*Int. Journal of Game Theory, Vol. 5, Issue 4, page 173–182. Physica-Verlag, Vienna.*

# On a Repeated Game Without a Recursive Structure

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*Abstract*: The solution is given here for the infinitely repeated two-person zero-sum games of incomplete information characterized by 2 × 2 games, with information matrices $\begin{pmatrix} a & b \\ b & b \end{pmatrix}$ for the first game and $\begin{pmatrix} b & b \\ b & a \end{pmatrix}$ for the second game.

## 1. Introduction

Two main classes of repeated two person zero-sum games with incomplete information are solved up to now:

– Games in which the information matrices matrices may depend on the player but not on the state of nature [*Mertens* and *Zamir, Mertens*].

– Games in which the information matrices do not depend on the players, may depend on the state of nature, with the additional assumption that each player recalls all prior moves [*Kohlberg* and *Zamir, Kohlberg*].

It seems that without those assumptions one loses the recursive structure that made those cases tractable.

Here an example is solved of a game not fulfilling those assumptions. It was mentioned as an open problem some six years ago [*Zamir*]:

There are two possible states of nature and accordingly two payoff matrices, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, the actual payoff matrix (i.e. the actual state of nature) is chosen once and for all by the referee (with probability $p$ for matrix $A$), and told to neither player. There are in addition two information matrices $H^A = \begin{pmatrix} a & b \\ b & b \end{pmatrix}$ and $H^B = \begin{pmatrix} b & b \\ b & a \end{pmatrix}$, $a$ and $b$ being two different letters. After each stage, if $T$ is the true payoff matrix ($A$ or $B$), and the players I and II played their pure strategies $i$ and $j$ respectively, the referee transfers $t_{ij}$ from player II's account to player I's and tell both players the letter $H^T_{ij}$. The players get no statement on their accounts before the end of the game. It is crucial here that the moves $i$ and $j$ are not stated explicitly by the referee. However, each player recalls his own move ($i$ or $j$) and all his own previous moves in addition to the information statements $H^T_{ij}$ made by the referee up to that stage.

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Notice that as soon as the letter $a$ is announced by the referee, the true matrix is revealed to both players.

The payoff in the infinitely repeated game is thought of as being $\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{k=1}^{n} t_{i_k j_k}\right)$, but is not defined due to the possible non existence of the limit. Nevertheless we will show that Min Max (and dually Max Min) of the infinite game exists in a well defined (and rather strong) sense:

Player II has an infinite game strategy that guarantees even in all sufficiently large finite games $E\left(\frac{1}{n}\sum_{k=1}^{n} t_{i_k j_k}\right) < \text{Min Max} + \epsilon$; conversely, for every infinite game strategy $\tau$ of player II, player I has an infinite game strategy $\sigma$ such that $\liminf_{n\to\infty} E\left(\frac{1}{n}\sum_{k=1}^{n} t_{i_k j_k}\right) >$ $> \text{Min Max} - \epsilon$.

For a proof of this result let us introduce a few conventions: We may obviously substract from the matrices $A$ and $B$ their values $\nu(A)$ and $\nu(B)$ respectively, which will substract from all payoffs the constant $p\nu(A) + (1-p)\nu(B)$. Hence we may assume without loss of generality that $\nu(A) = \nu(B) = 0$. We may multiply $A$ by $p$ and $B$ by $(1-p)$, and consider the payoff to be the sum of the payoffs that would be obtained if $A$ was the true matrix and if $B$ was the true matrix. We will do this in order to simplify slightly notations. Finally $x'$ will always stand for $1-x$.

1. We define the following auxiliary game $\bar{\Gamma}$:

| | $\tilde{L}$ | $\tilde{R}$ | $(\widetilde{\beta,\beta'})$ |
|---|---|---|---|
| $\tilde{T}$ | $b_{11}$ | $a_{12} + b_{12}$ | $\beta b_{11} + \beta' b_{12}$ |
| $\tilde{B}$ | $a_{21} + b_{21}$ | $a_{22}$ | $\beta a_{21} + \beta' a_{22}$ |
| $(1 \tilde{-} \epsilon)T$ | $b_{11}$ | $a_{12}$ | $0$ |
| $(1 \tilde{-} \epsilon)B$ | $b_{21}$ | $a_{22}$ | $0$ |
| $B_1$ | $b_{11}$ | $a_{12}$ | $\beta(\beta b_{11} + \beta' b_{12})$ |
| $T_1$ | $b_{21}$ | $a_{22}$ | $\beta'(\beta a_{21} + \beta' a_{22})$ |

Here $\tilde{L}$ (resp. $\tilde{R}$, $\tilde{T}$, $\tilde{B}$) stands for the strategy (of player II) of playing always Left (resp. Right, Top, Bottom); $(1 \tilde{-} \epsilon)\, T$ (resp. $(1 \tilde{-} \epsilon)\, B$) stands for the strategy of playing at every stage independently with probability $(1-\epsilon)$ Top (resp. $B$) and with probability $\epsilon$ Bottom (resp. $T$). $(\widetilde{\beta,\beta'})$ stands for strategy of playing at each stage and independently with probability $\beta$ Left and with probability $\beta'$ Right. Finally $T_1$ (resp. $B_1$) stands for a strategy consisting of playing once $T$ (resp. $B$) and all other times $B$ (resp. $T$). The entries $\bar{\Gamma}$ can be easily obtained as asymptotic payoffs corresponding to those strategies, using our previous conventions (and thinking of $\beta$ as strictly between 0 and 1).

Denote by $\bar{v}$ the value of $\bar{\Gamma}$. If we denote by Max Min $\Gamma$ and Min Max $\Gamma$ the max Min and Min Max value of our original game in the strong sense that we described we shall prove that Min Max $\Gamma = \bar{v}$ and that it may be different from Max Min $\Gamma$. To make these statements rigorous we need still two more definitions:

2. Let us define:

$$\bar{v}_\infty = \inf \{U \mid \exists N \exists \tau \text{ s.t. } \forall n \geqslant N \forall \sigma_n, \rho_n(\sigma_n, \tau) \leqslant U\}$$

$$\tilde{v}_\infty = \sup \{U \mid \forall \tau \exists \sigma \text{ s.t. } \lim_{n\to\infty} \inf \rho_n(\sigma, \tau) \geqslant U\}$$

where $\sigma$ (resp. $\tau$) stands for a strategy of player I (resp. II) in the infinite game while $\sigma_n$ (resp. $\tau_n$) stands for a strategy of player I (resp. II) in a game consisting of $n$ stages only. $\rho_n(\sigma, \tau)$ is the expected average payoff per stage in the first $n$ stages, given $\sigma$, $\tau$ and $p$; i.e. $\rho_n(\sigma, \tau) = E_{\sigma,\tau,p}\left(\frac{1}{n}\sum_{k=1}^{n} t_{i_k j_k}\right)$, where $T = (t_{ij})$ is the true payoff matrix chosen by the referee at the begining of the game.

Loosely speaking, $\bar{v}_\infty$ is the lowest value of lim sup $\rho_n$ that player II *can* guarantee in the infinite game while $\tilde{v}_\infty$ is the highest value of lim inf $\rho_n$ that *cannot* be guaranteed by player II. Clearly $\tilde{v}_\infty \leqslant \bar{v}_\infty$. In the next section we will prove that $\bar{v} = \bar{v}_\infty = \tilde{v}_\infty$, which establishes that $\bar{v}$ is Min Max $\Gamma$ in the above explained sense.

## 3. Proofs

For a strategy of player I in $\bar{\Gamma}$ let $(\alpha, \alpha')$ be the probability distribution induced on $(1 \tilde{-} \epsilon) T$ and $(1 \tilde{-} \epsilon) B$.

*Lemma 1.* For any $\alpha$ corresponding to any undominated optimal strategy of Player I in $\bar{\Gamma}$ that uses $(1 \tilde{-} \epsilon) T$ or $(1 \tilde{-} \epsilon) B$ with a positive probability, one of the following holds:

$$\alpha a_{12} \geqslant 0 \text{ and } \alpha' b_{21} \geqslant 0 \tag{3.1}$$

$$\alpha b_{11} \geqslant 0 \text{ and } \alpha' a_{22} \geqslant 0 \tag{3.2}$$

*Proof.* Assume that for some optimal $\alpha$ neither of (3.1) and (3.2) holds, so for instance $\alpha b_{11} < 0$ (the case $\alpha' a_{22} < 0$ is completely symmetric). Since $\alpha b_{11} < 0$ and $\nu(B) = 0$, we have $b_{21} \geqslant 0$ and thus also $\alpha a_{12} < 0$ ( since 3.1) does not hold). Since $\alpha a_{12} < 0$ and $\nu(A) = 0$, we have $a_{22} \geqslant 0$. But this implies that the strategy $(1 \tilde{-} \epsilon) B$ of player I strictly dominates his strategy $(1 \tilde{-} \epsilon) T$ in $\bar{\Gamma}$, and thus that $\alpha = 0$ which contradicts the assumption $\alpha b_{11} < 0$.

*Theorem 1.* $\tilde{v}_\infty \geqslant \bar{v}$

*Proof.* Consider an arbitrary strategy $\tau$ of player II and an arbitrary $\epsilon$ $(0 < \epsilon < 1)$. $\tau$ may be considered as a probability measure $P$ on the space $\Omega$ of all sequences of $L$ (left) and $R$ (right) with the understanding that as soon as the true matrix is revealed, player II switches to his optimal strategy in that matrix.

Let $p_1 = P(\tilde{L}), p_2 = P(\tilde{R}), p_3 = P(\Omega \setminus \{\tilde{L}, \tilde{R}\}) = 1 - p_1 - p_2$. Let $\Omega_\infty$ denote the subset of $\Omega$ consisting of sequences with infinitely many $L$ and infinitely many $R$. Let $\{L_i\}$ denote the subset of $\Omega$ with a finite non-zero number of $L$ in the sequence, and similarly $\{R_i\}$ is the subset of those sequences with a finite non-zero number of $R$. We shall refer to these finitely many $L$ or $R$ as the exceptional moves.

Define $N_1$ by:

Prob. [player II has not played all his exceptional moves before $N_1 \mid \{L_i\} \cup \{R_i\}] < \epsilon$

and $N_2$ by:

Prob. {number of $L$ and $R$ in the intervall $]N_1, N_2[$ are both at least

$$\frac{\ln \epsilon}{\ln(1-\epsilon)} \mid \Omega_\infty\} > 1 - \epsilon$$

with the understanding that whenever the conditioning set has zero probability, the corresponding integer takes its least possible value (1 or $N_1 + 1$).

It follows from the definitions that even if player I plays $(1 \tilde{-} \epsilon) B$ in $]N_1, N_2[$, matrix $A$ (if it is the true matrix) will be revealed with probability greater than $1 - 2\epsilon$, given $\Omega_\infty$, and also that:

Prob. [Both $L$ and $R$ appear before $N_2 \mid \Omega \setminus (\{\tilde{L}\} \cup \{\tilde{R}\})] > 1 - \epsilon$

Let $(q_1, q_2, \alpha q_3, \alpha' q_3, q_4, q_5)$ be an undominated optimal strategy of player I in $\bar{\Gamma}$. For any $k > N_2$, let $\sigma_k$ be the following strategy of player I:

- with probability $q_1$, play $\tilde{T}$
- with probability $q_2$, play $\tilde{B}$
- with probability $q_3$, choose $H$ with probability $\alpha$ and $S$ with probability $\alpha'$ and play:
  - if (3.1) holds: if $H : \tilde{B}$ up to $N_1$ and $(1 \tilde{-} \epsilon) T$ after $N_1$.
    if $S : \tilde{T}$ up to $N_1$ and $(1 \tilde{-} \epsilon) B$ after $N_1$.
  - if (3.2) holfs: if $H : \tilde{T}$ up to $N_1$ and $(1 \tilde{-} \epsilon) T$ after $N_1$.
    if $S : \tilde{B}$ up to $N_1$ and $(1 \tilde{-} \epsilon) B$ after $N_1$.
- with probability $q_4$, play a strategy $B_1$, with the time of playing $B$ chosen independently of all other choices and uniformly in $[1, k]$.
- with probability $q_5$, play a strategy $T_1$, with the time of playing $T$ chosen independently of all other choices and uniformly in $[1, k]$.

We have for all $n > k$,

$$1/n \leqslant N_1/n \leqslant N_2/n \leqslant k/n; \; 1/n\epsilon \leqslant k/n, \; 1/k \leqslant \epsilon. \tag{3.3}$$

Let $f_n = (1/n) \cdot$ (number of $L$ up to time $n$).

Let $M = \max\,[\max_{i,j} a_{ij} - \min_{i,j} a_{ij};\ \max_{i,j} b_{ij} - \min_{i,j} b_{ij}]$,
and let $0(\epsilon)$ stand for any quantity $x$ such that $|x| \leqslant M\epsilon$
similarly $0(1/k)$ stands for any $y$ such that $|y| \leqslant M/k$, etc.
Denote by $\rho_n(\sigma_k, \tau)$ the average payoff per stage resulting from strategies $\sigma_k$ and $\tau$,
we have that $\rho_n(\sigma_k, \tau)$ is the expectation of:

$$
\begin{aligned}
&q_1\,[p_1\,(b_{11} + 0(1/n)) + p_2\,(a_{12} + b_{12}) + p_3\,(f_n b_{11} + f'_n b_{12}) + 0(N_2/n) + 0(\epsilon)]\\
&+\ q_2\,[p_1\,(a_{21} + b_{21}) + p_2\,(a_{22} + 0(1/n)) + p_3\,(f_n a_{21} + f'_n a_{22} + 0(N_2/n) + 0(\epsilon))]\\
&+\ q_3\,[p_1\,\{\alpha b_{11} + \alpha' b_{21} + 0(N_1/n) + 2\,0(\epsilon) + (0(1/n\epsilon)\ \text{if (3.2) holds})\} +\\
&\quad p_2\,\{\alpha a_{12} + \alpha' a_{22} + 0(N_1/n) + 2\,0(\epsilon) + (0(1/n\epsilon)\ \text{if (3.2) holds})\} +\\
&\quad P(\Omega_\infty)\,(4\,0(\epsilon) + 0(N_2/n)) +\\
&\begin{cases}
\text{if (3.1) holds:}\\
P(\{L_i\})\,(\alpha a_{12} + 3\,0(\epsilon) + 0(N_1/n)) + p(\{R_i\})\,(\alpha' b_{21} + 3\,0(\epsilon) + 0(N_1/n))\\
\text{if (3.2) holds:}\\
P(\{L_i\})\,(\alpha' a_{22} + 3\,0(\epsilon) + 0(N_1/n) + 0(1/n\epsilon)) + P(\{R_i\})\,(\alpha b_{11}\\
\qquad + 3\,0(\epsilon) + 0(N_1/n) + 0(1/n\epsilon))
\end{cases}\\
&+\ q_4\,[p_1\,(b_{11} + 0(1/n)) + p_2\,(a_{12} + 0(k/n)) + p_3\,\{f_k\,(f_n b_{11} + f'_n b_{12}) + 0(1/k)\\
&\qquad + 0(\epsilon) + 0(k/n)\}]\\
&+\ q_5\,[p_1\,(b_{21} + 0(k/n)) + p_2\,(a_{22} + 0(1/n)) + p_3\,\{f'_k\,(f_n a_{21} + f'_n a_{22}) + 0(1/k)\\
&\qquad + 0(\epsilon) + 0(k/n)\}]
\end{aligned}
$$

Using relations (3.3) and Lemma 1 we get that for all $n > k$:

$$\rho_n(\sigma_k, \tau) \geqslant E\,(H(k, n, \tau, \omega)) - 4M\epsilon - 0(k/n),$$

where

$$
H(k, n, \tau, \omega) = \begin{bmatrix} q_1 \\ q_2 \\ \alpha q_3 \\ \alpha' q_3 \\ q_4 \\ q_5 \end{bmatrix} \cdot \begin{bmatrix} b_{11} & a_{12} + b_{12} & b_{11} & b_{12} \\ a_{21} + b_{21} & a_{22} & a_{21} & a_{22} \\ b_{11} & a_{12} & 0 & 0 \\ b_{21} & a_{22} & 0 & 0 \\ b_{11} & a_{12} & f_k b_{11} & f_k b_{12} \\ b_{21} & a_{22} & f'_k a_{21} & f'_k a_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ f_n p_3 \\ f'_n p_3 \end{bmatrix}
$$

Denote $E(H(k, n, \tau, \omega))$ by $\phi(f_k, f_n)$. The function $\phi : L_\infty \times L_\infty \to \underset{\sim}{R}$ is weakly continuous and affine in each variable separately on $L_\infty$ endowed with the weak topology $\sigma(L_\infty, L_1)$.

Let $C$ = closed convex hull of $\{f_i \mid i > N_2\}$ in $L_\infty - \sigma(L_\infty, L_1)$, and consider $\phi$ on $C \times C$: We have $\phi(f, f) \geqslant \bar{v}\ \forall f \in C$, indeed for $f_k = f_n = f$, $H(k, n, \tau, \omega) \geqslant \bar{v}$ holds for each value of $\omega$ since $(q_1, q_2, \alpha q_3, \alpha' q_3, q_4, q_5)$ is an optimal strategy of player I in $\bar{\Gamma}$. In addition $C$ is compact and convex for $\sigma(L_\infty, L_1)$ and $\phi$ is affine and continuous in each variable separately on $C$. It follows that $\phi$ has a saddle point, hence:

$$\exists\, g \in C \text{ s.t. } \forall f \in C : \phi(g, f) \geqslant \inf_f \sup_u \phi(u, f) \geqslant \inf_f \phi(f, f) \geqslant \bar{v}$$

Now $g$ is also in the closure of the convex hull of $\{f_i \mid i > N_2\}$ when $L_\infty$ is endowed with the Mackey topology $\tau(L_\infty, L_1)$ – due to the convexity of the set – (this is a well known result that follows from the Hahn-Banach theorem). Since on bounded sets of $L_\infty$ the Mackey topology $\tau(L_\infty, L_1)$ coincides with the topology of convergence in probability, it follows that there exist $\lambda_i$ $(1 \leqslant i \leqslant l,\ \lambda_i \geqslant 0,\ \sum_{i=1}^{l} \lambda_i = 1)$ and $k_i$ $(1 \leqslant i \leqslant l,\ k_i > N_2)$ such that:

$$P\left(\left|\sum_{i=1}^{l} \lambda_i f_{k_i} - g\right| \geqslant \epsilon\right) < \epsilon .$$

Let now $\sigma_{\epsilon,\tau}$ be the strategy of player I consisting of choosing at the start of the game a number $i$ $(1 \leqslant i \leqslant l)$ with probability $\lambda_i$, and thereafter using his strategy $\sigma_{k_i}$. Let also $K_M = \max\{k_i \mid 1 \leqslant i \leqslant l\}$ then we have:

$$\rho_n(\sigma_{\epsilon,\tau}, \tau) \geqslant \phi(g, f_n) - 6M\epsilon - 0(K_M/n) \text{ for all } n > K_M .$$

Thus: $\forall\, \tau$, strategy of player II, $\forall\, \epsilon$, $0 < \epsilon < 1$, $\exists\, \sigma_{\epsilon,\tau}$, strategy of player I, such that:

$$\lim_{n \to \infty} \inf \rho_n(\sigma_{\epsilon,\tau}, \tau) \geqslant \inf_{f \in C} \phi(g, f) - 6M\epsilon \geqslant \bar{v} - 6M\epsilon.$$

This completes the proof of Theorem 1.

*Lemma 2.* Player II has an optimal strategy in $\bar{\Gamma}$ using only a single value of $\beta$.

*Proof.* A priori player II's optimal strategy in $\bar{\Gamma}$ consists of a probability vector $(p_1, p_2, p_3)$ together with a probability measure $\mu$ on $[0, 1]$ to choose $\beta$. We want to show that player II has an optimal strategy in which $\mu$'s support is a single point in $[0, 1]$.

L 1) If $b_{11} \geqslant b_{12}$ and $a_{22} \geqslant a_{21}$ the result follows from the convexity in $\beta$ of the payoff function.

L 2) Otherwise we have either $b_{11} < b_{12}$ or $a_{22} < a_{21}$, by symmetry we may assume that $a_{22} < a_{21}$. Since $\nu(A) = 0$ it follows that $a_{22} \leqslant 0$.

L 2.1) If in addition $b_{11} \leqslant b_{12}$ the payoff function is concave in $\beta$ and thus $\mu$ is dominated by the probability on $\{0, 1\}$ that has the same mean. So without loss of generality we may assume that in this case $\mu(]0, 1[) = 0$. We get thus for $\bar{\Gamma}$ a $6 \times 4$ matrix with $\tilde{L}$, $\tilde{R}$, $\beta = 1$ and $\beta = 0$ as pure strategies for player II. The other strategies are eliminated by domination. In addition $\nu(B) = 0$ implies $b_{11} \leqslant 0$, and thus we conclude that rows $B_1$, and $T_1$ are dominated by $(1 \tilde{-} \epsilon) T$ and $(1 \tilde{-} \epsilon) B$ respectively. If either $b_{21} \leqslant 0$ or $a_{12} \leqslant 0$, one of the rows $\beta = 1$ or $\beta = 0$ is dominated by $\tilde{L}$ or $\tilde{R}$ respectively and the result follows. If either $a_{21} \leqslant 0$ or $b_{12} \leqslant 0$, say $a_{21} \leqslant 0$ then first $\tilde{B}$ is dominated by $(1 \tilde{-} \epsilon) B$ and then $\beta = 0$ is dominated by $\beta = 1$; the result follows again. Thus we may assume that $\mathrm{Min}(a_{12}, a_{21}, b_{12}, b_{21}) > 0$, it follows then from $\nu(A) = \nu(B) = 0$ that $b_{11} < 0, a_{22} < 0$.

$$\text{Let } R = \begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad C = \begin{pmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{pmatrix}$$

$$r = \mathrm{Val}(R), \qquad c = \mathrm{Val}(C)$$

L 2.1.1) If $r \leqslant 0$ and if we denote by $(\beta, \beta')$ the relative weights of the columns $\beta = 1$ and $\beta = 0$, then there exists an optimal $\beta$ for which $\beta b_{11} + \beta' b_{12} \leqslant 0$ and $\beta a_{21} + \beta' a_{22} \leqslant 0$ (if $c < 0$, the required $\beta$ is the relative weight of the last two columns in the equalizing strategy of player II, if $c \geqslant 0$ the value of the game is 0 and an optimal strategy of player II is $(0, 0, \beta, \beta')$ where $\beta$ is optimal in $R$ and hence satisfies the required inequalities). It follows that if in that optimal strategy, player II would replace the columns $\beta = 1$ and $\beta = 0$ by i.i.d. $(\beta, \tilde{\beta})$, rows $B_1$ and $T_1$ would still be dominated by $(1 \tilde{-} \epsilon) T$ and $(1 \tilde{-} \epsilon) B$ respectively and hence player II has in this case an optimal strategy using a single $\beta$.

L 2.1.2) If $r > 0$, the optimal mixture of the columns $\beta = 1$ and $\beta = 0$ is $(\beta, \beta')$, $\beta$ being optimal in $R$ and hence $\beta b_{11} + \beta' b_{12} > 0$ and $\beta a_{21} + \beta' a_{22} > 0$. It follows again that replacing the last two columns by i.i.d. $(\beta, \tilde{\beta}')$, rows $B_1$ and $T_1$ remain dominated, this time by $\tilde{T}$ and $\tilde{B}$ respectively, providing again a single $\beta$ optimal strategy for player II.

L 2.2) We are thus left with the case:

$$a_{22} < a_{21}, \ a_{22} \leqslant 0, \ b_{12} < b_{11}, \ b_{12} \leqslant 0.$$

Consider player II's optimal strategy in the game $\bar{\Gamma}$ without the rows $B_1$ and $T_1$; it obviously implies $\beta = 0$. For this $\beta$, $B_1$ is dominated by $(1 \tilde{-} \epsilon) T$ and $T_1$ by $(1 \tilde{-} \epsilon) B$ and thus this single $\beta$ strategy is also optimal in $\bar{\Gamma}$. This completes the proof of Lemma 2.

Notice that the strategies $\beta = 1$ and $\beta = 0$ in $\bar{\Gamma}$ should be interpreted as playing i.i.d. $(1 \tilde{-} \epsilon, \epsilon)$ and $(\epsilon, \tilde{1} - \epsilon)$ respectively. Thus in the single $\beta$ optimal strategy for player II established in Lemma 2 we may assume $0 < \beta < 1$.

*Theorem 2.* $\bar{v}_\infty \leqslant \bar{v}$.

*Proof.* We will show that whenever player II plays in $\Gamma$ one of his strategies $\tau$ in $\bar{\Gamma}$, consisting of a mixture of $\tilde{L}$, $\tilde{R}$ and one $(\beta, \tilde{\beta}')$ – with $0 < \beta < 1$ – any pure strategy of player I yields in $\Gamma_n$ a payoff dominated up to terms $0(1/n)$ by a convex combination of rows of $\bar{\Gamma}$. Since by Lemma 2 player II can guarantee $\bar{v}$ – up to $\epsilon$ – by such mixtures $\tau$ against rows of $\bar{\Gamma}$, the result will then follow.

If the pure strategy of player I is $\tilde{T}$ or $\tilde{B}$ then it is already a row of $\bar{\Gamma}$. Take any other pure strategy that begins say with $T$ (for strategies starting with $B$ the discussion is completely dual). Let $\omega_i = 1$ if $T$ occurs at time $i$ in the strategy and $\omega_i = 0$ otherwise. Let $f_n = \frac{1}{n} \sum_{i=1}^{n} \omega_i$, and $\omega_{t+1}$ be the first zero in the sequence $\{\omega_i\}$. Let $y = t/n$; we have $1/n \leqslant y \leqslant f_n \leqslant (n-1)/n$.

Let $D = \beta b_{11} + \beta' b_{12}$, $G = \beta a_{21} + \beta' a_{22}$,

$$X = \frac{1}{n} [\omega_1' + \omega_2' (1 - \beta\omega_1) + \ldots + \omega_n' (1 - \beta\omega_1) \ldots (1 - \beta\omega_{n-1})]$$

$$Y = \frac{1}{n} [\omega_1 + \omega_2 (1 - \beta'\omega_1') + \ldots + \omega_n (1 - \beta'\omega_1') \ldots (1 - \beta'\omega_{n-1}')] \ .$$

We have $f_n' \beta'^{nf_n} \leqslant X \leqslant f_n' \beta'^{ny}$ and

$$y + (f_n - y) \beta^{nf_n'} \leqslant Y \leqslant y + (f_n - y) \beta, \text{ neglecting terms } 0(1/n).$$

The strategy of player I obtains, up to $0(1/n)$; against $\tilde{L}$: $f_n b_{11} + f_n' b_{21}$; against $\tilde{R}$: $yb_{12} + f_n a_{12} + f_n' a_{22}$ and against $(\beta, \tilde{\beta}')$ : $GX + DY$. Majorizing this last term according to the sign of $G$ and $D$ we obtain (writing $f$ for $f_n$);

| against | $\tilde{L}$ | $\tilde{R}$ |
|---|---|---|
| a payoff $\leqslant$ | $fb_{11} + f'b_{21}$ | $yb_{12} + fa_{12} + f'a_{22}$ |

| against | $(\beta, \tilde{\beta}')$ with: | | | |
|---|---|---|---|---|
| | $G \geqslant 0, D \geqslant 0$ | $G < 0, D \geqslant 0$ | $G \geqslant 0, D < 0$ | $G < 0, D < 0$ |
| a payoff $\leqslant$ | $Gf'\beta'^{ny} + D(\beta f + \beta' y)$ | $Gf'\beta'^{nf} + D(\beta f + \beta' y)$ | $Gf'\beta'^{ny} + D(y + (f-y)\beta^{nf'})$ | $Gf'\beta'^{nf} + D(y + (f-y)\beta^{nf'})$ |

Since all terms are convex in $y$, we may replace $y$ by its extreme values $1/n$ and $f$. Neglecting terms $0(1/n)$ one gets thus:

| | $\tilde{L}$ | $\tilde{R}$ | $(\beta, \tilde{\beta}')$ with: | | | |
|---|---|---|---|---|---|---|
| | | | $G \geqslant 0, D \geqslant 0$ | $G < 0, D \geqslant 0$ | $G \geqslant 0, D < 0$ | $G < 0, D < 0$ |
| $y=1/n$ | $fb_{11} + f'b_{21}$ | $fa_{12} + f'a_{22}$ | $Gf'\beta' + Df\beta$ | $Gf'\beta'^{nf} + Df\beta$ | $Gf'\beta' + Df\beta^{nf'}$ | $Gf'\beta'^{nf} + Df\beta^{nf'}$ |
| $y=f$ | $fb_{11} + f'b_{21}$ | $f(a_{12} + b_{12}) + f'a_{22}$ | $Gf'\beta'^{nf} + Df$ | $Gf'\beta'^{nf} + Df$ | $Gf'\beta'^{nf} + Df$ | $Gf'\beta'^{nf} + Df$ |

$f\beta^{nf'}$ and $f'\beta'^{nf}$ are convex. When their coefficients are negative let us majorize them by zero. All functions get then linear or convex in $f$, so we may replace $f$ by its extreme values $1/n$ and $(1-1/n)$. Neglecting terms $0(1/n)$ one obtains:

| | $\tilde{L}$ | $\tilde{R}$ | $(\beta, \tilde{\beta}')$ with: | | | |
|---|---|---|---|---|---|---|
| | | | $G \geqslant 0, D \geqslant 0$ | $G < 0, D \geqslant 0$ | $G \geqslant 0, D < 0$ | $G < 0, D < 0$ |
| $y = 1/n, f = 1-1/n$ | $b_{11}$ | $a_{12}$ | $D\beta$ | $D\beta$ | 0 | 0 |
| $y = f = 1/n$ | $b_{21}$ | $a_{22}$ | $G\beta$ | 0 | $G\beta'$ | 0 |
| $y = f = 1-1/n$ | $b_{11}$ | $a_{12}+b_{12}$ | $D$ | $D$ | $D$ | $D$ |

We conclude that player I's strategy is dominated by the mixture of three similar strategies with $(y = 1/n, f = 1 - 1/n)$, $(y = f = 1/n)$ and $(y = f = 1 - 1/n)$, the weights being $f - y$, $f'$ and $y$ respectively. But this mixture is dominated by the convex combination with the same weights of the following rows of $\bar{\Gamma}$:

| case / weights | $G \geqslant 0, D \geqslant 0$ | $G < 0, D \geqslant 0$ | $G \geqslant 0, D < 0$ | $G < 0, D < 0$ |
|---|---|---|---|---|
| $f - y$ | $B_1$ | $B_1$ | $(1 - \tilde{\epsilon})T$ | $(1 - \tilde{\epsilon})T$ |
| $f'$ | $T_1$ | $(1 - \tilde{\epsilon})B$ | $T_1$ | $(1 - \tilde{\epsilon})B$ |
| $y$ | $\tilde{T}$ | $\tilde{T}$ | $\tilde{T}$ | $\tilde{T}$ |

This completes the proof of Theorem 2.

## 4. Conclusions

(i) $\bar{v}_\infty = \tilde{v}_\infty = \bar{v} = \text{Min Max } \Gamma$

(ii) Player II has an "$\epsilon$-MinMax" strategy of the type: With probability $p_1$ play always $L$, with probability $p_2$ play always $R$ and with probability $1 - p_1 - p_2$ play always i.i.d. with probability $\beta$, $L$ and with probability $\beta'$, $R$.

(iii) This strategy also guarantees that in any finite sufficiently long game the payoff is less than $\bar{v} + \epsilon$.

(iv) Dual results hold for player I.

(v) Analysis of the game $\bar{\Gamma}$ and its' dual $\underline{\Gamma}$ shows that the only cases where there is no value (i.e. $\bar{v} > \underline{v}$) are: $\{c \vee r < 0$ and $a_{21} > 0$ and either $a_{12} \wedge b_{12} > 0$ or $a_{12}(a_{21} - a_{22}) + b_{21} a_{22} < 0\}$
and its symmetries obtained by either permuting the games

$(a_{ij} \leftrightarrow b_{i'j'}$ where $1' = 2$ and $2' = 1)$ or permuting the players $(a_{ij} \leftrightarrow -\alpha_{ji}$, $b_{ij} \leftrightarrow -b_{ji})$ or both.

*An example* of a game without value is the following:

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

Optimal strategies in $\bar{\Gamma}$ are: For player I; (1/4, 1/4, 1/4, 1/4, 0, 0) and for player II; (1/4, 1/4, 1/2, (1/2, 1/2)) giving $\bar{v} = -1/2$. Optimal strategies in $\underline{\Gamma}$ are: For player I: (1/6, 1/6, 2/3, (1/2, 1/2)) and for player II; (1/6, 1/6, 0, 0, 1/3, 1/3) giving $\underline{v} = -2/3$.

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Received October, 1974
(revised version February, 1977)