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ALGEBRAIC CHARACTERIZATION OF INFINITE MARKOV CHAINS WHERE MOVEMENT TO THE RIGHT IS LIMITED TO ONE STEP

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Abstract

We consider an infinite Markov chain with states $E_0, E_1, \cdots$, such that $E_1, E_2, \cdots$ is not closed, and for $i \geqq 1$ movement to the right is limited by one step. Simple algebraic characterizations are given for persistency of all states, and, if E_0 is absorbing, simple expressions are given for the probabilities of staying forever among the transient states. Examples are furnished, and simple necessary conditions and sufficient conditions for the above characterizations are given.

ALGEBRAIC CHARACTERIZATION OF MARKOV CHAINS; PERSISTENCY; ABSORPTION PROBABILITIES

1. Main characterization theorem

Let $E_0, E_1, \cdots$ be the state space of an infinite Markov chain, with transition matrix $P = (p_{ij})$. Feller ((1968), pp. 401–403), gives simple proofs of the three following facts.

Theorem 1. *Let T be the set of transient states, and $E_i \in T$. The probabilities x_i, that, starting from E_i, the system stays forever among the transient states are given by the (componentwise) maximal solution of*

$$x_i = \sum_{E_j \in T} p_{ij} x_j \qquad E_i \in T, \tag{1}$$

such that $0 \leqq x_i \leqq 1$.

Criterion. In an irreducible Markov chain with states $E_0, E_1, \cdots$, the state E_0 is persistent if, and only if, the linear system

$$x_i = \sum_{j=1}^{\infty} p_{ij} x_j \qquad i \geqq 1 \tag{2}$$

admits of no solution with $0 \leqq x_i \leqq 1$, except $x_i = 0$ for all i.

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(Note that Equations (1) and (2) are the same if $T = \{E_1, E_2, \cdots\}$.)

Theorem 2. *Let C be a closed persistent set and T be the set of transient states. The probabilities y_i of ultimate absorption in C are given by the* (*componentwise*) *minimal non-negative solution of*

$$y_i = \sum_{E_j \in T} p_{ij} y_j + \sum_{E_j \in C} p_{ij} \quad E_i \in T. \tag{3}$$

In the present note we investigate in some detail the behavior of chains such that for $i \geqq 1$ transition to the right is possible only to the nearest neighbor, i.e., we assume

$$P = (p_{ij}) \text{ where } p_{i,i+1} > 0, i \geqq 1 \text{ and } p_{ij} = 0, j > i + 1. \tag{4}$$

We assume that the set $E_1, E_2, \cdots$ is *not closed*, and investigate when the states $E_1, E_2, \cdots$ are transient, and, if E_0 is an absorbing state (i.e. $p_{00} = 1$), we compute the probability, that, starting from E_i, the chain stays forever among the transient states. (The latter is equivalent to saying that the chain drifts to infinity, if E_i is identified with i.) One minus this probability is then the probability of an eventual absorption at E_0.

Assertion 1. *For P satisfying* (4), *any non-negative solution of* (2) *satisfies*

$$0 \leqq x_1 \leqq x_2 \leqq \cdots. \tag{5}$$

Proof. From (4) we have $x_1 = p_{11}x_1 + p_{12}x_2$, i.e. $x_2 = x_1(1 - p_{11})/p_{12} = x_1(p_{10} + p_{12})/p_{12} \geqq x_1$ where the last inequality is true for $x_1 \geqq 0$. Assume $0 \leqq x_1 \leqq \cdots \leqq x_k$. We shall show $x_{k+1} \geqq x_k$. By (2) $p_{k,k+1}x_{k+1} = (1 - p_{kk})x_k - \sum_{j=1}^{k-1} p_{kj}x_j \geqq (1 - p_{kk})x_k - x_k(\sum_{j=1}^{k-1} p_{kj}) = (p_{k0} + p_{k,k+1})x_k$ and the assertion follows.

Let $\mathbf{x}_k' = (x_1, \cdots, x_k)$ and let $\mathbf{x}_k$ be the corresponding column vector. Denote by $A_{k,k+1}$ the $k \times (k+1)$ matrix with elements satisfying for $i = 1, \cdots, k, j = 1, \cdots, k+1$

$$\begin{aligned} a_{ii} &= 1 - p_{ii} \\ a_{ij} &= -p_{ij} \qquad j \leqq i+1, \quad j \neq i \\ a_{ij} &= 0 \qquad\quad j > i+1. \end{aligned} \tag{6}$$

In terms of this notation Equation (2) becomes

$$\mathbf{0} = A_{k,k+1}\mathbf{x}_{k+1} \qquad k = 1, 2, \cdots. \tag{7}$$

Let A_k be the square matrix obtained from $A_{k,k+1}$ by crossing out its last column, and let D_k be the determinant of A_k. We have the following assertion.

Assertion 2. *For P satisfying* (4) *and any given value of* x_1 *the system* (2) *admits of the unique solution*

(8) $$x_{k+1} = \left(D_k \Big/ \prod_{i=1}^{k} p_{i,i+1}\right) x_1, \qquad k = 1, 2, \cdots.$$

Proof. Since the last column of $A_{k,k+1}$ has only its last element, $-p_{k,k+1}$, different from zero, (7) can be rewritten as

(9) $$(0, \cdots, 0, p_{k,k+1} x_{k+1})' = A_k \boldsymbol{x}_k.$$

The columns of A_k are independent, since otherwise there would exist $\boldsymbol{x}_k = (x_1, \cdots, x_k)'$ in which at least one component is positive, such that $A_k \boldsymbol{x}_k = (0, \cdots, 0)'$, and then $\boldsymbol{x}_{k+1} = (x_1, \cdots, x_k, 0)$ would be a solution of (9) contradicting Assertion 1. Thus A_k is regular. Multiplying both sides of (9) by A_k^{-1} and writing out explicitly the last row of this new equation, one obtains

$$a_{kk}^{(-)} p_{k,k+1} x_{k+1} = x_k,$$

where $a_{kk}^{(-)}$ is the k, kth element of A_k^{-1}. Now by definition of $a_{kk}^{(-)}$ in terms of minors, we have $a_{kk}^{(-)} = D_{k-1}/D_k$. Thus

$$x_{k+1} = x_k D_k / p_{k,k+1} D_{k-1}.$$

Now (8) follows directly by recursion, since it is true and easily directly established for $k = 1$.

Remark 1. (8) together with Assertion 1 implies that $D_k > 0$ for all k (since $D_1 = (1 - p_{11}) > 0$). This fact is not completely trivial to verify directly.

By the above assertions it follows that

(10) $$\lim_{k \to \infty} D_k \Big/ \prod_{i=1}^{k} p_{i,i+1} = L$$

exists, finite or infinite.

Theorem A. Let P satisfy (4). *If* $p_{00} < 1$ *then all states are persistent if and only if* $L = \infty$, *and all states are transient otherwise.*

If $p_{00} = 1$ *the probability of ultimate absorption in* E_0 *is one if and only if* $L = \infty$. *If* $L < \infty$, *the probability of staying forever among the transient states, starting from* E_k, *is* $D_{k-1}/(L \prod_{i=1}^{k-1} p_{i,i+1})$.

Proof. Immediate.

Remark 2. The values of (8) and (10) do not depend upon p_{ii}, $i = 1, 2, \cdots$, in the following sense.

Let P satisfy (4) and define $P^* = (p^*_{ij})$, by letting p^*_{0j} be arbitrary, and

(11) $$p^*_{i,j} = p_{i,j}/(1 - p_{ii}) \quad i \neq j \quad p^*_{i,i} = 0, \quad i \geqq 1, \ j \geqq 0.$$

If we denote by x^*_k the solution (2) with p_{ij} replaced by p^*_{ij}, and choose $x_1 = x^*_1$, then it follows immediately from (8) that $x_k = x^*_k$, $k \geqq 1$, and hence also the value of L does not change. We may therefore, whenever convenient, assume $p_{ii} = 0$, $i \geqq 1$, which renders all diagonal elements of A_k unity.

Clearly the fact whether L is finite or infinite, does not depend on the values of p_{ij} for small i, i.e. for $i \leqq s$, s fixed, provided (4) is satisfied.

2. Examples

Some simple examples, particular cases of which are well known, but derived by different methods, follow directly.

Example 1. $p_{ij} = 0$ for $j \notin \{0, i+1\}$, $i \geqq 1$. Here $D_k = 1$ and L is finite if and only if $\Pi_{i=1}^{\infty} p_{i,i+1} > 0$, i.e. if and only if $\Sigma_{i=1}^{\infty}(1 - p_{i,i+1}) = \Sigma_{i=1}^{\infty} p_{i,0} < \infty$. This is Example (c), Feller (1968), p. 400. If p_{ii} does not necessarily vanish, then by Remark 2, the above condition should be replaced by $\Sigma_{i=1}^{\infty} p_{i0}/(1 - p_{ii}) < \infty$, or equivalently by $\Sigma_{i=1}^{\infty} p_{i0}/p_{i,i+1} < \infty$.

Example 2. Let $t \geqq 1$ be fixed. Assume $p_{ij} = 0$ for $j \notin \{0, t, i, i+1\}$, and by Remark 2 we assume also $p_{i,i} = 0$. $D_k = 1$ for $k \leqq t$, and for $k > t$, computing D_k through its last row yields $D_k = p_{kt} \Pi_{j=t}^{k-1} p_{j,j+1} + D_{k-1}$, and by recursion

$$D_k = 1 - \sum_{i=t+1}^{k} p_{it} \prod_{j=t}^{i-1} p_{j,j+1}.$$

If $\Pi_{i=1}^{\infty} p_{i,i+1} > 0$ then clearly $L < \infty$. This product is positive if and only if $\Sigma_{i=1}^{\infty}(p_{i0} + p_{it}) < \infty$. We claim that the opposite is also true, i.e. $\Sigma_{i=1}^{\infty}(p_{i0} + p_{it}) = \infty$ implies $L = \infty$, unless $p_{i0} = 0$ for all $i \geqq t$. (Notice that the assumption that $E_1, E_2, \cdots$ is not closed rules out the possibility of $p_{i0} = 0$ for all $i \geqq t$.) Clearly $\Sigma_{i=1}^{\infty}(p_{i0} + p_{it}) = \infty$ implies $L = \infty$, unless $\lim_{k \to \infty} D_k = 0$. Since $p_{it} \leqq 1 - p_{i,i+1}$ it follows that

$$D_k \geqq 1 - \sum_{i=t+1}^{k} (1 - p_{i,i+1}) \prod_{j=t}^{i-1} p_{j,j+1} = 1 - p_{t,t+1} + \prod_{j=t}^{k} p_{j,j+1} \to 1 - p_{t,t+1}.$$

This limit is positive unless $p_{t,t+1} = 1$ (i.e. $p_{t,0} = 0$). If $p_{t,0} = 0$, then by our assumptions there exists an $s > t$ such that $p_{s0} > 0$ and then for $k \geqq s$ $D_k \geqq 1 + \Pi_{j=t}^{k} p_{j,j+1} + p_{s0} \Pi_{j=t}^{s-1} p_{j,j+1}$ which tends to a positive limit. If not all p_{ii} vanish, and there exist an $s \geqq t$ such that $p_{s0} > 0$, L is finite if and only if $\Sigma_{i=1}^{\infty}(p_{i0} + p_{it})/(1 - p_{ii}) < \infty$.

Example 3. $p_{ij} = 0$ for $j \notin \{0, i-1, i, i+1\}$, $i \geqq 1$ and by Remark 2 we assume also $p_{ii} = 0$. Computing D_k through its last row yields $D_k = D_{k-1} - p_{k,k-1} p_{k-1,k} D_{k-2}$. This yields a recursion relation which is not, in general easy to handle. For the special case $p_{i,0} = 0$ for $i > 1$, we have $p_{i,i-1} + p_{i,i+1} = 1$, and the above yields, if we set $x_1 = 1$

$$D_k - p_{k,k+1} D_{k-1} = p_{k,k-1}(D_{k-1} - p_{k-1,k} D_{k-2}) = \cdots = \prod_{i=1}^{k} p_{i,i-1}.$$

Thus

$$x_{k+1} - x_k = (D_k - p_{k,k+1}D_{k-1}) \Big/ \prod_{i=1}^{k} p_{i,i+1} = \prod_{i=1}^{k} (p_{i,i-1}/p_{i,i+1})$$

and $L < \infty$ if and only if $\Sigma_{k=1}^{\infty}\Pi_{i=1}^{k}(p_{i,i-1}/p_{i,i+1}) < \infty$. By Remark 2 this criteria remains unchanged also if $p_{ii} \neq 0$. This is the well-known birth and death process given as Example (d), p. 402 of Feller (1968).

Example 4. If the chain is a martingale, i.e. satisfies $\Sigma_{j=0}^{\infty} p_{ij} j = i$, $i \geqq 0$, then $x_i = i$ is a solution of (2). If (4) is satisfied, this is essentially a unique solution and hence $L = \infty$, which, since $p_{00} = 1$ implies absorption with probability one at E_0. This is therefore a simple particular case of the martingale convergence theorem.

3. Necessary and sufficient conditions for $L = \infty$

Several sufficient conditions for recurrence of general Markov chains have been treated in the literature. The best-known seems to be the following condition, which has recently been shown by Tweedie (1975) to be sufficient also for ultimate recurrence. (For definition of this concept, see Tweedie (1975). If a chain is irreducible, as is the case in the present discussion, ultimate recurrence is the same as recurrence.)

Theorem (Tweedie). *If there exists an integer N and a sequence* $\{x_j\}$ *with* $\infty > x_j \geqq 0$ *such that* $x_j \to \infty$ *as* $j \to \infty$, *and*

$$\sum_{j=0}^{\infty} p_{ij} x_j \leqq x_i \qquad (i \geqq N) \tag{12}$$

then the chain is ultimately recurrent.

For irreducible aperiodic chains this theorem is due to Pakes (1969) (see his Theorem 3), and with $N = 1$ it is due to Foster (1952) and (1953). (See Theorem 1, (1952) and Theorem 5, (1953).) Pakes and Foster prove a seemingly stronger theorem, since they do not require $x_i \geqq 0$. There is, however, no loss of generality in adding this requirement since it can always be achieved by shifting all x_i's by a constant.

For the chains considered in the present paper, the condition is also necessary. We have the following theorem.

Theorem B. Let P satisfy (4). *If* $p_{00} < 1$ *all states are persistent if and only if there exist a solution to the set of inequalities* (12), *for which* $x_j \to \infty$ *as* $j \to \infty$. *If* $p_{00} = 1$ *the probability of ultimate absorption in* E_0 *is one, if and only if such a solution exists.*

Proof. Sufficiency is obvious. By Theorem A a necessary condition for recurrence is $L = \lim_{j\to\infty} x_j / x_1 = \infty$, where x_j satisfy (2), and *a forteriori* satisfy (12).

Remark 3. Since we can, without loss of generality, assume that $x_i \geqq 0$, there is also no loss of generality in afterwards assuming $x_0 = 0$, since the inequalities (12) can at most be strengthened by this assumption. Thus Theorem B states that for P satisfying (4) the equalities in (2) of the criterion can be replaced by inequalities.

Foster (1952) gives some additional assumptions on the chain under which (12) is also necessary, but gives, in a footnote, an example of a simple chain satisfying (4), to show that his additional assumptions are not always needed to render (12) necessary.

A natural candidate for x_i in (12) is $x_i = i$. Thus one obtains the sufficient condition (see Tweedie (1975))

$$\sum_{j=1}^{\infty} jp_{ij} \leqq i \qquad (i \geqq N), \tag{13}$$

which, when E_i is identified with i, becomes a notion of eventual non-increasing expectation. (Compare with the notion of supermartingale.)

For chains satisfying (4) there is a simple way to get necessary conditions, and (different) sufficient conditions for $L = \infty$. It is intuitively clear that moving probability to the left, in each row separately, can only decrease the probability of staying in the transient states. This statement is made precise in the following assertion.

Assertion 3. *Let $P = (p_{ij})$ and $Q = (q_{ij})$ satisfy* (4), *and*

$$\sum_{t=0}^{j} p_{it} \leqq \sum_{t=0}^{j} q_{it} \quad j \geqq 0, \; i \geqq 1. \tag{14}$$

Denote by x_i and x_i^*, $i = 1, 2, \cdots$, the solutions of (2) *for P and Q respectively, and $x_1 = x_1^*$. Then $x_i \leqq x_i^*$, $i = 1, 2, \cdots$, and hence $L = \infty$ implies $L^* = \infty$, and $L^* < \infty$ implies $L < \infty$, where L^* is the value of* (10) *defined for Q.*

Proof. Let $z_k = x_k - x_{k-1}$ $(x_0 = 0)$, $k = 1, 2, \cdots$ and define z_k^* correspondingly. The assertion follows if we show $z_k \leqq z_k^*$, $k = 1, 2, \cdots$, and actually this latter statement is somewhat stronger. Transcribing (2) in terms of the z_i's, we have

$$p_{k,k+1} \sum_{i=1}^{k+1} z_i = (1 - p_{kk}) \sum_{i=1}^{k} z_i - \sum_{i=1}^{k-1} p_{ki} \sum_{j=1}^{i} z_j$$

which after some rearrangement yields

$$p_{k,k+1} z_{k+1} = \sum_{i=1}^{k} z_i \sum_{j=0}^{i-1} p_{kj}. \tag{15}$$

Since (14) implies $p_{k,k+1} \geqq q_{k,k+1}$ the assertion follows directly by induction.

Conclusion 1. This assertion yields simple proofs of many inequalities about D_k, direct proofs of which are more difficult. For example,

$$D_k \leqq D_{k-1}(1-p_{kk}),\, k>1, \tag{16}$$

is obtained by setting $q_{ij}=p_{ij}$ for $i<k$ and all j, and $q_{k0}=\Sigma_{j=0}^{k-1}p_{kj}$, $q_{kk}=p_{kk}$, $q_{k,k+1}=p_{k,k+1}$ and $q_{kj}=0$ for all other j. Then (14) holds, and denoting the D-values of Q by D_i^*, $i \geqq 1$, we have $D_k^*=(1-q_{kk})D_{k-1}^*=(1-p_{kk})D_{k-1}$. Now $x_{k+1} \leqq x_{k+1}^*$ together with (8) imply (16).

In contrast to (16), Assertion 1 together with (8) yield $D_k \geqq p_{k,k+1}D_{k-1}$. From (16) follows immediately

$$D_1 \geqq D_2 \geqq \cdots \text{ and } D_k \leqq \prod_{i=1}^{k}(1-p_{ii}). \tag{17}$$

Assertion 3 and Examples 1 and 2 yield the following conclusion.

Conclusion 2.

(a) *A sufficient condition for* $L=\infty$ *is that for some* $t \geqq 0$,

$$\sum_{i=1}^{\infty}\sum_{j=0}^{t} p_{ij}/(1-p_{ii})=\infty.$$

(b) *A sufficient condition for* $L<\infty$ *is* $\Sigma_{i=1}^{\infty}[1-p_{i,i+1}/(1-p_{ii})]<\infty$.

For chains satisfying (4) summation in (13) goes to $i+1$ only, and thus it may seem that (13) is a more natural condition than (a) of Conclusion 2. Indeed, it seems difficult to violate (13) and still have $L=\infty$. The following example shows that this is possible, however, also in cases which are not completely trivial.

Example 5. For $i \geqq 2$ $p_{i0}=(2i)^{-1}$, $p_{ij}=(2i(i-1))^{-1}$ for $j=1,\cdots,i-1$, $p_{ii}=0$, $p_{i,i+1}=1-i^{-1}$. Then $\Sigma_{j=1}^{i+1} jp_{ij}=i+\frac{1}{4}-i^{-1}$ so (13) is violated for all $i>4$. Nevertheless $\Sigma_{i=1}^{\infty}p_{i0}=\infty$, so (a) of Conclusion 2 holds (with $t=0$), and $L=\infty$.

The results of this note can be generalized to similar results, when the chain is restricted to move to the right no more than any *fixed* number $t \geqq 1$ of steps, i.e., when $p_{ij}=0$ for all $j>i+t$, $i \geqq 1$.

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