Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 60, No. 2, September 1977

A Duality Theorem on a Pair of Simultaneous Functional Equations

JEAN FRANÇOIS MERTENS AND SHMUEL ZAMIR

Université Catholique de Louvain, Louvain, Belgium, and Hebrew University, Jerusalem, Israel

Submitted by Kenneth L. Cooke

Given P and Q convex compact sets in R^k and R^a, respectively, and u a continuous real valued function on $P \times Q$, we consider the following pair of dual problems: PROBLEM I—Minimize f so that $f: P \times Q \to R$ and $f \geqslant \mathrm{Cav}_p \mathrm{Vex}_q \times \max(u, f)$. PROBLEM II—Maximize g so that $g: P \times Q \to R$ and $g \leqslant \mathrm{Vex}_q \times \mathrm{Cav}_p \min(u, g)$. Here Cav_p is the operation of concavification of a function with respect to the variable $p \in P$ (for each fixed $q \in Q$). Similarly, Vex_q is the operation of convexification with respect to $q \in Q$. Maximum and minimum are taken here in the partial ordering of pointwise comparison: $f \leqslant g$ means $f(p, q) \leqslant g(p, q)\ \forall (p, q) \in P \times Q$. It is proved here that both problems have the same solution which is also the unique simultaneous solution of the following pair of functional equations: (i) $f = \mathrm{Vex}_q \max(u, f)$. (ii) $f = \mathrm{Cav}_p \min(u, f)$. The problem arises in game theory, but the proof here is purely analytical and makes no use of game-theoretical concepts.

1. INTRODUCTION

A certain problem in game theory gives rise to a pair of simultaneous functional equations involving the operations of concavification and convexification of a function. Using game-theoretical arguments and techniques it was proved in [1] that this set of equations has a unique solution. It was pointed out by several readers of [1] that the result, as a general result on functional equations or on convex programming, may be of some interest and should be provable without any reference to game-theoretical context or techniques. This is, in fact, done in this paper in which we state and prove the result from first principles.

2. NOTATIONS AND STATEMENT OF THE THEOREMS

Let P be a compact and convex set in the k-dimensional Euclidean space R^k. Let Q be a compact and convex set in R^a. Let u be a continuous real-valued function on $P \times Q$. The sets P and Q and the function u are given and fixed

ISSN 0022-247X

throughout the whole paper. We denote by F the set of all real-valued functions on $P \times Q$, i.e., $F = \{f \mid f: P \times Q \to R\}$. Unless something else is specified, the word "function" will mean an element of F; $\min\{f \mid \cdots\}$ will mean $\min \{f \in F \mid \cdots\}$, etc. A function f is said to be *concave* with respect to p (w.r.t. p) if $\lambda f(p_1, q) + (1 - \lambda) f(p_2, q) \leqslant f(\lambda p_1 + (1 - \lambda) p_2, q)$, for all p_1 and p_2 in P, $q \in Q$, and $\lambda \in [0, 1]$. Similarly, f is said to be *convex* w.r.t. q if $\lambda f(p, q_1) + (1 - \lambda) f(p, q_2) \geqslant f(p, \lambda q_1 + (1 - \lambda) q_2)$ for all $p \in P$, q_1 and q_2 in Q, and $\lambda \in [0, 1]$.

DEFINITION. Let $f \in F$. The *concavification* of f is denoted by $\mathrm{Cav}_p f$ and is defined by

$$\operatorname*{Cav}_p f = \min\{g \in F \mid g \text{ is concave w.r.t. } p \text{ and } g(p, q) \geqslant f(p, q)\ \forall (p, q) \in P \times Q\}$$

The *convexification* of f is denoted by $\mathrm{Vex}_q f$ and defined by

$$\operatorname*{Vex}_q f = \max\{g \in F \mid g \text{ is convex w.r.t. } q \text{ and } g(p, q) \leqslant f(p, q)\ \forall (p, q) \in P \times Q\}.$$

"min" and "max" always mean here a pointwise minimization and maximization, respectively, of the functions under consideration.

In the notations of [2, p. 36] $\mathrm{Vex}_q f$ is the *convex hull* of f, $\mathrm{conv} f$, when f is viewed as a function of q only, for each $p \in P$. Similarly for $\mathrm{Cav}_p f$. The slightly different notations we use here are somewhat more convenient when one uses both operations simultaneously on the same function of two variables.

If we consider Cav_p and Vex_q as two operations dual to each other, then the following two problems may be considered as a pair of dual problems:

PROBLEM I. Minimize f subject to

$$f \geqslant \operatorname*{Cav}_p \operatorname*{Vex}_q \max(u, f). \tag{2.1}$$

PROBLEM II. Maximize g subject to

$$g \leqslant \operatorname*{Vex}_q \operatorname*{Cav}_p \min(u, g). \tag{2.2}$$

THEOREM 2.1. *Both Problems I and II have solutions and the two solutions are equal.*

THEOREM 2.2. *The common solution of Problems I and II is also a simultaneous solution, and the only simultaneous solution, of the following two functional equations:*

$$f = \operatorname*{Vex}_q \max(u, f). \tag{2.3}$$

$$f = \operatorname*{Cav}_p \min(u, f) \tag{2.4}$$

3. Proofs

In this section we proceed in a sequence of steps that will lead to the proofs of the two theorems.

Denote by F_1 the set of feasible functions for Problem I (i.e., functions satisfying (2.1) and by F_2 the set of feasible functions for Ptroblem II (i.e., functions satisfying (2.2)).

PROPOSITION 3.1. $F_1 \neq \phi$ and $F_2 \neq \phi$.

Proof. Let m and M be, respectively, the minimum and the maximum of u on $P \times Q$. Let $f_0 \equiv M$ and $g_0 \equiv m$ then $f_0 \in F_1$ and $g_0 \in F_2$.

Denote $\underline{v} = \inf\{f \mid f \in F_1\}$ and $\bar{v} = \sup\{g \mid g \in F_2\}$

PROPOSITION 3.2. $\underline{v} \in F_1$ and $\bar{v} \in F_2$.

Proof. For any $f \in F_1$ one has

$$f \geqslant \operatorname*{Cav}_p \operatorname*{Vex}_q \max(u, f) \geqslant \operatorname*{Cav}_p \operatorname*{Vex}_q \max(u, \underline{v}).$$

Hence

$$\inf\{f \mid f \in F_1\} \geqslant \operatorname*{Cav}_p \operatorname*{Vex}_q \max(u, \underline{v}),$$

which proves $\underline{v} \in F_1$. The second statement, $\bar{v} \in F_1$, is proved similarly.

COROLLARY 3.3. *$\underline{v} = \min\{f \mid f \in F_1\}$ and $\bar{v} = \max\{g \mid g \in F_2\}$, and hence $\underline{v}$ and $\bar{v}$ are the solutions of Problems I and II, respectively.*

PROPOSITION 3.4.

$$\underline{v} = \operatorname*{Cav}_p \operatorname*{Vex}_q \max(u, \underline{v}), \tag{3.1}$$

and

$$\bar{v} = \operatorname*{Vex}_q \operatorname*{Cav}_p \min(u, \bar{v}). \tag{3.2}$$

Proof. Assume that for $(p_0, q_0) \in P \times Q$ we had

$$\underline{v}(p_0, q_0) = (\operatorname*{Cav}_p \operatorname*{Vex}_q(u, \underline{v}))\,(p_0, q_0) + \epsilon; \qquad \text{where} \quad \epsilon > 0.$$

Define $\underline{\underline{v}}$ by

$$\begin{aligned} \underline{\underline{v}}(p, q) &= \underline{v}(p, q) - \epsilon \qquad && \text{if} \quad (p, q) = (p_0, q_0) \\ &= \underline{v}(p, q) && \text{if} \quad (p, q) \neq (p_0, q_0). \end{aligned}$$

Clearly, $\mathrm{Cav}_p \mathrm{Vex}_q \max(u, \underline{v}) \geqslant \mathrm{Cav}\,\mathrm{Vex} \max(u, \underline{\underline{v}})$; hence,

$$\begin{aligned} \underline{\underline{v}}(p_0, q_0) = \underline{v}(p_0, q_0) - \epsilon &= (\underset{p}{\mathrm{Cav}}\ \underset{q}{\mathrm{Vex}} \max(u, \underline{v}))\,(p_0, q_0) \\ &\geqslant (\underset{p}{\mathrm{Cav}}\ \underset{q}{\mathrm{Vex}} \max(u, \underline{\underline{v}}))\,(p_0, q_0). \end{aligned} \tag{3.3}$$

For $(p, q) \neq (p_0, q_0)$ we have

$$\begin{aligned} \underline{\underline{v}}(p, q) = \underline{v}(p, q) &\geqslant (\underset{p}{\mathrm{Cav}}\ \underset{q}{\mathrm{Vex}} \max(u, \underline{v}))\,(p, q) \\ &\geqslant (\underset{p}{\mathrm{Cav}}\ \underset{q}{\mathrm{Vex}} \max(u, \underline{\underline{v}}))\,(p, q). \end{aligned} \tag{3.4}$$

(3.3) and (3.4) imply that $\underline{\underline{v}} \in F_1$, in contradiction to the definition of $\underline{v}$ since $\underline{\underline{v}}(p_0, q_0) < \underline{v}(p_0, q_0)$. This proves (3.1). Statement (3.2) is proved similarly.

LEMMA 3.5. *For any $f \in F$, each of* $\mathrm{Cav}_p \mathrm{Vex}_q f$ *and* $\mathrm{Vex}_q \mathrm{Cav}_p f$ *is both concave w.r.t. p and convex w.r.t. q.*[1]

Proof. Let us prove this for $\mathrm{Cav}_p \mathrm{Vex}_q f$. The proof for $\mathrm{Vex}_q \mathrm{Cav}_p f$ is obtained in the same way using the obvious duality between Cav_p and Vex_q, min and max.

Clearly $\mathrm{Cav}_p \mathrm{Vex}_q f$ is concave w.r.t. p; to see that it is also convex w.r.t q we first notice that for any fixed p

$$\begin{aligned} (\underset{p}{\mathrm{Cav}}\ \underset{q}{\mathrm{Vex}} f)(p, q) = \mathrm{Sup}(\lambda(\underset{q}{\mathrm{Vex}} f)(p_1, q) &+ (1 - \lambda)(\underset{q}{\mathrm{Vex}} f)(p_2, q)), \\ 0 \leqslant \lambda \leqslant 1,\ \lambda p_1 &+ (1 - \lambda) p_2 = p,\ p_1 \in P,\ p_2 \in P. \end{aligned} \tag{3.5}$$

$(\mathrm{Vex}_q f)(p_i, q)$ is clearly convex (w.r.t. q). The convex combination of two convex functions is convex, and the Sup of convex functions is again a convex function. It follows from (3.5) that for each p, $(\mathrm{Cav}_p \mathrm{Vex}_q f)(p, q)$ is convex w.r.t. q. This completes the proof of the lemma.

COROLLARY 3.6. *Each of $\underline{v}$ and $\bar{v}$ is both concave w.r.t. p and convex w.r.t. q.*

LEMMA 3.7.

$$\underline{v} = \underset{q}{\mathrm{Vex}} \max(u, \underline{v}), \tag{3.6}$$

$$\bar{v} = \underset{p}{\mathrm{Cav}} \min(u, \bar{v}). \tag{3.7}$$

Proof. By (3.1) we have

$$\underline{v} = \underset{p}{\mathrm{Cav}}\ \underset{q}{\mathrm{Vex}} \max(u, \underline{v}) \geqslant \underset{q}{\mathrm{Vex}} \max(u, \underline{v}).$$

[1] Referring to [2, p. 349] again, we claim that both Cav Vex f and Vex Cav f are concave–convex functions on $P \times Q$.

On the other hand, $\underline{v} \leqslant \max(u, \underline{v})$ and $\underline{v}$ is convex w.r.t. q implies $\underline{v} \leqslant \operatorname{Vex}_q \max(u, \underline{v})$, hence $\underline{v} = \operatorname{Vex}_q \max(u, \underline{v})$ which is (3.6). The dual statement (3.7) is proved in the same way.

Define now two sequences of functions $\{\underline{u}_n\}$ and $\{\bar{u}_n\}$ by $\underline{u}_0 \equiv -\infty$ and $\bar{u}_0 \equiv +\infty$, and

$$\underline{u}_{n+1} = \operatorname*{Cav}_p \operatorname*{Vex}_q \max(u, \underline{u}_n) \qquad n = 1, 2, \ldots, \tag{3.8}$$

$$\bar{u}_{n+1} = \operatorname*{Vex}_q \operatorname*{Cav}_p \min(u, \bar{u}_n) \qquad n = 1, 2, \ldots. \tag{3.9}$$

PROPOSITION 3.8. $\{\underline{u}_n\}$ is an increasing sequence, uniformly converging to a finite continuous function $\underline{u}$. $\{\bar{u}_n\}$ is a decreasing sequence uniformly converging to a finite continuous limit $\bar{u}$.

Proof. Using (3.8) and (3.9) it is easily proved by induction that $\{\underline{u}_n\}$ is increasing ,$\{\bar{u}_n\}$ is decreasing $|\underline{u}_n| \leqslant K$, and $|\bar{u}_n| \leqslant K$, $n = 1, 2, \ldots$, where K is such that $|u(p, q)| \leqslant K \; \forall (p, q) \in P \times Q$.

The operations of Cav_p, Vex_q, max, and min preserve or diminish the modulus of uniform continuity; therefore $\underline{u}_n$, $\bar{u}_n$ have the same modulus of uniform continuity as u. It follows that both sequences converge uniformly to finite continuous functions which we denote by $\underline{u}$ and $\bar{u}$, respectively: $\underline{u}_n \uparrow \underline{u}$ and $\bar{u}_n \downarrow \bar{u}$.

PROPOSITION 3.9.

$$\underline{u} \geqslant \underline{v}, \tag{3.10}$$

$$\bar{u} \leqslant \bar{v}. \tag{3.11}$$

Proof. From (3.8) we get at the limit $\underline{u} = \operatorname{Cav}_p \operatorname{Vex}_q \max(u, \underline{u})$, which implies $\underline{u} \in F_1$, and hence $\underline{u} \geqslant \underline{v}$ by the definition of $\bar{v}$. (3.11) is proved in the same way.

PROPOSITION 3.10.

$$\underline{u} = \underline{v}, \tag{3.12}$$

$$\bar{u} = \bar{v}. \tag{3.13}$$

Proof. Let us first prove by induction that $\underline{u}_n \leqslant \underline{v}$. In fact, $\underline{u}_0 \equiv -\infty < \underline{v}$ (since $\underline{v}$ is finite) and

$$(\underline{u}_n \leqslant \underline{v}) \Rightarrow \underline{u}_{n+1} = \operatorname*{Cav}_p \operatorname*{Vex}_q \max(u, \underline{u}_n) \leqslant \operatorname*{Cav}_p \operatorname*{Vex}_q \max(u, \underline{v}) = \underline{v}.$$

It follows that $\underline{u} = \lim_{n\to\infty} \underline{u}_n \leqslant \underline{v}$ which when combined with (3.10) yields (3.12). A similar proof applies for (3.13).

LEMMA 3.11. $\bar{v} \leqslant \underline{v}$.

Proof. Let $\delta = \max_{(p,q)\in P\times Q} (\bar{v}(p, q) - \underline{v}(p, q))$. This maximum is attained since $\bar{v}$ and $\underline{v}$ are continuous and $P \times Q$ is compact.

Assume that the lemma is false, i.e., $\delta > 0$. Let

$$D = \{(p, q) \in P \times Q \mid \bar{v}(p, q) - \underline{v}(p, q) = \delta\}.$$

Clearly, D is compact, so let (p_0, q_0) be an extreme point of D. It is not possible that both

$$\max\{u(p_0, q_0), \underline{v}(p_0, q_0)\} = (\operatorname*{Vex}_q \max\{u, \underline{v}\})(p_0, q_0)$$

and

$$\min\{u(p_0, q_0), \bar{v}(p_0, q_0)\} = (\operatorname*{Cav}_p \min\{u, \bar{v}\})(p_0, q_0),$$

because then (3.6) and (3.7) would be inconsistent;

$$(3.6) \Rightarrow \underline{v}(p_0, q_0) = \max\{u(p_0, q_0), \underline{v}(p_0, q_0)\} \Rightarrow \underline{v}(p_0, q_0) \geqslant u(p_0, q_0),$$
$$(3.7) \Rightarrow \bar{v}(p_0, q_0) = \min\{u(p_0, q_0), \bar{v}(p_0, q_0)\} \Rightarrow \bar{v}(p_0, q_0) \leqslant u(p_0, q_0),$$

which is impossible since $\bar{v}(p_0, q_0) = \underline{v}(p_0, q_0) + \delta > \underline{v}(p_0, q_0)$. We conclude that at least one of the operations Vex_q in (3.6) and Cav_p in (3.7) is nontrivial at (p_0, q_0). Assume for instance that Vex_q is not trivial, i.e., if we set $w = \max(u, \underline{v})$ then, using Caratheodory's theorem , we have

$$(\operatorname*{Vex}_q w(p, q))(p_0, q_0) = \sum_{i=1}^{s+1} \lambda_i w(p_0, q_i),$$

where $\lambda_i \geqslant 0, \sum_{i=1}^{s+1} \lambda_i = 1, \sum_{i=1}^{s+1} \lambda_i q_i = q_0$, and at least for one i, $q_i \neq q_0$. Then

$$\underline{v}(p_0, q_0) = (\operatorname*{Vex}_q w(p, q))(p_0, q_0) = \sum_{i=1}^{s+1} \lambda_i w(p_0, q_i) \geqslant \sum_{i=1}^{s+1} \lambda_i \underline{v}(p_0, q_i).$$

Also since $\bar{v}$ is convex w.r.t. q (Corollary 3.6):

$$\bar{v}(p_0, q_0) \leqslant \sum_{i=1}^{s+1} \lambda_i \bar{v}(p_0, q_i).$$

Hence

$$\delta = \bar{v}(p_0, q_0) - \underline{v}(p_0, q_0) \leqslant \sum_{i=1}^{s+1} \lambda_i(\bar{v}(p_0, q_i) - \underline{v}(p_0, q_i)).$$

This implies that $\bar{v}(p_0, q_i) - \underline{v}(p_0, q_i) = \delta$ for $i = 1,..., s + 1$, since otherwise one would have $\bar{v}(p_0, q_i) - \underline{v}(p_0, q_i) > \delta$ for some i, which is impossible. We get therefore $(p_0, q_i) \in D$ for $i = 1, 2,..., s + 1$, $\sum_{i=1}^{s+1} (p_0, q_i) = (p_0, q_0)$, and for at least one i $(p_0, q_i) \neq (p_0, q_0)$. This contradicts the fact that (p_0, q_0) is an extreme point of D. We conclude that D is a compact set with no extreme points, therefore, $D = \phi$, a contradiction that proves $\delta \leqslant 0$. The other case (when Cav_p in (3.7) is not trivial) is treated in the same way and this completes the proof of the lemma.

COROLLARY 3.12. *There is at most one function in the intersection* $F_1 \cap F_2$

Proof. Follows from Proposition 3.11 and the definitions of $\underline{v}$ and $\bar{v}$.
Note now that by our definitions: $\underline{u}_1 = \mathrm{Cav}_p \mathrm{Vex}_q u$.

PROPOSITION 3.13. $\bar{v} \geqslant \underline{u}_1$.

Proof. Let $f = \mathrm{Vex}_q u$. Since $u \geqslant \mathrm{Vex}_q u$, we have

$$\mathop{\mathrm{Cav}}_p \min(u, f) = \mathop{\mathrm{Cav}}_p f = \mathop{\mathrm{Cav}}_p \mathop{\mathrm{Vex}}_q u = \underline{u}_1 .$$

Hence by Lemma 3.5

$$\mathop{\mathrm{Vex}}_q \mathop{\mathrm{Cav}}_p \min(u, f) = \mathop{\mathrm{Vex}}_q (\mathop{\mathrm{Cav}}_p \mathop{\mathrm{Vex}}_q u) = \mathop{\mathrm{Cav}}_p \mathop{\mathrm{Vex}}_q u \geqslant \mathop{\mathrm{Vex}}_q u = f.$$

So $f \in F_2$ and therefore $\bar{v} \geqslant f = \mathrm{Vex}_q u$. But since $\bar{v}$ is concave w.r.t. p we have also $\bar{v} \geqslant \mathrm{Cav}_p \mathrm{Vex}_q u = \underline{u}_1$.

LEMMA 3.14. *If u in Problem I and II is replaced by* $U = \max(u, \underline{u}_1)$, *then the two new problems will have the same solutions as the original problems.*

Proof. Let us keep the same notations for the new problems using capital U and V instead of u and v, respectively. So for instance the solutions of the new problems I and II are $\underline{V}$ and $\bar{V}$, respectively. By Proposition 3.10

$$\lim \underline{U}_n = \underline{V} \qquad \text{and} \qquad \lim \bar{U}_n = \bar{V}. \tag{3.14}$$

The lemma states $\underline{V} = \underline{v}$ and $\bar{V} = \bar{v}$. To prove this observe first that

$$\underline{U}_1 = \mathop{\mathrm{Cav}}_p \mathop{\mathrm{Vex}}_q U = \mathop{\mathrm{Cav}}_p \mathop{\mathrm{Vex}}_q \max(u, \underline{u}_1) = \underline{u}_2 .$$

Assume that $\underline{U}_n = \underline{u}_{n+1}$, then

$$\underline{U}_{n+1} = \mathop{\mathrm{Cav}}_p \mathop{\mathrm{Vex}}_q \max(U, \underline{U}_n) = \mathop{\mathrm{Cav}}_p \mathop{\mathrm{Vex}}_q \max(\max(u, \underline{u}_1), \underline{u}_{n+1}).$$

Since $\underline{u}_{n+1} \geqslant \underline{u}_1$ (Proposition 3.8) we get

$$\underline{U}_{n+1} = \mathop{\mathrm{Cav}}_p \mathop{\mathrm{Vex}}_q \max(u, \underline{u}_{n+1}) = \underline{u}_{n+2} .$$

We have thus proved that

$$\underline{U}_n = \underline{u}_{n+1}\,; \qquad n = 1, 2,\ldots \tag{3.15}$$

It follows from this that

$$\underline{V} = \lim \underline{U}_n = \lim \underline{u}_n = \underline{v}.$$

To prove the second part of the lemma, namely, $\overline{V} = \bar{v}$, it is clearly sufficient to show that $\overline{U}_n = \bar{u}_n$ for $n = 1, 2,\ldots$ This we prove by induction on n: $\overline{U}_0 = \bar{u}_0 \equiv \infty$. Assume $\overline{U}_n = \bar{u}_n$ then

$$\begin{aligned}\overline{U}_{n+1} &= \operatorname*{Vex}_q \operatorname*{Cav}_p \min(U, \overline{U}_n)\\ &= \operatorname*{Vex}_q \operatorname*{Cav}_p \min(\max(u, \underline{u}_1), \bar{u}_n)\\ &= \operatorname*{Vex}_q \operatorname*{Cav}_p \max[\min(\underline{u}_1 \bar{u}_n), \min(u, \bar{u}_n)].\end{aligned}$$

Since by Propositions (3.8) and (3.13) $\bar{u}_n \geqslant \bar{v} \geqslant \underline{u}_1$, we get

$$\overline{U}_{n+1} = \operatorname*{Vex}_q \operatorname*{Cav}_p \max[\underline{u}_1\,, \min(u, \bar{u}_n)]. \tag{3.16}$$

We claim that

$$\operatorname*{Cav}_p \max[\underline{u}_1\,, \min(u, \bar{u}_n)] = \operatorname*{Cav}_p \min(u, \bar{u}_n). \tag{3.17}$$

In fact, if we denote the function on the left side by f and that on the right by g, then obviously f is concave w.r.t. p and $f \geqslant \min(u, \bar{u}_n)$. Since clearly $f \geqslant g$ we need only show that $g \geqslant \underline{u}_1$ and this is done by the following:

$$g \geqslant \operatorname*{Vex}_q \operatorname*{Cav}_p \min(u, \bar{u}_n) = \bar{u}_{n+1} \geqslant \bar{v} \geqslant \underline{u}_1\,.$$

By (3.16) and (3.17) we get finally

$$\overline{U}_{n+1} = \operatorname*{Vex}_q \operatorname*{Cav}_p \min(u, \bar{u}_n) = \bar{u}_{n+1}\,,$$

which completes the induction step and hence the proof of the lemma.

Proof of Theorem 2.1. So far we know that Problems I and II have solutions $\underline{v}$ and $\bar{v}$, respectively (Corollary 3.3), such that $\bar{v} \leqslant \underline{v}$ (Lemma 3.11). It remains to prove $\bar{v} \geqslant \underline{v}$. We do that as follows: By Lemma 3.14 and Proposition 3.13 we get $\bar{v} = \overline{V} \geqslant \underline{U}_1 = \underline{u}_2$. Iterating this procedure of replacing u by $\max(u, \underline{u}_1)$ we get $\bar{v} \geqslant \underline{u}_3\,, \bar{v} \geqslant \underline{u}_4\,,$ and so on. In general $\bar{v} \geqslant \underline{u}_n\,, n = 1, 2,\ldots$ Hence $\bar{v} \geqslant \lim \underline{u}_n = \underline{v}$, and our proof is completed.

Proof of Theorem 2.2. Denoting by v the common solution of problems I and II (i.e., $v = \bar{v} = \underline{v}$) we have by Lemma 3.7 that v is a simultaneous solution of (2.3) and (2.4). To prove that this is the only solution let f be any solution of both (2.3) and (2.4). It follows from the equations that f is both concave w.r.t. p and convex w.r.t. q; hence,

$$f = \operatorname*{Cav}_{p} \operatorname*{Vex}_{q} \max(u, f), \tag{3.18}$$

$$f = \operatorname*{Vex}_{q} \operatorname*{Cav}_{p} \min(u, f). \tag{3.19}$$

Equation (3.18) implies $f \in F_1$ and, consequently, $f \geqslant \underline{v} = v$, while (3.19) implies $f \in F_2$ and, consequently, $f \leqslant \bar{v} = v$. This completes the proof.

Remark. In [1] the system (2.3) and (2.4) was solved for a few numerical examples. In many cases the problem can be reduced to solving the differential equations defining the points (p, q) for which $f(p, q) = u(p, q)$.

Acknowledgments

We are deeply indebted to E. Kohlberg, T. Bewley, and L. S. Shapley for suggesting the problem and for some useful discussions. We would also like to thank the referee for his helpful suggestions.

References

1. J. F. Mertens and S. Zamir, The value of two-person zero-sum repeated games with lack of information on both sides, *Internat. J. Game Theory* 1, No. 1 (1971), 39–64.
2. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, N.J., 1970.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium