrt{n}}\right)$ $\forall p \in (0,1)$ were characterized by Mertens and Zamir as the games for which $\sqrt{n}(v_n(p) - Cav\,u(p)) \to \Phi(p)$, where $\Phi(p)$ is an appropriately scaled normal density function. (For details see e.g. Mertens and Zamir [6]).

We shall refer to this class of games as "*normal games*".

For any strategy σ of PI in $G_\infty(p)$ (or in $G_n(p)$), define a sequence of random variables:[4]

$$P_1 \equiv p$$

and for each $n > 1$,

$$P_n := P_{p,\sigma,\tau}(k = 1 \mid h_n).$$

That is, given h_n, σ and τ, P_n is the conditional probability that at stage 0 chance chose $k = 1$. It can be shown that the distribution P_n is independent of τ since τ is independent of k. The reason for this is that PI's belief as to the matrix that was chosen is dependent on PII's strategy *only through PII's actions*. Therefore, *given the history*, PI's belief is no longer dependent on PII's strategy. It can also be shown that $\mathscr{P}^\infty = \{P_n\}_{n=1}^\infty$ is a martingale. Hence, any strategy of PI in $G_\infty(p)$, yields an infinite martingale $\mathscr{P}^\infty = \{P_n\}_{n=1}^\infty$ satisfying $0 \le P_n \le 1$, $\forall n$.

For all n and all $m \ge n$ let:

$$V_n(\mathscr{P}^m) = E\sum_{k=1}^{n} |P_{k+1} - P_k|$$

be the nth variation of $\mathscr{P}^m = \{P_k\}_{k=1}^m$. The nth variation is a measure for the expected amount of information revealed by PI up to (and including) stage n, when he uses the strategy σ. Note that the definition for $V_n(\mathscr{P}^m)$ holds also for $m = \infty$.

The variation $V_n(\mathscr{P}^m)$ plays a key role in the analysis, since the extra gain of PI (beyond $Cav\,u(p)$) is constrained by the amount of information he reveals. More precisely: it is proved (see p. 224 of [3]) that there is $c > 0$, such that

$$\inf_\tau \gamma_n(\sigma,\tau) \le \frac{1}{n}\sum_{k=1}^{n} Eu(P_k) + \frac{c}{n} V_n(\mathscr{P}^m) \le Cav\,u(p) + \frac{c}{n} V_n(\mathscr{P}^m) \tag{3}$$

for all $n \le m$ and all σ (a strategy in $G_m(p)$, $m \ge n$).

By the Cauchy-Schwarts inequality and the fact that $\mathscr{P}^\infty$ is a uniformly bounded martingale, one can prove that[5]

$$V_n(\mathscr{P}^m) \le \sqrt{pp'}\sqrt{n}, \quad \text{for all } n,\ (m \ge n). \tag{4}$$

Remark 3.8. *It follows from (3) and (4) that* $e(p) \le O^*\left(\frac{1}{\sqrt{n}}\right)$.

[4] For background see e.g. p. 189 of [4] and p. 122 of [8].

[5] See e.g. proposition 3.8 p. 122 of [8].

It was proved (see Mertens and Zamir [5]) that $O^*\left(\frac{1}{\sqrt{n}}\right)$ is the *least* upper bound for the order of magnitude of $e(p)$. In other words there exists a game in which $e_n(p) \geq \frac{bpp'}{\sqrt{n}}$, for some $b > 0$, $\forall n$ and $\forall p \in (0,1)$.

Hence from (3) it follows that there is a $c > 0$ such that for each n there is an n-length martingale of probabilities $\mathscr{P}^n$ satisfying that

$$V_n(\mathscr{P}^n) \geq c\sqrt{n}.$$

Mertens and Zamir also proved[6] that for any *infinite* uniformly bounded martingale $\mathscr{P}^\infty$, $\lim_{n\to\infty}\left\{\frac{V_n(\mathscr{P}^\infty)}{\sqrt{n}}\right\} = 0$.

Result 3.9. *From the above we get that if $v_s(p)$ exists, then it satisfies: $0 \leq v_s(p) < O^*(n^{1/2})$.*

From this point we proceed as follows:

- Although there isn't any *infinite* uniformly bounded martingale $\mathfrak{X}^\infty$ satisfying $V_n(\mathfrak{X}^\infty) \geq a\sqrt{n}$, $\forall n$ for some $a > 0$, we prove in Part 4 that $\sqrt{n}$ can be reached asymptotically. That is, for every $\varepsilon > 0$, we will construct an infinite martingale $\mathfrak{X}_\varepsilon^\infty$ satisfying

  $$V_n(\mathfrak{X}_\varepsilon^\infty) \geq cn^{1/2-\varepsilon}, \quad \forall n$$

  for some $c > 0$.
- In Part 5 we construct for any $\varepsilon > 0$, a strategy σ_ε in $G_\infty(p)$, that yields an infinite martingale $\mathscr{P}_\varepsilon^\infty$ that coincides with $\mathfrak{X}_\varepsilon^\infty$ in some interval (l,u) in $[0,1]$.
- In Part 6 we prove that by using the strategy constructed in part 5 in the normal games, PI guarantees a rate of convergence of $O^*\left(\frac{1}{\sqrt{n}}\right)$. We prove that in these games there is a value $v_s(p)$ for $SG_\infty(p)$ $\forall p \in (0,1)$, and that $v_s(p) = O^*\left(\frac{1}{\sqrt{n}}\right)$.
- We conclude by showing in Part 7 a class of games that for all $0 < p < 1$, does *not* have a value for $SG_\infty(p)$.

4. On the variation of uniformly bounded infinite martingales

As mentioned earlier, Mertens and Zamir proved[7] that for any *infinite* uniformly bounded martingale $\mathscr{P}^\infty$, $\lim_{n\to\infty}\left\{\frac{V_n(\mathscr{P}^\infty)}{\sqrt{n}}\right\} = 0$.

Our first result states that although the n-stage variation of a uniformly bounded infinite martingale is strictly smaller than $O^*(\sqrt{n})$, it can be of $O^*(n^{1/2-\varepsilon})$ for arbitrarily small $\varepsilon > 0$.

Theorem 4.1. *For any $\varepsilon > 0$ there is a $c > 0$ and a uniformly bounded ∞-martingale $\mathscr{P}_\varepsilon^\infty = \{P_n\}_{n=1}^\infty$ satisfying*

$$V_n(\mathscr{P}_\varepsilon^\infty) \geq cn^{1/2-\varepsilon}, \quad \forall n.$$

[6] For details see Theorem 2.4 p. 255 of [5].

[7] see Theorem 2.4 p. 255 of [5].

From the discussion above it is clear that the behavior of infinite bounded martingales plays a central role in our model; therefore, it is not surprising that our result will be based on the following theorem.

Theorem 4.2. *For any $\varepsilon > 0$, $\eta > 0$ and $0 \leq l < p < u \leq 1$, there is $c > 0$ and a martingale $\mathfrak{X}_\varepsilon^\infty = \{X_n\}_{n=1}^\infty$ with $EX_1 = p$ that satisfies:*

(a) $P(l < X_n < u, \forall n) > 1 - \eta$
(b) $V_n(\mathfrak{X}_\varepsilon^\infty) \geq cn^{1/2-\varepsilon}\ \forall n.$

Furthermore:

$$P\left\{\sum_{k=1}^{n} |X_{k+1} - X_k| \geq cn^{1/2-\varepsilon},\ \forall n\right\} = 1.$$

Proof: We construct a martingale that satisfies (a) and (b).

For a given $0 < \theta < 1$, let Y_k, $k = 1, 2 \ldots,$ be i.i.d. random variables, defined by:

$$P(Y_k = \theta) = \theta' \quad \text{and} \quad P(Y_k = -\theta') = \theta$$

(where $\theta' = 1 - \theta$).

The required martingale $\mathfrak{X}_\varepsilon^\infty$ is now defined as follows:

$X_1 \equiv p$, and for all $n > 1$:

$$X_n := X_{n-1} + \frac{Y_{n_0+n}}{(n_0+n)^{1/2+\varepsilon}},$$

where $n_0 = n(\varepsilon, p, l, u)$ is a constant that we choose so that $\mathfrak{X}_\varepsilon^\infty$ satisfies (a) and (b). To prove that there is an n_0 such that $\mathfrak{X}_\varepsilon^\infty$ satisfies (a) and (b), we shall prove the following two lemmas.

Lemma 4.3. $X_\infty = \lim_{n\to\infty} X_n$ *exists and it satisfies:*

- *X_∞ is finite almost surely.*
- *$EX_\infty = p$.*
- *The convergence of X_n to X_∞ is almost uniform; that is, for all $\eta, \delta > 0$, there is an N such that*

$$P(|X_n - X_\infty| < \delta, \forall n > N) \geq 1 - \eta.$$

Proof:

$$Var(X_n) = Var\left\{\sum_{k=n_0+2}^{n_0+n} \frac{Y_k}{k^{1/2+\varepsilon}}\right\} = \sum_{k=n_0+2}^{n_0+n} \frac{\theta\theta'}{k^{1+2\varepsilon}} \leq M_\varepsilon,$$

where $M_\varepsilon = \sum_{k=1}^\infty \frac{\theta\theta'}{K^{1+2\varepsilon}}$.

Note that $E|X_n| \leq \sqrt{EX_n^2} = \sqrt{E^2X_n + Var\,X_n} = \sqrt{p^2 + Var\,X_n} \leq 1 + M_\varepsilon$.

Since $E|X_n| \leq M_\varepsilon + 1 \ \forall n$, we get that by the martingale Convergence Theorem,[8] $X_\infty = \lim_{n\to\infty} X_n$ exists and is finite, and $EX_\infty = p$.

Using Egoroff's theorem[9], we get that the convergence of X_n to X_∞ is almost uniform. □

Lemma 4.4. *For any $\delta, \eta > 0$, there is an $N = N(\delta, \eta)$, such that for any $n_0 > N$, the process $\mathfrak{X}_\varepsilon^\infty$ constructed using n_0 will satisfy*

$$P(|X_n - X_\infty| < \delta, \forall n) \geq 1 - \eta.$$

Proof: Consider the above-defined process with $n_0 = 0$; that is, for each $n > 1$

$$\tilde{X}_n = p + \sum_{k=2}^{n} \frac{Y_k}{k^{1/2+\varepsilon}} \tag{5}$$

$$\tilde{X}_\infty = p + \sum_{k=2}^{\infty} \frac{Y_k}{k^{1/2+\varepsilon}} \tag{6}$$

By Lemma 4.3 we have that for any $\delta > 0$ and $\eta > 0$, there is an $N = N(\delta, \eta)$ such that

$$P(|\tilde{X}_n - \tilde{X}_\infty| < \delta, \ \forall n > N) \geq 1 - \eta. \tag{7}$$

Now for any n_0: $X_\infty - X_n \equiv \tilde{X}_\infty - \tilde{X}_{n+n_0}$, so we have

$$\begin{aligned} P(|X_\infty - X_n| < \delta, \forall n) &= P(|\tilde{X}_\infty - \tilde{X}_{n+n_0}| < \delta, \forall n) \\ &= P(|\tilde{X}_\infty - \tilde{X}_n| < \delta, \ \forall n > n_0). \end{aligned}$$

So for any such process defined with $n_0 > N$, where N satisfies (7), we get:

$$\begin{aligned} P(|X_\infty - X_n| < \delta, \ \forall n) &= P(|\tilde{X}_\infty - \tilde{X}_n| < \delta, \ \forall n > n_0) \\ &\geq P(|\tilde{X}_\infty - \tilde{X}_n| < \delta, \ \forall n > N) \\ &\geq 1 - \eta, \end{aligned}$$

and with that we have proved lemma 4.4. □

To complete the definition of the martingale $\mathfrak{X}_\varepsilon^\infty$, we now define n_0 as follows.

Given ε, p, l and u, such that $0 \leq l < p < u \leq 1$, define $\delta^* = \min\left\{\frac{p-l}{2}, \frac{u-p}{2}\right\}$. For fixed $\eta > 0$, let n_0 be the minimal $N(\delta^*, \eta)$ that satisfies the inequality of lemma 4.4 for $\delta = \delta^*$.

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[8] see e.g. p. 244 of [7].

[9] see e.g. p. 88 of [2].

- We now prove that the martingale $\mathfrak{X}_\varepsilon^\infty$ satisfies (a):

$$\begin{aligned} P(l < X_n < u,\ \forall n) &\geq P(|X_n - p| < 2\delta^*,\ \forall n) \\ &\geq P(|X_n - X_\infty| + |X_\infty - p| < 2\delta^*,\ \forall n) \\ &\geq P(|X_n - X_\infty| < \delta^*\ \forall n,\ \text{and}\ |X_\infty - X_1| < \delta^*) \\ &= P(|X_n - X_\infty| < \delta^*,\ \forall n) \geq 1 - \eta. \end{aligned}$$

- To prove that $\mathfrak{X}_\varepsilon^\infty$ satisfies (b):
  Denote $\min(\theta, \theta')$ by α.

$$\sum_{k=1}^{n} |X_{k+1} - X_k| = \sum_{k=n_0+2}^{n_0+n+1} \frac{|Y_k|}{k^{1/2+\varepsilon}} \geq \sum_{k=n_0+2}^{n_0+n} \frac{\alpha}{k^{1/2+\varepsilon}} \geq cn^{1/2-\varepsilon}$$

  for some $c > 0$; hence

$$P\left\{\sum_{k=1}^{n} |X_{k+1} - X_k| \geq cn^{1/2-\varepsilon},\ \forall n\right\} = 1.$$

  In particular, $V_n(\mathfrak{X}_\varepsilon^\infty) \geq cn^{1/2-\varepsilon}$, $\forall n$, which concludes the proof of Theorem 4.2. □

Proof of Theorem 4.1: Define $\mathscr{P}_\varepsilon^\infty$ as follows: For any $0 \leq l < p < u \leq 1$, we defined, the constant $n_0 = n(\varepsilon, p, l, u)$ in the proof of Theorem 4.2. Since ε and p are fixed, we abbreviate this by $n_0(l, u)$. For a given $0 < \theta < 1$, let Y_k, $k = 1, 2 \ldots$, be i.i.d. random variables, defined by:

$$P(Y_k = \theta) = \theta' \quad \text{and} \quad P(Y_k = -\theta') = \theta$$

(where $\theta' = 1 - \theta$).
The required martingale $\mathscr{P}_\varepsilon^\infty$ is now defined as follows:
$P_1 \equiv p$.

- As long as $P_{n-1} \in (l, u)$; then:

$$P_n := P_{n-1} + \frac{Y_{n_0+n}}{(n_0 + n)^{1/2+\varepsilon}}.$$

- If there is N such that $P_N \notin (l, u)$, then for all $n \geq N : P_n \equiv P_N$; in other words, $\mathscr{P}_\varepsilon^\infty$ absorbs outside (l, u).

Since in (l, u) $\mathscr{P}_\varepsilon^\infty = \{P_n\}_{n=1}^\infty$ coincides with the $\mathfrak{X}_\varepsilon^\infty = \{X_n\}_{n=1}^\infty$ defined in the proof of Theorem 4.2, then:

$$P(l < P_n < u,\ \forall n) = P(l < X_n < u,\ \forall n) \geq 1 - \eta.$$

In particular, for all n: $P(l < P_m < u,\ \forall m \leq n) \geq 1 - \eta$.

And so:

$$V_n(\mathscr{P}_\varepsilon^\infty) = E\sum_{k=1}^{n}|P_{k+1} - P_k|$$

$$\geq E\left\{\sum_{k=1}^{n}|P_{k+1} - P_k| \,|\, l < P_m < u, \forall m \leq n\right\}\cdot P(l < P_m < u,\ \forall m \leq n)$$

$$\geq E\sum_{k=n_0+2}^{n_0+n}\frac{|Y_k|}{(n_0(l,u)+k)^{1/2+\varepsilon}}\cdot(1-\eta)$$

$$\geq \sum_{k=n_0+2}^{n_0+n}\frac{\alpha(1-\eta)}{(n_0(l,u)+k)^{1/2+\varepsilon}} \geq cn^{1/2-\varepsilon},$$

for some $c > 0$. □

Remark 4.5. *Theorem 4.1 can be generalized as follows (and proved in the same manner):*

Theorem 4.6. *For all positive functions $h(x)$ satisfying that $\sum_{k=1}^{\infty} h^2(k)$ is finite, there is $c > 0$ and an infinite uniformly bounded martingale $\chi_h^\infty = \{X_n\}_{n=1}^\infty$, with $EX_1 = p$ that satisfies:*

$$V_n(\chi_h^\infty) \geq c\int_0^n h(x)\,dx, \quad \forall n.$$

Example: $h(x) = \frac{1}{x^{1/2}(\ln x)^\alpha}$ for any $\alpha > \frac{1}{2}$. Note that the order of magnitude of h is strictly greater than $O^*\left(\frac{1}{x^{1/2+\varepsilon}}\right)$ for all $\varepsilon > 0$.

5. Constructing a strategy for PI with maximal variation

As mentioned earlier:

$$\inf_\tau \gamma_n(\sigma,\tau) \leq Cav\, u(p) + \frac{c}{n}V_n(\mathscr{P}_\varepsilon^\infty), \quad \forall n.$$

Therefore, a strategy for PI that will give him the highest $O^*(\inf_\tau \gamma_n(\sigma,\tau))$ must have maximal $O^*(V_n(\mathscr{P}_\varepsilon^\infty))$, where $\mathscr{P}_\varepsilon^\infty$ is the martingale of the conditional probabilities derived from σ. Note that any $0 < \theta < 1$ defines the following strategy for PI in $D(p)$: For any p Play $p(T) = \theta$ and $p(B) = \theta'$.

We concentrate on the normal games. These games have the following characteristics:

- PI has a strategy θ, which is optimal for *all* p in $D(p)$ and in $G_\infty(p)$.

- $Cav\, u(p) \equiv u(p) \equiv 0$, which implies that, from any stage n on, PI can guarantee $Cav\, u(p)$ (since $u(P_n) \equiv Cav\, u(p) \equiv 0$.) Thus he can deviate from the non-revealing optimal strategy θ while guaranteeing the asymptotic value $Cav\, u(p)$.

In order to use his extra information PI would like to deviate from θ. He would like to do so in such a way that he would gain as close as possible to $cn^{-1/2}$. In particular, he has to make sure that with positive probability, $\mathscr{P}_\varepsilon^\infty$ never absorbs to zero or one. The way to achieve that is to create a "safe zone" (l, u) inside $(0, 1)$ and to construct a strategy that yields a martingale $\mathscr{P}_\varepsilon^\infty$ which, as long as $\mathscr{P}_\varepsilon^\infty$ is inside this zone, satisfies:

$$V_n(\mathscr{P}_\varepsilon^\infty) \geq cn^{1/2-\varepsilon}, \quad \forall n$$

for some $c > 0$, (by exploiting our construction in the proof of Theorem 4.1.) Outside that zone, $\mathscr{P}_\varepsilon^\infty$ is absorbed as PI will play the non-revealing optimal strategy θ, and guarantee $Cav\, u(p) = 0$ from there on. Theorem 4.1 guarantees that with positive probability $\mathscr{P}_\varepsilon^\infty$ will always stay in the safe zone and hence achieve the maximal variation.

In Part 4, we defined the abbreviation: $n_0(l, u)$, for fixed ε and p. We now define the safe zone (l, u) as follows:

- If $\frac{1}{\sqrt{n_0(0,1)}} < p < 1 - \frac{1}{\sqrt{n_0(0,1)}}$, then $l = \frac{1}{\sqrt{n_0(0,1)}}$ and $u = 1 - \frac{1}{\sqrt{n_0(0,1)}}$.
- Otherwise $l = \frac{1}{\sqrt{n_0(0,1)+M}}$ and $u = 1 - \frac{1}{\sqrt{n_0(0,1)+M}}$, where M is the smallest integer that satisfies $p \in (l, u)$.

We also define the following sequence $\varphi(n)$ as follows.

- If $\frac{1}{\sqrt{n_0(0,1)}} < p < 1 - \frac{1}{\sqrt{n_0(0,1)}}$, then: $\varphi(n) = (n_0(l, u) + n)^{1/2+\varepsilon}$.
- Otherwise, $\varphi(n) = (n_0(l, u) + n + M)^{1/2+\varepsilon}$, where M is defined as above.

For any $\varepsilon > 0$, we will construct a strategy σ_ε for PI such that the sequence of conditional probabilities yielded by it will be the same as $\mathscr{P}_\varepsilon^\infty$ for that ε. The only information about the history that PI will use at stage n is the conditional probability P_n, and so by abuse of notation we denote $s_n^k(P_n)$ as the probability that PI will choose T at stage n, given $K = k$ and P_n.

Definition of σ_ε*:* Given $0 < \theta < 1$, for any stage $n = 1, 2 \ldots,$

- If $l < P_n < u$, then:

$$s_n^1(P_n) = \theta + \frac{\theta\theta'}{\varphi(n)P_n} \quad s_n^2(P_n) = \theta - \frac{\theta\theta'}{\varphi(n)P_n'}.$$

- Otherwise, $s_n^k(P_n) = s(P_n)$ for $k = 1, 2$, where $s(P_n)$ is an optimal strategy of PI in $D(P_n) = P_nA_1 + (1 - P_n)A_2$.

Remark 5.1. *Note that for any two intervals such that* $(a, b) \subseteq (c, d)$*, we have* $n_0(a, b) \geq n_0(c, d)$*, since by the definition of* δ^* *we get that* $\delta^*(a, b) \leq \delta^*(c, d)$ *(where* $\delta^*(x, y)$ *is* δ^* *that correspond to the case of interval* (x, y)*). Hence, any* N *that satisfies (7) for* $\delta^*(a, b)$*, satisfies (7) for* $\delta^*(c, d)$*. Since we* *defined* $n_0(c,d)$ *as the* minimal N *that satisfies (7) for* $\delta^*(c,d)$*, we get that* $n_0(a,b) \geq n_0(c,d)$.

Lemma 5.2. *The* σ_ε *defined above is well defined, meaning; For all* k, n, P_n *satisfies* $0 \leq s_n^k(P_n) \leq 1$.

Proof: First, if $l = \frac{1}{\sqrt{n_0(0,1)}}$ and $u = 1 - \frac{1}{\sqrt{n_0(0,1)}}$,

- If $P_n \notin (l,u)$, then it is obvious since $s_n^k(P_n)$ is an optimal strategy of PI in $D(P_n)$.
- If $P_n \in (l,u)$ then:

$$s_n^1(P_n) = \theta + \frac{\theta\theta'}{\varphi(n)P_n}.$$

We have to prove that $\theta + \frac{\theta\theta'}{\varphi(n)P_n} \leq 1$. This is equivalent to $\frac{\theta\theta'}{\varphi(n)P_n} \leq \theta'$, which is equivalent to $\theta \leq \varphi(n)P_n$. Now $\varphi(n) = (n_0(l,u)+n)^{1/2+\varepsilon}$, so,

$$\varphi(n)P_n \geq (n_0(l,u)+n)^{1/2+\varepsilon} l > \left(\frac{n_0(l,u)}{n_0(0,1)}\right)^{1/2}.$$

By Remark 5.1: $\frac{n_0(l,u)}{n_0(0,1)} \geq 1$, so we get:

$$\varphi(n)P_n \geq 1 > \theta.$$

In the same way we prove that $0 \leq s_n^2(P_n) \leq 1$.

Secondly, if $l = \frac{1}{\sqrt{n_0(0,1)+M}}$ $u = 1 - \frac{1}{\sqrt{n_0(0,1)+M}}$, where M is defined above, then the proof of this lemma is again straightforward. □

Lemma 5.3. *The strategy* σ_ε *yields a martingale* $\mathscr{P}_\varepsilon^\infty = \{P_n\}_{n=1}^\infty$*, which coincides with* $\mathscr{P}_\varepsilon^\infty$ *defined in the proof of Theorem 4.1.*

Proof: $\{P_n\}_{n=1}^\infty$ is a martingale that satisfies:

$$P_{n+1} = \begin{cases} \dfrac{s_n^1(P_n)}{\bar{s}_n(P_n)} P_n, & \text{if } i_n = T \\[2ex] \dfrac{s_n^{1\prime}(P_n)}{\bar{s}_n'(P_n)} P_n, & \text{if } i_n = B \end{cases} \tag{8}$$

(recall $\bar{s}_n(P_n) = P_n s_n^1(P_n) + P_n' s_n^2(P_n)$, where $P_n' = 1 - P_n$) so $P(i_n = T) = \bar{s}_n(P_n)$ and $P(i_n = B) = \bar{s}_n'(P_n)$ and so:

- If $P_n \in (l,u)$, then by definition:

$$\bar{s}_n(P_n) = P_n\left(\theta + \frac{\theta\theta'}{\varphi(n)P_n}\right) + P_n'\left(\theta - \frac{\theta\theta'}{\varphi(n)P_n'}\right) = \theta.$$

And so by (8), if $P_n \in (l, u)$, then:

$$P_{n+1} = \begin{cases} \frac{\left(\theta + \frac{\theta\theta'}{\varphi(n)P_n}\right)P_n}{\theta}, & \text{If } i_n = T \\ \frac{\left(\theta' - \frac{\theta\theta'}{\varphi(n)P_n}\right)P_n}{\theta'}, & \text{If } i_n = B. \end{cases}$$

That is: $P\left(P_{n+1} = P_n + \frac{\theta'}{\varphi(n)}\right) = P(i_n = T) = \theta$ and:

$$P\left(P_{n+1} = P_n - \frac{\theta}{\varphi(n)}\right) = P(i_n = B) = \theta'.$$

In other words, if $P_n \in (l, u)$, then:

$$P_{n+1} = P_n + \frac{Y_n}{(n_0(l,u) + n)^{1/2+\varepsilon}}.$$

- If $P_n \notin (l, u)$, then $s_n^k(P_n) = s(P_n)$, $k = 1, 2$. That is σ_ε is then non revealing, and so: $P_m = P_n \ \forall n$; i.e., P_n is absorbed outside (l, u), which concludes the proof of lemma 5.3. □

6. The speed of convergence in the normal games

We will now use σ_ε which we constructed in Section 5 for the normal games. We will show that by using it, PI guarantees maximal speed of convergence. That is, there is a $c > 0$ such that for all τ: $\gamma_n(\sigma_\varepsilon, \tau) - Cav\, u(p) \geq \frac{c}{n^{1/2+\varepsilon}}$, $\forall n$.

Theorem 6.1. *In the normal games there is a value $v_s(p)$ for $SG_\infty(p)$, for all $0 < p < 1$, and $v_s(p) = O^*\left(\frac{1}{\sqrt{n}}\right)$.*

The normal games were characterized by Mertens and Zamir,[10] who showed that a normal game has a presentation of:

$$A_1 = \begin{pmatrix} \theta' a & -\theta' a' \\ -\theta a & \theta a' \end{pmatrix} \quad A_2 = \begin{pmatrix} \theta' b & -\theta' b' \\ -\theta b & \theta b' \end{pmatrix}$$

$0 < \theta, a, b < 1$ and without loss of generality $a > b$.

We will prove the theorem by proving two lemmas:

Lemma 6.2. *In the normal games: PI can guarantee $O^*\left(\frac{1}{\sqrt{n}}\right)$ in $SG_\infty(p)$ for all $0 < p < 1$.*

Proof: For $\varepsilon > 0$, use σ_ε which was defined in Part 5.

At stage n:

- If $P_n \in (l, u)$, and PII is using (t, t'), then the payoff for this stage is:

[10] For details see Theorem 2.1 of [6].

$$\left[P_n(s_n^1(P_n), s_n^{1\prime}(P_n))\begin{pmatrix} \theta' a & -\theta' a' \\ -\theta a & \theta a' \end{pmatrix}\right.$$

$$\left. + P_n'(s_n^2(p_n), s_n^{2\prime}(P_n))\begin{pmatrix} \theta' b & -\theta' b' \\ -\theta b & \theta b' \end{pmatrix}\right]\begin{pmatrix} t \\ t' \end{pmatrix}$$

$$= \frac{\theta\theta'}{\varphi(n)}(a-b)(t+t') = \frac{\theta\theta'(a-b)}{\varphi(n)}.$$

This is true for all t, and for all $l < P_n < u$, so when σ_ε is used by PI, the payoff g_n for stage n satisfies:

$$E(g_n \mid l < P_n < u) = \frac{\theta\theta'(a-b)}{\varphi(n)}, \quad \forall t. \tag{9}$$

- Now, if $P_n \notin (l, u)$ then $s_n^k(P_n) = s(P_n) = \theta$, so:

$$E(g_n \mid P_n \notin (l, u)) = 0 \quad \forall t.$$

Hence for all n, τ:

$$\gamma_n(\sigma_\varepsilon, \tau) = \frac{1}{n}\sum_{k=1}^{n} E(g_k(\sigma_\varepsilon, \tau))$$

$$= \frac{1}{n}E\left\{\sum_{k=1}^{n}(g_k(\sigma_\varepsilon, \tau)) \mid l < P_m < u, \ \forall m \le n\right\}(1-\eta)$$

and so by equation (9):

$$\gamma_n(\sigma_\varepsilon, \tau) = \left[\frac{1}{n}\sum_{k=1}^{n}\frac{\theta\theta'(a-b)}{\varphi(k)}\right](1-\eta)$$

$$= \frac{(1-\eta)\theta\theta'(a-b)}{n}\sum_{k=1}^{n}\frac{1}{\varphi(k)} \ge \frac{c}{n^{1/2+\varepsilon}}, \quad \text{for some } c > 0.$$

Hence, by Definition 3.3, this concludes the proof of lemma 6.2. □

Lemma 6.3. *PII can guarantee* $O^*\left(\frac{1}{\sqrt{n}}\right)$ *in* $SG_\infty(p)$ *for all* $0 < p < 1$.

Proof: PII has a strategy τ_B, based on Blackwell's approachability theorem (see e.g. Aumann and Maschler p. 225 of [3]), which guarantees that in any $G_\infty(p)$ (not just in the normal games,) he need not pay more than $Cav\, u(p) + \frac{a}{\sqrt{n}}$, for some $0 < a$, for all $0 < p < 1$. □

Proof of Theorem 6.1: By the definition of $v_s(p)$, Lemmas 6.2 and 6.3 imply that there is a value $v_s(p)$ for $SG_\infty(p)$ in the normal games, and that: $v_s(p) = O^*\left(\frac{1}{\sqrt{n}}\right)\ \forall p \in (0, 1)$. □

7. A case in which the game $SG_\infty(p)$, does not have a value

Since for all p, $v_n(p)$ is monotonically decreasing function in n (see Proposition 3.19 in [8]) and $v_n(p) \geq Cav\, u(p)\ \forall n$, then $v_1(p) = Cav\, u(p)$ implies $v_n(p) = Cav\, u(p)\ \forall n$, and thus $e_n(p) = 0$, $\forall n$. Hence, in this special case PI cannot gain any benefit that exceeds $Cav\, u(p)$ by using his extra knowledge, and therefore we consider this game to be trivial.

Definition 7.1. *If $v_1(p) = Cav\, u(p)$, then we say that $G_\infty(p)$ is a* trivial game.

It is easy to see that if $G_\infty(p)$ is trivial, then $v_s(p)$ exists and $v_s(p) = O^*(\mathbf{0})$. Note that $G_\infty(0)$ and $G_\infty(1)$ are always trivial.

Theorem 7.2. *If $u(p)$ is strictly concave on $[0,1]$ and $G_\infty(p)$ is not trivial, then the game $SG_\infty(p)$ does not have a value.*

Games with strictly concave $u(p)$ represent cases in which PI prefers the situation that none of the players know which is the game played, rather than the situation that both of them *do* know. To see that, note that when none of the players know which game is being played then they play $D(p)$ and the value is $u(p)$. If both players know which game is being played, then both can play optimal in *that* game, so the value is:

$$pv_1 + p'v_2,$$

where v_1, v_2 are the values of A_1, A_2, respectively. (Note that $u(1) = v_1$ and $u(0) = v_2$.) By the strict concavity of $u(p)$, we have:

$$u(p) > pu(1) + p'u(0) = pv_1 + p'v_2, \quad \text{for all } 0 < p < 1.$$

So in such games we would expect PI to be conservative in his use of information, in order not to reveal it to PII. It turns out that in $G_\infty(p)$, PI should *never* use his information.

Lemma 7.3. *If $u(p)$ is strictly concave on $[0,1]$, then if PI guarantees f in $SG_\infty(p)$, then $f = O^*(\mathbf{0})$.*

Proof: Since $u(p)$ is strictly concave, then for all p:

$$Cav\, u(p) = u(p).$$

(1) If σ is a non-revealing (NR) strategy, that is for all n: $s_n^1(h_n) \equiv s_n^2(h_n)$ then for all n:

$$\inf_\tau \gamma_n(\sigma,\tau) \leq u(p) = Cav\, u(p).$$

Thus the NR optimal strategy in $G_\infty(p)$, consisting of playing repeatedly an optimal strategy in $D(p)$, guarantees $\mathbf{0}$ in $SG_\infty(p)$.

(2) We claim that any other strategy σ is not (even) optimal in $G_\infty(p)$. We do that by proving that there is an N and $\delta > 0$ such that $\inf_\tau \gamma_n(\sigma,\tau) < u(p) - \delta$. $\forall n > N$.

Let $\hat{n}$ be the first stage such that $s_n^1(h_n) \not\equiv s_n^2(h_n)$.
For all n:

$$\inf_\tau \gamma_n(\sigma, \tau) \le \frac{1}{n}\sum_{i=1}^{n} Eu(P_i) + \frac{c}{\sqrt{n}}$$

(see (3) and (4) in Part 3).

- For all $n \le \hat{n}$: $P_i \equiv p$.
- For $n = \hat{n}$: $P_{\hat{n}+1} \not\equiv P_{\hat{n}}$.
  So by the strict concavity of $u(p)$:

$$E(u(P_{\hat{n}+1} \mid P_{\hat{n}})) < u(E(P_{\hat{n}+1} \mid P_{\hat{n}})) = u(P_{\hat{n}}) = u(p).$$

- Let $-\delta = Eu(P_{\hat{n}+1}) - u(p)$.
  $\{u(P_n)\}$ is a *super-martingale*, since, using Jensen's inequality:

$$E(u(P_n) \mid P_{n-1}) \le u((EP_n \mid P_{n-1})) = u(P_{n-1}).$$

So for all $n > \hat{n}$: $Eu(P_n) - u(p) \le -\delta$. Hence:

$$\inf_\tau \gamma_n(\sigma, \tau) \le \frac{1}{n}\left[\sum_{i=1}^{\hat{n}} Eu(P_i) + \sum_{\hat{n}+1}^{n} Eu(P_i)\right] + \frac{c}{\sqrt{n}}$$

$$\inf_\tau \gamma_n(\sigma, \tau) \le \frac{1}{n}\left[\sum_{i=1}^{\hat{n}} u(p) + \sum_{\hat{n}+1}^{n}(u(p) - \delta))\right] + \frac{c}{\sqrt{n}}$$

$$\inf_\tau \gamma_n(\sigma, \tau) \le u(p) - \delta \cdot \frac{n - \hat{n}}{n} + \frac{c}{\sqrt{n}}.$$

Thus: $$\inf_\tau \gamma_n(\sigma, \tau) - u(p) \le -\delta\left(1 - \frac{\hat{n}}{n}\right) + \frac{c}{\sqrt{n}}.$$

For n large enough, the right side of the last inequality is strictly smaller than zero, so σ is not an optimal strategy in $G_\infty(p)$, which concludes the proof of Lemma 7.3. □

Lemma 7.4. *If $u(p)$ is concave on $[0, 1]$ and $G_\infty(p)$ is not trivial, then if PII guarantees g in $SG_\infty(p)$, then $g \ge O^*\left(\frac{1}{n}\right)$.*

Proof: If the game is not trivial, then PI can play σ_n^* in $G_n(p)$ defined as follows. For the first $n-1$ stages play for every realization of h_m, $m \le n$, optimal in $D(p)$. Thus, up to stage $(n-1)$:

$$\inf_\tau \gamma_{n-1}(\sigma_n^*, \tau) = u(p).$$

At stage n, play optimally in $G_1(p)$, to guarantee in this stage: $v_1(p) > u(p)$. Denote $v_1(p) - u(p)$, by $c(p)$. Then PI can guarantee:

$$\inf_\tau \gamma_n(\sigma_n^*, \tau) = u(p) + \frac{c(p)}{n}.$$

So, for any strategy τ_n of PII in $G_n(p)$:

$$\gamma_n(\sigma_n^*, \tau_n) \geq u(p) + \frac{c(p)}{n}.$$

Since $u(p)$ is concave, for all p: $u(p) = Cav\, u(p)$, and so:

$$\gamma_n(\sigma_n^*, \tau_n) \geq Cav\, u(p) + \frac{c(p)}{n}.$$

Now PII cannot play better in $G_\infty(p)$ than in any $G_n(p)$, (see Proposition 3.1 in part 3), which implies that if PII guarantees g in $SG_\infty(p)$, then:

$$g \geq O^*\left(\frac{1}{n}\right). \qquad \square$$

Proof of Theorem 7.2: By Lemmas 7.3 and 7.4 and by the Definition of $v_s(p)$, we get that the game $SG_\infty(p)$ does not have a value. $\square$

We conclude with an example of a game in which for all $0 < p < 1$, $v_s(p)$ does not exist and the gap between any f, g that PI and PII can respectively guarantee in $SG_\infty(p)$ is bounded away from zero by $\frac{\ln n}{n}$.

Let:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This game was presented by Aumann and Maschler and it was proved by Zamir, (see Theorem 4 of [9]) that $e(p) = O^*\left(\frac{\ln n}{n}\right)$, $\forall p \in (0,1)$. For this game $u(p) = p(1-p)$, which is a strict concave function and hence by lemma 7.3, if PI can guarantee f in $SG_\infty(p)$, then: $f = O^*(\mathbf{0})$.

On the other hand, PII cannot do better in $G_\infty(p)$ than in any $G_n(p)$; hence if PII can guarantee g in $SG_\infty(p)$ then: $g \geq O^*\left(\frac{\ln n}{n}\right)$.

References

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