A Duality Theorem on a Pair of Simultaneous Functional Equations

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Abstract

Given P and Q convex compact sets in R^k and R^s, respectively, and u a continuous real valued function on $P \times Q$, we consider the following pair of dual problems: PROBLEM I – Minimize f so that f: $P \times Q$->R and f >= $Cav_p Vex_q \times max(u,f)$. PROBLEM II – Maximize g so that g: $P \times Q$->R and g £ $Vex_q \times Cav_p$ $max(u,g)$. Here, Cav_p is the operation of concavification of a function with respect to the variable $p \in P$ (for each fixed $q \in Q$). Similarly, Vex_q is the operation of convexification with respect to $q \in Q$. Maximum and minimum are taken here in the partial ordering of pointwise comparison: f £ g means $f(p,q)$ £ $g(p,q)$ " $(p,q) \in P \times Q$. It is proved here that both problems have the same solution which is also the unique simultaneous solution of the following pair of functional equations: (i) $f = Vex_q$ $max(u,f)$. (ii) $f = Cav_p$ $min(u,f)$. The problem arises in game theory, but the proof here is purely analytical and makes no use of game-theoretical concepts.