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First–price auctions when the ranking of valuations is common knowledge

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Abstract. We consider a first–price auction when the ranking of bidders' private valuations is common knowledge among bidders. This new informational framework is motivated by several applications, from procurement to privatization. It induces a particular asymmetric auction model with affiliated private values that has several interesting properties but raises serious technical complications. We prove existence and uniqueness of equilibrium in pure strategies and show that the seller's revenue is generally higher in a first–price than in second–price and English auctions, in contrast to the ranking in the affiliated values model by Milgrom and Weber. This also implies that in first–price auctions, providing information concerning the ranking of valuations among bidders tends to increase the seller's expected revenue.

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1 Introduction

The theoretical auction literature has very strong results for the symmetric independent private values framework, some of which have been extended and

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modified for the symmetric affiliated values model.[1] However, in many applications the symmetry assumption is not plausible. Therefore, the literature has begun to analyze different kinds of asymmetries between bidders.

Adhering to the independent, private values framework, several authors have assumed that bidders' valuations are drawn from different probability distributions which are *common knowledge* among them. Already in the early literature, Vickrey (1961) considers auctions with two bidders where one knows the other's valuation with certainty, and Griesmer et al. (1967) analyze first-price auctions with two bidders whose valuations are uniformly distributed over different supports. More recently, Plum (1992) analyzes the two bidder case for arbitrary continuous distribution functions, proves that the first-price auction has a unique pure strategy equilibrium with strictly monotone increasing bid functions, and computes the equilibrium strategies for a class of parametric distribution functions. Using the same framework, Maskin and Riley (2000a) explain the properties of several asymmetric two bidder examples, assuming a strong stochastic order between distribution functions (stronger than first-order stochastic dominance). Maskin and Riley (2000b), Reny (1999) and Simon and Zame (2000) analyze existence of pure strategy equilibria in first-price auctions, and Lebrun (1999) analyzes existence and uniqueness of pure strategy equilibria for the n-bidder case.[2]

In this paper we consider another kind of asymmetry among bidders. We assume that bidders do not only know their own valuation, but also know the ranking of valuations. With two bidders, this means that the identity of the bidder with the highest valuation is common knowledge. This assumption induces an asymmetry among bidders that cannot be subsumed under the approach by Plum, Lebrun, and Maskin and Riley, because after knowing their ranking and their own valuation, each bidder computes conditional probabilities of other bidders' valuations that are not common knowledge among them, and indeed induces distinct results.

To motivate the kind of asymmetry assumed in this paper, consider a procurement auction in which bidders know who is the strongest bidder. This information may be due to experience accumulated in other bidding occasions or from industrial espionage. Similarly, in art auctions bidders often revise their strategies after they learn that some wealthy collector participates who is known to have a higher valuation. And in privatization or takeover bidding, participants often have access to information about each others' financial resources or other idiosyncratic features that affect bidders' valuations.

[1] For basic results of standard auction models see the surveys by McAfee and McMillan (1987), Milgrom (1989), Matthews (1995), and Wolfstetter (1996).

[2] The different proofs of existence employ very different methods. Maskin and Riley (2000b) use topological methods developed by Dasgupta and Maskin (1986). Plum (1992) and Lebrun (1999) establish directly that a solution to a suitable set of differential equations exists. Reny (1999) employs his concept of "payoff secure" games. And Simon and Zame (2000) view the tie-breaking rule as part of the solution of the game, and show that there is always some tie–breaking rule for which an equilibrium exists.

Also note that the solution of an auction game induced by our model, where the ranking of valuations is common knowledge among bidders, is an essential ingredient of the analysis of some important multi–stage bidding games. This is relevant in auction settings with several rounds of bidding before a transaction takes place. Another case are auctions in which several identical units are sold sequentially to the same set of bidders and, after each round, bidders find out who the winner was but do not observe the winning bid. For example, in Italy the formerly state owned industrial conglomerate ENI[3] was privatized using precisely such a procedure with essentially two rounds. In the first round bidders submitted sealed bids and reorganization plans. Then, the auctioneer screened out the lowest bidders, informed the remaining bidders whether their bid was the highest or not, without revealing the highest bid, and finalized the sale in a first–price auction, with the proviso that bids could not be lowered at this second stage. Clearly, in the analysis of this and similar auction games, one has to solve all subgames, including the subgames in which bidders have revealed the ranking of their valuations through their first–round bids, regardless of whether these subgames happen to occur on the equilibrium path of that game.

Although we start our model as one where valuations are independent, and the distribution of valuations is common knowledge among bidders, after having incorporated the information about ranking, the resulting conditional distributions are not common knowledge. Yet, the resulting environment can be analyzed as a game with common knowledge of the distribution on types provided that one assumes that valuations are drawn from a commonly known joint probability distribution with a triangular support (in case of two bidders), and where the higher valuation is assigned to one particular bidder (say bidder 1).

Technically speaking, knowing the ranking of valuations induces a particular stochastic dependence or affiliation of valuations, even if the original context was that of independent valuations. Indeed, it leads to an affiliated values model in the spirit of Milgrom and Weber (1982), however, without their crucial assumption of a symmetric distribution of signals. Therefore, our model can be viewed on one hand as a particular extension of the Milgrom and Weber model that considers a specific asymmetric distribution — triangular, and on the other hand as an extension of the asymmetric models of Maskin and Riley (2000a), Lebrun, and Plum to particular asymmetric values which are *not independent*.

Our main results are as follows:

– The first-price auction has a unique equilibrium in pure strategies with strictly monotone increasing bid functions.
– The first-price auction is generally inefficient (with positive probability), and the well-known ranking of auction forms based on Milgrom and Weber's (1982) symmetric affiliated values model can be reversed.
– Assuming a uniform prior distribution function, the low valuation bidder bids more aggressively than the high valuation bidder, and both bid more aggressively than in the associated symmetric game.

[3] ENI stands for *Italian Oil and Energy Corporation.* For a detailed account of the breaking–up of ENI see Caffarelli (1998).

– Assuming a uniform prior distribution function, the first-price auction is more profitable for the seller. As for the bidders, the low valuation bidder also prefers the first-price auction, whereas the high valuation bidder prefers the second-price auction. However, *ex ante*, before the ranking of valuations is known, bidders prefer the second-price auction.

Our results imply that the information structure that we address may be attractive for the auctioneer; therefore, an interesting question is whether the auctioneer can induce or exploit it strategically. This is in fact the topic of work in progress by Kaplan and Zamir (2000).

Some of these results, such as the potential inefficiency of the first-price auction and the failure of revenue equivalence, extend results obtained in the asymmetric auction framework developed by Plum, Maskin and Riley, and Lebrun to another asymmetric auction environment. However, other results, such as bidders' preference for auction formats, differ.

The plan of the paper is as follows. In Sect. 2 we present the model and explain its relationship to the affiliated values model. In Sect. 3 we prove existence and uniqueness of equilibrium in pure strategies; the proof is by construction. In general, the game does not have an analytic solution even in the case of simple distributions. In Sect. 4 we elaborate on the ranking of auction formats by the seller and by bidders, and compare our results with those obtained in the independent, asymmetric private values model by Plum, Maskin and Riley, and Lebrun. Some of the more technical arguments are spelled out in the Appendix.

2 The model

Consider a first–price auction where an indivisible good is auctioned to two risk neutral bidders. The seller's reserve price is equal to zero. Valuations v are realizations of a random variable V, independently drawn from a differentiable probability distribution function $G(v)$ with density $g := G'$ which is strictly positive on the support $[0, 1]$.

Valuations are privately observed. In addition, each bidder knows whether his valuation is the higher or lower of the two. Furthermore, the ranking of valuations is common knowledge among bidders.

Denote bidders by H and L, where H stands for the bidder with the higher and L for the bidder with lower valuation. Let b_H and b_L be the respective bid functions and $\phi := b_L^{-1}$, $\sigma := b_H^{-1}$ the associated inverse bid functions. We restrict the analysis to equilibria in pure and strictly monotone increasing strategies.

As a technical requirement, we assume that the density of G around 0 is positive and bounded away from 0.

An alternative interpretation. The present model is equivalent to the following game in which the players are treated asymmetrically: V_1 and V_2 are independently drawn (according to the distribution G), then $V_H := \max\{V_1, V_2\}$, is assigned to bidder 1 (who is called bidder H) while $V_L := \min\{V_1, V_2\}$ is assigned to bidder 2 (who is called bidder L). Equivalently, the bidders' valuations V_H, V_L are drawn from a joint probability distribution with probability density $f : \mathscr{V} \to \mathscr{R}$, with the triangular support $\mathscr{V} := \{(v_H, v_L) \in [0,1]^2 \mid v_L \leq v_H\}$, which is equal to

$$f(v_H, v_L) = 2g(v_H)g(v_L), \tag{1}$$

Of course, V_H, V_L are stochastically dependent. In fact, from (1) it follows immediately that the affiliation inequality (see Milgrom and Weber 1982)[4]

$$f(\overline{v} \vee \overline{v}')f(\overline{v} \wedge \overline{v}') \geq f(\overline{v})f(\overline{v}') \tag{2}$$

is satisfied with equality for all vectors of valuations $\overline{v}, \overline{v}' \in \mathscr{V}$. Therefore, the present model induces a particular auction game with affiliated valuations. The crucial difference to Milgrom and Weber (1982) is that the distribution f is not symmetric.

Although the distribution f is common knowledge among bidders, the mutual beliefs about each other's valuation *are not* common knowledge (as they were not also in the original interpretation, since V_H and V_L are not independent).

Equilibrium conditions. Suppose bidder H has valuation v and bids x. If the rival bidder plays the strict monotone increasing strategy b_L, the probability that H wins is

$$\begin{aligned} \Pr\{H \text{ wins}\} &= \Pr\{b_L(V) < x \mid V < v\} \\ &= \frac{\Pr\{V < \min\{\phi(x), v\}\}}{G(v)}, \end{aligned} \tag{3}$$

where $\phi = b_L^{-1}$. Therefore, the expected payoff of bidder H with valuation v if he bids x is

$$\Pi_H(x; v) = \frac{\Pr\{V < \min\{\phi(x), v\}\}}{G(v)}(v - x). \tag{4}$$

To compute the best reply of H, note that in equilibrium $\min\{\phi(x), v\} = \phi(x)$ because otherwise H could lower his bid and still win with certainty. Consequently, the best reply is obtained by solving

$$\max_x G(\phi(x))(v - x). \tag{5}$$

It is readily seen that there is a unique local maximum to this problem thus, differentiating (5) with respect to x, and using the fact that in equilibrium x is equal to $b_H(v)$ (or equivalently, $v = \sigma(x) = b_h^{-1}(x)$), one obtains the differential equation

$$\phi'(x)g(\phi(x))\,(\sigma(x) - x)) = G(\phi(x)). \tag{6}$$

Next, consider bidder L with valuation v who bids x. The probability of winning is

[4] $\overline{v} \vee \overline{v}'$ denotes the componentwise *maximum* and $\overline{v} \wedge \overline{v}'$ the componentwise *minimum* of $(\overline{v}, \overline{v}')$.

$$\begin{aligned}\Pr\{L\,\text{wins}\} &= \Pr\{b_H(V) < x \mid V > v\} \\ &= \frac{\Pr\{V > v \text{ and } \sigma(x) > V\}}{1 - G(v)}. \end{aligned} \tag{7}$$

Note that, in equilibrium, the bid x must satisfy $\sigma(x) > v$, because otherwise, in order to have a positive probability of winning, L would have to raise his bid. Therefore,

$$\Pr\{L\,\text{wins}\} = \frac{G(\sigma(x)) - G(v)}{1 - G(v)}. \tag{8}$$

Computing the best reply, as before, one obtains the differential equation

$$g(\sigma(x))\sigma'(x)(\phi(x) - x) = G(\sigma(x)) - G(\phi(x)). \tag{9}$$

Two boundary conditions apply:

$$\begin{aligned}&\sigma(0) = \phi(0) = 0 \quad \text{and} \\ &\sigma(b) = \phi(b) = 1 \quad \text{for some } b \leq 1.\end{aligned} \tag{10}$$

The first boundary condition in (10) follows from the fact that in equilibrium a bidder with $v = 0$ does not make a positive bid i.e., $b_H(0) = b_L(0) = 0$. The other boundary condition is due to the fact that in equilibrium the maximum bid, b (that of valuation $v = 1$), must be the same for both bidders, because if it differed, the bidder with the higher bid could lower it, still win the object with probability 1, and thus strictly increase his expected payoff. Consequently,

$$b_H(1) = b_L(1) = b. \tag{11}$$

3 Equilibrium

In this section we show that the game has a unique equilibrium in pure strategies. We also note that this solution is generally not an analytic solution (in the sense that the bidding functions have no expansion about 0 to power series with rational powers).

In order to prove existence and uniqueness of equilibrium in pure strategies for arbitrary probability distributions consider the following system:

$$\begin{aligned}&\phi'(x) = \frac{G(\phi(x))}{g(\phi(x))(\sigma(x) - x)} \\ &\sigma'(x) = \frac{G(\sigma(x)) - G(\phi(x))}{g(\sigma(x))(\phi(x) - x)} \\ &\exists b \in (0,1) \quad \text{such that} \quad \phi(b) = \sigma(b) = 1 \\ &\sigma(0) = \phi(0) = 0.\end{aligned} \tag{12}$$

We prove that this system, to which we refer as the *constrained system*, has a unique solution. The proof follows from a sequence of Lemmas.

The main idea of the proof is to start from an arbitrary boundary point b, as defined by the first boundary condition in (12), and move along the trajectories governed by the differential equations in (12). In what follows, we refer to this system as the *partially constrained system* (that is the system (12) without the boundary condition at 0). Note, the partially constrained system can be written in the form

$$\frac{dG(\phi(x))}{dx} = \frac{G(\phi(x))}{\sigma(x) - x} \tag{13}$$

$$\frac{dG(\sigma(x))}{dx} = \frac{G(\sigma(x)) - G(\phi(x))}{\phi(x) - x} \tag{14}$$

$$\phi(b) = \sigma(b) = 1. \tag{15}$$

By a standard property of ordinary differential equations[5], for every $b \in (0, 1)$ the partially constrained system (13)–(15) has a unique solution which we denote by ϕ_b and σ_b. We show that there is exactly one b at which the second boundary condition in (12) is also satisfied i.e., $\phi_b(0) = \sigma_b(0) = 0$.

We distinguish between two kinds of solutions to which we refer as Ω_1, Ω_2:

Definition 1. *A solution of the partially constrained system belongs to* Ω_1 *if* $\phi_b(x) > x$ *for all* $x \in (0, b]$*; it belongs to* Ω_2 *if* $\phi_b(x) = x$ *for some* $x \in [0, b)$.

Lemma 1. *Both sets* Ω_1 *and* Ω_2 *are not empty.*

Proof. The proof is in two parts: (i) We show that for b sufficiently small $(\phi_b, \sigma_b) \in \Omega_1$, while (ii) for b sufficiently close to 1, $(\phi_b, \sigma_b) \in \Omega_2$.

(i) By our assumption $\mu := \min_x g(x) > 0$. We claim that for $b = \mu/4$ we have $(\phi_b, \sigma_b) \in \Omega_1$. In fact we show that, for this value of b, both functions ϕ_b and σ_b are above the line $\ell :\ \ y = 1/2 + (2/\mu)x$ (see Fig. 1). By (12), the derivatives of the functions are bounded by

$$\phi_b'(x) \leq 1/(\sigma_b(x) - x)\mu \ \text{ and } \ \sigma_b'(x) \leq 1/(\phi_b(x) - x)\mu \ .$$

In particular $\phi_b'(b) \leq \frac{1}{(1-b)\mu}$ and $\sigma_b'(b) \leq \frac{1}{(1-b)\mu}$ which is smaller than $2/\mu$, the slope of ℓ , and hence both functions are above ℓ in the neighborhood of $b = \mu/4$ (see Fig. 1).

Assume, contrary to our claim, that the functions do not lie entirely above ℓ. Let x_0 be the largest crossing points of one of the two functions, say ϕ_b with ℓ. Note that since $\mu \leq 1$, the slope of ℓ is at least 2, so $\sigma_b(x_0) - x_0 > 1/2$ and hence $\phi_b'(x_0) < 2/\mu$ which means that $\phi_b(x)$ is below ℓ at the right of x_0 which is in contradiction to the fact that x_0 is the largest crossing point of ϕ_b.

(ii) Let $c := \min_{\frac{1}{2} \leq x \leq 1} \frac{G(x)}{g(x)}$, then, by (12), $\phi_b'(x) > c/(1-x)$ for all $b > 1/2$ and $x > 1/2$. Take $x_1 > 1/2$ such that $c/(1 - x_1) > 2$ and $b = (1 + x_1)/2$ (see Fig. 1), then

$$\phi_b(x_1) < \phi_b(b) - 2(b - x_1) = x_1,$$

[5] See e.g., Corduneanu (1971).

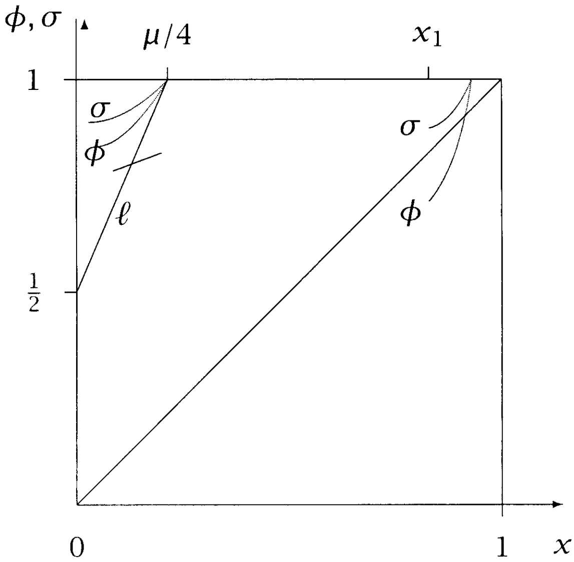

Fig. 1. The solutions of the partially constrained system for small and large values of b

and hence $(\phi_b, \sigma_b) \in \Omega_2$. □

For any $b \in (0, 1)$ define $x_b \in [0, b)$ by

$$x_b = \begin{cases} 0 & \text{if } \phi_b(x) > x \ \forall x \in [0, b) \\ \max\{x \mid \phi_b(x) = x\} & \text{otherwise.} \end{cases}$$

Lemma 2. *$\sigma_b(x) > \phi_b(x)$, for all $b \in (0, 1)$ and for all $x \in (x_b, b)$.*

Proof. By the two differential equations and the first boundary condition in (12), $\phi_b'(b) > 0$ and $\sigma_b'(b) = 0$. Therefore, $\sigma_b(x) > \phi_b(x)$ at least in a neighborhood of b. If our claim is false then let $\bar{x}$ be the largest x in (x_b, b) such that $\sigma_b(x) = \phi_b(x)$. Then again $\phi_b'(\bar{x}) > 0$ (since $\phi_b(x) > x$ in (x_b, b) by definition of x_b), and $\sigma_b'(\bar{x}) = 0$ and hence, $\sigma_b(\bar{x}+\epsilon) < \phi_b(\bar{x}+\epsilon)$ for sufficiently small ϵ, in contradiction to the definition of $\bar{x}$. □

For $b \in (0, 1)$ let $I_b = \{x \in (x_b, b] | \phi_b(x) \geq x\}$.

Lemma 3. *If $b' > b$, then $\phi_b(x) > \phi_{b'}(x)$ and $\sigma_b(x) > \sigma_{b'}(x)$, for all x in $I_b \cap I_{b'}$.*

Proof. By the differential equations, $\sigma_b(x)$, $\phi_b(x)$ are strictly increasing in I_b so $\phi_b(b) = 1 = \phi_{b'}(b') > \phi_{b'}(b)$, and similarly $\sigma_b(b) = 1 = \sigma_{b'}(b') > \sigma_{b'}(b)$. Hence, the assertion is valid in some (left) neighborhood of b. Assume, it does not hold everywhere in $I_b \cap I_{b'}$. Then, there exists a largest (closest to b) crossing point z of either the ϕ functions or of the σ functions. If this last crossing is of the ϕ functions only, we have $\phi_b(z) = \phi_{b'}(z)$ and $\sigma_b(z) > \sigma_{b'}(z)$ and therefore,

$$\phi_b'(z) = \frac{G(\phi_b(z))}{g(\phi_b(z))(\sigma_b(z) - z)} < \frac{G(\phi_{b'}(z))}{g(\phi_{b'}(z))(\sigma_{b'}(z) - z)} = \phi_{b'}'(z), \tag{16}$$

implying that in the right neighborhood of z we have $\phi_b(x) < \phi_{b'}(x)$, in contradiction to the fact that z is the largest crossing point. Similarly, we can rule out the possibility that $\sigma_b(z) = \sigma_{b'}(z)$, and $\phi_b(z) > \phi_{b'}(z)$. Finally, we rule out the possibility that z is the crossing point both of the ϕ functions and of the σ functions i.e., $\sigma_b(z) = \sigma_{b'}(z)$ and $\phi_b(z) = \phi_{b'}(z)$. In fact this means that the two differential equations in (12), with the boundary condition at z, have two distinct solutions in $[z, b]$ namely (ϕ_b, σ_b) and $(\phi_{b'}, \sigma_{b'})$, violating the uniqueness of the solution to such a system (guaranteed by standard results). □

Lemma 4. *There exists a unique b^* that induces a solution of the partially constrained system which belongs to both sets Ω_1 and Ω_2. This solution satisfies $\phi_{b^*}(0) = 0$ and $\phi_{b^*}(x) > x$ $\forall x \in (0, b^*]$.*

Proof. For convenience we shall write $b \in \Omega_i$ for $(\phi_b, \sigma_b) \in \Omega_i$. The monotonicity properties established in Lemmas 2 and 3 imply that if $b \in \Omega_1$ and $b' < b$, then also $b' \in \Omega_1$. Similarly, if $b \in \Omega_2$, and $b' > b$, then also $b' \in \Omega_2$. Let $b^* := \sup\{b \mid b \in \Omega_1\}$. We claim that this is the desired b^*. In fact if $b^* \in \Omega_1 \cap \Omega_2$, then $\phi_{b^*}(0) = 0$ and $\phi_{b^*}(x) > x$ for all $x > 0$. We have thus to rule out the possibility $b^* \notin \Omega_1 \cap \Omega_2$ which (since $\Omega_1 \cup \Omega_2 = [0, 1]$) consists of two cases:

Case 1. $b^* \in \Omega_1$ and $b^* \notin \Omega_2$. Then, $\phi_{b^*}(x) > x$ for all $x \in (0, b^*]$, and hence (by continuity) $\phi_{b^*}(0) > 0$. Since the solution $(\phi_{b^*}, \sigma_{b^*})$ is C^∞, a known result about stability of smooth solutions to ordinary differential equations with respect to changes in initial conditions, implies the existence of nearby solutions (see Corduneanu (1971), Theorem 3.4). In the present case this implies that there exists a $b > b^*$ and yet $b \in \Omega_1$, in contradiction to the definition b^*.

Case 2. $b^* \in \Omega_2$, $b^* \notin \Omega_1$. Then, by definition, there exists $0 < x_{b^*} < b^*$ where $\phi_{b^*}(x_{b^*}) = x_{b^*}$ (recall that, by its definition, x_{b^*} is the largest value in $(0, b^*)$ satisfying this equality). Hence by Lemma 2, $\sigma_{b^*}(x_{b^*}) \geq x_{b^*}$ since the opposite inequality would imply (by continuity) $\sigma_{b^*}(x_{b^*} + \epsilon) < \phi(x_{b^*} + \epsilon)$, in contradiction to Lemma 2. We now show that this inequality leads to a contradiction:

(i) If $\sigma_{b^*}(x_{b^*}) > x_{b^*}$ then (by the differential equation) $\phi_{b^*}'(x_{b^*})$ is finite, and therefore $\phi_{b^*}(x)$ is C^1 on $[x_{b^*}, b^*]$, and for x in the neighborhood $[x_{b^*}, x_{b^*} + \delta]$, it can be expressed in the form $\phi_{b^*}(x) = x_{b^*} + \gamma(\tilde{x} - x_{b^*})$, where $\tilde{x} \in [x_{b^*}, x]$. But then it follows from the second differential equation in (12) that the singularity of $\sigma_{b^*}(x)$ at x_{b^*} is not integrable, and therefore σ_{b^*} cannot attain a finite value at x_{b^*}, a contradiction.

(ii) The only case that remains to be considered is $\sigma_{b^*}(x_{b^*}) = \phi_{b^*}(x_{b^*}) = x_{b^*}$ for some fixed $x_{b^*} > 0$. We prove that $(\phi_{b^*}, \sigma_{b^*})$ are C^0 in $[x_{b^*}, b^*]$: In fact these functions are smooth in $(x_{b^*}, b^*]$, monotone increasing and bounded from below by x_{b^*} in $[x_{b^*}, b^*]$. Therefore, for any monotone sequence $x_n \downarrow x_{b^*}$ we have also the convergence: $\phi_{b^*}(x_n) \to \phi_*$ and $\sigma_{b^*}(x_n) \to \sigma_*$, with $x_{b^*} \leq \phi_* \leq \sigma_*$. If $\phi_* > x_{b^*}$, then we could pass to the limit $x \downarrow x_{b^*}$ in the differential equations (12) and thus, obtain a solution starting at b^* which is different from $(\phi_{b^*}, \sigma_{b^*})$, in contradiction to the uniqueness theorem. The case $\phi_* = x_{b^*} < \sigma_*$ is ruled out by the same argument used in (Case 1), and we conclude that $x_{b^*} = \phi_* = \sigma_*$, establishing the continuity.

The continuity implies that for a fixed small $\epsilon > 0$ (to be chosen later), there is a small $\eta(\epsilon) > 0$ such that $\phi_{b^*}(x_{b^*}+\epsilon)-x_{b^*} < \eta(\epsilon)$, and $\sigma_{b^*}(x_{b^*}+\epsilon)-x_{b^*} < \eta(\epsilon)$ (and hence also $\phi_{b^*}(x_{b^*}+\epsilon)-(x_{b^*}+\epsilon) < \eta(\epsilon)$, and $\sigma_{b^*}(x_{b^*}+\epsilon)-(x_{b^*}+\epsilon) < \eta(\epsilon)$), with $\eta(\epsilon) \to 0$ as $\epsilon \to 0$.

Consider now the partially constrained system on $[x_{b^*} + \epsilon, b^*]$. Since the solution is smooth there, we can find $b < b^*$ (and very close to it), so that $\phi_b(x_{b^*} + \epsilon) - x_{b^*} < \delta(\epsilon)$ and $\sigma_b(x_{b^*} + \epsilon) - x_{b^*} < \delta(\epsilon)$, with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. But then $\phi'_b(x_{b^*} + \epsilon) > c/\delta(\epsilon)$ (for some $c > 0$), and since σ_b decreases as we move to the left of $x_{b^*} + \epsilon$, the inequality $\phi'_b(x) > c/\delta(\epsilon)$ still holds in some small (but ϵ – independent) interval to the left of x_{b^*}. Since $\phi_b(x_{b^*})$ is $\eta(\epsilon)$–close to the diagonal $y = x$, and its derivative is $\delta^{-1}(\epsilon)$ large, it must cross the diagonal at some point $0 < \bar{x} < x_{b^*}$ and thus $b \notin \Omega_1$, contradicting the definition $b^* := \sup\{b \mid b \in \Omega_1\}$. □

Combining the above Lemmas we prove that:

Theorem 1. *The auction game has a unique pure strategy equilibrium.*

Proof. We need to show that the full system in (12) has a unique solution. By Lemma 4 we know already that $(\phi_{b^*}, \sigma_{b^*})$ solve the partially constrained system and satisfies $\phi_{b^*}(0) = 0$. It remains to be shown that $\sigma_{b^*}(0) = 0$. Recall that $\sigma_{b^*}(x) > x$ for $x > 0$. Assume $\sigma_{b^*}(0) > 0$. Then, by the first differential equation in (12), $\phi'_{b^*}(0) = 0$ (because $G(0) = 0$ and the denominator is positive), and therefore $\phi_{b^*}(x) < x$ in some small interval $[0, \epsilon)$, in contradiction with Lemma 4. Therefore, $(\phi_{b^*}, \sigma_{b^*})$ satisfy also the second boundary condition in (12) as well. Uniqueness follows from the uniqueness of b^*. □

We now point out that the model does not have a closed–form solution even if the distribution of valuations is uniform.

Assume G is uniform on $[0, 1]$. Rewriting (6), (9) and (10), we obtain

$$
\begin{aligned}
\phi'(x) &= \frac{\phi(x)}{\sigma(x) - x} \\
\sigma'(x) &= \frac{\sigma(x) - \phi(x)}{\phi(x) - x} \\
\phi(0) &= \sigma(0) = 0 \\
\phi(b) &= \sigma(b) = 1 \quad \text{for some} \quad b \leq 1.
\end{aligned}
\tag{17}
$$

Proposition 1. *System (17) does not have an analytic solution.*

Proof. The proof is in the Appendix. □

4 Equilibrium properties

How does the introduction of common knowledge concerning the ranking of valuations among bidders affect equilibrium bid functions? Does the bidder with the lower valuation always bid more aggressively? Does it affect the efficiency of the first–price auction? And how does it change the ranking of the seller's expected revenue and bidders' expected payoff in first– and second–price auctions? In the following we draw comparisons to the symmetric independent and the symmetric affiliated private values models, and also to the models of asymmetric auctions by Plum (1992), Lebrun (1999), and Maskin and Riley (2000a).

4.1 Inefficiency of the first–price auction

Although Lemma 2 was established as part of the proof of Theorem 1, it is of independent interest. It shows that the low valuation bidder bids higher than his rival:

$$b_L(v) > b_H(v) \quad \forall v \in (0,1).$$

The straightforward intuitive explanation is that otherwise the low valuation bidder would stand no chance of winning, which cannot be part of an equilibrium. An important consequence of this relationship between the bid functions is:

Corollary 1. *If the ranking of valuations is common knowledge, the object is awarded to the bidder with the lower valuation with positive probability; therefore, the first–price auction is inefficient.*

Proof. For any $x \in (0,1)$ we have $\sigma(x) > \phi(x)$. Using the monotonicity of these functions, this implies that any Low bidder with valuation higher than $\phi(x)$ bids more than x while any High bidder with valuation smaller than $\sigma(x)$ bids less than x. It follows that whenever both valuations are in the interval $(\phi(x), \sigma(x))$, which occurs with positive probability, the low valuation bidder wins the auction and gets the object. □

The inefficiency is due to the fact that the two bid functions are apart, and hence the distance between σ and ϕ is related to the "degree" of inefficiency. For example, if we measure the inefficiency by the probability that bidder L, with the lower valuation, gets the object then it can be easily verified that:

$$\Pr\{\text{the object is awarded to } L\} = 2\int_0^b \left(G(\sigma(x)) - G(\phi(x))\right)\sigma'(x)g(\sigma(x))\,dx\,.$$

For the uniform distribution, this measure simplifies to

$$2\int_0^b (\sigma(x) - \phi(x))\,\sigma'(x)\,dx\,.$$

As it has been already noticed in the auction literature, the inefficiency of the first-price auction is closely related to the asymmetry of bidders, and it is of course also a feature of the asymmetric auction model by Plum (1992), Lebrun (1999) and Maskin and Riley (2000a).

4.2 Strategy comparison

Does the present model result in more aggressive bidding by one or both bidders, relative to the standard symmetric independent private value model? Proposition 2 shows that both bidders *may* bid more aggressively.

Proposition 2. *If valuations are drawn from a uniform distribution, then, the bid functions of the high and the low valuation bidders are above the equilibrium bid function $b(v)$ in the symmetric independent private values case.*

$$b(v) < b_H(v) < b_L(v), \quad \forall v \in (0, 1) \tag{18}$$

Proof. See Appendix. □

The equilibrium bid functions for the uniform distribution case are given in Fig. 2 where we see also the bid function of the symmetric model $b(v) = v/2$; These functions are from a numerical solution of the differential equations (17).

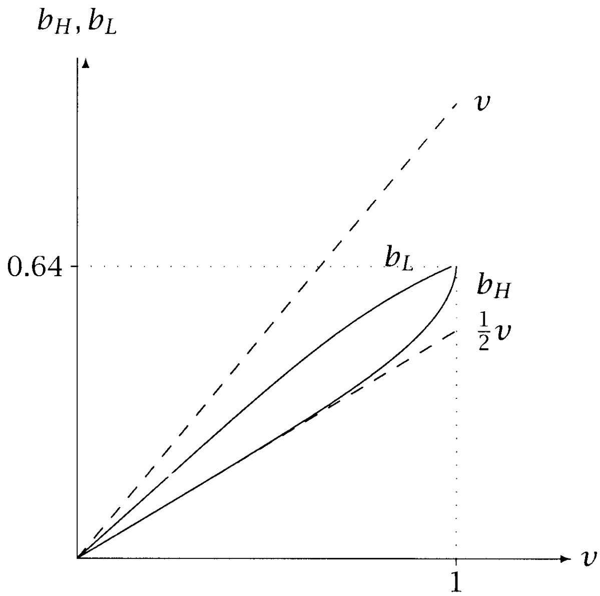

Fig. 2. Equilibrium bid functions for the uniform distribution case

The numerical integration of this system requires some care because of the singularity at the origin. We used an asymptotic expansion given by (19), (20), (23) and (24) in Appendix A to compute the value of ϕ and σ at $dx > 0$ for an arbitrary choice of the free parameter α. We then divided $[dx, 1]$ into small subintervals of size dx, and integrated the partially constrained system obtained from (17) by ignoring the second boundary condition, by a finite difference scheme. We started with a small positive α and repeated the procedure outlined above with increasing values of this parameter, until we hit a numerical solution that also satisfies the second boundary condition in (17). To ensure that the solution obtained is not a numerical artifact, we performed similar calculations with different choices of dx. The solution is plotted in Fig. 2.

It is natural to ask about the generality of the qualitative properties of the bidding functions stated in Proposition 2. Note first that by Lemma 2, the inequality $b_H(v) < b_L(v)$ holds for any distribution G. In ongoing work, Philippe Fevrier proved that the inequalities $b(v) < b_L(v)$ and $b_H(v) < b_L(v)$ hold for any distribution and any number of bidders if the identity of the highest valuation is common knowledge among bidders. That is, informing a bidder that he or she does not have the highest valuation results in more aggressive bidding compared to the standard model and more aggressive than the bidder informed to have the highest valuation. However, the inequality $b(v) < b_H(v)$, does not always hold. Indeed, there are distributions for which in equilibrium $b_H(v) < b(v)$ for some values of v. Such examples were generated numerically and an existence proof is given in the Appendix under the subtitle “counterexample”.

We mention that the inequalities $b_L(v) > b_H(v), \forall v$, are also observed in Maskin and Riley (2000a), for their notion of strong (our H) and weak (our L) bidder (see their Proposition 3.5). However, in their framework it cannot occur that the strong bidder bids also more aggressively than in the associated symmetric game, as we have it in the uniform distribution case.

4.3 Revenue ranking

As is well–known, in the symmetric independent private values model first– and second–price auctions are payoff equivalent. However, if independence is replaced by affiliation, as proposed by Milgrom and Weber (1982), the seller’s expected revenue is higher in a second–price auction. In the present model, this revenue ranking can be reversed.[6] This may contribute to explain why procurements are usually conducted as first–price auctions.

Proposition 3. *If valuations are drawn from a uniform distribution, the seller’s expected revenue is higher in a first–price auction than in a second–price (or open English) auction; the low valuation bidder (L) also prefers the first-price auction, whereas the high valuation bidder (H) prefers the second price auction. However,* ex ante*, before the ranking of valuations is known, bidders unanimously prefer the second-price auction.*

Proof. Recall the ranking of bid functions from Proposition 2. It follows immediately that the winning bid is always higher than in the standard symmetric case, for each configuration of valuations, even though the item is not always awarded to the bidder with the highest valuation. Therefore, the seller’s revenue is also higher than in the symmetric independent private values model, for each configuration of valuations. Since in a second–price auction bidding is unaffected by

[6] Milgrom and Weber show that the open English auction is even better for the seller. Although with two bidders, the open English and the second–price auction are equivalent, the actual reversed ordering in fact occurs as our result for the uniform distribution holds for any number of bidders n when it is common knowledge who has the highest valuation.

the assumed common knowledge, it follows that a first–price auction generates higher revenue to the seller.

Bidder L prefers the first-price auction, because in the second-price auction he stands no chance of winning the auction. In turn, bidder H prefers the second-price auction, because compared to the first-price auction he wins more often (in fact he always wins in the second-price auction) and consistently pays less in expectation, for each of his valuations.[7] However, *ex ante*, before the ranking of valuation is known, bidders unambiguously prefer the second-price auction because the seller's expected revenue is higher and the entire expected surplus is lower, due to inefficiency. □

The result that the seller prefers the first-price auction to the second-price auction was obtained also in the models by Plum (1992), Lebrun (1999) and Maskin and Riley (2000a). However, bidders' ranking of auction formats differs from what was obtained in our model. Indeed, assuming uniform distributions, both bidders may prefer the second-price auction in their asymmetric auctions framework. At first glance this seems to contradict Proposition 3.6 in Maskin and Riley (2000a) where it is claimed that bidders rank auctions in the same way as in our model. Notice, however, that uniform distributions do not satisfy the "conditional stochastic dominance condition" which is assumed in that Proposition, even though uniform distributions are a perfect example of first-oder stochastic dominance. Using Plum's (1992) explicit solution of equilibrium strategies, one can compute bidders' equilibrium payoffs and confirm[8] that both weak and strong bidders may prefer the second-price auction.[9]

We have also computed the equilibrium expected revenues for a variety of other probability distributions and always found that making the ranking known among bidders raises the seller's expected revenue and that, if the ranking of valuations is common knowledge among bidders, the first–price auction gives the seller a higher expected revenue than the second–price or English auction. We therefore conjecture that these two properties, derived for the case of the uniform distribution, are valid for a large class of probability distributions. Since the present model can be viewed as an *asymmetric* affiliated private values auction, this implies that the superiority of the second–price or English auction derived by Milgrom and Weber (1982) does not survive the introduction of asymmetry to the affiliated private values model.

[7] Another argument proving the preference of H is: As the expected sellers' payoff is higher in the first-price auction than in the second-price auction, the sum of the two bidders payoffs is lower in the first-price auction. And since the payoff of L is higher in the first-price auction, it must be that the expected payoff of H is lower in the first-price auction than in the second-price auction.

[8] An example are uniformly distributed valuations on the support $[50, 150]$, resp. $[50, 200]$.

[9] In footnote 16 on p. 425 of their paper, Maskin and Riley (2000a) write: "Under the weaker assumption of first-order stochastic dominance, it can be shown that the ranking by buyers continues to hold for all those buyers with sufficiently high valuations." This is, evidently, incorrect without additional assumptions on the distributions .

5 Conclusions

The present paper has modified the standard symmetric private value models, with and without affiliation, by assuming that bidders know the rank order of their valuations. This modification is relevant in many real–world auctions. We presented a constructive proof of existence and uniqueness of equilibrium in pure strategies for the first–price sealed bid auction with two bidders. In this equilibrium the low valuation bidder always bids higher (than a high valuation bidder with the same value). Consequently, the first–price auction is not efficient since it occurs with positive probability that the object is sold to the bidder with the lower valuation. In the case of uniformly distributed valuations, we showed that *both* bidders bid higher than in the standard symmetric case (i.e. without the common knowledge of ranking). Therefore, the seller's revenue is higher for all configurations of valuations. Noting that bidding the true value is a dominant strategy in second-price auction also when the ranking is common knowledge, this indicates that a revenue maximizer seller should prefer the first–price to the second–price auction. This reverses the well–known revenue ranking in private values auctions with and without affiliation.

Appendix

Proof of Proposition 1. Evidently, $\phi(x) = \frac{4}{3}x$ and $\sigma(x) = 2x$ is an analytic solution of the two differential equations in (17) which satisfy the first boundary condition but violates the second one. We now show that this is the only analytic solution of the system in (17) without the second boundary condition.

Since both differential equations in (17) are singular at $x = 0$, to obtain $\sigma'(0)$ and $\phi'(0)$ we apply L'Hôpital's rule which gives $\sigma'(0) = 2$ and $\phi'(0) = \frac{4}{3}$. Since every solution around 0 can be expressed as an asymptotic expansion in x, the power series expansions of $\phi(x)$ and $\sigma(x)$ around 0 are

$$\phi(x) = \frac{4}{3}x + \alpha x^k + \ldots \quad (19)$$

$$\sigma(x) = 2x + \beta x^r + \ldots . \quad (20)$$

where $k, r > 1$ are the first powers whose coefficients are non–zero.

Substitute (19) and (20) into the first differential equation in (17), rearrange terms, and one obtains

$$\frac{4}{3}x + \alpha k x^k + \frac{4}{3}\beta x^r + \beta\alpha k x^{k+r-1} + \ldots = \frac{4}{3}x + \alpha x^k + \ldots . \quad (21)$$

If $r > k > 1$ then $\alpha = 0$, which is a contradiction. If $k > r > 1$ then $\beta = 0$, again a contradiction. Therefore, $k = r$ which in turn implies

$$\alpha(k-1) + \frac{4}{3}\beta = 0. \quad (22)$$

Applying similar reasoning to (19), (20) and the second differential equation in (17), one obtains

$$(\frac{k}{3} - 1)\beta + 3\alpha = 0. \tag{23}$$

These two linear equations in α, β have a non–trivial solution if and only if the determinant of coefficients vanishes, which implies $4 - (k-1)(\frac{k}{3} - 1) = 0$, i.e.

$$k = 2 + \sqrt{13}. \tag{24}$$

This proves the Lemma since for a solution to be analytic, the exponents must be integers. □

Proof of Proposition 2. The proof of the proposition is obtained as part of a proof of existence of a unique solution of constrained system (17). As a by–product we thus have two quite different proofs of existence and uniqueness for the uniform distribution case.

In general, a system like (17) has a solution if the corresponding partially constrained system (i.e., if the second boundary condition is ignored) has infinitely many solutions. We will show that this is indeed the case.

Restricting attention to the partially constrained system in (17) we observe the following:

(i) $\sigma_2 = \frac{4}{3}x$, $\phi_2 = 2x$ is a solution.
(ii) every solution must satisfy $\sigma'(0) = 2$, $\phi'(0) = \frac{4}{3}$ (using L'Hôpital's rule and the differential equations).
(iii) the asymptotic behavior of every solution, near $x = 0$, is described by the following power series expansions which are obtained by inserting k from (24) into (19), (20)

$$\phi(x) = \frac{4}{3}x + \alpha x^{2+\sqrt{13}} + \ldots \tag{25}$$

$$\sigma(x) = 2x + \beta x^{2+\sqrt{13}} + \ldots . \tag{26}$$

Inserting $k = 2 + \sqrt{13}$ into (22) and (23) yields two linearly dependent equations and therefore they have one parameter family of solutions:

$$\beta(\alpha) = -\frac{3\alpha(\sqrt{13}+1)}{4}, \quad \text{for all } \alpha \neq 0. \tag{27}$$

Define the functions $q(x) := \phi(x)/x$ and $h(x) := \sigma(x)/x$. Rewriting the two first differential equation in (17) in terms of $q(x)$ and $h(x)$ gives

$$xq'(x) = \frac{q(x)}{h(x) - 1} - q(x) \tag{28}$$

$$xh'(x) = \frac{h(x) - q(x)}{q(x) - 1} - h(x) \tag{29}$$

and dividing (28) by (29) yields

$$\frac{dq}{dh} = \frac{q-1}{h-1}\ \frac{q(2-h)}{h(2-q)-q}. \tag{30}$$

Note that (28) and (29) are invariant to changes in the scale of x, i.e. if (h,q) is a solution, so is $(h_c, q_c) = (h(cx), q(cx))$, for all $c > 0$. If we interpret the variable x as 'time' then c is the 'speed' of motion along the trajectory of the solution in the (h,q) plane. We shall use this phase–plane to show that there exist infinitely many solutions to (17) if the second boundary condition is ignored, and then show that one and only one of them satisfies the second boundary condition in (17).

Although the following argument is self contained, it is based on methods explained in Boyce and DiPrima (1992, Ch. 9) where more detailed discussion and examples can be found.

First we make the dynamic system (28) and (29) an autonomous system by changing variable from x to t; letting $x = e^t$, $\tilde{q}(t) := q(e^t)$ and $\tilde{h}(t) := h(e^t)$ we obtain

$$\tilde{q}'(t) = \frac{\tilde{q}(t)}{\tilde{h}(t)-1} - \tilde{q}(t) \tag{31}$$

$$\tilde{h}'(t) = \frac{\tilde{h}(t)-\tilde{q}(t)}{\tilde{q}(t)-1} - \tilde{h}(t) \tag{32}$$

Since dividing these two equations yields the same differential equation (30), the two dynamic systems $\{(28), (29)\}$ and $\{(31), (32)\}$ have *the same trajectories* in the phase plane (q,h) or $(\tilde{q},\tilde{h})$; they are governed by the differential equation. This phase diagram is given in Fig. 3.

The point $\omega := (2, \frac{4}{3})$ corresponds to the first boundary condition in (17); at $x = 0$ in the system $\{(28), (29)\}$ or $t = -\infty$ in the system $\{(31), (32)\}$. It is a critical point of both systems which means that at this point $(\tilde{h}', \tilde{q}')(-\infty) = (0,0)$ and $(h', q')(0) = (0,0)$. The local behavior of (30) can be studied by first finding the possible directions of the trajectories emanating from the critical point ω in the phase space. To do this we consider the linear approximation to the dynamic system around ω: Let

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} := \begin{pmatrix} q - \frac{4}{3} \\ h - 2 \end{pmatrix} = \begin{pmatrix} \tilde{q} - \frac{4}{3} \\ \tilde{h} - 2 \end{pmatrix},$$

be the vector of distance from ω, then the linear approximation of the dynamic system ($\{(28), (29)\}$) or $\{(31), (32)\}$) yields:

$$\begin{pmatrix} \delta_1' \\ \delta_2' \end{pmatrix} = \begin{pmatrix} 0 & -\frac{4}{3} \\ -9 & 2 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + O(\delta^2).$$

The matrix of coefficients in this equation has two eigenvalues:

1. $r_1 = (1 - \sqrt{13}) \approx -2.6$ which corresponds to the eigenvector $\xi_1 = \begin{pmatrix} 1+\sqrt{13} \\ 9 \end{pmatrix}$.

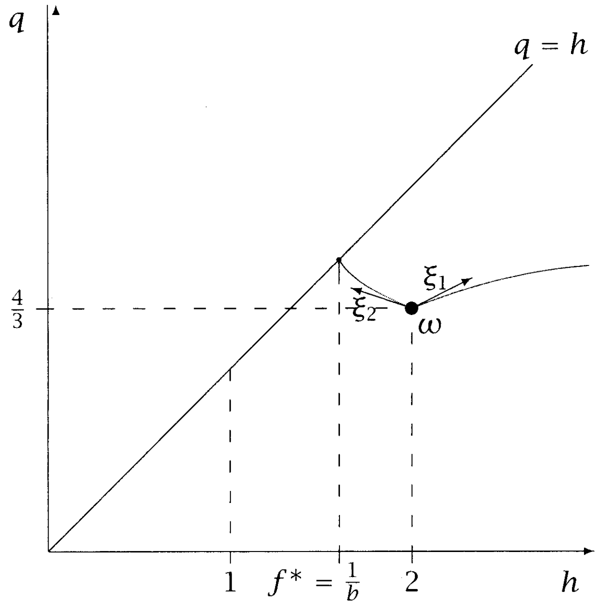

Fig. 3. Trajectories in the phase plane

2. $r_2 = (1+\sqrt{13}) \approx 4.6$ which corresponds to the eigenvector $\xi_2 = \begin{pmatrix} 1-\sqrt{13} \\ 9 \end{pmatrix}$.

The two directions ξ_1 and ξ_2 are indicated in Fig. 3. The trajectory at the direction of ξ_1 is irrelevant to our dynamic system: Since $r_1 < 0$, a solution of the form $\delta = \xi_1 \exp(r_1 t)$ does not satisfy $\delta \to 0$ as $t \to -\infty$ (which is the first boundary condition at $x = 0$). On the other hand, the trajectory starting at ω in the direction ξ_2 is the "unstable manifold" of the system, that is, this is the solution that leaves the critical point ω as t increases from $-\infty$ (or as x increases from 0). This corresponds to a negative value of β in (20) (see also (27)). The slope of this trajectory cannot become positive before it hits the line $q = h$, because a reversal requires that it becomes zero somewhere, which cannot occur as long as $h < 2$ (see Fig. 3). Therefore, the trajectory must cross the line $q = h$ at some point that we denote by f^*. While there is a unique trajectory of this sort, it corresponds to infinitely many solutions of the system $\{(28), (29)\}$ (or to the two differential equations in (17)) that differ in their "speed" c. To choose the "right" speed we choose c for which

$$q_c(\frac{1}{f^*}) = h_c(\frac{1}{f^*}) = f^*. \tag{33}$$

Since other solutions move either faster or slower along the same trajectory, there is a unique solution that satisfies (33). Taking now $b^* = 1/f^*$ we have:

$$\phi_{b^*}(b^*) = b^* h_c(\frac{1}{f^*}) = \frac{1}{f^*} \cdot f^* = 1$$

and

$$\sigma_{b^*}(b^*) = b^* q_c(\frac{1}{f^*}) = \frac{1}{f^*} \cdot f^* = 1$$

Thus, we establish a unique solution to the full system (17) with $b^* \in (\frac{1}{2}, 1)$ (since $f^* \in (1, 2)$).

As we proved, along the equilibrium trajectory the function h is always smaller than 2 which implies that $h(x) = \frac{\sigma(x)}{x} < 2$, and therefore $\sigma(x) < 2x$. and by Lemma 2, $\phi(x) < \sigma(x) < 2x$. Recalling that $2x$ is the inverse of the symmetric case bidding function $b(v) = v/2$, we conclude that $b(v) < b_H(v) < b_L(v)$ □

Counterexample. Consider the constrained system (12) with the class of distributions for which we proved the existence and uniqueness of solution namely, distributions G which have a Taylor expansion around 0:

$$G(x) = \alpha x + \beta x^2 + \dots,$$

with $\alpha > 0$. This implies that the density q has the expansion (around 0):

$$q(x) = \alpha + 2\beta x + \dots.$$

Recall that in the symmetric case, the inverse bidding function $\rho := b^{-1}$ is determined by the following differential equation (which can be easily derived directly),

$$\rho'(x) = \frac{G(\rho(x))}{q(\rho(x))(\rho(x) - x)} \quad \text{with the boundary condition} \quad \rho(0) = 0. \tag{34}$$

First it can be shown although, the inverse bidding functions may be not analytic at 0, they do have the first two terms of Taylor expansion (linear and quadratic; non rational powers must be higher than 2). Using the Taylor expansion for G in equation (34), we obtain the following asymptotic expansion of $\rho(x)$ near $x = 0$,

$$\rho(x) = 2x + Cx^2 + o(x^2) \quad \text{with} \quad C = -\frac{4\beta}{3\alpha}.$$

On the other hand carrying out an asymptotic expansion for $\phi(x)$ and $\sigma(x)$ (near $x = 0$), solving the differential equations of the system (12), we obtain that both in $\phi(x)$ and $\sigma(x)$ the second term (after the linear term) is quadratic (again, non–rational powers must be higher than 2):

$$\begin{aligned} \phi(x) &= \frac{4}{3}x + Ax^2 + o(x^2) \\ \sigma(x) &= 2x + Bx^2 + o(x^2) \end{aligned}$$

with

$$A + \frac{4}{3}B = -\frac{16\beta}{9\alpha}$$
$$3A - \frac{1}{3}B = -\frac{4\beta}{9\alpha}.$$

This implies

$$B = -\frac{44\beta}{39\alpha},$$

and therefore for $\beta > 0$ we have $B > C$, and hence $\sigma(x) > \rho(x)$ at least for some interval near $x = 0$ that is, $b_H(v) < b(v)$ at least in some interval of valuations near $v = 0$. □

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