A Duality Theorem on a Pair of Simultaneous Functional Equations

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Abstract

Given $P$ and $Q$ convex compact sets in $\mathbb{R}^k$ and $\mathbb{R}^s$, respectively, and $u$ a continuous real valued function on $P \times Q$, we consider the following pair of dual problems:

PROBLEM I – Minimize $f$ so that $f: P \times Q \to \mathbb{R}$ and $f \geq \text{Cav}_p \text{Vex}_q \times \max(u,f)$.

PROBLEM II – Maximize $g$ so that $g: P \times Q \to \mathbb{R}$ and $g \leq \text{Vex}_q \times \text{Cav}_p \max(u,g)$. Here, $\text{Cav}_p$ is the operation of concavification of a function with respect to the variable $p \in P$ (for each fixed $q \in Q$). Similarly, $\text{Vex}_q$ is the operation of convexification with respect to $q \in Q$. Maximum and minimum are taken here in the partial ordering of pointwise comparison: $f \leq g$ means $f(p,q) \leq g(p,q) \ \forall (p,q) \in P \times Q$. It is proved here that both problems have the same solution which is also the unique simultaneous solution of the following pair of functional equations: (i) $f = \text{Vex}_q \max(u,f)$. (ii) $f = \text{Cav}_p \min(u,f)$. The problem arises in game theory, but the proof here is purely analytical and makes no use of game-theoretical concepts.