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Common belief and common knowledge*

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> But game theory as we presently know it can not proceed without the fulcrum of common knowledge.
>
> Robert Wilson (1985, p. 2).

In the universal belief space [Mertens and Zamir (1985)] which incorporated all situations of incomplete information concerning a state space S, we define a 'knowledge operator' in terms of beliefs. From this operator we derive (in the usual way) the concept of common knowledge and the result is: An event E is common knowledge if and only if it is a belief subspace. Recalling that any game model, with complete or incomplete information, is a belief subspace, this result may be regarded as a considerable weakening of the *common knowledge assumption* that is: If we adopt the universal belief space as a general framework model for incomplete information games, then the statement 'the game (i.e. the belief subspace) is Common Knowledge' is a formal provable statement *within* the model. Since a belief subspace may or may not be consistent (in Harsanyi's sense), it follows that with this definition, and unlike in Aumann's model, players *may agree to disagree.*

1. Introduction

Following Aumann's seminal work *Agreeing to Disagree* (1976), the notions of knowledge and common knowledge have been extensively studied in the game-theoretic and mathematical economics literature. Their importance is stressed by Robert Wilson in the above citation, and in the recent book of Fudenberg and Tirole (1991). Games of incomplete information, however, have been usually studied via the Bayesian approach of Harsanyi (1967–1968). The basic ingredient of this approach is the set of *beliefs* of the players, i.e. each player's (subjective) probability distribution on the unknown state of nature and on the beliefs of the other players.

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Knowledge and belief together, were first studied by Aumann (1976). Let us recall his main ideas, for future reference.

- The *state space* is an arbitrary set Ω (which he assumed to be finite).
- *Knowledge* is defined through an auxiliary notion of information structure. The information structure of player i is an exogenous partition of Ω. If state $\omega \in \Omega$ occurs, player i knows that the true state could be any state in $\pi_i(\omega)$, the element of his partition which contains ω. Player i then knows, at state ω, that event $E \subset \Omega$ occurred if E contains $\pi_i(\omega)$.
- *Beliefs* of player i at state ω are represented by a probability measure p_i^ω on $\pi_i(\omega)$. To get the result of that paper, Aumann had to assume that this probability measure is derived from some common probability measure p on Ω by the relation $p_i^\omega(E) = p(E \mid \pi_i(\omega))$, for any player i and any event $E \subset \Omega$. This is referred to as the assumption of *common priors.*
- The 'description of the game', i.e. the state space Ω, the partitions π_i and the beliefs p_i^ω is common knowledge in an informal sense. This is referred to as the *common knowledge assumption.*

We sketch now our approach to knowledge and belief in games of incomplete information with n players by relating it to Aumann's ideas.

- The state space Ω is not an arbitrary set. Given a set S of *states of nature* (i.e. parameter set), Ω is the universal belief space generated by S and n, as defined by Mertens and Zamir (1985). The space Ω can be written as $X^n \times S$, where X is the univeral type space. A point $\omega = (x_1, \ldots, x_n, s)$ in Ω is a complete description of the state of nature s and the 'state of mind' (also called 'type') x_i of each player i. Each x_i is naturally associated with a probability measure on $X^{n-1} \times S$, i.e. with the beliefs of i about the state of nature and other players' state of mind.
- The beliefs are given by a point in X^n.
- Knowledge is defined through the auxiliary notion of a possibility operator which, unlike the partitions in Aumann's model, is not exogenous, but defined in terms of beliefs. More precisely, $\pi_i(\omega)$ is now derived from Ω as the support of the beliefs of player i at ω. Knowledge and common knowledge are then defined as in Aumann (1976). Our first result is that an event E is common knowledge at each and every of its points if the only if it is closed under the possibility operators of all agents [equivalently, E is a *belief subspace*, in the terminology of Mertens and Zamir (1985)]. Belief subspaces can be finite (and small); hence the finiteness lost by taking Ω to be the universal state space, can be regained by making assumptions about the players prior beliefs.
- The statement 'the description of the game is common knowledge' is now a formal statement within the model. To see this, recall that the game played in a state of the world ω is the smallest belief subspace $E(\omega)$ containing ω.

Therefore, according to our first result, $E(\omega)$ if (formally) common knowledge at each and every of its points.

The paper is organized as follows: section 2 introduces the main definitions and established the relations between common knowledge and common belief. Section 3 introduces an alternative definition of common knowledge, subjective common knowledge, proves its equivalence to the original definition and utilizes it to prove that the event 'E is common knowledge' is itself common knowledge. Section 4 relates our approach to the literature.

2. Common knowledge and common belief

Consider as set $I=\{1,\dots,n\}$ of n players facing uncertainty about the true state of nature. Let S be the set of possible states. Mertens and Zamir (1985) show that if S is compact and Hausdorff, there is a compact Hausdorff space X and n continuous isomorphisms:

$$f_i: X \to \Delta(S \times X^{n-1}), \quad i \in I, \tag{1}$$

where $\Delta(K)$ denotes the set of Borel regular probability measures on K. The interpretation is that an n-tuple $(x_1, x_2, \dots, x_n) \in X^I$ represents a possible state of beliefs; for $i \in I$ the beliefs of player i are represented by x_i. If $E \subset S \times X^{I \setminus \{i\}}$, then $f_i(x_i)(E)$ is the probability which agent i assigns to the event E when his beliefs are represented by x_i. We define

$$\Omega = X^I \times S.$$

A typical point in Ω is $\omega=(x,s)$ where $x=(x_1,x_2,\dots,x_n)\in X^I$ and $s \in S$. We think of ω as a *state of the world*, i.e. a complete description of the state of nature s and the state of agents' beliefs x about the state of nature and about each others' beliefs. The component x_i of x may be called the *state of mind* of player i at the state of the world ω. We denote by X_i the ith factor in X^I. Clearly all X_i for $i \in I$ are copies of the same space X.

Before stating our result we need some more notation and definitions:

1. Denoted by 2^Ω the power set of Ω. The *possibility operator of player i* is the function $P_i: \Omega \to 2^\Omega$ defined by

$$P_i(\omega) = \{x_i\} \times \operatorname{Supp}(f_i(x_i)),$$

where $\operatorname{Supp}(\mu)$ is the support of the probability measure μ. In words, the set of states considered possible by player i at ω is, roughly, the product of i's type at ω with the smallest closed set of states of nature and states of other's minds that is assigned probability one by i at ω.

2. We say that agent i *knows* event $E \subset \Omega$ at $\omega \in \Omega$ if

$P_i(\omega) \subset E$.

3. The knowledge operator of player i is the function $K_i: 2^\Omega \to 2^\Omega$ defined by

$$K_i(E) = \{\omega \in \Omega \,|\, P_i(\omega) \subset E\} = P_i^{-1}(E).$$

In words: $K_i(E)$ is the set of all states of the world in which player i knows E.

4. The common knowledge operator $K: 2^\Omega \to 2^\Omega$ is defined in two steps: Denote by B the set of all finite sequences of elements of I. For each $b = (i_1, \ldots, i_m)$ in B define K_b to be the composition of the knowledge operators $K_{i_1}, \ldots, K_{i_m}$. Then K is given by

$$K(E) = \bigcap_{b \in B} K_b(E), \quad E \subset \Omega.$$

5. An event $E \subset \Omega$ is *common knowledge at* $\omega \in \Omega$ if $\omega \in K(E)$.

6. An event $E \subset \Omega$ is common knowledge at each and every one of its points if $E \subset K(E)$.

7. The (joint) possibility operator P is the function $P: \Omega \to 2^\Omega$ defined by

$$P(\omega) = \bigcup_{i \in I} P_i(\omega).$$

8. An event $E \subset \Omega$ is *belief-closed* if $P(E) \subset E$, i.e. if at each $\omega \in E$, the set E contains the possible sets of all players. That is, any minimal set of states assigned probability one by at least one agent is included in E [see Mertens and Zamir (1985), the notion of *belief subspace*].

We are now ready to state and prove our main result:

Theorem 2.1. An event $E \subset \Omega$ *is belief-closed if and only if it is common knowledge at each and every one of its points. That is:*

$$P(E) \subset E \quad \textit{if and only if} \quad E \subset K(E).$$

Proof. Sufficiency ('if'). Suppose that E is common knowledge at each and every one of its points, then

$$\begin{array}{llll} & E \subset K(E) & & \text{(given)} \\ \Rightarrow & E \subset K_i(E) & \forall i \in I & \text{(def. 4)} \\ \Rightarrow & P_i(E) \subset P_i[K_i(E)] \subset E & \forall i \in I & \text{(def. 3)} \\ \Rightarrow & P(E) = \bigcup_{i \in I} P_i(E) \subset E & & \text{(def. 7).} \end{array}$$

Hence E is belief-closed.

Necessity ('only if'). Suppose that E is belief-closed. We shall show that

$E \subset K(E)$. To do this we prove by induction on the length of the string, m, that $P_{i_1},,,P_{i_m}(E) \subset E$ for all $i_1,\dots,i_m$. In fact $P_i(E) \subset E \, \forall i \in I$ (since $P(E) = \bigcup_{i\in I} P_i(E) \subset E$).

Suppose that $P_{i_1},\dots,P_{i_m}(E) \subset E$, then

$$P_{i_1}P_{i_2}\dots P_{i_{m+1}}(E) \subset P_{i_1}(E) \subset E,$$

completing the inductive proof. If follows (def. 3) that

$$E \subset K_{i_m}\dots K_{i_1}(E).$$

Since this is true for any finite chain $i_1,\dots,i_m$, we have $E \subset K(E)$ (by def. 4), completing the proof of the theorem. Q.E.D.

3. An alternative definition of common knowledge

The idea behind the possibility operator P_i defined in the previous section is that at ω, player i considers a state of the world to be possible if (roughly speaking) it is in the support of his own beliefs. An alternative definition would add to these states also the states that i himself assigns zero possibility, but some other player j (of a type that player i considers possible) includes in the support of his beliefs. This is captured by the following:

Definition 3.1. (*i*) Let $\mathbb{N}$ be the set of positive natural numbers. $T_i : X_i \times \mathbb{N} \to 2^{\Omega}$ is the function defined inductively by [recall that $P_i(\omega)$ depends only on the coordinate x_i of ω]

$$T_i(x_i, 1) = P_i(\omega),$$
$$T_i(x_i, k+1) = T_i(x_i, k) \cup P[T_i(x_i, k)] \quad \text{for} \quad k \geqq 1.$$

(*ii*) For $i \in I$ define the possibility operator of player i to be $\hat{P}_i : \Omega \to 2^{\Omega}$ given by

$$\hat{P}_i(\omega) = \bigcup_{t=0}^{\infty} P^t P_i(\omega) = \bigcup_{k=1}^{\infty} T_i(x_i, k),$$

where as usual P^t stands for t successive applications of the operator P.

(*iii*) The (joint) possibility operator $\hat{P}$ is defined by

$$\hat{P}(\omega) = \bigcup_{i\in I} \hat{P}_i(\omega).$$

(*iv*) The knowledge operator $\hat{K}$ corresponding to $\hat{P}$ is defined in the usual manner by

$$\hat{K}(E)=\{\omega\in\Omega\,|\,\hat{P}(\omega)\subset E\}=\hat{P}^{-1}(E).$$

(Note that this is also the common knowledge operator, since it does not depend on i).

We call $\hat{P}_i$ the subjective possibility operator of player i. $\hat{P}_i(\omega)$ is 'all the world' from the point of view of player i at ω. The corresponding common knowledge operator $\hat{K}$ is also called subjective and, by the same token, the common knowledge operator K of def. 4 is called objective. Note that $\hat{K}$, as opposed to K does not involve iteration and composition of individual knowledge operators (since this was taken care of in the definition of $\hat{P}_i$). This may be of advantage in some proofs if we can show that $K=\hat{K}$, i.e., that subjective and objective definitions of common knowledge coincide. This in fact turns out to be the case.

Theorem 3.2. $\hat{K}(E)=K(E)$, *for any* $E\subset\Omega$.

It will be more convenient to prove this by proving the equality of the two corresponding possibility operators. Denote by $\check{P}$ the possibility operator corresponding to common knowledge operator K, i.e. it satisfies

$$K(E)=\{\omega\in\Omega\,|\,\check{P}(\omega)\subset E\}=\check{P}^{-1}(E),$$

which is equivalent to

$$\check{P}(\omega)=\bigcup_{m=1}^{\infty}\bigcup_{i_l\in I}P_{i_1}\dots P_{i_m}(\omega). \tag{2}$$

We shall prove that

$$\hat{P}(\omega)=\check{P}(\omega)\quad\forall\omega\in\Omega.$$

To do that let us prove a few lemmas which will be also used for further results.

Lemma 3.3. For all $i\in I$ *and* $\omega\in\Omega$,

$$P\hat{P}_i(\omega)\subset\hat{P}_i(\omega).$$

Proof. By definition of $\hat{P}_i$ [Definition 3.1(ii)],

$$P\,\hat{P}_i(\omega)\subset\bigcup_{k=1}^{\infty}P[T_i(x_i,k)].$$

So it is enough to prove (by induction) that $P[T_i(x_i,k)] \subset \hat{P}_i(\omega)$ for all k. In fact

$$P[T_i(x_i,1)] \subset T_i(x_i,2)] \subset \hat{P}_i(\omega).$$

The last inclusion is again by Definition 3.1(ii). Assume that $P[T_i(x_i,k)] \subset \hat{P}_i(\omega)$, then

$$P[T_i(x_i,k+1)] \subset T_i(x_i,k+2)] \subset \hat{P}_i(\omega). \qquad \text{Q.E.D.}$$

Lemma 3.4. *For all i and j in I and $\omega \in \Omega$,*

$$\hat{P}_j\hat{P}_i(\omega) \subset \hat{P}_i(\omega).$$

Proof. By Definition 3.1(ii) and def. 7,

$$\hat{P}_j\hat{P}_i(\omega) = \bigcup_{t=0}^{\infty} P^t P_j[P_i(\omega)]$$

$$\subset \bigcup_{t=0}^{\infty} P^{t+1}P_i(\omega) = \bigcup_{t=1}^{\infty} P^t P_i(\omega).$$

By Lemma 3.3 each of the sets in the union is a subset of $\hat{P}_i(\omega)$. Q.E.D.

Lemma 3.5. *For all $\omega \in \Omega$,*

$$P\check{P}(\omega) \subset \check{P}(\omega).$$

Proof. Using the explicit expression for $\check{P}$ [eq. (2)] we have

$$P_j\check{P}(\omega) = P_j\left(\bigcup_{m=1}^{\infty} \bigcup_{i_l \in I} P_{i_1} \ldots P_{i_m}(\omega)\right)$$

$$\subset \bigcup_{m=1}^{\infty} \bigcup_{i_l \in I} P_j P_{i_1} \ldots P_{i_m}(\omega) \subset \check{P}(\omega).$$

Hence by def. 7,

$$P\check{P}(\omega) \subset \bigcup_{j \in I} P_j\check{P}(\omega) \subset \check{P}(\omega). \qquad \text{Q.E.D.}$$

Lemma 3.6. *For all $\omega \in \Omega$,*

$$P\hat{P}(\omega) \subset \hat{P}(\omega).$$

Proof. By Definition 3.1(iii) and Lemma 3.3,

$$P_j\hat{P}(\omega) \subset \bigcup_{i\in I} P_j\hat{P}(\omega) \subset \bigcup_{i\in I} P\hat{P}(\omega) = \hat{P}(\omega).$$

Hence $P\hat{P}(\omega) \subset \hat{P}(\omega)$. Q.E.D.

Proof of Theorem 3.2. We want to prove that $\hat{P}(\omega) = \check{P}(\omega)$ for all $\omega \in \Omega$.

First we prove inductively that $T_i(x_i, k) \subset \check{P}(\omega)$ holds for all ω, $i \in I$ and k. In fact by Definition 3.1(i), $T_i(x_i, 1) = P_i(\omega) \subset \check{P}(\omega)$, and if $T_i(x_i, k) \subset \check{P}(\omega)$ then, using Lemma 3.5,

$$T_i(x_i, k+1) = T_i(x_i, k) \cup P[T_i(x_i, k)]$$

$$\subset \check{P}(\omega) \cup P\check{P}(\omega) \subset \check{P}(\omega).$$

It follows by Definition 3.1(i) that $\hat{P}(\omega) \subset \check{P}(\omega)$. To prove the inverse inclusion [using eq. (2)] it suffices to show for all m and for all $i_t \in I$, $t = 1, \dots, m$, $P_{i_m} \dots P_{i_m}(\omega) \subset \hat{P}(\omega)$. This is again proved by induction on m. For $m = 1$ it follows from Definition 3.1 that $P_i(\omega) \subset \hat{P}(\omega)$ for all $i \in I$. Assume the claim is true for any sequence of length m and consider any sequence $i_1, \dots, i_m, i_{m+1}$. By the induction hypothesis and Lemma 3.6,

$$P_{i_1} \dots P_{i_m} P_{i_{m+1}}(\omega) \subset P_{i_1}[\hat{P}(\omega)] \subset \hat{P}(\omega).$$

This completes the induction step and the proof of Theorem 3.2. Q.E.D.

We complete our discussion of subjective possibility and knowledge with two more results. The first, provides a condition on the state ω which guarantees that all players agree, at ω, about what is subjectively possible. This is an extension of Proposition 4.4 in Mertens and Zamir (1985).

Theorem 3.7. If $\omega \in \hat{P}_i(\omega)$ for all $i \in I$, then $\hat{P}_i(\omega) = \hat{P}_j(\omega)$ for all i and j and consequently $\hat{P}_i(\omega) = \hat{P}(\omega)$ for all $i \in I$.

Proof. We shall prove that

$$\omega \in \hat{P}_j(\omega) \Rightarrow \hat{P}_i(\omega) \subset \hat{P}_j(\omega) \quad \forall i \in I,$$

which of course implies the desired result. To see this observe first that by Lemma 3.3 $\omega \in \hat{P}_j(\omega)$ implies $P_i(\omega) \subset \hat{P}_j(\omega)$ for all i. Again, by t applications of Lemma 3.3 we get that for any integer $t \geq 0$,

$$P_t P_i(\omega) \subset P^t \hat{P}_j(\omega) \subset \hat{P}_j(\omega).$$

Hence by Definition 3.1,

$$\hat{P}_i(\omega) = \bigcup_{t=0}^{\infty} P^t P_i(\omega) \subset \hat{P}_j(\omega). \qquad \text{Q.E.D.}$$

Remark. The condition $\omega \in \hat{P}_i(\omega)$ is a very weak condition on the beliefs. It says that player i cannot be so wrong in his beliefs so as not to include the true state of the world in his subjectively possible world. Formally this is implied by consistency: the condition of the theorm is satisfied if the players' beliefs are consistent in Harsanyi's sense [Harsanyi (1967–1968)], that is if they are derived from a common prior given each player's information. This condition of consistency is assumed in most of the works in this field. For examples of inconsistent beliefs which satisfy the condition of Theorem 3.7 see Mertens and Zamir (1985) or Mertens et al. (1991).

The second result, while interesting by itself, is also a modest demonstration of the usefulness of Theorem 3.2 in proofs. At any state $\omega \in K(E)$, the event E is common knowledge. Is this fact itself a common knowledge? If the answer were 'no', our definition of K based on countably many compositions and on intersections of the individual knowledge operators, would not capture the intutive notion of common knowledge at each and every of its points, and transfinite induction would be needed in the definition of K. Fortunately, however, the answer is 'yes'.

Theorem 3.8. For any event E in Ω,

$$K(E) \subset K[K(E)].$$

Proof. By Theorem 3.2,

$$\omega \in K(E) \Leftrightarrow \bigcup_{i \in I} \hat{P}_i(\omega) \subset E.$$

Therefore,

$$\omega \in K[K(E)] \Leftrightarrow \bigcup_{i \in I} \hat{P}_i(\omega) \subset K(E)$$

$$\Leftrightarrow \bigcup_{j \in I} \bigcup_{i \in I} \hat{P}_j \hat{P}_i(\omega) \subset E.$$

Hence, to show that $\omega \in K(E) \Rightarrow \omega \in K[K(E)]$, it suffices to show that

$$\bigcup_{i \in I} \hat{P}_i(\omega) \subset E \Rightarrow \bigcup_{j \in I} \bigcup_{i \in I} \hat{P}_j \hat{P}_i(\omega) \subset E,$$

or equivalently

$$\bigcup_{j \in I} \bigcup_{i \in I} \hat{P}_j \hat{P}_i(\omega) \subset \bigcup_{i \in I} \hat{P}_i(\omega).$$

In fact by Lemma 3.4, $\hat{P}_j\hat{P}_i(\omega) \subset \hat{P}_i(\omega)$. Taking union over $i \in I$ and then over $j \in I$ yields the required inclusion. Q.E.D.

4. Relations to existing literature

In this section, we compare our notion of knowledge and beliefs to those already in the literature, mainly with those of Aumann (1976), Brandenburger and Dekel (1985), Shin (1988) and Samet (1987).

1. In the literature, it is explicitly assumed that $\omega \in P_i(\omega)$, i.e. what agents see contains the true state.

In our approach, ω might not be in $P_i(\omega)$ for some or even for all $u \in I$: agents might erroneously assign zero probability to the true state. To see this, consider a simple situation with two agents each of which can be of two different types $x_1 \neq x_2$, and such that

$$f_1(x_1)(E) = \begin{cases} 1 & S \times \{x_1\} \subset E \\ 0 & \text{otherwise} \end{cases}.$$

In words, agent 1 believes with probability one that agent 2's type, namely x_2, is equal to his own, namely x_1; this is clearly not true in all states of the world.

2. In the literature, the collection of sets $\{P_i(\omega) | \omega \in \Omega\}$ is assumed to be a partition of Ω [Aumann (1976)] or to satisfy, in addition to $\omega \in P_i(\omega)$, the property [Samet (1987)]: If $\omega' \in P_i(\omega)$ then $P_i(\omega') \subset P_i(\omega)$.

In our approach, neither of these properties is necessarily true. To see this, notice that if $x_i \neq x_i'$ then $P_i(\omega)$ and $P_i(\omega')$ can stand in any possible relation since x_i and x_i' represent different beliefs of agent i and $P_i(\omega)$ depends only on x_i by construction. The reason for the difference is that if P_i represents an agent's perceptive abilities, one needs to assume that the agent is not completely misguided by his senses, i.e. that what he sees bears some relation to what actually happens. Beliefs, however, can be completely wrong.

3. In Brandenburger and Dekel (1985), agent i knows E at ω if his beliefs are such that the conditional probability of E given $P_i(\omega)$ is one; in our notation this is $f_i(x_i)(E | P_i(\omega)) = 1$, or

$$\frac{f_i(x_i)(E \cap P_i(\omega))}{f_i(x_i)(P_i(\omega))} = 1.$$

In our case this is trivially true, since i knows E at ω iff $P_i(\omega) \subset E$. It is possible, though, that $P_i(\omega) \not\subset E$ while $f_i(x_i)(E | P_i(\omega)) = 1$ [take for instance E which is not closed, let $P_i(\omega)$ be the closure of E and let $f_i(x_i)$ be a continuous distribution]. Hence in our approach, belief with *conditional* probability one is necessary but not sufficient for knowledge.

4. In the literature, Ω is usually endowed with a common prior $\psi \in \Delta(\Omega)$. In other words, *beliefs are assumed to be consistent* (in Harsanyi's sense), i.e. to be generated from a common prior ψ by conditioning on each agent's type. In our notation, this means that the only admissible beliefs are those $x \in X^I$ that satisfy, for all $i \in I$, $f_i(x_i)(E) = \psi(E|x_i)$, for any E which is a Borel subset of $S \times X^{I \setminus \{i\}}$. There is nothing in the construction of the universal beliefs space to indicate why this should be assumed. The universal belief space accommodates beliefs which do not satisfy this restriction, and Mertens and Zamir (1985, example 4.3) have provided examples of such *inconsistent beliefs*.

5. The assumptions of a common prior and of the sets $\{P_i(\omega)|\omega \in \Omega\}$ being partitions with $\omega \in P_i(\omega)$ enabled Aumann to prove that players *cannot agree to disagree* [Aumann (1976)], i.e. it cannot be common knowledge that agents disagree on the posterior probabilities they assign to any given event E after ω has occurred. In our notation, if

$$F_{i,\alpha} = \{\omega \in \Omega \,|\, f_i(x_i)(E|P_i(\omega)) = \alpha\},$$

then $\omega \in K(F_{i\alpha} \cap F_{j\beta})$ implies $\alpha = \beta$. This is not necessarily true in our approach; in our model, *players can agree to disagree*. As a consequence, the no-speculation results of Milgrom and Stokey (1982) and Tirole (1982) need not hold, even if the description of the game is common knowledge.

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