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Formulation of Bayesian Analysis for Games with Incomplete Information

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Abstract: A formal model is given of Harsanyi's infinite hierarchies of beliefs. It is shown that the model closes with some Bayesian game with incomplete information, and that any such game can be approximated by one with a finite number of states of world.

1. Introduction

In analyzing a game with incomplete information, i.e. games in which players are uncertain about all the parameters defining the strategy spaces and the payoff functions, one is led naturally to handle "an infinite hierarchy of beliefs" for each player: His beliefs (i.e. subjective probabilities) on the parameters of the games, his beliefs on the beliefs of the other players on the parameters of the games, his beliefs on the other players' beliefs on his own beliefs on the parameters of the games, his beliefs on the other players' beliefs on his own beliefs on their beliefs on the parameters of the games, etc...

In an attempt to overcome the difficulty of having to work with infinite sequences of mutual beliefs. *Harsanyi* [1967 – 1968] introduced the concept of *type* which proved to be very useful in making games with incomplete information much more manageable. Harsanyi's idea was to summarize all parameters and beliefs concerning a certain player, by one vector which he calls the *attribute* vector. In his words [see *Harsanyi*, 1967, p. 171]: "... we can regard the vector c_i as representing certain physical, social, and psychological *attributes* of player *i himself* in that it summarizes some crucial parameters of player *i*'s own payoff function U_i as well as the main parameters of his beliefs about his social and physical environment ... the rules of the game as such allow any given player *i* to belong to any one of a number of possible *types*, correspond-

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Finally, in Section 5, we define a game in strategic form determined by the beliefs space (or subspace). This will be typically a game "with vector payoffs", but the Nash equilibria are well defined. For a consistent beliefs subset, the Nash equilibria will be the same as those of a certain extensive form game in which the state of the world is chosen according to the (uniquely determined) probability distribution, and each player is informed on what is his own type. This is Harsanyi's theorem [*Harsanyi*, 1967, part II, p. 321] which is in the background of most models of games with incomplete information.

It should be pointed out that works in the same direction were done by *Böge et al.* who, being interested mainly in the equilibrium points of games with incomplete information, incorporated the strategy choices of the players as part of the space of parameters on which the infinite hierarchy of beliefs is built.

2. The Universal Beliefs Space Y

The main objective of this section is to prove Theorem 2.9 which establishes the existence of a space of infinite hierarchy of beliefs. We consider a situation of incomplete information involving a set of players $I = \{1, \ldots, n\}$, the members of which are uncertain about the parameters of the game they are playing which may be any element of some set S (we may think of a point of S as a full listing of the strategy spaces and the payoff functions). We shall refer to S as the *parameter-space.*

Assumption: S is a compact space.

Remark: To see that this assumption is not too restrictive, let us see how, in a typical and rather general model if incomplete information , the space S will in fact be compact: Observe that S has most generally to include all the parameters of the game including the parameters of the utility functions of the player. So let S_0 be the set of possible values of all the parameters of the game. Clearly S_0 may be assumed (by enlarging it if necessary) to be compact. For each player i let A^i be his action set (enlarged so as to become independent of $S \in S_0$). The set of outcomes can then be identified with the set $C = S_0 \times \overset{n}{\underset{i=1}{\times}} A^i$ and is compact if A^i are compact. The Von-Neumann Morgenstern utility function of player i is a (continuous) real map $u_i: C \to R$, which we may want to assume to be bounded (for instance by applying the Von Neumann-Morgenstern theory to all countable lotteries, in order to avoid the St. Petersburg paradox). Hence we may take $u_i: C \to [0, 1]$ and the set of all possible games is then $S = S_0 \times [[0, 1]^C]^n$ which is compact. A special case is of course that in which S_0 and A^i are finite then S will be in addition metrisable.

For any compact space X, $\Pi(X)$ will denote the compact space of probability measures on X, endowed with the weak* topology. [It is clearly closed in the set of all measures of norm $\leqslant 1$ since the function 1 is continuous-, and the set is by Riesz theorem the unit ball of the dual of $C(X)$, hence weak*-compact by Alaoglu's theorem.]

Definition 2.1: A coherent beliefs hierarchy of level K ($K = 1, 2, \ldots$) is a sequence $(C_0, C_1, \ldots, C_K)$ where:

1) C_0 is a compact subset of S and for $k = 1, \ldots, K$, C_k is a compact subset of $C_{k-1} \times [\Pi(C_{k-1})]^n$ (as topological spaces). (We denote by ρ_{k-1} and t^i the projections of C_k on C_{k-1} and the i-th copy of $\Pi(C_{k-1})$ respectively.)

$$\rho_{k-1}(C_k) = C_{k-1}; k = 1, \ldots, K. \tag{2}$$

$$\forall c_k \in C_k \text{ let } c_{k-1} = \rho_{k-1}(c_k), \text{ then } \forall i: \tag{3}$$

H1) the marginal distribution of $t^i(c_k)$ on C_{k-2} is $t^i(c_{k-1})$;

H2) the marginal distribution of $t^i(c_k)$ on the i-th copy of $\Pi(C_{k-2})$ is the unit mass at $t^i(c_{k-1}) = t^i(\rho_{k-1}(c_k))$.

$$\forall i, \forall t \in t^i(C_k); k = 1, \ldots, K, t\{\rho_{k-1}[(t^i)^{-1}(t)]\} = 1. \tag{4}$$

We interpret C_k as a set of beliefs up to level k and thus a point in C_k consists of hierarchy of beliefs up to level $(k-1)$ (i.e. a point in C_{k-1}) and for each player i a probability distribution t^i_k on hierarchies of beliefs up to level $(k-1)$ (i.e. $t^i_k \in \Pi(C_{k-1})$). Condition H1 says that player i's k-level beliefs coincide with his $(k-1)$ level beliefs in whatever concerns hierarchies up to level $(k-2)$. Condition H2 says that player i knows his own previous order beliefs.

In the next definition we formalize the properties of the space of states of the world C we would like to obtain: Any point $c \in C$ determines uniquely a set of parameters $s \in S$ and the type t^i of each player. The type t^i is a probability distribution on C which is coherent in the sense that each player knows his own type. In other words if $t^i \in \Pi(C)$ is a certain type of player i, then in all points in the support of t^i (Supp (t^i)) player i is of type t^i. This motivates the following.

Definition 2.2: An S-based abstract beliefs space (BL-space) is an $(n+3)$ tuple $(C, S, f, (t^i)_{i=1}^n)$ where C is a compact set, S is some compact space, f is a continuous mapping $f: C \to S$ and t^i, $i = 1, \ldots, n$, are continuous mappings $t^i: C \to \Pi(C)$ (with respect to the weak* topology) satisfying:

(*) $\tilde{c} \in C$ and $\tilde{c} \in \text{Sup}(t^i(c)) \Rightarrow t^i(\tilde{c}) = t^i(c)$.

When no confusion may result we shall denote the BL-space simply by C.

The space C is a space in which each point $c \in C$ contains a full description not only of the state of nature $s \in S$ but also of all beliefs, beliefs on beliefs etc. on S. In fact if we interpret t^i as player i's (subjective) probability distribution on C, then combined with f it defines a probability distribution on S, which is the first level beliefs of player i. But t^i also defines a probability distribution on $(t^j)_{j \neq i}$, and hence on the first level beliefs of the other players. This may be called the second level beliefs of player i. Proceeding inductively we find that with each $c \in C$ is associated one infinite hierarchy of beliefs for each player. The condition (*) is a consistency condition which says basically that a player i assigns positive probability (in the discrete case) only to points of C in which he has the same beliefs. In other words he is certain of his own beliefs.

Let us write now formally the above mentioned observation:
Given S we define the spaces X_k, T_k, by

$$X_0 = S$$

$$T_k = \Pi\,(X_{k-1})$$

$$X_k = X_{k-1} \times [T_k]^n = S \times \mathop{\times}_{l=1}^{k} [T_l]^n; \quad k = 1, 2, \ldots$$

Define also $X = S \times \mathop{\times}_{l=1}^{\infty} [T_l]^n$, which is a well defined compact space when so is S.

Note that X is generated by S and whenever we want to specify the generating space we shall write $X(S)$. We shall denote a typical point in $X(S)$ as $x = (s, t_1^1, \ldots, t_1^n, \ldots, t_k^1, \ldots, t_k^n, \ldots)$, where for each i and each k $t_k^i \in T_k^i = \Pi\,(X_{k-1})$.

If $\varphi\colon C \to \tilde{C}$ is a continuous mapping between two compact spaces C and $\tilde{C}$, we denote by $\hat{\varphi}$ the mapping $\Pi\,(C) \to \Pi\,(\tilde{C})$ canonically induced by φ, namely the mapping $\hat{\varphi}\colon \Pi\,(C) \to \Pi\,(\tilde{C})$ which maps $\mu \in \Pi\,(C)$ to $\hat{\mu} \in \Pi\,(\tilde{C})$ such that for any continuous function f on $\tilde{C}$, $\int_{\tilde{C}} f(\tilde{c})\,d\hat{\mu} = \int_C (f \circ \varphi)(c)\,d\mu$.

To any S-based abstract BL-space $(C, S, f, (t^i)_{i=1}^n)$ we define now a certain natural continuous mapping $h : C \to X(S)$. This will be done by defining for each $k = 0, ., 2, \ldots$ a mapping $h_k : C \to X_k$ such that

$$k \leqslant l \Rightarrow \rho_k\,(h_l\,(c)) = h_k\,(c) \qquad \forall\, c \in C,$$

in other words, $h_k\,(c)$ is the projection of $h\,(c)$ on X_k.

The mappings h_k are defined inductively as follows: $h_0\,(c) = f\,(c)$. Assume $h_k : C \to X_k$ is defined then we want to define $h_{k+1} : C \to X_{k+1}$. Take any $c \in C$ and let $h_k\,(c) = (s, t_1^1, \ldots, t_1^n, \ldots, t_k^1, \ldots, t_k^n) \in X_k$ then $h_{k+1}\,(c) = (s, t_1^1, \ldots, t_1^n, \ldots, t_k^1, \ldots, t_k^n, t_{k+1}^1, \ldots, t_{k+1}^n) \in X_{k+1}$ where $\forall\, i$,

$t^i_{k+1} = \hat{h}_k \circ t^i : C \to \Pi(X_k) = T_{k+1}$, and $\hat{h}_k$ is the mapping $\hat{h}_k : \Pi(C) \to \Pi(X_k)$ canonically induced by h_k.

It follows that the so defined $h : C \to X(S)$ is continuous. Let $H = h(C) \subseteq X(S)$. When we want to emphasize the underlying S we shall write $H(S)$. By construction, the image $h(c)$ contains all possible information concerning S and beliefs on S. Therefore it is intuitively pretty clear that $h(c) \neq h(c')$ for $c \neq c'$ unless c and c' are identical in whatever concerns S and differ only by something which is redundant to S and to the beliefs structure on S.

To define this notion of nonredundancy more formally, given an BL-space $(C, S, f, (t^i)_{i=1}^n)$ let $\mathcal{F}$ be the smallest σ-field (of subsets of C) for which f is measurable and $\forall i, (t^i(c))(B)$ is measurable $\forall B \in \mathcal{F}$.

Definition 2.4: A BL-space $(C, S, f, (t^i)_{i=1}^n)$ is said to satisfy the *non-redundancy* condition (NR-condition) if the σ-field $\mathcal{F}$ separates each two distinct points in C.

By our previous discussion we thus have:

Proposition 2.5: If an S-based abstract BL-space $(C, S, f, (t^i)_{i=1}^n)$ satisfies the NR-condition then the mapping $h : C \to H$ is also one to one hence it is an isomorphism.

In dealing with BL-spaces we would like to consider homeomorphisms between BL-spaces which (in addition to their topological properties) will also preserve the beliefs structure. These mappings will be called *BL-morphisms* and we proceed now to define them formally.

Definition 2.6: A beliefs morphism (BL-morphism) from a BL-space $(C, S, f, (t^i)_{i=1}^n)$ to a BL-space $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ is a pair (φ, φ') where φ' is a continuous mapping of C onto $\tilde{C}$ and φ is a continuous mapping of S onto $\tilde{S}$ such that for each i, $i = 1, \ldots, n$, the following diagram comutes:

$$
\begin{tikzcd}
S \arrow[r, "\varphi"] & \tilde{S} \\
C \arrow[u, "f"] \arrow[r, "\varphi'"] \arrow[d, "t^i"'] & \tilde{C} \arrow[u, "\tilde{f}"'] \arrow[d, "\tilde{t}^i"] \\
\Pi(C) \arrow[r, "\hat{\varphi}'"] & \Pi(\tilde{C})
\end{tikzcd}
$$

where $\hat{\varphi}'$ is the mapping $\hat{\varphi}' : \Pi(C) \to \Pi(\tilde{C})$ canonically induced by φ'.

Definition 2.7: A BL-morphism (φ, φ') from $(C, S, f, (t^i)_{i=1}^n)$ to $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ is called a BL-isomorphism if the inverse mappings φ^{-1} and $(\varphi')^{-1}$ exist and $(\varphi^{-1}, (\varphi')^{-1})$ is a BL-morphism from $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ to $(C, S, f, (t^i)_{i=1}^n)$. The two BL-spaces are said to be *BL-isomorphic.*

Some thought on the diagramm of Definition 2.6 leads us to the observation that if (φ, φ') is a BL-morphism from C to $\tilde{C}$ then there is actually one essential mapping and not two since φ' seems to be determined by φ via the above diagram. This is in fact true provided $\tilde{C}$ satisfies the NR-condition:

Lemma 2.8: If (φ, φ') is a BL-morphism from $(C, S, f, (t^i)_{i=1}^n)$ to $(\tilde{C}, \tilde{S}, \tilde{f}, (\tilde{t}^i)_{i=1}^n)$ and if the latter satisfies the NR-condition, then φ' is uniquely determined by φ.

Proof: Using our notation $h: C \to X(S)$ and $\tilde{h}: \tilde{C} \to X(\tilde{S})$ we denote by $h \circ \varphi: C \to X(\varphi(S)) \subseteq X(\tilde{S})$ the mapping which maps $c \in C$ to $h(c)$ in which the underlying S is replaced by $\varphi(S)$. The fact that the diagram of Definition 2.6 computes implies that $\forall\, c \in C$ we have $\tilde{h}(\varphi'(c)) = (h \circ \varphi)(c) \in X(\tilde{S})$. Since $\tilde{C}$ satisfies the NR-condition $\tilde{h}$ is one to one (by Proposition 2.5) and hence invertible. Therefore:

$$\varphi'(c) = \tilde{h}^{-1}(h \circ \varphi)(c).$$

In words, the idea of the proof is that φ combined with the diagram determines for each $c \in C$ uniquely the infinite hierarchy $\tilde{h}(c')$ associated with $c' = \varphi'(c)$, and hence it determines uniquely c' itself since $\tilde{C}$ satisfies the NR-condition.

Remark: In view of Lemma 2.8 we shall shorten our notation and terminology and speak of BL-morphism φ from BL-space C to BL-space $\tilde{C}$. This is the BL-morphism induced by the mapping $\varphi: S \to \tilde{S}$.

We are now ready to state the main theorem of this section.

Theorem 2.9: For any compact S and positive integer n there are spaces $\mathcal{Y}$ and $\mathcal{T}$ such that:

$$\left.\begin{array}{l} 1)\ \mathcal{Y} = S \times [\mathcal{T}]^n \\ 2)\ \mathcal{T} = \Pi(S \times [\mathcal{T}]^{n-1}) \end{array}\right\}\ \textit{up to BL-morphisms.}$$

3) There are compact spaces $\{Y_k\}_{k=0}^{\infty}$ s.t. $\forall\, k\ (Y_0, Y_1, \ldots, Y_k)$ is a coherent beliefs hierarchy and $\mathcal{Y}$ is the projective limit $\{Y_k\}_{k=0}^{\infty}$ (with respect to the natural projection $\rho_{k-1}: Y_k \to Y_{k-1}$. We denote by ρ_k also the projection of $\mathcal{Y}$ on Y_k).

4) $\mathcal{Y}$ is an S-based BL-space (with the projections $f: \mathcal{Y} \to S$ and $t^i: \mathcal{Y} \to \mathcal{T}^i$).

5) Any S-based abstract BL-space, which satisfies the NR-condition, is canonically BL-homeomorphic to a compact subset of $\mathcal{Y}$ (which will be called a BL-subspace of $\mathcal{Y}$).

6) For any coherent beliefs hierarchy $(C_0, C_1, \ldots, C_K)$ there is a BL-subscpace C of Y s.t. $\rho_k(C) = C_k$, $k = 0, \ldots, K$.

7) Any $\tilde{Y}$ and $\tilde{T}$ which satisfy 1) and 2) or 4) and 5) can be mapped continuously onto Y and T respectively. This map induces a BL-homeomorphism between Y and a BL-subspace of $\tilde{Y}$. Any $\tilde{Y}$ which satisfies 3) and 6) can be BL-morphically mapped onto Y.

Y will be called the Universal BL-space generated by S (and n) and T will be called the Universal type space generated by S (and n).

Proof: We shall prove the theorem by constructing the sequence $\{Y_k\}_{k=0}^{\infty}$ in (3) and define Y as its projective limit and T^i as the projection of Y on player i's coordinates. Then we shall prove that these Y and T satisfy the required properties.

Construction of Y

Define the sequence of spaces $\{Y_k\}_{k=0}^{\infty}$ as follows:

$Y_0 = S$ and for $k = 1, 2, \ldots$

(2.1) $Y_k = \{y_k \in Y_{k-1} \times [\Pi(Y_{k-1})]^n \mid (a)\ \forall i$ the marginal distribution of $t^i(y_k)$ on Y_{k-2} is $t^i(y_{k-1})$ and (b) the marginal distribution of $t^i(y_k)$ on the i-th copy of $\Pi(Y_{k-2})$ is the unit mass at $t^i(y_{k-1})\}$.

As we have already noted if X is compact, then $\Pi(X)$ is also compact. Note also that the conditions (a) and (b) in the definition of Y_k are closed conditions. It follows that if Y_{k-1} is compact, then Y_k is also compact. Since $Y_0 = S$ is compact, it follows inductively that Y_k is compact $\forall k$. Let Y be the projective limit of $\{Y_k\}_{k=0}^{\infty}$ with respect to the natural projections $\rho_{k-1}: Y_k \to Y_{k-1}$. Y is a well defined compact set.

Now by definition of Y_k we have that $\forall k$, $(Y_0, \ldots, Y_k)$ satisfy automatically all properties of a coherent beliefs hierarchy (Definition 2.1) except for condition (2), namely that $\rho_k(Y_{k+1}) = Y_k$, $k = 0, 1, \ldots$ This we prove now:

Proposition 2.10: $\rho_k(Y_{k+1}) = Y_k$, $\quad k = 0, 1, 2, \ldots$

This proposition has the following immediate corollaries.

Corollary 2.11:

i) $\quad \forall k, (Y_0, Y_1, \ldots, Y_k)$ *is a coherent beliefs hierarchy;*

(ii) $\quad \forall k, \rho_k(Y) = Y_k$, *in particular* $Y \neq \emptyset$.

The proof of Proposition 2.10 will follow from the following.

Lemma 2.12: Let A and B be compact sets, D a compact subset of $A \times B$ and $q \in \Pi(A)$. A necessary and sufficient condition for the existence of $p \in \Pi(D)$ whose marginal distribution on A is q, is that $q(D_A) = 1$, where D_A is the projection of D on A.

Proof: Since $D \subset D_A \times B$, the necessity is obvious. To prove the sufficiency assume $q(D_A) = 1$. Define $L_q(f) = \int f dq$. L_q is a linear functional defined on $C(D_A)$, (the linear space of continuous real functions on D_A). If we consider a function on D_A as a function on D, by the natural definition $F(a, b) = f(a) \ \forall (a, b) \in D$, and write $L_q(F) = \int F dq$, Lq is then a linear functional defined on a linear subspace of $C(D)$. This is clearly a positive functional with $\| L_q \| = 1$. By Hahn-Banach extension theorem L_q can be extended to a positive linear functional L of norm 1 on $C(D)$. Finally by Riesz representation theorem there is a probability measure $p \in \Pi(D)$, s.t. $L(f) = \int_D f dp \ \forall f \in C(D)$. This p is the required extension of q.

Proof of Proposition 2.10: We prove the proposition inductively on k. It holds for $k = 0$ since $Y_0 = S$ and $Y_1 = S \times [\Pi(S)]^n$, thus $\rho_0(Y_1) = Y_0$. Assume that $\rho_{k-1}(Y_k) = Y_{k-1}$ and let us prove that $\rho_k(Y_{k+1}) = Y_k$. In other words we have to show that any point $y \in Y_k$ can be extended to a point $(y, t_{k+1}^1, \ldots, t_{k+1}^n) \in Y_{k+1}$. So we have to establish the existence of an n-tuple $t_{k+1}^1, \ldots, t_{k+1}^n$ of probability distributions $t_{k+1}^i \in \Pi(Y_k)$ satisfying conditions (a) and (b) in the definition of Y_k, namely that the marginal disitribution on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$ is $t_k^i \times \delta_{t^i(y)}$, where $\delta_{t^i(y)}$ is the element $\Pi(Y_{k-1})$ which assigns mass 1 to $t^i(y)$. We have thus to show that each of these marginals can be extended to a probability distribution t_{k+1}^i on $Y_{k-1} \times [\Pi(Y_{k-1})]_1 \times \ldots \times [\Pi(Y_{k-1})]_n$ supported by its subset Y_k i.e. $t_{k+1}^i(Y_k) = 1$. Using Lemma 2.12 it remains to prove that

$$\text{Sup}\,(t^i(y)) \times \{t^i(y)\} = \text{Supp}\,[t^i(y) \times \delta_{t^i(y)}] \subseteq \text{projection of } Y_k \text{ on } Y_{k-1} \times [\Pi(Y_{k-1})]_i .$$

So let $(\tilde{y}_{k-1}, t^i(y)) \in \text{Supp}\, t^i(y) \times \{t^i(y)\}$ i.e. $\tilde{y}_{k-1} \in \text{Supp}\, t^i(y) \subseteq Y_{k-1}$. Since $t^i(y)$ assigns probability 1 to $t^i(\rho_{k-1}(y))$ it follows that $t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))$. Since by induction hypothesis $\rho_{k-1}(Y_k) = Y_{k-1}$, there is an extension $(\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, \tilde{t}_k^n) \in Y_k$. We claim that if in this point we replace $\tilde{t}_k^i$ by $t^i(y)$ we obtain a point which is also in Y_k, proving that $(\tilde{y}_{k-1}, t^i(y))$ is in the projection of Y_k on $Y_{k-1} \times [\Pi(Y_{k-1})]_i$ and thus completing the proof.

To see that $(\tilde{y}_{k-1}, \tilde{t}_k^1, \ldots, t^i(y), \ldots, \tilde{t}_k^n) \in Y_k$, note that all conditions concerning $\tilde{t}_k^j$, $j \neq i$ are satisfied since $(\tilde{y}_{k-1}, \tilde{t}_k, \ldots, \tilde{t}_k^n) \in Y_k$. The conditions concerning $t^i(y)$ are satisfied since these are the conditions required for $y \in Y_k$ (recalling that $t^i(\tilde{y}_{k-1}) = t^i(\rho_{k-1}(y))$). This completes the proof of Proposition 2.10.

Remark: Note that when $y \in Y_k$ is such that all distributions t_j^i are of finite support, the extension of y to a point in Y_{k+1} is straightforward and an extension, also with finite support, can be pointed at explicitely.

For any $y = (y_0, y_1, \ldots) \in \mathcal{Y}$ and for each $i \in N$, consider the sequence of probabilities $t^i(y_1), t^i(y_2), \ldots$ on $Y_0, Y_1, Y_2, \ldots$ respectively. By the definition of $\{Y_k\}_{k=0}^{\infty}$, this sequence satisfies that $\forall k$, the marginal of $t^i(y_{k+1})$ on Y_{k-1} is $t^i(y_k)$. Since also $\rho_k(\mathcal{Y}) = Y_k \; \forall k$, it follows that for any continuous real function f_K on $\mathcal{Y}$ which depends only on K coordinates, the sequence of integrals $(\int f_K \, d\, t^i(k))_{k=1}^{\infty}$ is well defined and constant for $k \geqslant K+1$. Therefore the sequence $(t^i(y_k))_{k=1}^{\infty}$ defines a linear positive functional of norm 1 on the space of all such functions f_K and hence on the closure of this space which is the space of all continuous functions on $\mathcal{Y}$. By Riesz representation theorem there is a uniquely determined probability measure in $\Pi(\mathcal{Y})$ which represents this linear functional.

Definition 2.13:

(i) For each $y \in \mathcal{Y}$ and $\forall i \in N$, define by $t^i(y)$ the probability distribution on $\mathcal{Y}$ determined by y in the above described way

(ii) Let $\mathcal{T}^i = t^i(\mathcal{Y}) \subsetneq \Pi(\mathcal{Y})$.

Remark: Note that the mappings t^i are continuous.

Clearly all $\mathcal{T}^i$ are copies of the same space which we denote by $\mathcal{T}$.

The spaces $\mathcal{Y}$ and $\mathcal{T}$ are respectively the universal beliefs space and the universal type space generated by S (and n), and the rest of this section is devoted to prove that these $\mathcal{Y}$ and $\mathcal{T}$ in fact satisfy the properties claimed in Theorem 2.9. So far we have that 3) is satisfied by construction.

Property 1: $\mathcal{Y} = S \times [\mathop{\times}\limits_{i=1}^{n} \mathcal{T}^i]$ *(homeomorphically).*

Proof: First let us establish a one to one mapping between the two sets. Each $y \in \mathcal{Y}$ determines uniquely some $s \in S$ (namely $s = \rho_0(y)$), also by definition of $\mathcal{T}$, y determines uniquely $t^i(y) \in \mathcal{T}^i \; \forall i$. This establishes a mapping $f: \mathcal{Y} \to S \times [\mathop{\times}\limits_{i=1}^{n} \mathcal{T}^i]$.

On the other hand by its definition $\mathcal{Y}$ can be represented as:

$$Y = \{y_0, (t^1(y_k))_{k=1}^{\infty}, \ldots, (t^n(y_k))_{k=1}^{\infty} \mid y_0 \in S \; \forall k \; \forall i$$
$$t^i(y_k) \in \Pi(Y_{k-1}) \text{ and conditions a) and b) of formula (2.1)}$$
$$\text{are satisfied}\}.$$

But for certain i the conditions $t^i(y_k) \in \Pi(y_{k-1})$, a) and b) $\forall k$ are conditions only on the sequence $(t^i(y_k))_{k=1}^{\infty}$, which are satisfied by the sequence $(t_k^i)_{k=1}^{\infty}$ on $\{Y_k\}_{k=0}^{\infty}$ derived from any $t^i \in T^i$. Thus any point in $S \times [\overset{n}{\underset{i=1}{\times}} T^i]$ determines uniquely a sequence $(y_0, y_1, \ldots)$ corresponding to some $y \in Y$. So we have a mapping $g: S \times [\overset{n}{\underset{i=1}{\times}} T^i] \to Y$ which is easily verified to be the inverse of f.

Now note that by Stone-Weierstrass theorem, any continuous function on Y can be approximated by continuous functions on Y_k. This implies that the mappings t^i: $Y \to \Pi(Y)$ are continuous and hence T^i is compact $\forall i$ (since Y is compact). Also clearly the projection ρ_0: $Y \to S$ is continuous. So the mapping $\rho_0 \overset{n}{\underset{i=1}{\times}} t^i$: $Y \to S \times$ $\times (\overset{n}{\underset{i=1}{\times}} T^i)$ is one to one and continuous, and therefore it is a homeomorphism since Y is compact and $S \times (\overset{n}{\underset{i=1}{\times}} T^i)$ is a Hausdorff space.

The following lemma establishes an important property of the mappings t^i which will be needed for the rest of the proof.

Lemma 2.14: $\forall i \; \forall y \in Y$ *if* $\tilde{y} \in \text{Supp}(t^i(y))$, *then* $t^i(\tilde{y}) = t^i(y)$.

Proof: Let $(t_1^i, t_2^i, \ldots)$ and $(\tilde{t}_1^i, \tilde{t}_2^i, \ldots)$ be the sequences of marginal distributions of $t^i(y)$ and $t^i(\tilde{y})$ respectively on $Y_0, Y_1, \ldots$ $\tilde{y} \in \text{Supp}(t^i(y))$ implies that $\forall k$ the support of the marginal distribution of $t_t^i \in \Pi(Y_{k-1})$ on $\overset{k-1}{\underset{l=0}{\times}} [\Pi(Y_l)]_i$ contains $(\tilde{t}_1^i, \ldots, \tilde{t}_{k-1}^i)$. But since $y \in Y$ it follows by using repeatedly properties a) and b) of (2.1) that the marginal distribution of t_k^i on $\overset{k-1}{\underset{l=0}{\times}} [\Pi(Y_l)]_i$ assigns probability 1 to $(t_1^i, \ldots, t_{k-1}^i)$. Therefore $(\tilde{t}_1^i, \ldots, \tilde{t}_{k-1}^i) = (t_1^i, \ldots, t_{k-1}^i)$ $\forall k$ and thus $t^i(\tilde{y}) = t^i(y)$.

As an immediate consequence of Lemma 2.14, the continuity of t^i and of the projection $Y \to S$, we have:

Property 4: Y is an S-based abstract BL-Space.

Property 2: $T = \Pi(S \times [T]^{n-1})$ *(homeomorphically).*

Proof: We shall prove that $\forall\, i$, T^i is homeomorphic to $\Pi\,(S \times (\mathop{\times}_{j \neq i} T^j))$. Each $t^i \in T^i$ is an element in $\Pi\,(Y)$, hence in $\Pi\,(S \times (\mathop{\times}_{j=1}^{n} T^j))$ (by Property 1). But by Lemma 2.14, $(s, \tilde{t}^1, \ldots, \tilde{t}^n) \in \operatorname{Supp}(t^i) \Rightarrow \tilde{t}^i = t^i$. Therefore there is natural mapping f^i of T^i to $\Pi\,(S \times (\mathop{\times}_{j \neq i} T^j))$ which maps each $t^i \in T^i$ to its marginal on $S \times (\mathop{\times}_{j \neq i} T^j)$.

We want to show now that this f^i is homeomorphism: T^i being compact and $S \times (\mathop{\times}_{j \neq i} T^j)$ being Hausdorff, it is sufficient to prove that f^i is one to one and onto. For this we shall exhibit the inverse mapping of f^i: Given $\mu \in \Pi\,(S \times (\mathop{\times}_{j \neq i} T^j))$ we want to show the existence of $y \in Y$ s.t. the marginal of $t^i(y)$ on T^i is a unit mass at $t^i(y)$ and on $S \times (\mathop{\times}_{j \neq i} T^j)$ is μ. By Property 1) it is enough to define a sequence $(t_1^i, t_2^i, \ldots)$ of marginal distributions on $Y_0, Y_1, \ldots$, respectively which will satisfy conditions a) and b) of (2.1) $\forall\, k$ and which define an element of $\Pi\,(Y)$ having the correct marginal distributions.

For each $k \geqslant 1$, let μ_k be the marginal distribution of μ on $(S \times \mathop{\times}_{l=0}^{k-1} \mathop{\times}_{j \neq i} [\Pi\,(Y_l)]_j)$ (that is the factor space of Y_k which does not involve coordinate i). Let $t_1^i = \rho_0(\mu) =$ the marginal distribution of μ on S and define inductively

$t_k^i \in \Pi\,(Y_{k-1})$ by: $t_k^i = \mu_k \times \delta_{(t_1^i, \ldots, t_{k-1}^i)}$, $k \geqslant 2$. It follows readily from the construction that $(t_k^i)_{k=1}^{\infty}$ has the required properties. This completes the proof of Property 2.

Definition 2.15: A closed subset $\tilde{C}$ of Y which satisfies

$$\forall\, y \in \tilde{C},\ \forall\ i \in I,\ t^i(y) \text{ is supported by } \tilde{C} \tag{2.2}$$

will be called *beliefs closed (BL*-closed*)* or a *beliefs subspace (BL*-subspace*)* of Y.

Property 5: The S-based abstract BL-spaces are the BL-closed subspaces of Y (by BL-morphism), under the non-redundancy condition (Definition 2.4).

Proof: Let C be an S-based abstract BL-subspace. We shall define a mapping y: $C \to Y$ by defined for each $c \in C$ an $y(c) = (s, (t_k^1)_{k=1}^{\infty}, \ldots, (t_k^n)_{k=1}^{\infty})$ where $s \in S$ and each of the sequences $(t_k^i)_{k=1}^{\infty}$ is a sequence of distributions on $(Y_k)_{k=0}^{\infty}$ respectively, satisfying conditions a), b) (of 2.1) for all $k \geqslant 1$: Remark that any point $(s, (t_k^1)_{k=1}^{K}, \ldots, (t_k^n)_{k=1}^{K})$ which satisfy a) and b) $\forall\, i$ and $\forall\, k$ determines uniquely a

point in Y_k, therefore by defining $y_K(c) = (s, (t_k^1)_{k=1}^K, \ldots, (t_k^n)_{k=1}^K)$ $\forall c \in C$ we are defining a mapping $y_K : C \to Y_K$ and hence an induced mapping $\hat{y}_K : \Pi(C) \to \Pi(Y_K)$. We construct these mappings inductively on K: $\forall c \in C$ let $y_0(c) = f(c) \in S$ and for $K = 1, 2, \ldots$, define $\forall i$, $t_K^i = \hat{y}_{K-1} \circ t^i : C \to \Pi(Y_{K-1})$, where $\hat{y}_{k-1}$ is the mapping $\hat{y}_{k-1} : \Pi(C) \to \Pi(Y_{k-1})$ canonically induced by y_{k-1}.
Using condition (*) of Definition (2.2), it follows that $\forall c \in C$ the above defined $y(c)$ in fact satisfies the required condition and hence corresponds to a point in Y. Furthermore, since f and t^i are continuous, it follows inductively that y_K $\forall K$ are continuous mappings and hence the defined $y: C \to Y$ is continuous. If we denote $y(C) = \tilde{C} \subseteq Y$ then it is clear from the cobstruction that $\tilde{C}$ satisfies (2.2) i.e. it is a BL-subspace of Y. At this point we have to notice the following proposition whose proof follows readily from the definitions:

Proposition 2.16: Any BL-closed subset of Y is an S-based abstract BL-space (with respect to the projections of $Y = S \times (\overset{n}{\underset{i=1}{\times}} T^i)$ on its factor spaces).

Using the terminology of Lemma 2.8 and the remark that follows it, the mapping $y: C \to \tilde{C}$ we constructed is the BL-morphism from C to Y induced by the identity on S (since Y clearly satisfies the NR-condition). Using the same notation the above constructed y is clearly invertible, and hence BL-isomorphism between $H = h(C)$ (the space of infinite hierarchies generated by C) and $\tilde{C}$. Therefore if C satisfies the NR-condition we use Proposition 2.5 to deduce that $y: C \to \tilde{C}$ is a BL-isomorphism. This concludes the proof of property 5.

Property 6: For any coherent beliefs hierarchy $(C_0, C_1, \ldots, C_K)$ there is a BL-subspace C of Y s.t. $\rho_k(C) = C_k$, $k = 0, \ldots, K$.

Proof: By condition (4) of Definition 2.1, $\forall t_K^i \in t^i(C_K)$:

$$\operatorname{Supp}(t_K^i \times \delta_{t_K^i}) \subseteq \text{Projection of } C_K \text{ on } C_{K-1} \times [\Pi(C_{K-1})]_i.$$

It follows (for instance by Lemma 2.12) that there is an extension of t_K^i to a probability distribution $\tilde{t}^i$ on $C_K \subseteq C_{K-1} \times [\Pi(C_{K-1})]_1 \times \ldots \times [\Pi(C_{K-1})]_n$. Take all possible such extensions for each $t_K^i \in t^i(C_K)$, $\forall i$, to define C_{K+1}. Prove that $C_0, \ldots, C_K, C_{K+1}$ is a coherent beliefs hierarchy of level $K+1$, and proceed inductively as in the construction of Y to construct a limiting $C \subseteq Y$ which be the required BL-subspace.

Property 7: The minimality properties of Y *and* T.

- If $\tilde{Y}$ and $\tilde{T}$ satisfy 1) and 2), then $\tilde{Y}$ is an S-based abstract BL-space therefore by the proof of 5), it can be mapped BL-morphically onto some BL-subspace $\tilde{C}$ of Y. By 1), $\rho_0(\tilde{C}) = S$ and inductively (using 1) and 2)) $\rho_k(\tilde{C}) = Y_k \ \forall k$ hence $\tilde{C} = Y$. The mapping from $\tilde{T}$ onto T is induced accordingly.
- Assume that $\tilde{Y}$ and $\tilde{T}$ satisfies 4) and 5). By 5) since the Y we constructed satisfy the NR-conditions, there is a compact $\tilde{C} \subseteq \tilde{Y}$ and a BL-morphism $\varphi: Y \to \tilde{C}$ which induces the identity on S. On the other hand, by 4) $\tilde{Y}$ is an S-based BL-subspace, it follows from the proof of 5) that there is a BL-morphism ψ from $\tilde{Y}$ to a BL-subspace of Y which also induces the identity on S, therefore the composed BL-morphisms $\psi \circ \varphi: Y \to Y$ must be the identity and hence $\psi = \varphi^{-1}$ and Y is BL-isomorphic to the BL-subspace $\tilde{C}$ of $\tilde{Y}$. The mapping of $\tilde{T}$ onto T is induced in the natural way.
- If $\tilde{Y}$ an S-based BL-space which satisfy 6), then since the $(Y_0, Y_1, \ldots)$ we defined is a coherent beliefs hierarchy, there is a BL-subspace $\tilde{C}$ of $\tilde{Y}$, s.t. $\rho_k(\tilde{C}) = Y_k$, $\forall k$, thus $\tilde{C}$ is BL-homeomorphic to the projective limit of $(Y_0, Y_1, \ldots$ namely Y. By the same argument, $\tilde{Y}$ satisfy 3) and Y satisfy 6) imply that $\tilde{Y}$ is BL-homeomorphic to a BL-subspace of Y. Since the two BL-morphism induce the identity on S we obtain the required result.

This concludes the proof of Theorem 2.9.

Remark 2.17: A very common situation of incomplete information is that in which in addition to incomplete information about S each player has some ***private information*** which *may depend on the state of nature*. For instance if each player know his own utility function. Can such a situation be incorporated in our model? In other words can we construct a BL-subspace in which each player knows his private information and it is a common knowledge that such is the situation? This in fact can be done as follows: Let $h_i: S \to H_i$ be the private information function of player i which assigns to each state $s \in S$ the element $h_i(s)$ of some space H_i. We would like to construct a BL-subspace $C \subseteq Y$, with the property: $\forall i \ \forall y \in C$, the distribution of $h_i \circ \rho_0$ under $t^i(y)$ is a unit mass at $h_i \circ \rho_0(y)$. To do this let $C_0 = S$ and

$$C_1 = \{(s, t_1, \ldots, t_n) \mid s \in S,\ t_i \in \Pi\,(h_i^{-1}(h_i(s)));\quad i = 1, \ldots, n\}$$

(C_0, C_1) is trivially a beliefs hierarchy which can therefore be closed to a BL-subspace by property 6. This BL-subspace will have the required property.

Remark 2.18: If S is finite or countable or a standard Borel space, all our results are purely measure theoretic: indeed the set of probabilities Π on a standard Borel space S is again a standard Borel space (with σ-field generated by $\{\pi \mid \pi(B) \geqslant \alpha\}$; B is a

Borel set in S and $\alpha \in R$) and this σ-field is the same as the one we derive from the weak* topology.

To carry also the results on BL-subspaces and on abstract BL-spaces to the measure theoretic setup, one has first to rewrite the above proofs for the case where those concepts would be defined as analytic spaces instead of compact spaces. We are quite convinced that except for technical complication, this extension of the theory causes no serious difficulty.

3. Approximation of a BL-Subspace by a Finite BL-Subspace

In this section we prove the following approximation theorem.

Theorem 3.1: For any closed BL-subspace C of $\mathcal{Y}$ and any finite open cover $\mathcal{O}$ of $\mathcal{Y}$, there is a finite BL-subspace C^* of $\mathcal{Y}$ s.t.

(i) $C \subseteq \cup \{0 \in \mathcal{O} \mid 0 \cap C^* \neq \emptyset\}$

(ii) $C^* \subseteq \cup \{0 \in \mathcal{O} \mid 0 \cap C \neq \emptyset\}$.

In other words, Theorem 3.1 states that:

The finite BL-subspaces of $\mathcal{Y}$ are dense in the set of all BL-subspaces of $\mathcal{Y}$, in the Hausdorff topology on closed subsets on $\mathcal{Y}$.

To prove this theorem we use the following known result (see e.g. Kelley, General Topology 6.33, p. 199).

Lemma 3.2: Let X be a compact space. For any finite open cover $\mathcal{O}$ of X there is a neighborhood $\mathcal{V}$ of the diagonal in $X \times X$ s.t. $\forall x \in X, \exists\ 0_x \in \mathcal{O}$ which satisfies: $(x, y) \in \mathcal{V} \Rightarrow y \in 0_x$.

Remark 3.3: Clearly $\mathcal{V}$ in Lemma 3.2 can be taken to be a basic neighborhood of the diagonal (for instance one which is generated by a finite open cover of X).

Lemma 3.4: For any finite open cover $\mathcal{O}$ of a compact space X there is a finite open cover $\mathcal{W}$ s.t. $\forall x, y, z \in X$, if (x,y) are $\mathcal{W}$-close and (y, z) are $\mathcal{W}$ close, then (x, z) are $\mathcal{O}$-close.

Proof: By Remark 3.3 let $\mathcal{V}$ be a neighorhood of the diagonal $X \times X$ satisfying the conclusion of Lemma 3.2 and which is generated by some finite open cover $\mathcal{W}$ of X. This $\mathcal{W}$ satisfies the required property.

Notation: We shall denote by Ref $(\mathcal{O})$ all such finite open covers $\mathcal{W}$ given by Lemma 3.4.

Lemma 3.5: Let K be a compact space. Given a finite open cover $\mathcal{O}$ of $\Pi(K)$, then there is a finite set of continuous functions $f_1, \ldots, f_n$ on K s.t. $\forall \nu \in \Pi(K), \exists\ 0_\nu \in \mathcal{O}$ such that $|\mu(f_j) - \nu(f_j)| \leqslant 1, \forall j = 1, \ldots, n$ implies $\mu \in 0_\nu$.

Proof: Using Lemma 3.2 for $X = \Pi(K)$ let V be a neighborhood of the diagonal in $\Pi(K) \times \Pi(K)$ satisfying the conclusions of the lemma. In view of Remark 3.3, V can be taken to be a basic open neighborhood of the diagonal, i.e. of the form:

$$V = \{(\nu, \mu) \mid |\nu(f_j) - \mu(f_j)| \leqslant 1; j = 1, \ldots, n\},$$

where $f_1, \ldots, f_n$ are continuous functions on K. This finite set of functions satisfy the required properties.

Lemma 3.6: Given a finite open cover $\mathcal{O}$ of $\Pi(K)$, then there is a finite open cover $\mathcal{U}$ of K with the property that $\forall \mu \in \Pi(K)\ \exists\, 0 \in \mathcal{O}$ s.t. if $S: K \to K$ is a measurable mapping for which $\forall x \in K$, $(x, S(x)) \in U \times U$ for some $U \in \mathcal{U}$, then $((\mu, S(\mu)) \in 0 \times 0$.

Proof: Let $f_1, \ldots, f_n$ be the continuous functions determined by Lemma 3.5 for the finite cover $\mathcal{O}$. Let $\mathcal{U}$ be a finite open cover of K s.t.:

$$(x, y) \in U \times U \text{ for some } U \in \mathcal{U} \text{ implies } |f_j(x) - f_j(y)| \leqslant 1,\ j = 1, \ldots, n.$$

We claim that this finite open cover $\mathcal{U}$ is the required one. In fact, let $\mu \in \Pi(K)$, let $0 \in \mathcal{O}$ be the open set containing μ and satisfying the conclusion of Lemma 3.5 and take such a measurable mapping φ, then $\forall j = 1, \ldots, n$:

$$|\mu(f_j) - \varphi(\mu)(f_j)| = |\mu(f_j) - \mu(f_j \circ \varphi)| \leqslant \mu(\max_{x \in K} |f_j(x) - f_j(\varphi(x))|)$$

But $\forall x \in K$, $(x, \varphi(x)) \in U \times U$ for some $U \in \mathcal{U}$ and hence $|f_j(x) - f_j(\varphi(x))| \leqslant 1, \forall j$ It follows that $|\mu(f_j) - \varphi(\mu)(f_j)| \leqslant 1$ for $j = 1, \ldots, n$ which imply by the definition of f_j that $(\mu, \varphi(\mu)) \in 0 \times 0$.

Lemma 3.7: Let $X = \bigtimes_{i=1}^{n} X_i$ where $\forall i$, X_i is a compact space. For any finite open cover $\mathcal{O}$ of X there are finite open covers $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of $X_1, \ldots, X_n$ respectively s.t. $\mathcal{V} = \bigtimes_{i=1}^{n} \mathcal{V}_i$ is an open cover of X which is finer than $\mathcal{O}$.

Proof: Let $\mathcal{O} = \{0_1, \ldots, 0_n\}$ and let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a rectangular open cover which is finer than $\mathcal{O}$. As usual denote by ρ_i the projection $X \to X_i$ and $\forall x_i \in X_i$ let $V_{x_i} = \cap \{\rho_i(U_j) \mid x_i \in \rho_i(U_j)\}$. Then take the finite covers $\mathcal{V}_i = \{V_{x_i} \mid x_i \in X_i\}$.

Notation: We shall denote by $\mathcal{RP}(\mathcal{O}, X_1, \ldots, X_n)$ the set of all such product covers refining $\mathcal{O}$, provided by Lemma 3.7.

Having done these preparations we proceed now to prove the main result of this section, Theorem 3.1.

By Theorem 1.3 we write $Y = \mathop{\mathsf{X}}_{i=0}^{n} T^i$, where $T^0 = S$ and for $i = 1, \ldots, n$ $T^i = t^i(Y)$ is the type set of player i.

Consider $A = \{O^\alpha\}$, the increasing net of all finite open covers of Y with the partial order: $O^\alpha \leqslant O^\beta$ iff O^α refines O^β. When no confusion may result we will denote the elements of A by $\alpha, \beta, \ldots$, instead $O^\alpha, O^\beta, \ldots$ Accordingly we will write $\alpha \leqslant \beta$ instead of $O^\alpha \leqslant O^\beta$.

Let C be a closed BL-subspace of Y. $\forall\, \alpha \in A$ let $(O_0, \ldots, O_n) \in RP(O^\alpha, T^0, \ldots, T^n)$ and $\forall\, i$ let $P^i = (P^i_1, \ldots, P^i_n)$ be a measurable partition of T^i s.t. $P \in P^i \Rightarrow \exists\, 0 \in O_i, P \subseteq 0$. Such $(n+1)$tuple of partitions $P = (P^0, \ldots, P^n)$ will be *finer than the open cover* α.

$\forall\, i, i = 0, 1, \ldots, n, \forall\, j, j = 1, \ldots, n_i$, let t^i_j be any fixed point in $P^i_j \cap \rho^i(C)$ if this intersection is non empty and any point in P^i_j otherwise. $\forall\, i$ let $X^i = \{t^i_j\}$ and let $X = \mathop{\mathsf{X}}_{i=0}^{n} X^i$.

Define the mapping $\varphi_0 : Y \to Y$ by

$$\varphi_0(t^0, \ldots, t^n) = (\bar{t}^0, \ldots, \bar{t}^n)$$

where $\forall\, i\, \exists\, j : t^i \in P^i_j$ and $\bar{t}^i = t^i_j$.

Clearly $\rho_0(Y) = X \subseteq Y$. Remark also that $\bar{t}^i$ depends only on t^i, therefore φ_0 defines also uniquely mappings $T^i \to T^i$ which will all be denoted by φ_0 to avoid additional notation.

For $i \geqslant 1$ and for each $t^i \in X^i$, define the following probability distribution $P^i_{t^i}$ on X by $P^i_{t^i}(x) = t^i(\varphi_0^{-1}(x))$. Remark that $P^i_{t^i}(t^0_{j_0}, \ldots, t^n_{j_n}) > 0 \Rightarrow t^i_{j_i} = t^i$.

If we denote $\forall\, i$ by P^i the mapping $t^i \to P^i_{t^i}$ from X^i to $\Pi(X)$, then by our definition $(X, P^1, \ldots, P^n)$ is some S-based abstract BL-space and so is also $(\tilde{X}, P^1, \ldots, P^n)$ where $\tilde{X} = \varphi_0(C)$. By Proposition 3.5, it is homeomorphic to some (finite) BL-subspace of Y which we will denote by $\tilde{C}_\alpha$. Since $(X, P^1, \ldots, P^n)$ is determined solely by φ_0, we have a mapping $\psi_{\varphi_0} : X \to Y$ such that:

$$\tilde{C}_\alpha = \psi_{\varphi_0}(\tilde{X}) = (\psi_{\varphi_0} \circ \varphi_0)(C).$$

Proposition 3.8: $\tilde{C}_\alpha$ converges to C (in the Hausdorff topology on closed subsets of Y).

Proof: Considering the mapping $\varphi_\alpha = \psi_{\varphi_0} \circ \varphi_0 : C \to \tilde{C}_\alpha$, note that φ_α is not determined uniquely by α but also by the special choice of the finite measurable partitions $\mathcal{P}^0, \ldots, \mathcal{P}^n$ and by the special choice of the points $\{t_j^i\}$. So let ϕ_α be the set of all such mappings φ_α i.e.,

$$\phi_\alpha = \{\varphi \mid \text{There is a partition } \mathcal{P} = (\mathcal{P}^0, \ldots, \mathcal{P}^n) \text{ finer than } \alpha \text{ and a choice of } \{t_j^i\} \text{ that yield } \varphi\}.$$

It is sufficient to prove that ϕ_α converges uniformly to the identity mapping on C, i.e.

$$\forall \alpha \in \mathrm{A}\ \exists\ \beta \in \mathrm{A} \text{ such that } \forall \varphi \in \phi_\beta\ \forall x \in C,\ \varphi(x) \text{ is } \mathcal{O}^\alpha\text{-close to } x.$$

For the next argument we recall the definition of $\mathcal{Y}$ as the (projective) limit of Y_k and write a generic point in $\mathcal{Y}$ as $y = (s, t_1, \ldots, t_k, \ldots)$ where $s \in S$ and $\forall\, k$, $t_k = (t_k^1, \ldots, t_k^n)$, $t_k^i \in \Pi\,(Y_{k-1})\ \forall\, i$. We shall refer to t_k as the k-th coordinate of y (s being the 0-coordinate) and $\forall\, k \geqslant 0$ define:

$$\mathrm{A}_k = \{\alpha \in \mathrm{A} \mid \forall\, 0 \in \mathcal{O}^\alpha, 0 \text{ is defined in terms of the first } k \text{ coordinates}\}.$$

Since any cover $\mathcal{O}^\alpha$ is refined by some cover $\widetilde{\mathcal{O}}^\alpha$ involving only a finite number of coordinates, it is sufficient to prove that:

$$(*)\qquad \forall\, \alpha \in \mathrm{A}_k, \exists\, \beta \in \mathrm{A}: \forall\, \gamma \geqslant \beta, \forall\, \varphi \in \phi_\gamma, \forall\, x \in C; \varphi(x) \text{ is } \mathcal{O}^\alpha\text{-close to } x.$$

We shall prove (*) by induction on k:

For $k = 0$ the statement is obvious from our definitions, taking $\beta = \alpha$. Assume that (*) is true for k and let us prove it for $k + 1$: Let $\alpha \in \mathrm{A}_{k+1}$ and $\forall\, i \geqslant 1$ let $\mathcal{V}_i$ be a finite open cover of T_{k+1}^i and let $\mathcal{V}_0$ be a finite open cover of Y_k such that

$\mathcal{V}_0 \times \bigtimes_{i=1}^{n} \mathcal{V}_i$ is finer than $\mathcal{O}^\alpha$.

$\forall\, i$, let $\widetilde{\mathcal{V}}_i \in \mathcal{R}\mathrm{ef}\,(\mathcal{V}_i)$ and $\forall\, i \geqslant 1$, let $\mathcal{W}_i$ be a finite open cover of Y_k such that for any measurable $\mathcal{W}_i$-shift ψ of $\mathcal{Y}$ (i.e., $\forall\, y \in Y_k$, $(y, \psi(y)) \in W \times W$ for some $W \in \mathcal{W}_i$) and $\forall\, t^i \in T^i$, $\psi(t^i)$ is $\mathcal{V}_i$-close to t^i (see Lemma 4.2). Let $\overline{\mathcal{V}}_0$ be any common refinement of $(\widetilde{\mathcal{V}}_0, \mathcal{W}_1, \ldots, \mathcal{W}_n)$ and let $\widetilde{\mathcal{V}} = \overline{\mathcal{V}}_0 \times \bigtimes_{i=1}^{n} \widetilde{\mathcal{V}}_i$.

Finally, if we denote by $\beta_{\overline{\mathcal{V}}_0}$ the $\beta \in \mathrm{A}$ satisfying (*) for $\overline{\mathcal{V}}_0 \in \mathrm{A}_k$ (by induction hypothesis), the required β which corresponds to the given $\alpha \in \mathrm{A}_{k+1}$ is $\beta = \max\,(\beta_{\overline{\mathcal{V}}_0}, \widetilde{\mathcal{V}})$. Let us prove that this β in fact satisfies the property stated in (*). For $x \in Y_{k+1}$, it will be convenient to use the notation $t^i(x)$ for $\rho^i(x)$.

Let $\gamma \geqslant \beta$ and let $\varphi \in \phi_\gamma$, $x \in C$, we have to show that $\varphi(x)$ is $\mathcal{O}^\alpha$-close to x.

By definition of $\beta_{\tilde{V}_0}$ we have that $\varphi(x)$ is $\tilde{V}_0$-close to x, therefore there remains to show that $\forall i$, $t^i(\varphi(x))$ is V_i-close to $t^i(x)$.

Since $\gamma \geqslant \beta \geqslant \tilde{V}$ we know that if $\varphi = \psi_{\varphi_0} \circ \varphi_0$, then $\varphi_0(x)$ is $\tilde{V}$-close to x. Thus $\forall t^i \in T^i, \varphi_0(t^i)$ is $\tilde{V}_i$-close to t^i. Extend φ (defined on C) to φ: $Y \to Y$ by defining $\varphi(x) = x$ for $x \notin C$. We have then that $\varphi(x)$ is $\tilde{V}_0$-close to x $\forall x \in Y$ and hence $\forall i \geqslant 1, \forall t^i \in T^i, t^i \circ \varphi^{-1}$ is $\tilde{V}_i$-close to t^i (see definition of W_i). Thus $\forall t^i \in \rho^i(C)$, t^i and $t^i \circ \varphi^{-1}$ are two probability distributions on C and on $\tilde{C}_\alpha$ respectively which are $\tilde{V}_i$-close. In particular for $t^i = t^i_j \in \rho^i(C)$, $t^i_j \circ \varphi^{-1} = P^i_{t^i_j}$ is $\tilde{V}$-close to t^i_j. Therefore $\forall x \in C$, $t^i(\varphi_0(x)) \circ \varphi^{-1} = P^i_{t^i(\varphi_0(x))}$ is a probability distribution on $\tilde{C}_\alpha$ which is $\tilde{V}_i$-close to $t^i(\varphi_0(x))$ which is on the other hand $\tilde{V}_i$-close to $t^i(x)$ (on C; since $\varphi_0(x)$ is V-close to $x \ \forall x \in C$).

Since by definition of $\tilde{C}_\alpha$ $P_{t^i(\varphi_0(x))} = t^i(\varphi(x))$ and since $\tilde{V}_i \in \text{Ref}(V_i)$ we conclude that $t^i(\varphi(x))$ is V_i-close to $t^i(x)$, completing the proof of Theorem 3.1.

4. Consistency

Summing up the structure developed so far: We started from a compact set S of possible games and we constructed from it the universal BL-space Y generated by S. This may be thought of as the space of "states of the world" in the sense that each point $y \in Y$ defines completely all levels of beliefs and mutual beliefs for all the players. At each state $y \in Y$, player i certainly knows his own (subjective) probability (distribution $t^i(y)$) on Y. We shall also denote this distribution by P^i_y.

Nothing was said so far as to *what is* the actual state of the world? According to what procedure is it determined? What are the relations, if any, between the beliefs of the different players? Following Harsanyi we ask: Are there situations in which the subjective beliefs of the players, namely P^i_y, are equal to the conditional probabilities, given each player's private information, derived from some "prior" probability distribution P on Y? Can one characterize those points in y for which this is in fact the case? In this section, we answer these questions and in the next section we discuss their game theoretical relevance.

Let Y be a closed BL-subspace of Y.

Definition 4.1: A probability distribution $P \in \Pi(Y)$ is said to be *consistent* if:

$$P = \int_Y P^i_y \, dP \qquad \forall i, i = 1, \dots, n. \tag{4.1}$$

The following proposition proves that this definition in fact captures the intuitive meaning of consistency we have in mind, namely: If P is consistent then for each player i, his subjective probability P^i_y equals the conditional P-probability given his type. In other words, P may be regarded as a prior distribution on Y having the P^i_y as posteriors. Formally, with the appropriate measurability structure on Y and on $\Pi(Y)$, let $T(t^i)$ be the sub σ-field of measurable sets in $\Pi(Y)$ generated by the projection t^i, then

Proposition 4.2: If $P \in \Pi(Y)$ is consistent, then:

$$P^i_y(A) = P(A \mid T(t^i)) \;\forall y, \forall i, \forall A\text{-borel subset of } Y. \tag{4.2}$$

Proof: To see the idea more clearly we shall first prove the proposition for the simple case in which Y is finite and then provide a proof for the general case which asks for more careful measurability considerations.

Proof for finite Y:

The projections t^i: $Y \to \Pi(Y)$ define for each i a partition T^i of any subset $y \subset Y$ into subsets of various types of player i, namely

$$T^i(y) = \{\tilde{y} \in Y \mid P^i_{\tilde{y}} = P^i_y\} = (t^i)^{-1}(t^i(y)).$$

With this notation, the statement of Lemma 2.14 can be rewritten as

$$\forall i, \forall y \in Y, \operatorname{Supp}(P^i_y) \subset T^i(y). \tag{4.3}$$

When Y is finite, (4.2) becomes

$$P^i_y(A) = P(A \mid T^i(y)) \;\;\forall y \in Y, \forall A \subset Y. \tag{4.4}$$

Actually we want to prove this whenever it has any meaning, namely whenever $P(T^i(y)) > 0$. (This will be satisfied if $y \in \operatorname{Supp}(P)$.) Now we write (4.1) as

$$P(A) = \sum_{\tilde{y} \in Y} P^i_{\tilde{y}}(A) P(\tilde{y}) \qquad \forall A \subset Y.$$

So

$$P(A \cap T^i(y)) = \sum_{\tilde{y} \in Y} P^i_{\tilde{y}}(A \cap T^i(y)) P(\tilde{y}).$$

But by (4.3), Supp $(P^i_{\tilde{y}}) \subset T^i(\tilde{y})$ hence

$$P^i_{\tilde{y}}(A \cap T^i(y)) = \begin{cases} 0 & \tilde{y} \notin T^i(y) \\ P^i_y(A \cap T^i(y)) & \tilde{y} \in T^i(y). \end{cases}$$

Also, again by (4.3), $P^i_y(A \cap T^i(y)) = P^i_y(A)$, so

$$P(A \cap T^i(y)) = P^i_y(A)\,P(T^i(y)),$$

which is

$$P^i_y(A) = P(A \mid T^i(y)),$$

what has to be proved.

Proof in the general case:

Notice first that by the regularity of the measures P and P^i_y, and the continuity of $t^i: Y \to P^i_y$, equation (4.1) extends from the continuous functions on Y to all upper-semicontinuous functions on Y, and therefore, first by a monotone class argument to all bounded functions, and finally from those to all bounded universally measurable functions f on Y, by bracketing f between two borel functions $\underline{f} \leqslant f \leqslant \overline{f}$ with the same integral w.r.t. P.

Remark also that this argument implies that P^i_y applied to a Baire (resp. Borel. resp. universally measurable) function yields a similar function.

Thus, letting F (t^i) stand for any of those σ-fields on $T^i = t^i(Y)$ we know that P^i_y is a transition probability from T^i to Y, and there remains to show that, for any measurable set A in Y, $P^i_j(A)$ is the conditional expectation of I_A (the indicator function of A) given F (t^i), i.e. that for any $B \in$ F (t^i)

$$\int_B P^i_y(A)\,dP(y) = \int_B I_A\,dP(y).$$

The right-hand side is equal to $P(A \cap B)$ so that this equation will follow from 4.1 – applied to the measurable set $A \cap B$ – if we show that $I_B P^i_y(A) = P^i_y(A \cap B)$. This follows readily from the fact that t^i is constant on the support of P^i_y, so that this full support is either in B or disjoint from B. This concludes the proof of Proposition 4.2.

Clearly the first question to be asked is: Does a consistent distribution exist for every BL-subspace? The following example answers this question negatively.

Example 4.3: Consider a situation of two players each of which has two types. The BL-subspace Y has thus four points corresponding to the four possible couples of types:

$$Y = \begin{bmatrix} 11 & 12 \\ 21 & 22 \end{bmatrix}$$

At each point of Y (an entry of the matrix), the first coordinate denotes the type of player 1 and the second is that of player 2. Similarly we denote the subjective probabilities of the players by:

$$\begin{bmatrix} p_1 & 1-p_1 \\ p_2 & 1-p_2 \end{bmatrix} \text{ for player 1 and } \begin{bmatrix} q_1 & q_2 \\ 1-q_1 & 1-q_2 \end{bmatrix} \text{for player 2, i.e.}$$

player 1's probability distribution on the types of player 2 is $(p_1, 1-p_1)$ is he is of first type and $(p_2, 1-p_2)$ if he is of second type. Similarly for player 2.

We write a general element of $\Pi(Y)$ as $P = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$ $\alpha_{ij} \geqslant 0$, $\Sigma\alpha_{ij} = 1$).

For P to be consistent it has to satisfy:

$$\frac{\alpha_{11}}{\alpha_{12}} = \frac{p_1}{1-p_1} \text{ and } \frac{\alpha_{11}}{\alpha_{21}} = \frac{q_1}{1-q_1}, \text{ hence } \frac{\alpha_{12}}{\alpha_{21}} = \frac{1-p_1}{p_1} \cdot \frac{q_1}{1-q_1} \underset{\text{Def}}{=} f(p_1, q_1)$$

also:

$$\frac{\alpha_{22}}{\alpha_{12}} = \frac{1-q_2}{q_2} \text{ and } \frac{\alpha_{22}}{\alpha_{21}}\,\frac{1-p_2}{p_2}, \text{ hence } \frac{\alpha_{12}}{\alpha_{21}} = \frac{q_2}{1-q_2} \cdot \frac{1-p_2}{p_2} = f(1-q_2, 1-p_2).$$

So unless $\frac{1-p_1}{p_1} \cdot \frac{q_1}{1-q_1} = \frac{q_2}{1-q_2} \cdot \frac{1-p_2}{p_2}$, which is generally not the case, there is no consistent distribution on Y.

We proceed to show that given a BL-subspace Y (or equivalently an abstract BL-space, see (5) of Theorem 2.9), there is a natural way to identify what we shall call *finite consistent subset of* Y.

Assume that, in view of our approximation results, we consider a *finite* BL-subspace Y. For each state of the world $y \in Y$ and for each player i define

$$C^i_{y,1} = \mathrm{Supp}\,(P^i_y),$$

and inductively

$$C^i_{y,k+1} = C^i_{y,k} \cup \Big[\bigcup_{\tilde{y} \in C^i_{y,k}} \bigcup_j \mathrm{Supp}\,(P^j_{\tilde{y}})\Big], \quad k = 1, 2, \ldots$$

We have $C^i_{y,1} \subset C^i_{y,2} \subset \ldots$, and since Y is finite, a limiting set will be reached which we shall denote by C^i_y. This is *according to player i's beliefs*, the minimal BL-subspace containing the real state of the world and it satisfies: $z \in C^i_y \Rightarrow C^i_z \subseteq C^i_y$. Of course, C^i_y may not be really a BL-subspace; it may even fail to contain y. However we have:

Proposition 4.4: If $y \in \text{Supp}(P)$, for some consistent P with finite support, then: $C^i_y = C^j_y$ for all i and j. Denoting this set by C_y then $y \in C_y$ (and hence C_y is in fact the minimal BL-Subspace containing y).

Proof: First observe that

$$y \in \text{Supp}(P) \Rightarrow y \in \text{Supp}(P^i_y) \subset \text{Supp}(P) \quad \forall\, i. \tag{4.5}$$

In fact, by Proposition 4.2, $P^i_y(y) = P(y \mid T^i(y)) > 0$ since $P(y) > 0$, proving that $y \in \text{Supp}(P^i_y)$. The second inclusion in (4.5) is also obtained by the same equality:

$$P^i_y(z) > 0 \Rightarrow P(z \mid T^i(y)) > 0 \Rightarrow P(z) > 0 \Rightarrow z \in \text{Supp}(P).$$

By (4.5), if we let $C^i_{y,0} = \{y\}$, then $C^i_{y,k+1} = \bigcup_{\tilde{y} \in C^i_{y,k}} \bigcup_j \text{Supp}(P^j_{\tilde{y}})$. Obviously it follows by induction that $C^i_{y,k}$ is the same for all i, and hence so is C^i_y, proving the proposition.

Note that in the situation described in the Proposition, C_y is a *common knowledge*, i.e. it can be computed by each player and by an outside observer *only* from knowing the set Y.

The following proposition shows that not only that for each y in the support of some consistent distribution, C_y is uniquely determined and is a common knowledge, but that there is a uniquely determined probability distribution on C_y which is also a common knowledge.

Proposition 4.5: For any consistent P of finite support and for any y and i, either $P(C^i_y) = 0$, or $P(\cdot \mid C^i_y)$ is uniquely determined by Y.

Proof: By the consistency of P it follows from Proposition 4.2 that

$$P(z) > 0 \text{ and } y \in \text{Supp}(P^i_z) \Rightarrow \frac{P(y)}{P(z)} = \frac{P^i_z(y)}{P^i_z(z)} > 0. \tag{4.6}$$

Proceed now by induction on k: Assume that either $P(\tilde{y}) = 0 \;\forall\, \tilde{y} \in C^i_{y,k}$ or $\frac{P(\tilde{y})}{P(z)} > 0$ is uniquely defined by $Y \;\forall\, \tilde{y}, z \in C^i_{y,k}$. By (4.6) we then have that the same statement is true also for $C^i_{y,k+1}$. Since the statement is trivially true for C_0 it is true for C^i_y.

Note that if $P(C^i_y) = 0$ for some consistent P then also $P'(C^i_y) = 0$ for any other consistent distribution P'.

Definition 4.6: A BL-subspace C on which there exists a consistent distribution P with Supp $P = C$, will be called a *consistent BL-subspace* or shortly a C-subspace. Any $y \in C$ will be called a *consistent state of the world* (otherwise it is said to be inconsistent).

A combination of Proposition 4.4 and 4.5 yields:

Corollary 4.7: A state y is consistent if it is in the support of some consistent distribution P. If y is a consistent state of the world, then the C-subspace containing it is $C = C^i_y$ $\forall$ i, and the consistent distribution on C is uniquely determined (by C) and is a common knowledge.

In view of Corollary 4.7, it makes sense to think of a consistent distribution as a *prior distribution*, not only because it is so, mathematically speaking, but also because it may be assumed to be known by the players as it is usually assumed in the Bayesian approach.

The question of consistency of the state of the world can now be presented as the problem of testing the following hypothesis:

H_c: *"The actual state of the world y is consistent."*

By corollary 4.7, if H_c is true, then each player i will reach the *same* set $C^i_y \underset{\text{def}}{=} C$ and the *same* consistent distribution on it, $P(\cdot \mid C)$, hence:

If player i finds no consistent P on the C^i_y he computed, he may *reject* H_c with no *possible error being committed.*

What if player i finds a consistent P defined on his C^i_y? Should he accept H_c? Clearly such a C^i_y with the consistent P on it is a C-subspace (by definition). The only question is whether it contains the real state of the world y. If $y \in$ Supp (P^i_y) then by definition of C^i_y in fact $y \in C^i_y$ and hence H_c is true.

In other words a player i that finds a consistent C^i_y and accepts H_c commits an error *only* if $y \notin$ Supp (P^i_y), i.e. only if he assigns probability 0 (in the finite case) to the real state of the world. So, in particular his subjective probability of commiting an error is 0. So we have:

The hypothesis H_c is testable by each player with 0-subjective probability of error. If H_c is accepted, then the corresponding C-subspace and the consistent probability distribution is computable by each player.

Especially if we make the rather weak assumption that each player assigns positive subjective probability to any neighborhood of the real state of the world (i.e.

$y \in \text{Supp}\,(P^i_y\ \forall\ i)$, then we have:

Theorem 4.8: *Whether the real state of the world is in a C-subspace or not is a common knowledge. If it is, then the C-subspace containing it and the consistent distribution on it (the priors) are also common knowledge.*

Remark: It should be emphasized that the consistent prior distribution if there is any, is a common knowledge derived *only* from the beliefs of each player on others beliefs and not from a "new" type of beliefs on the mechanism selecting the state of the world.

The case $y \notin \text{Supp}\,P^i_y$ describes a highly "inconsistent" belief in any reasonable meaning of this word. If players are so much mistaken in their beliefs so as to consider "impossible" (i.e. has probability 0) the real state of the world, then (objectively) wrong conclusions are quite expected as the following examples show.

Example 4.9: There are two players each of which has two types, thus $Y = \begin{bmatrix} 11 & 12 \\ 21 & 22 \end{bmatrix}$.
The subjective probabilities of each player on the types of the others are given by:

$$\begin{array}{lcc} & \text{Player 2 type 1} & \text{Player 2 type 2} \\ & \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \text{Player 1, type 1 } (1, 0) & \left[11 \right. & \left. 12 \right] \\ \text{Player 2, type 2 } (2/3, 1/3) & \left[21 \right. & \left. 22 \right] \end{array}$$

If the actual state of the world is $y = 12$, then $\text{Supp}\,(P^1_y) = \{11\}$, $\text{Supp}\,(P^2_y) = \{22\}$. Both players will find the C-set $\{11, 21, 22\}$ with the (only) consistent probability distribution $(1/2, 1/3, 1/6)$. So by accepting H_c the players will be commiting (type II) error.

Note that for the state $y = 12$, in fact $y \notin \text{Supp}\,P^i_y$ for $i = 1, 2$, as it should be since both players committed an error (although each player believes with probability 1 that he is right in accepting it). Note however that inspite of its being inconsistent, the state $y = 12$ led both players to the same consistent subset $C = \{11, 21, 22\}$. The next example shows that even this is not guaranteed in an inconsistent state.

Example 4.10: Consider the previous example with different subjective probabilities, namely:

$$\begin{array}{cc} & \begin{matrix}\begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}0\\1\end{bmatrix}\end{matrix} \\ \begin{matrix}[1,0]\\ [0,1]\end{matrix} & \begin{bmatrix}11 & 12\\ 21 & 22\end{bmatrix}\end{array}$$

If $y = 12$, player 1 will find the consistent set $C = \{11\}$ with $P(11 \mid 12) = 1$, while player 2 will find $\tilde{C} = \{22\}$ with $\tilde{P}(22 \mid 12) = 1$.

Example 4.11: Y consists of 20 states with 4 types of player 1 and 5 types of player 2. Using the same notation as in the previous example Y is given by

$$\begin{array}{cc} P^2_y & \begin{matrix} 0 & \frac{3}{5} & 0 & 0 & 0\\ 1 & \frac{2}{5} & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{2}{3}\end{matrix} \\ P^1_y & \\ \begin{matrix}(1, 0, 0, 0, 0)\\ (\frac{1}{3}, \frac{2}{3}, 0, 0, 0)\\ (0, 0, 0, \frac{1}{4}, \frac{3}{4})\\ (0, 0, 0, \frac{1}{4}, \frac{3}{4})\end{matrix} & \begin{bmatrix} 11 & 12 & 13 & 14 & 15\\ 21 & 22 & 23 & 24 & 25\\ 31 & 32 & 33 & 34 & 35\\ 41 & 42 & 43 & 44 & 45\end{bmatrix}\end{array}$$

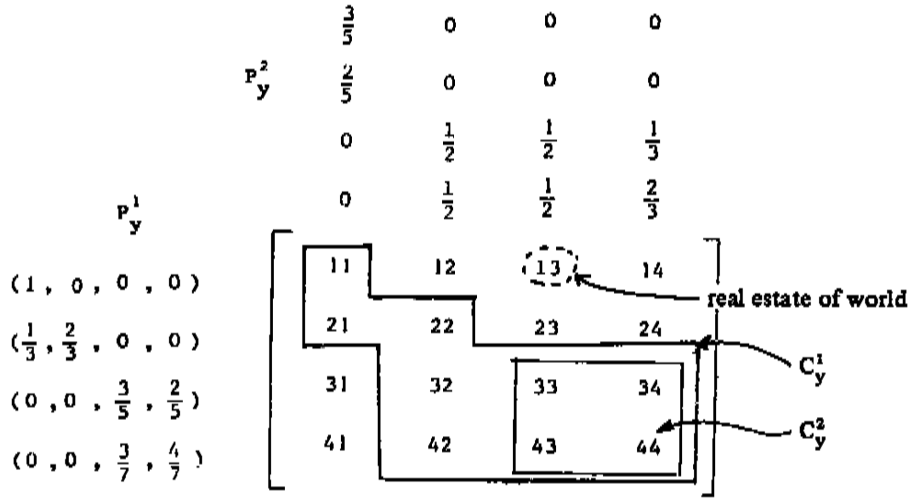
real state of world

C^1_y

C^2_y

If the actual state of nature is $y = 13$, player 1 will find $C^1_y = \{21, 22, 12\}$ with the consistent probability distribution $P = (1/6, 1/3, 1/2)$ hence he will reach the "wrong" conclusion of accepting H_c. Player 2 will find $C^2_y = \{33, 34, 43, 44, 45\}$ with no consistent P on it. He will therefore correctly reject H_c. Note that $y \notin P^1_y; y \notin P^2_y$. Unlike in previous examples player 2 reached a correct negative conclusion although $y \notin P^2_y$, but this is just a matter of accident.

Example 4.12: Consider the following BL-subspace with 16 states and 4 types for each player.



If the state is $y = 13$, then y is inconsistent and we expect player 1 to come to this conclusion. In fact, player 1 will compute $C_y^1 = \{11, 21, 22, 32, 33, 34, 42, 43, 44\}$ but no consistent distribution on it. (The verification of this is rather simple via Proposition 4.2: For any consistent P we must have $P(11) = 0$ since $P_{13}^2(11) = 0$ but also $P(11) > 0$ since $P_{13}^1(11) > 0$.) So player 1 will in fact conclude that he is in an inconsistent state. On the other hand player 2 will compute $C_y^2 = \{33, 34, 43, 44\}$ with the consistent distribution $P = (1/4, 1/6, 1/4, 1/3)$ on it.

5. Nash Equilibria

Up to now we constructed and discussed the universal BL-space and its BL-supspaces in a game situation with incomplete information. We proceed now to define a game based on $\mathcal{Y}$ (or on any abstract BL-space Y). For this we have obviously to add few ingredients:

– $\forall\, i$, player i has an action set A^i (without loss of generality this may be assumed to be independent of player i's type. One can achieve this by taking as A^i the product of the type dependent action sets over all his types).

– $\forall\, i$, $\forall\, y \in \mathcal{Y}$, there is a utility function u_y^i:

$$u_y^i : \bigtimes_{j=1}^{n} A^j \to R.$$

We define first a *vector-payoff* game in which:

- The players set is $I = \{1, 2, \ldots, n\}$
- The strategy set S^i of player i is the set of mappings

  $\sigma^i : Y \to A^i$ which is $\mathcal{T}^i$-measurable.

- The payoff to player i resulting from an n-tuple of strategies $\sigma = (\sigma^1, \ldots, \sigma^n)$ is the vector payoff:

  $$u_i = (u_{t^i})_{t^i \in T^i}$$

  (i.e. a payoff for each type t^i) where:

  $$u_{t^i}(\sigma) = \int (u^i_{\tilde{y}}(\sigma(\tilde{y})))\, dt^i(\tilde{y}).$$

Note that u_{t^i} is $\mathcal{T}^i$-measurable as it should be. Although this is not a game in the usual sense, the concept of *Nash-Equilibrium* (N.E.) can be defined in the usual way, namely: $\sigma = (\sigma^1, \ldots, \sigma^n)$ is N.E. if $\forall i$, $\forall t^i \in T^i$, $u_{t^i}(\sigma) \geq u_{t^i}(\sigma \mid \tilde{\sigma}^i)\ \forall \tilde{\sigma}^i \in S^i$, where $(\sigma \mid \tilde{\sigma}^i) = (\sigma^1, \ldots, \sigma^{i-1}, \tilde{\sigma}^i, \sigma^{i+1}, \ldots, \sigma^n)$.

Remark 5.1: When Y is a finite BL-subspace, the above defined game is an n-person game in which the payoff for player i is a vector with number of coordinates equal to the number of types of player i (namely $|T^i|$). It is easily seen that as far as N.E. are concerned this game is equivalent to what is called by Harsanyi "Selten game G^{**}" [see *Harsanyi*, 1967, 1968, Section 15, p. 496]. This is an ordinary $|T^1| \times |T^2| \times \ldots \times |T^n|$ person game in which each "player" $t^i \in T^i$ selects a strategy and *then* selects his $(n-1)$ partners, one from each T^j, $j \neq i$ according to his subjective probability distribution.

Remark 5.2: When Y is finite we can define an *ordinary game in strategic form* which is the same as the one we defined above but instead of vector payoffs we define the payoffs for player i to be $\bar{u}_i = \sum_{t^i \in T^i} \gamma_{t^i} u_{t^i}$ where $\forall t^i \in T^i$, γ_{t^i} is a strictly positive constant. Clearly, *independently of the constants* γ_{t^i} we choose, this game has the sam N.E. points as our vector payoff game (and hence as the corresponding Selten game). In particular if we take $\forall i$, γ_{t^i} to satisfy $\sum_{t^i \in T^i} \gamma_{t^i} = 1$ we get a game equivalent to that suggested by Aumann and Maschler for the inconsistent case [*Aumann/Maschle* p. 341]. As far as N.E. points are concerned their game is *independent* of the paramete γ_i. Also all these games have the same N.E. points as that suggested by Selten.

For a consistent subset C one has the following theorem, due to Harsanyi, that allows us, in looking for N.E. to replace the strategic form game by a certain extensive form game:

Theorem 5.3 (Harsanyi): Let *C be a consistent subset of Y. Let P be the consistent distribution on C. Then the strategic form vector payoff game defined by C has the same N.E. points as the following game in extensive form:*

– *A chance move chooses $y \in C$ according to P then each player is informed of P^i_y.*

– *$\forall$ i, player i then chooses $s^i \in A^i$ and receives a payoff u^i_y $(s, \ldots, s^n)$.*

Proof: It follows readily from the definition of the games, the definition of N.E. and the fact that Supp $(P) = C$.

Remark: Harsanyi calls this extensive form game "a game in standard form".

Remark 5.4: By analyzing the situation defined by C via the extensive form game, unlike Harsanyi, we *do not* claim that the players should in any way believe in P as a prior probability distribution on C. The introduction of P is just a matter of mathematical convenience. It serves to find the original N.E. points naturally defined by C via subjective probabilities.

Furthermore, by Corollary 4.7, since the "priors" are common knowledge, the above described game in extensive form is also a common knowledge which gives even more justification for using it in analyzing the situation of incomplete information.

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