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A 2-PERSON GAME WITH LACK OF INFORMATION ON $1\frac{1}{2}$ SIDES*

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We consider a repeated 2-person 0-sum game with incomplete information about the pay-off matrix. Player I (maximizer) knows the real pay-off matrix but he is uncertain about the beliefs of his opponent. We show that in this case the Aumann–Maschler results on incomplete information on one side no longer hold. In particular such a game will not have a value in general, in spite of the fact that one player is fully informed about the state of nature.

1. Introduction. In the literature on 2-person 0-sum repeated games of incomplete information ([1], [2], [3]) a distinction is made between *games with incomplete information on one side* in which one of the players is fully informed about the pay-off matrix and *games with incomplete information on two sides* in which both players are uncertain about the real pay-off matrix. In this note we consider what looks at first sight to be intermediate case (suggesting the name "incomplete information on $1\frac{1}{2}$ sides"): one player is fully informed about the true state of nature but he is uncertain about the beliefs of his opponent about the state of nature. Although formally such games turn out to be equivalent to games of incomplete information on two sides, we think that they merit special attention for at least two reasons:

(1) They underline the distinction between *state of nature*, which is in this case the true pay-off matrix, and *state of the world*, which is the state of nature and the state of all beliefs and mutual beliefs concerning the state of nature (see [4]). Here we have a situation in which one player has complete information about the state of nature (namely the pay-offs) but incomplete information about the state of the world, since he has incomplete information about the beliefs of the other player. As we shall see, these games will not have a value in general and hence behave mathematically as games of incomplete information on two sides. The conclusion to remember is therefore: when speaking about a game of incomplete information in which a certain player has a complete information, it has to be clear that he has a complete information about the *state of the world* (and not only on the state of nature) and only then may we apply the various results on incomplete information on one side.

In reference to [4], the structure of hierarchy of beliefs, the case we will be considering is of the simplest generic type: the hierarchy of beliefs consists of one level after which all becomes a common knowledge.

(2) The games considered here are mathematically equivalent to games of incomplete information on two sides of what is called *the dependent case* (see [2]), that is the beliefs of each player on the types of the other player depend on his own type. This shows that the dependent case is not only conceivable but seems to be rather the typical case: whenever the states of the world or the "types" in question contain also the beliefs, as is typically the situation, the dependent case seems inavoidable.

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2. The game.

2.1. Let A and B be two $|I| \times |J|$ zero-sum pay-off matrices. For each $\lambda, r, s \in [0,1]^3$ we define a game $G(\lambda, r, s)$ as follows:

Step 00. Chance chooses $\mathbf{t} \in \{r, s\}$ with $\operatorname{Prob}(\mathbf{t} = r) = \lambda$ and this choice is told to Player II only.

Step 0. Chance chooses $\mathbf{C} \in \{A, B\}$ with $\operatorname{Prob}(\mathbf{C} = A) = \mathbf{t}$ and this choice is told to Player I only.

Step 1. Player I (resp. Player II) selects some move $i_1 \in I$ (resp. $j_1 \in J$) and this choice is announced to both players. For $m = 1, \ldots, n$:

Step m. Both players knowing in addition to their private information (from Steps 00 and 0) the previous history, i.e. $(i_1, j_1, \ldots, i_{m-1}, j_{m-1})$, select some move and this pair (i_m, j_m) is announced to both players. $G_n(\lambda, r, s)$ is the n-repeated game with pay-off for Player I (the maximizer) given by: $n^{-1}\sum_{m=1}^{n} \mathbf{C}_{i_m j_m}$ and G_∞ is the infinitely repeated game. All the previous description including this sentence is common knowledge.

The game can be represented as follows:

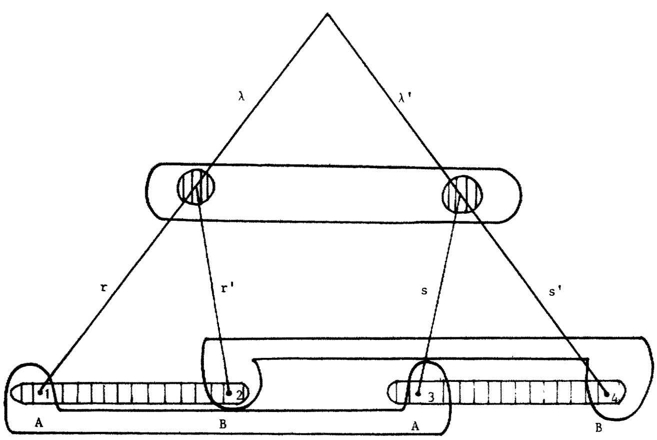

FIGURE 1

where x' stands for $1 - x$, $\forall x \in [0,1]$.

Hence Player I is told the true pay-off matrix without knowing the (objective) belief of PII about the true pay-off matrix.

Obviously these games belong to the special class of games with lack of information on both sides, dependent case, as introduced in [2].

In fact, using the notations there we have:

$K = \{1,2,3,4\}$ (states of the world),

$$\left.\begin{aligned} K_{\mathrm{I}} &= \{(1,3),(2,4)\} \\ K_{\mathrm{II}} &= \{(1,2),(3,4)\} \end{aligned}\right\} \text{(initial information partitions),}$$

$q = (q_1, q_2, q_3, q_4) = (\lambda r, \lambda r', \lambda' s, \lambda' s')$ (prior probability distribution),

$u(q) = \operatorname{val}((q_1 + q_3)A + (q_2 + q_4)B)$ (value of the nonrevealing game),

$\Pi_{\mathrm{I}}(q) = \{\alpha q_1, \beta q_2, \alpha q_3, \beta q_4;\ \alpha \geqslant 0,\ \beta \geqslant 0,\ \alpha(q_1 + q_3) + \beta(q_2 + q_4) = 1\}$,

$\Pi_{\mathrm{II}}(q) = \{\gamma q_1, \gamma q_2, \delta q_3, \delta q_4;\ \gamma \geqslant 0,\ \delta \geqslant 0,\ \gamma(q_1 + q_2) + \delta(q_3 + q_4) = 1\}$ (sets of concavification and convexification).

It follows then from [2] and [3] that:

Minmax G_∞ exists and equals $\operatorname{Vex}_{\mathrm{II}}\operatorname{Cav}_{\mathrm{I}} u(q)$,

Maxmin G_∞ exists and equals $\operatorname{Cav}_{\mathrm{I}}\operatorname{Vex}_{\mathrm{II}} u(q)$,

$v = \lim \mathrm{val}(G_n)$ exists and is the only simultaneous solution of

$$w(q) = \operatorname*{Vex}_{\mathrm{II}} \max\{u(q), w(q)\}, \tag{1}$$

$$w(q) = \operatorname*{Cav}_{\mathrm{I}} \min\{u(q), w(q)\}. \tag{2}$$

Let us first compute $\mathrm{Cav}_{\mathrm{I}} u$ and $\mathrm{Vex}_{\mathrm{II}} u$. Define $\tilde{u}(p) = \mathrm{val}(pA + p'B)$ for $p \in [0,1]$ and $\rho = q_1 + q_3$.

2.2.

$$\begin{aligned}\operatorname*{Cav}_{\mathrm{I}} u(q) &= \sup_{t,q^1,q^2} \left\{ tu(q^1) + t'u(q^2);\, t \in [0,1],\, tq^1 + t'q^2 = q,\, q^i \in \Pi_{\mathrm{I}}(q);\, i = 1,2 \right\} \\ &= \sup_{t,\rho^1,\rho^2} \left\{ t\tilde{u}(\rho^1) + t'\tilde{u}(\rho^2);\, t \in [0,1],\, t\rho^1 + t'\rho^2 = \rho,\, (*)\ \rho^i + \beta^i\rho' = 1;\, i = 1,2 \right\}\end{aligned}$$

where $\rho^i = q_1^i + q_3^i$; $i = 1,2$ and $\rho' = 1 - \rho = q_2 + q_4$.

If $\rho' = q_2 + q_4 = 0$, $\Pi_{\mathrm{I}}(q) = \{q\}$ and $\mathrm{Prob}(\mathbf{C} = B) = 0$ and otherwise for any $\rho^i \in [0,1]$ there exists β^i such that $(*)$ holds; $i = 1,2$. Hence:

$$\operatorname*{Cav}_{\mathrm{I}} u(q) = \sup_{t,\rho^1,\rho^2} \left\{ t\tilde{u}(\rho^1) + t'\tilde{u}(\rho^2);\, t \in [0,1],\, t\rho^1 + t\rho^2 = \rho,\, \rho^i \in [0,1]\ i = 1,2 \right\}.$$

Thus $\mathrm{Cav}_{\mathrm{I}} u(q) = \mathrm{Cav}\, \tilde{u}(q_1 + q_3)$, where Cav is the usual operator on [0, 1].

Note that this is the value of the usual Aumann–Maschler game with incomplete information on one side (see [1]), where Player II is not told **t**.

2.3.

$$\begin{aligned}\operatorname*{Vex}_{\mathrm{II}} u(q) &= \inf_{t,q^1,q^2} \left\{ tu(q^1) + t'u(q^2);\, t \in [0,1], tq^1 + t'q^2 = q,\, q^i \in \Pi_{\mathrm{II}}(q)\ i = 1,2 \right\} \\ &= \inf_{t,\rho^1,\rho^2} \Big\{ t\tilde{u}(\rho^1) + t'\tilde{u}(\rho^2);\, t\rho^1 + t'\rho^2 = \rho,\, t \in [0,1] \\ &\qquad (**)\ \rho^i + \gamma^i q_2 + \delta^i q_4 = 1,\, i = 1,2 \Big\}.\end{aligned}$$

(Note that $tq^1 + t'q^2 = q \Leftrightarrow t\rho^1 + t'\rho^2 = \rho$ whenever $\lambda\lambda'(r - s) \neq 0$.)

$(**)$ gives as extreme values for ρ^i: $\pi(q) = q_1(q_1 + q_2)$ and $\tau(q) = q_3/(q_3 + q_4)$.

Letting $|\pi(q), \tau(q)|$ be the interval of points between these values, we obtain: $\mathrm{Vex}_{\mathrm{II}} u(q) = \mathrm{Vex}_{|\pi(q),\tau(q)|} \tilde{u}(q_1 + q_3)$, where $\mathrm{Vex}_{|a,b|} f$ is the greatest function less than f and convex on $|a, b|$. (Note that $\mathrm{Vex}_{\mathrm{II}} u(q)$ is not a function of ρ only.)

2.4. Since $\mathrm{Cav}_{\mathrm{I}} u(q)$ depends only on ρ and is concave on [0, 1], we obtain:

$$\begin{aligned}\operatorname*{Vex}_{\mathrm{II}} \operatorname*{Cav}_{\mathrm{I}} u(q) &= \mu\, \mathrm{Cav}\, \tilde{u}(\pi(q)) + \mu'\, \mathrm{Cav}\, \tilde{u}(\tau(q)) \quad \text{with} \quad \mu\pi(q) + \mu'\tau(q) = \rho, \\ &= \lambda\, \mathrm{Cav}\, \tilde{u}(r) + \lambda'\, \mathrm{Cav}\, \tilde{u}(s).\end{aligned}$$

Note that the min max is the value of the game where Player I is also informed of **t**.

2.5. Finally for the max min we get:

$$\begin{aligned}\operatorname*{Cav}_{\mathrm{I}} \operatorname*{Vex}_{\mathrm{II}} u(q) = \sup_{t,q^1,q^2} \Big\{ & t\, \mathrm{Vex}_{|\pi(q^1),\tau(q^1)|} \tilde{u}(\rho^1) + t'\, \mathrm{Vex}_{|\pi(q^2),\tau(q^2)|} \tilde{u}(\rho^2);\, t \in [0,1], \\ & tq^1 + t'q^2 = q,\, q^i \in \Pi_{\mathrm{I}}(q);\, i = 1,2 \Big\}.\end{aligned}$$

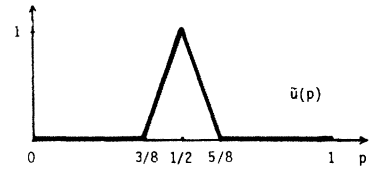

FIGURE 2. The function $\tilde{u}$.

3. Example. Let

$$A = \begin{pmatrix} 5 & -3 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -3 & 5 \\ 0 & 0 \end{pmatrix}.$$

We moreover assume $s = 1$, i.e., $q_4 = 0$ and $\tau(\cdot) \equiv 1$.

3.1. Let us first look at the point $r = \frac{1}{2}$, $\lambda = \frac{1}{2}$.

From 2.4 we obtain: $\min\max G(\frac{1}{2}, \frac{1}{2}, 1) = \frac{1}{2}\operatorname{Cav}\tilde{u}(\frac{1}{2}) = \frac{1}{2}$ and from 2.5:

$$\max\min G\left(\tfrac{1}{2}, \tfrac{1}{2}, 1\right) = \sup_{t, q^1, q^2} \Big\{ t \operatorname{Vex}_{|\pi(q^1),1|}\tilde{u}(\rho^1) + t' \operatorname{Vex}_{|\pi(q^2),1|}\tilde{u}(\rho^2);$$

$$t \in [0,1],\ t\rho^1 + t'\rho^2 = \tfrac{3}{4},\ q^i \in \Pi_{\mathrm{I}}(q)\ i = 1,2 \Big\}.$$

Note that $\rho^i \geqslant \frac{5}{8}$ implies $\operatorname{Vex}_{|\pi(q^i),1|}\tilde{u}(\rho^i) = 0$.

Moreover, since $q^i \in \Pi_{\mathrm{I}}(q)$, $\rho^i < \frac{5}{8}$ implies $\pi(q^i) < \frac{3}{8}$ hence the Vex is still 0. Thus $\max\min G(\frac{1}{2}, \frac{1}{2}, 1) = 0$.

The interpretation is the following: in order to obtain a strictly positive pay-off, Player I has to generate posteriors lying in $[\frac{3}{8}, \frac{5}{8}]$, i.e., to increase the probability of B. But then for $\mathbf{t} = r$, the conditional probability of A (namely $\pi(\rho^i)$) will be less than $\frac{3}{8}$ and Player II will obtain 0 by playing either L (if $\mathbf{t} = r$) or R (if $\mathbf{t} = s$).

3.2. *Construction of the* min max *and the* max min. Note that $\Pi_{\mathrm{I}}(q)$ is the line through q and $(0,1,0)$. Hence $\operatorname{Cav}_{\mathrm{I}} f$ is the smallest function greater than f and concave on each of these lines. (Similarly $\Pi_{\mathrm{II}}(q)$ is the line through q and $(0,0,1)$.) Thus we obtain the various functions which are described in the following figures.

Here we represent each function by a pair of figures: On the left we draw the simplex $Q = \{(q_2, q_3) \mid q_2 \geqslant 0,\ q_3 \geqslant 0\ q_2 + q_3 \leqslant 1\}$ cut into various regions. On each region (or line) we either write the value of the function or we draw an arrow (which corresponds to Π_{I} for Π_{II}) to indicate that the function in question is linear in the

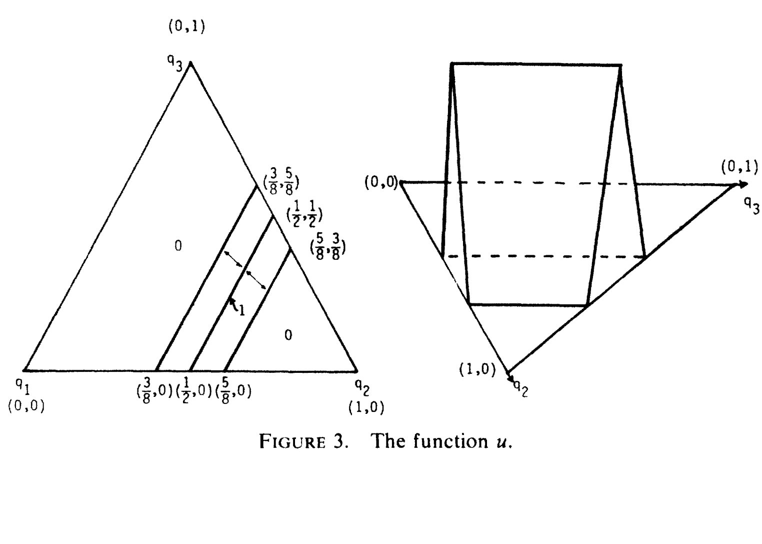

FIGURE 3. The function u.

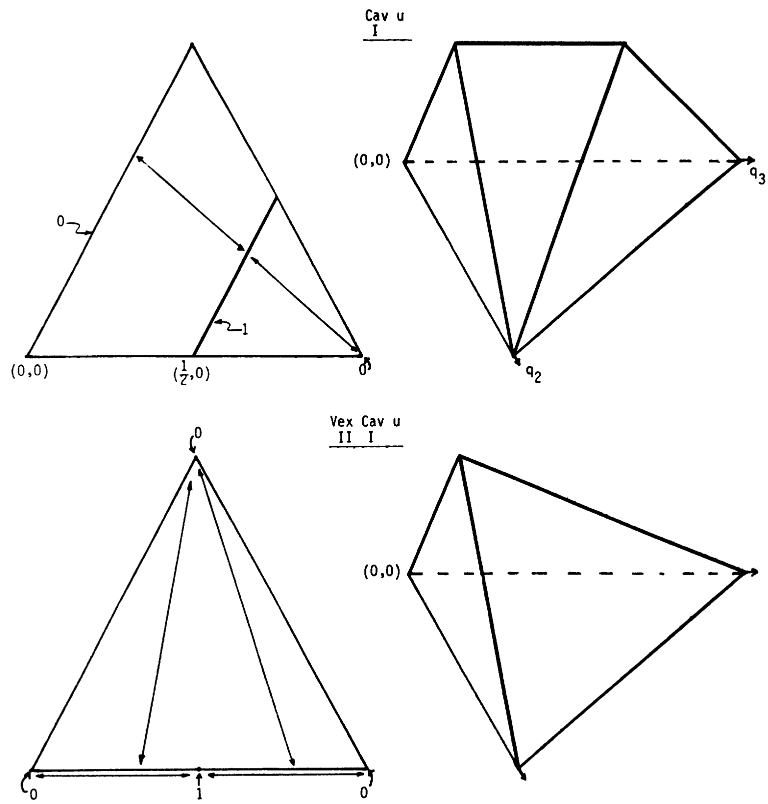

FIGURE 4. The functions $\mathrm{Cav}_{\mathrm{I}}u$ and $\mathrm{Vex}_{\mathrm{II}}\mathrm{Cav}_{\mathrm{I}}u$.

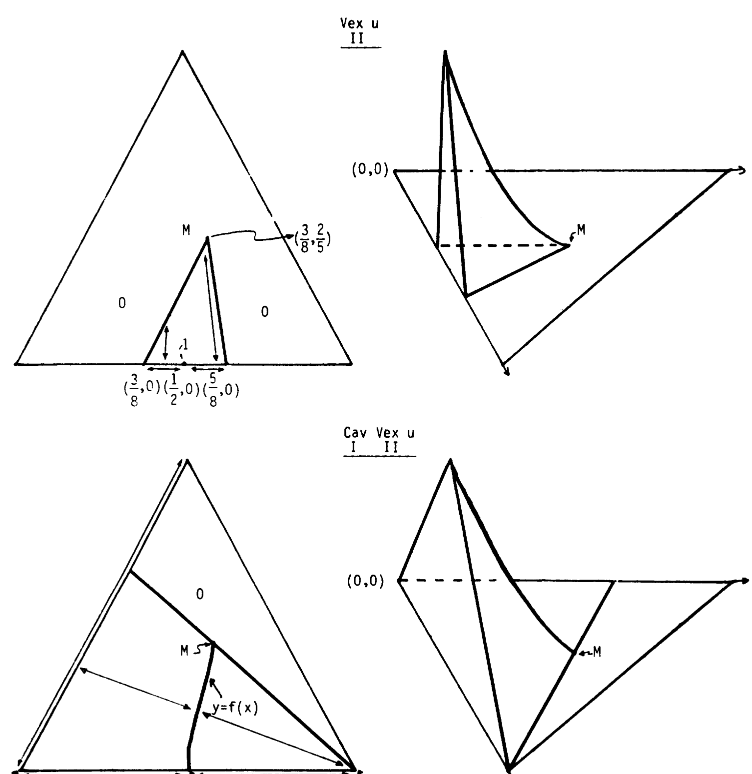

FIGURE 5. The functions $\mathrm{Vex}_{\mathrm{II}}u$ and $\mathrm{Cav}_{\mathrm{I}}\mathrm{Vex}_{\mathrm{II}}u$.

direction of the arrow. On the right-hand figure we give the three-dimensional shape of the function.

REMARKS. It is easy to see the graph of f is in the right of the line from $(\frac{1}{2},\frac{1}{2})$ to $(0,1)$ and coincides with this line near $(\frac{1}{2},\frac{1}{2})$.

So we have: $\mathrm{Cav}_{\mathrm{I}}\mathrm{Vex}_{\mathrm{II}} \neq \mathrm{Vex}_{\mathrm{II}}\mathrm{Cav}_{\mathrm{I}}$ everywhere on the interior of the simplex.

3.3. In order to study v, we first compare v and u (recall that $\max\min \leqslant v \leqslant \min\max$).

It is generally true for the solution v of (1) and (2) that (see [2]):

—On the region in which $v > u$, v is I-linear.

—On the region in which $v < u$, v is II-linear.

In view of this, in order to find v it suffices to divide the simplex Q into the three subregions in which: $v > u$, $v < u$ or $v = u$ (see Figure 6).

(a) Regions where $v \geqslant u$. First observe that $v \geqslant \mathrm{Cav}_{\mathrm{I}}\mathrm{Vex}_{\mathrm{II}}u \geqslant 0$ everywhere hence $v \geqslant u$ on region I, and $v > 0$ where $\mathrm{Cav}_{\mathrm{I}}\mathrm{Vex}_{\mathrm{II}}u > 0$. It follows that $v > u$ on region II.

We have also $v > u$ on region III in which $\mathrm{Vex}_{\mathrm{II}}u = u$, thus $\mathrm{Cav}_{\mathrm{I}}\mathrm{Vex}_{\mathrm{II}}u > u$ which implies $v > u$.

(b) Regions where $v < u$. Since $v \leqslant \mathrm{Vex}_{\mathrm{II}}\mathrm{Cav}_{\mathrm{I}}u$, $v < u$ on the line $q_2 = \frac{1}{2}$ (except for $q_3 = 0$) and on the open segment $](\frac{3}{8},\frac{5}{8}),(\frac{5}{8},\frac{3}{8})[$.

(c) Regions where $v = u$. This region consists of the following parts of the boundary of Q: the segments $[(0,0),(0,1)], [(0,1),(\frac{3}{8},\frac{5}{8})], [(\frac{5}{8},\frac{3}{8}),(1,0)]$ and the point $(\frac{1}{2},0)$. (The value of v on the boundary is readily computed by the Aumann–Maschler result on incomplete information on one side.)

From this and from (a) and (b) the region $v = u$ must contain also two lines both starting from $(\frac{1}{2},0)$, one on the left side of $q_2 = \frac{1}{2}$ and reaching $(\frac{3}{8},\frac{5}{8})$, the other on the right side reaching $(\frac{5}{8},\frac{3}{8})$.

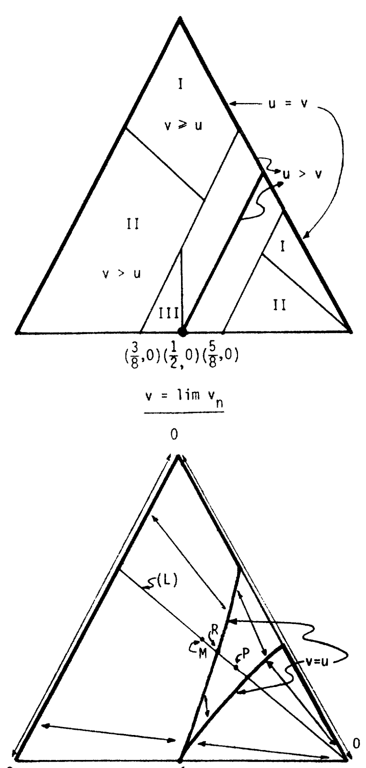

FIGURE 6. Partition of Q according to the relation between v and u.

With the help of the last two lines of $v = u$ mentioned in (c), the description of v can be thus summarized by:
$v < u$—hence v is II-linear inside these lines,
$v > u$—hence v is I-linear outside them.

3.4. *Comparison of* v *and* min max, max min. Consider a point P on the line (L) through $(1,0)$ and M where $u > v$ (this is precisely $\Pi_{\mathrm{I}}(M)$). Note that at this point Cav Vex = 0. Now we claim that $v(P) > 0$, otherwise there are two points P_1, P_2 where $u(P_i) = v(P_i) = 0$ and $P_i \in \Pi_{\mathrm{II}}(P)$. But then one of these points lies on $q_2 = \frac{5}{8}$ and strictly below (L), where the Cav Vex is strictly positive, hence v also: a contradiction.

Now let R be the point on the line (L) with $q_2 < \frac{1}{2}$ and $u = v$. The min max at R is greater than or equal to v. Now on (L) v is linear from R to the point on $q_2 = 0$, but min max is strictly increasing up to the line $[(\frac{1}{2}, 0), (0, 1)]$ hence the min max is strictly greater than v near R on the left.

Th fact that Cav Vex $\neq$ Vex Cav everywhere in the interior of the simplex now implies that there is a game $G(\lambda, r, 1)$ with: max min $< v <$ min max.

References

[1] Aumann, R. J., and Maschler, M. (1966). Game Theoretic Aspects of Gradual Disarmament. Chapter V of *Development of Utility Theory for Arms Control and Disarmament.* Report to the U.S. A.C.D.A., Contract S.T. 80, Mathematica, Princeton.

[2] Mertens, J. F. and Zamir, S. (1971). The Value of Two-Person Zero-Sum Game with Lack of Information on Both Sides. *Internat. J. Game Theory* **1** 39–64.

[3] ——— and ———. (1980). Min max and Max min of Repeated Games with Incomplete Information. *Internat. J. Game Theory* **9** 201–215.

[4] ——— and ———. (1982). On Harsanyi's Notions of Types and Consistency in Games of Incomplete Information. CORE D. P. 8230.

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