Extending the Condorcet Jury Theorem to a general dependent jury

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Extending the Condorcet Jury Theorem to a general dependent jury

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Abstract   We investigate necessary and sufficient conditions for the existence of Bayesian-Nash equilibria that satisfy the *Condorcet Jury Theorem* (*CJT*). In the Bayesian game G_n among n jurors, we allow for arbitrary distribution on the types of jurors. In particular, any kind of dependency is possible. If each juror i has a "constant strategy", σ^i (that is, a strategy that is independent of the size $n \geq i$ of the jury), such that $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n \ldots)$ satisfies the *CJT*, then by McLennan (Am Political Sci Rev 92:413–419, 1998) there exists a Bayesian-Nash equilibrium that also satisfies the *CJT*. We translate the *CJT* condition on sequences of constant strategies into the following problem:

(**) For a given sequence of binary random variables $X = (X^1, X^2, \ldots, X^n, \ldots)$ with joint distribution P, does the distribution P satisfy the asymptotic part of the *CJT*?

We provide sufficient conditions and two general (distinct) necessary conditions for (**). We give a complete solution to this problem when X is a sequence of exchangeable binary random variables.

1 Introduction

The simplest way to present our problem is by quoting Condorcet's classic result (see Young 1997):

Theorem 1 (*CJT*–Condorcet 1785) *Let n voters (n odd) choose between two alternatives that have equal likelihood of being correct a priori. Assume that voters make their*

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judgements independently and that each has the same probability p of being correct
($\frac{1}{2} < p < 1$). Then, the probability that the group makes the correct judgement using
simple majority rule is

$$\sum_{h=(n+1)/2}^{n} [n!/h!(n-h)!]p^h(1-p)^{n-h}$$

which approaches 1 as n becomes large.

We build on some of the literature on this issue in the last 30 years. First, we notice
that Nitzan and Paroush (1982) and Shapley and Grofman (1984) allow for unequal
competencies of the juries. They replace the simple majority committee by weighted
majority simple games to maintain the optimality of the voting rule.

Second, we notice that many papers are on the dependency among jurors. Among
these papers are Shapley and Grofman (1984), Boland et al. (1989), Ladha (1992,
1993, 1995), Berg (1993a,b), Dietrich and List (2004), Berend and Sapir (2007), and
Dietrich (2008). It is widely understood and accepted that the votes of the jurors are
often correlated. For example, group deliberation prior to voting is viewed, justifiably,
as undermining independence (Grofman et al. 1983; Ladha 1992, 1995; Estlund 1994;
Dietrich and List 2004). In particular, Dietrich (2008) argues that independence cannot
be fully justified in the Condorcet jury model.

Finally, we mention the seminal paper of Austen-Smith and Banks (1996) which
incorporated strategic analysis into the Condorcet jury model. This paper had many
followers, in particular McLennan (1998) and Duggan and Martinelli (2001) which
investigated the Condorcet Jury Theorem (*CJT*) for Bayesian-Nash equilibria (BNE).

In this work, we investigate the *CJT* for BNE. Unlike Austen-Smith and Banks
(1996), we do not assume that the *types* of the voters are independent (given the
state of nature). Indeed, we assume arbitrary dependency among (the types of) jurors.
As far as we could ascertain, McLennan (1998) is the only paper that studies the
CJT for BNE assuming dependency among the jurors. In fact, we rely heavily on
McLennan's work; the game among n jurors, is a Bayesian game G_n in which all the
players have the same payoff function which is the probability of *correct decision*.
Therefore, any n-tuple of strategies $\sigma_n = (\sigma_n^1, \ldots, \sigma_n^n)$ that maximizes the common
payoff is a BNE (McLennan 1998, Theorem 1). Now consider an infinite sequence
of such strategies $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots)$ that are BNE for the sequence of games
$G_1, G_2, \ldots, G_n, \ldots$ with a jury of growing size. If there exists any other sequence of
strategies $\tau = (\tau_1, \tau_2, \ldots, \tau_n, \ldots)$ (not necessarily BNE), that satisfies the *CJT*, then
the original sequence σ is a sequence (of BNE) that also satisfies the *CJT*. Thus, we
may focus on the following problem:

(*)   For a given sequence of Bayesian games $G_1, G_2, \ldots, G_n, \ldots$ with an increasing
set of jurors, find some sequence of strategies $\tau = (\tau_1, \tau_2, \ldots, \tau_n, \ldots)$ where τ_n
is an n-tuple of strategies for the game G_n, so that the sequence $(\tau_n)_{n=1}^{\infty}$ satisfies
the *CJT*.

In view of the generality and the complexity of our model, we limit ourselves to
sequences τ of "constant" strategies; that is, we assume that $\tau_n^i = \tau_m^i$ if $1 \leq i \leq m \leq$

$n < \infty$. This means that the strategy τ_n^i of a specific juror i does not change when the size of the jury increases. We shall refer to such a sequence as a "constant sequence." [1] We prove that verifying the *CJT* for a constant sequence is equivalent to the following problem:

(**) For a given sequence of binary random variables $X = (X^1, X^2, \ldots, X^n, \ldots)$ with joint distribution P, find whether or not the distribution P satisfies the *CJT*.

Note that prior to Austen-Smith and Banks (1996), the analysis of the Condorcet jury problem had focused on problem (**). One general result is that of Berend and Paroush (1998) which characterizes the independent sequences of binary random variables that satisfy the *CJT*.

In this paper, we find sufficient conditions for (**). Then, we supply two general necessary conditions. However, we do not have a complete characterization of the solution to (**). We do have full characterization (necessary and sufficient conditions) for sequences of *exchangeable* random variables.

Our basic model is introduced in Sect. 2. The full characterization for the case of exchangeable variables is given in Sect. 3. In Sect. 4, we give sufficient conditions for the *CJT*. In Sect. 5, we develop necessary conditions for the validity of the *CJT* in two different planes of parameters of the distribution. In Sect. 6, we prove that these necessary conditions are not sufficient. In Sect. 7, we introduce the notion of *interlacing* of two sequences, which proves to be a useful tool to construct new classes of distributions that satisfy the *CJT*. In particular, we construct rich classes of non-exchangeable sequences that satisfy the *CJT*. We conclude in Sect. 8. Two proofs are given in the Appendix. In the last part of the appendix, we clarify the relationship between the *CJT* and the *Law of Large Numbers* (*LLN*). Basically, we show that these are two different properties that do not imply each other in spite of their superficial similarity.

2 The basic model

We generalize Condorcet's model by presenting it as a game with incomplete information in the following way: let $I = \{1, 2, \ldots, n\}$ be a set of jurors and let D be the defendant. There are two *states of nature*: g—the defendant is guilty, and z—the defendant is innocent. Thus, $\Theta = \{g, z\}$ is the set of states of nature. Each juror has two available actions: c—to convict the defendant, and a—to acquit the defendant; thus $A = \{a, c\}$ is the action set of each of the jurors. Before voting, each juror gets a private random signal $t_j^i \in T^i = \{t_1^i, \ldots, t_{k_i}^i\}$. In the terminology of games with incomplete information, T^i is the *type set* of juror i. The private signals of the jurors may be dependent and may, of course, depend on the state of nature. Again, in the style of games with incomplete information, let $\Omega_n = \Theta \times T^1 \times, \ldots, \times T^n$ be the set of the *states of the world*. That is, a state of the world $\omega = (\theta, t^1, \ldots, t^n)$ consists of the state of nature and the list of types of the n jurors. Let $p^{(n)}$ be the probability

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[1] The restriction to constant strategies is needed only for the existence results. The sufficient condition as well as the necessary conditions are valid for any infinite sequence of strategies. See Remark 1

Fig. 1 The probability distribution $p^{(2)}$
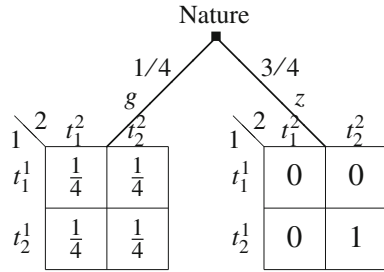
distribution (i.e., a common prior) on Ω_n. This is the joint probability distribution of the state of nature and the signals (types) of all jurors. We assume that the action taken by the finite society of jurors $I = \{1, 2, \ldots, n\}$.i.e., the jury verdict, is determined by the voting rule $V : A^I \to A$, which is the *simple majority* rule (with some tie-breaking procedure such as coin tossing). Finally, to complete the description of the game, we let all jurors have the same payoff function $u : \Theta \times A \to \mathbb{R}$ namely,

$$u(g, c) = u(z, a) = 1 \quad \text{and} \quad u(g, a) = u(z, c) = 0, \quad \forall i \in I$$

This concludes the definition of a game, which we denote by G_n. A (pure) strategy of juror $i \in I$ in G_n is a function $s^i : T^i \to A$. We denote by S^i the set of all pure strategies of juror $i \in I$ and by $S = S^1 \times, \ldots, \times S^n$ the set of strategy profiles of the society. The (common) ex-ante payoff for each juror, when the strategy vector $s = (s^1, \ldots, s^n) \in S$ is used, is $E_u = Eu(\theta, V(s^1(t^1), \ldots, s^n(t^n)))$, where θ is the true state of nature. Note that E_u is precisely the probability of correct decision by I when the strategy vector s is used.

Example 1 In the original Condorcet theorem, we have $T^i = \{t^i_g, t^i_z\}$; $p^{(n)}(g) = p^{(n)}(z) = 1/2$ and the types are conditionally independent given the state of nature; each has a probability $p > 1/2$ of getting the correct signal. That is,

$$p^{(n)}(t^i_g|g) = p^{(n)}(t^i_z|z) = p > \frac{1}{2}$$

Condorcet further assumed that all the jurors vote *informatively*, that is, use the strategy $s^i(t^i_z) = a$ and $s^i(t^i_g) = c$. In this case, the probability of correct voting, by each juror, is p, and as the signals are (conditionally) independent, the *CJT* follows (for example, by the *Law of Large Numbers*).

Figure 1 illustrates our construction in the case $n = 2$. In this example, according to $p^{(2)}$ the state of nature is chosen with unequal probabilities for the two states: $p^{(2)}(g) = 1/4$ and $p^{(2)}(z) = 3/4$ and then the types of the two jurors are chosen according to a joint probability distribution that depends on the state of nature.

Following the seminal work of Austen-Smith and Banks (1996), we intend to study the *CJT* via the Bayesian-Nash Equilibria (BNE) of the game G_n. However, unlike in the case of (conditionally) independent signals, there is no obvious way to find the

relevant BNE in the general case of arbitrary dependence. Therefore, our approach will be indirect. Before describing our techniques, we first enlarge the set of strategies of the jurors by adding the possibility of mixed strategies. Indeed, it was shown by Wit (1998) that the introduction of mixed strategies may help the realization of the *CJT*.

A mixed strategy[2] for juror $i \in I$, in the game G_n, is a function $\sigma_n^i : T^i \to \Delta(A)$, where $\Delta(A)$ is the set of probability distributions on A. Denote by $\sum_n^i$ the set of all mixed strategies of juror i and by $\sum_n = \sum_n^1 \times \cdots \times \sum_n^n$ the set of mixed strategy vectors (profiles) in the game G_n. The (common) ex-ante pay-off for each juror, when the strategy vector $\sigma_n = (\sigma_n^1, \ldots, \sigma_n^n) \in \sum_n$ is used, is $E_u = Eu(\theta, V(\sigma_n^1(t^1), \ldots, \sigma_n^n(t^n)))$, where θ is the true state of nature. Again, E_u is precisely the probability of correct decision by I when the strategy vector σ is played.

We shall now find a more explicit expression for the payoff E_u. Given a strategy vector $\sigma_n = (\sigma_n^1, \ldots, \sigma_n^n) \in \sum_n$, we denote by $X_n^i(\sigma_n^i) : \Theta \times T^i \to \{0, 1\}$ the indicator of the set of correct voting of juror i when using the mixed strategy σ^i. That is,

$$X_n^i(\sigma_n^i; \theta_n, t_n^i) = \begin{cases} 1 & \text{if } \theta_n = g \text{ and } \sigma_n^i(t_n^i) = c \text{ or } \theta_n = z \text{ and } \sigma_n^i(t_n^i) = a \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where by a slight abuse of notation we denoted by $\sigma_n^i(t_n^i)$ the realized pure action when juror i of type t_n^i uses mixed strategy σ_n^i. Given a strategy vector $\sigma_n = (\sigma_n^1, \ldots, \sigma_n^n)$, the probability distribution $p^{(n)}$ on Ω_n induces a joint probability distribution on the vector of binary random variables $(X_n^1, X_n^2, \ldots, X_n^n)$ which we denote by $p_{\sigma_n}^{(n)}$. Assume now that n is odd; then E_u is given by

$$E_u = p_{\sigma_n}^{(n)} \left(\sum_{i=1}^n X_n^i > \frac{n}{2} \right).$$

Guided by Condorcet, we are looking for limit theorems as the the size of the jury increases. Formally, as n goes to infinity we obtain an increasing sequence of "worlds", $(\Omega_n)_{n=1}^\infty$, such that for all n, the projection of Ω_{n+1} on Ω_n is the whole Ω_n. The corresponding sequence of probability distributions is $(p^{(n)})_{n=1}^\infty$ and we assume that for every n, the marginal distribution of $p^{(n+1)}$ on Ω_n is $p^{(n)}$. It follows from the Kolmogorov extension theorem (see Loève 1963, p. 93) that this defines a unique probability measure P on the (projective, or *inverse*) limit

$$\Omega = \lim_{\infty \leftarrow n} \Omega_n = \Theta \times T^1 \times \cdots \times T^n \ldots$$

such that, for all n, the marginal distribution of P on Ω_n is $p^{(n)}$.

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[2] As a matter of fact, the strategy we define here is a *behavior strategy*, but as the game is clearly a game with *perfect recall*, it follows from Kuhn's theorem (1953) that any mixed strategy has a payoff equivalent behavior strategy. Thus we (ab)use the term "mixed strategy" which is more familiar in this literature.

Let $(\sigma_n)_{n=1}^{\infty}$ be an infinite sequence of strategy vectors for an increasing jury. We say that $(\sigma_n)_{n=1}^{\infty}$ satisfies the (asymptotic part of) *CJT* if

$$\lim_{n \to \infty} p_{\sigma_n}^{(n)} \left(\sum_{i=1}^{n} X_n^i(\sigma_n^i) > \frac{n}{2} \right) = 1. \tag{2}$$

Our aim in this work is to find sufficient conditions for the existence of a sequence of BNE $(\sigma_n)_{n=1}^{\infty}$ that satisfy the (asymptotic part of) *CJT*. As far as we know, the only existing result on this general problem is that of Berend and Paroush (1998), which deals only with independent jurors. For that, we make use of the following result due to McLennan for games with common interest (which is our case):

Theorem 2 (McLennan 1998) *For n = 1, 2, ..., if*

$$\sigma_n^* = (\sigma_n^{*1}, \ldots, \sigma_n^{*n}) \in \arg \max_{(\sigma_n^1, \ldots, \sigma_n^n))} E_u(\theta, V(\sigma_n^1(t^1), \ldots, \sigma_n^n(t^n))), \tag{3}$$

then σ_n^* is a Bayesian-Nash Equilibrium of the game G_n

This is an immediate application of Theorem 1 in McLennan (1998), which implies that σ_n^* is a Nash equilibrium of the type-agent representation of G_n. Since by Theorem 2, a Bayesian-Nash Equilibrium of G_n maximizes the probability of correct decision, then clearly, if there exists any sequence of strategy vectors $(\sigma_n)_{n=1}^{\infty}$ that satisfies the asymptotic part of *CJT*, (2), then there is also a sequence $(\sigma_n^*)_{n=1}^{\infty}$ of BNE that satisfies (2), the asymptotic part of *CJT*.

Our approach in this paper is to provide such a sequence that satisfies the *CJT*. In particular, we shall consider infinite sequences of mixed strategy vectors that are constant with respect to the number of players, that is, $(\sigma_n)_{n=1}^{\infty}$ such that if $n \geq m$ then $\sigma_n^i = \sigma_m^i$ for all $i \leq m$. Such a constant sequence can be represented as one infinite sequence of strategies $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n, \ldots)$, where σ^i is the strategy of juror i in all juries that he is a member of (i.e. in all games G_n with $n \geq i$). Whenever we find such a constant sequence that satisfies the *CJT*, it follow, as we argued, that there is a sequence $(\sigma_n^*)_{n=1}^{\infty}$ of BNE that satisfies (2), the asymptotic part of *CJT*. A constant sequence $(\sigma_n)_{n=1}^{\infty}$ can be interpreted as a sequence of an increasing jury in which the strategies of the jury members do not change as the jury increases. In addition to their plausibility, we restrict our attention to constant sequences because of the complexity of our model. As we shall see, even with this restriction, we get some interesting results. The following two examples demonstrate the advantage of the game theoretical model over the traditional probabilistic model[3].

Example 2 (Reverse voting) Suppose that given the state of nature, each juror has two types, t_g and t_z, and the signals of the voters are i.i.d. with $p(t_g \mid g) = p(t_z \mid z) = p < 1/2$. Clearly, in the probabilistic model with *informative voting*[4] such a jury

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[3] We are grateful to an anonymous referee for drawing our attention to these examples

[4] In informative voting, each juror votes according to his/her signal: type t_g juror votes to convict and type t_z juror votes to acquit.

will not satisfy the *CJT*. However, if we consider the strategy σ given by: $\sigma(t_g) = a$ (that is, acquit with probability 1) and $\sigma(t_z) = c$ (convict with probability 1), then the sequence of constant strategies $\sigma = (\sigma, \sigma, \ldots, \sigma, \ldots)$ will satisfy the *CJT* and consequently, there exists a sequence $(\sigma_n^*)_{n=1}^{\infty}$ of BNE that satisfy (2), the asymptotic part of *CJT*.

Example 3 (Random voting) In a model with equal probability to the two states of nature ($p(g) = p(z) = 1/2$), suppose that a fraction α of the jury ($0 < \alpha < 1/2$) receive i.i.d. signals with probability $p > 1/2$ of being correct, that is $p(t_g^i \mid g) = p(t_z^i \mid z) = p > 1/2$. The rest, a $(1-\alpha)$ fraction of the jury, receive the wrong signal, that is $p(t_g^i \mid g) = p(t_z^i \mid z) = 0$. Again, in the probabilistic model with informative voting, such a jury will not satisfy the *CJT*. However, if only the well informed jurors vote informatively while the rest of the jurors vote randomly (convict with probability $1/2$ and acquit with probability $1/2$), such strategy vector will satisfy the *CJT*. Consequently, this game also has an infinite sequence $(\sigma_n^*)_{n=1}^{\infty}$ of BNE that satisfy the asymptotic part of *CJT*.

Remark 1 As far as we can see, the assumption of constant strategies will be needed only for our existence results (Theorem 5, Corollary 8, and Theorem 16). For the sufficient condition, as well as for the two necessary conditions, we need neither the restriction to constant strategies, nor the assumption on the stationarity of the probabilities $p^{(n)}$ (of G_n). The proofs are the same, with the appropriate adjustment of notations; mainly, for a general sequence of strategies $(\sigma_n)_{n=1}^{\infty}$, the corresponding sequence X of binary random variables, is the sequence of n-vectors of random variables $(X_n^1, \ldots, X_n^n)$ corresponding to the game G_n and the strategy vector σ_n; that is, $X = (X_1^1; X_2^1, X_2^2; \ldots; X_n^1, \ldots, X_n^n; \ldots)$. The *CJT* property is defined, as usual, by Eq. 2.

A constant sequence of mixed strategies $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n, \ldots)$ naturally yields a sequence of binary random variables $X = (X^1, X^2, \ldots, X^n, \ldots)$ where $X^i := X_n^i(\sigma_n^i; \theta_n, t_n^i)$ is the indicator variable of correct voting of juror i defined in (1), and is independent of n since the strategy is constant. As the *CJT* is expressed in terms of X, we shall mostly be working with this infinite sequence of binary random variables. In fact, working with the infinite sequences X is equivalent to working with the underlying infinite sequences of games and strategy vectors: on the one hand, as we said, a sequence of games $(G_n)_{n=1}^{\infty}$ and an infinite sequence of constant strategies $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n, \ldots)$, yield an infinite sequence X of binary random variables. On the other hand, as we show in Appendix 9.1, for any infinite sequence of binary random variables X there is a sequence of games $(G_n)_{n=1}^{\infty}$ and an infinite sequence of constant strategies $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n, \ldots)$ that yield this X as the infinite sequence of the indicators of correct voting.

Let us now briefly remark on the non-asymptotic part of the *CJT* (see Ben-Yashar and Paroush 2000). An infinite sequence of mixed strategy vectors $\sigma_n = (\sigma_n^1, \ldots, \sigma_n^n)$, $n = 1, 2, \ldots$, is said to be *consistent with the majority rule* if for $n = 1, 2, \ldots$,

$$p_{\sigma_n}^{(n)}\left(\sum_{i=1}^{n} X_n^i(\sigma^i) > \frac{n}{2}\right) > p_{\sigma_n}^{(n)}(X_n^i(\sigma^i) = 1); \quad i = 1, \ldots, n$$

$$p_{\sigma_{n+1}}^{(n+1)}\left(\sum_{i=1}^{n+1} X_{n+1}^i(\sigma_{n+1}^i) > \frac{n+1}{2}\right) \geq p_{\sigma_n}^{(n)}\left(\sum_{i=1}^{n} X_n^i(\sigma_n^i) > \frac{n}{2}\right); \quad n = 1, 2, \ldots.$$

In view of the complexity of our model, we shall not investigate non-asymptotic consistency with majority rule of infinite sequences of strategies, and shall study only the asymptotic part of the *CJT*.

3 Exchangeable variables

In this section, we fully characterize the distributions of sequences $X = (X^1, X^2, \ldots, X^n, \ldots)$ of *exchangeable* random binary variables that satisfy the *CJT*. Let us first introduce some notation:

Given a sequence of binary random variables $X = (X^1, X^2, \ldots, X^n, \ldots)$ with joint distribution P, denote $p^i = E(X^i)$, $Var(X^i) = E(X^i - p^i)^2$ and $Cov(X^i, X^j) = E[(X^i - p^i)(X^j - p^j)]$, for $i \neq j$, where E denotes, as usual, the expectation operator. Also, let $\overline{p}_n = (p^1 + p^2, \ldots + p^n)/n$ and $\overline{X}_n = (X^1 + X^2, \ldots + X^n)/n$. Next, we recall:

Definition 1 A sequence of random variables $X = (X^1, X^2, \ldots, X^n, \ldots)$ is *exchangeable* if for every n and every permutation $(k_1, \ldots, k_n)$ of $(1, \ldots, n)$, the finite sequence $(X^{k_1}, \ldots, X^{k_n})$ has the same n-dimensional probability distribution as $(X^1, \ldots, X^n)$.

In our context, this property may be interpreted as *anonymity* of the jurors; the names and the location in the list of jurors does not affect the distribution of correct voting. Note that this does not rule out correlation between the distributions of the "correct voting" among jurors.

We shall make use of the following characterization theorem according to de Finetti[5] (see, e.g., Feller 1966, vol II, p. 225).

Theorem 3 *A sequence of binary random variables* $X = (X^1, X^2, \ldots, X^n, \ldots)$ *is exchangeable if and only if there is a probability distribution F on* $[0, 1]$ *such that for every n,*

$$Pr(X^1 = \cdots = X^k = 1, \ X^{k+1} = \ldots = X^n = 0) = \int_0^1 \rho^k (1-\rho)^{n-k} dF \quad (4)$$

$$Pr(X^1 + \cdots + X^n = k) = \binom{n}{k} \int_0^1 \rho^k (1-\rho)^{n-k} dF \quad (5)$$

In words, de-Finetti's theorem says that binary exchangeable variables are *conditionally* i.i.d.: Given the value of ρ, the variables are i.i.d. Bernoulli random variables

with parameter ρ. In our underlying model, the parameter ρ can be interpreted as *public information* regarding the defendant (all available evidence, witnesses etc.). Given this public information, the distribution of "correct voting" is the same for all jurors and independent among jurors.

Using de Finetti's theorem, we can characterize the distributions of sequences of exchangeable binary random variables by their expectation and the asymptotic variance of $\overline{X}_n$.

Theorem 4 *Let $X = (X^1, X^2, \ldots, X^n, \ldots)$ be a sequence of exchangeable binary random variables and let F be the corresponding distribution function in de Finetti's theorem. Then,*

$$\underline{y} := \lim_{n \to \infty} E(\overline{X}_n - u)^2 = V(F), \tag{6}$$

where

$$u = \int_0^1 \rho \, dF \quad \text{and} \quad V(F) = \int_0^1 (\rho - u)^2 dF.$$

Proof We have

$$u = E(X^i) = Pr(X^i = 1) = \int_0^1 x \, dF; \; V(X^i) = u(1 - u)$$

and for $i \neq j$,

$$Cov(X^i, X^j) = Pr(X^i = X^j = 1) - u^2 = \int_0^1 x^2 \, dF - u^2 = V(F).$$

Therefore,

$$E(\overline{X}_n - u)^2 = E\left(\frac{1}{n} \sum_1^n (X^i - u)\right)^2$$

$$= \frac{1}{n^2} \sum_1^n V(X^i) + \frac{1}{n^2} \sum_{i \neq j} Cov(X^i, X^j)$$

$$= \frac{nu(1 - u)}{n^2} + \frac{n(n - 1)}{n^2} V(F),$$

which implies Eq. 6. $\qquad\square$

We can now state the characterization theorem:

Theorem 5 *A sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ of binary exchangeable random variables with a corresponding distribution $F(\rho)$ satisfies the CJT if and only if*

$$Pr\left(\frac{1}{2} < \rho \leq 1\right) = 1, \tag{7}$$

that is, if and only if a support of F is in the semi-open interval $(1/2, 1]$.

Proof The "only if" part follows from the fact that any sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ of binary *i.i.d.* random variables with expectation $E(X^i) = \rho \leq 1/2$, violates the *CJT* (by the Berend and Paroush's necessary condition).

To prove that a sequence satisfying condition (7) also satisfies the *CJT*, note that for $0 < \varepsilon < 1/4$,

$$Pr\left(\overline{X}_n > \frac{1}{2}\right) \geq Pr\left(\rho \geq \frac{1}{2} + 2\varepsilon\right) Pr\left(\overline{X}_n > \frac{1}{2} \,\middle|\, \rho \geq \frac{1}{2} + 2\varepsilon\right). \tag{8}$$

For the second term in (8), we have:

$$Pr\left(\overline{X}_n > \frac{1}{2} \,\middle|\, \rho \geq \frac{1}{2} + 2\varepsilon\right) = \sum_{k > \frac{n}{2}} Pr\left(X^1 + \ldots + X^k = k \,\middle|\, \rho \geq \frac{1}{2} + 2\varepsilon\right) \tag{9}$$

$$= \sum_{k > \frac{n}{2}} \binom{n}{k} \int_{\frac{1}{2}+2\varepsilon}^{1} \rho^k (1-\rho)^{n-k} \mathrm{d}F \tag{10}$$

$$= \int_{\frac{1}{2}+2\varepsilon}^{1} \left[\sum_{k > \frac{n}{2}} \binom{n}{k} \rho^k (1-\rho)^{n-k}\right] \mathrm{d}F \tag{11}$$

$$:= \int_{\frac{1}{2}+2\varepsilon}^{1} S_n(\rho) \,\mathrm{d}F \tag{12}$$

Now, using Chebyshev's inequality we have:

$$S_n(\rho) = Pr\left(\overline{X}_n > \frac{1}{2} \,\middle|\, \rho\right) \geq Pr\left(\overline{X}_n > \frac{1}{2} + \varepsilon \,\middle|\, \rho\right) \tag{13}$$

$$\geq 1 - \frac{V(\overline{X}_n | \rho)}{(\rho - \frac{1}{2} - \varepsilon)^2} = 1 - \frac{\rho(1-\rho)}{n(\rho - \frac{1}{2} - \varepsilon)^2} \tag{14}$$

Since the last expression in (14) converges to 1 uniformly on $[1/2+2\varepsilon, 1]$ as $n \to \infty$, taking the limit $n \to \infty$ of (12) and using (14) we have:

$$\lim_{n \to \infty} Pr\left(\overline{X}_n > \frac{1}{2} \,\middle|\, \rho \geq \frac{1}{2} + 2\varepsilon\right) \geq \int_{\frac{1}{2}+2\varepsilon}^{1} dF = Pr\left(\rho \geq \frac{1}{2} + 2\varepsilon\right). \quad (15)$$

From (8) and (15), we have that for any fixed $\varepsilon > 0$,

$$\lim_{n \to \infty} Pr\left(\overline{X}_n > \frac{1}{2}\right) \geq \left[Pr\left(\rho \geq \frac{1}{2} + 2\varepsilon\right)\right]^2. \quad (16)$$

Since (16) must hold for all $1/4 > \varepsilon > 0$, and since $Pr\left(\frac{1}{2} < \rho \leq 1\right) = 1$, we conclude that

$$\lim_{n \to \infty} Pr\left(\overline{X}_n > \frac{1}{2}\right) = 1, \quad (17)$$

i.e., the sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ satisfies the *CJT*. □

To draw the consequences of Theorem 5, we prove first the following proposition which enables us, for testing the validity of the *CJT*, to use the easily computed parameters u and $V(F)$, rather than the unknown distribution F.

Proposition 1 *Any distribution F of a variable ρ in $[1/2, 1]$ satisfies*

$$V(F) \leq \left(u - \frac{1}{2}\right)(1 - u), \quad (18)$$

where $u = E(F)$, and equality holds in (18) only for F for which

$$Pr\left(\rho = \frac{1}{2}\right) = 2(1 - u) \quad \text{and} \quad Pr(\rho = 1) = 2u - 1. \quad (19)$$

Proof We want to show that

$$\int_{1/2}^{1} \rho^2 dF(\rho) - u^2 \leq \left(u - \frac{1}{2}\right)(1 - u), \quad (20)$$

or, equivalently,

$$\int_{1/2}^{1} \rho^2 dF(\rho) - \frac{3}{2}u + \frac{1}{2} \leq 0. \quad (21)$$

Replacing $u = \int_{1/2}^{1} \rho \, dF(\rho)$ and $\frac{1}{2} = \int_{1/2}^{1} \frac{1}{2} \, dF(\rho)$, inequality (20) is equivalent to

$$\int\limits_{1/2}^{1} (\rho^2 - \frac{3}{2}\rho + \frac{1}{2})\, \mathrm{d}F(\rho) := \int\limits_{1/2}^{1} g(\rho)\, \mathrm{d}F(\rho) \le 0. \tag{22}$$

The parabola $g(\rho)$ is convex and satisfies $g(1/2) = g(1) = 0$ and $g(\rho) < 0$ for all $1/2 < \rho < 1$, which proves (22). Furthermore, equality to 0 in (22) is obtained only when F is such that $Pr(1/2 < \rho < 1) = 0$, and combined with $u = E(F)$ this implies (19). $\qquad\square$

The next Proposition provides a sort of inverse to proposition 1.

Proposition 2 *For $(u, w) = (1, 0)$ and for any pair (u, w) where $1/2 < u < 1$ and $0 \le w < (u-1/2)(1-u)$, there is a distribution $F(\rho)$ on $(1/2, 1]$ such that $E(F) = u$ and $V(F) = w$.*

Proof For $(u, w) = (1, 0)$ the claim is trivially true (with the distribution $Pr(\rho = 1) = 1$). Given (u, w), for any y satisfying $1/2 < y \le u < 1$ define the distribution F_y for which

$$Pr(\rho = y) = (1 - u)/(1 - y) \quad \text{and} \quad Pr(\rho = 1) = (u - y)/(1 - y).$$

This distribution satisfies $E(F_y) = u$ and it remains to show that we can choose y so that $V(F_y) = w$. Indeed,

$$V(F_y) = \frac{1 - u}{1 - y}\, y^2 + \frac{u - y}{1 - y} - u^2.$$

For a given $u < 1$, this is a continuous function of y satisfying both $\lim_{y \to u} V(F_y) = 0$ and $\lim_{y \to 1/2} V(F_y) = (u - 1/2)(1 - u)$. Therefore, for $0 \le w < (u - 1/2)(1 - u)$, there is a value y^* for which $V(F_{y^*}) = w$. $\qquad\square$

3.1 Presentation in $\mathbb{R}^2$

In studying the validity of the *CJT* for a sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ with general joint distribution (X, P), rather than working with the whole covariance structure of the sequence, we shall see that many results can be obtained by examining just two parameters of the distribution namely, $(\underline{p}, \underline{y})$ defined by.

$$\underline{p} := \liminf_{n \to \infty} \overline{p}_n \tag{23}$$

$$\underline{y} := \liminf_{n \to \infty} E(\overline{X}_n - \overline{p}_n)^2 \tag{24}$$

(Note that this definition of $\underline{y}$ is consistent with that given in Eq. 6 for exchangeable variables; a case in which the limit exists.)

Therefore, we shall study the *CJT* property of a sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ through its projection on the plane $\mathbb{R}^2$ in which we shall denote the coordinates by (u, w). We first identify the range of this mapping:

Proposition 3 *For every pair (X, P), the corresponding parameters $(\underline{p}, \underline{y})$ satisfy $\underline{y} \leq \underline{p}(1 - \underline{p})$.*

Proof Given a sequence of binary random variables X with its joint distribution P, we first observe that for any $i \neq j$,

$$Cov(X^i, X^j) = E(X^i X^j) - p^i p^j \leq \min(p^i, p^j) - p^i p^j.$$

Therefore,

$$E(\overline{X}_n - \overline{p}_n)^2 = \frac{1}{n^2} \left\{ \sum_{i=1}^{n} \sum_{j \neq i} Cov(X^i, X^j) + \sum_{i=1}^{n} p^i(1 - p^i) \right\} \tag{25}$$

$$\leq \frac{1}{n^2} \left\{ \sum_{i=1}^{n} \sum_{j \neq i} [\min(p^i, p^j) - p^i p^j] + \sum_{i=1}^{n} p^i(1 - p^i) \right\}. \tag{26}$$

We claim that the maximum of the last expression (26), under the condition $\sum_{i=1}^{n} p^i = n\overline{p}_n$, is $\overline{p}_n(1 - \overline{p}_n)$. This is attained when $p^1 = \cdots = p^n = \overline{p}_n$. To see that this is indeed the maximum, assume to the contrary that the maximum is attained at $\tilde{p} = (\tilde{p}^1, \cdots, \tilde{p}^n)$ with $\tilde{p}^i \neq \tilde{p}^j$ for some i and j. Without loss of generality assume that: $\tilde{p}^1 \leq \tilde{p}^2 \leq \cdots \leq \tilde{p}^n$ with $\tilde{p}^1 < \tilde{p}^j$ and $\tilde{p}^1 = \tilde{p}^\ell$ for $\ell < j$. Let $0 < \varepsilon < (\tilde{p}^j - \tilde{p}^1)/2$ and define $p^* = (p^{*1}, \cdots, p^{*n})$ by $p^{*1} = \tilde{p}^1 + \varepsilon$, $p^{*j} = \tilde{p}^j - \varepsilon$, and $p^{*\ell} = \tilde{p}^\ell$ for $\ell \notin \{1, j\}$. A tedious, but straightforward, computation shows that the expression (26) is higher for p^* than for $\tilde{p}$, in contradiction to the assumption that it is maximized at $\tilde{p}$. We conclude that

$$E(\overline{X}_n - \overline{p}_n)^2 \leq \overline{p}_n(1 - \overline{p}_n).$$

Let now $(\overline{p}_{n_k})_{k=1}^{\infty}$ be a subsequence converging to $\underline{p}$; then

$$\underline{y} = \liminf_{n \to \infty} E(\overline{X}_n - \overline{p}_n)^2 \leq \liminf_{k \to \infty} E(\overline{X}_{n_k} - \overline{p}_{n_k})^2$$

$$\leq \liminf_{k \to \infty} \overline{p}_{n_k}(1 - \overline{p}_{n_k}) = \underline{p}(1 - \underline{p}).$$

$\square$

This leads to:

Theorem 6 *The range of the mapping $(X, P) \rightarrow (\underline{p}, \underline{y})$ is (see Fig. 2)*

$$FE_2 = \{(u, w) | 0 \leq u \leq 1, \ 0 \leq w \leq u(1 - u)\} \tag{27}$$

That is, for any pair (X, P), we have $(\underline{p}, \underline{y}) \in FE_2$ and for any $(u, w) \in FE_2$ there is a pair (X, P) for which $\underline{p} = u$ and $\underline{y} = w$.
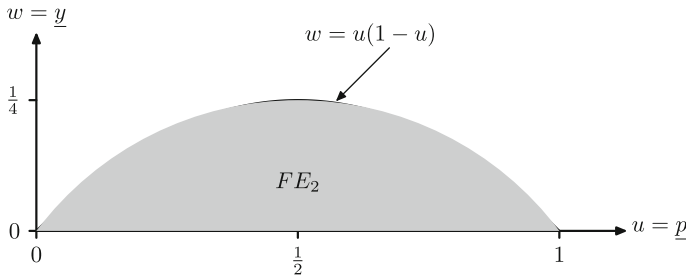
Fig. 2 The feasible set FE_2

Proof The first part follows from Proposition 3 (since clearly $\underline{y} \geq 0$). For the second part, observe first, as we have remarked in the proof of Proposition 3, that for the pair (X, P) in which $P\{X^1 = X^2 = \ldots = 1\} = u$ and $P\{X^1 = X^2 = \ldots = 0\} = 1 - u$ we have $p^1 = p^2 = \cdots = p^n = \overline{p}_n = u$ and hence $\underline{p} = u$. Also, for all $n = 1, 2, \ldots$,

$$E(\overline{X}_n - \overline{p}_n)^2 = E(\overline{X}_n - u)^2 = u(1 - u) \quad \text{and hence}$$
$$\underline{y} = \liminf_{n\to\infty} E(\overline{X}_n - \overline{p}_n)^2 = u(1 - u),$$

which means that any point on the parabola $w = u(1 - u)$ is attainable as an image of a pair (X, P). Next, note that for $u \in [0, 1]$, the pair $(Y, \tilde{P})$ in which $(Y_i)_{i=1}^{\infty}$ are i.i.d. with $\tilde{P}\{Y_i = 1\} = u$ and $\tilde{P}\{Y_i = 0\} = 1 - u$ is mapped to $(\underline{p}, \underline{y}) = (u, 0)$ since

$$\underline{y} = \liminf_{n\to\infty} E(\overline{X}_n - \overline{p}_n)^2 = \liminf_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} E(\overline{X}_i - u)^2 = \liminf_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} u(1 - u)$$

$$= \liminf_{n\to\infty} \frac{u(1 - u)}{n} = 0.$$

It remains to prove that all interior points of FE_2 are attainable. Let (u, w) be such an interior point, that is, $0 < u < 1$ and $0 < w < u(1 - u)$. Define the pair (Z, Q) to be the above-defined pair (X, P) with probability $w/u(1 - u)$ and the above-defined $(Y, \tilde{P})$ with probability $1 - w/u(1 - u)$. It is readily seen that this pair is mapped to

$$\frac{w}{u(1 - u)}(u, u(1 - u)) + \left(1 - \frac{w}{u(1 - u)}\right)(u, 0) = (u, w). \qquad \square$$

The geometric expression of Theorem 5, combined with Theorem 3, Proposition 1, and Proposition 2, can now be stated as follows: in the $\mathbb{R}^2$ plane of $(\underline{p}, \underline{y})$ let

$$A = \left\{ (\underline{p}, \underline{y}) \;\middle|\; \frac{1}{2} < \underline{p} \leq 1; \;\; \text{and} \;\; \underline{y} < \left(\underline{p} - \frac{1}{2}\right)(1 - \underline{p}) \right\} \bigcup \{(1, 0)\} \qquad (28)$$

This is the region strictly below the small parabola in Fig. 3, excluding $(1/2, 0)$ and adding $(1, 0)$.
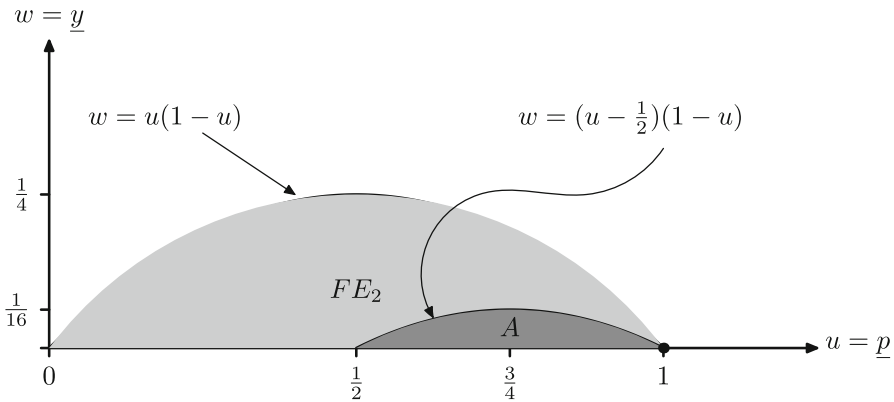
Fig. 3 The *CJT* region for exchangeable variables

Theorem 7 *1. Any exchangeable sequence of binary random variables that satisfy the CJT corresponds to $(\underline{p}, \underline{y}) \in A$.*

2. *To any $(\underline{p}, \underline{y}) \in A$ there exists an exchangeable sequence of binary random variables with parameters $(\underline{p}, \underline{y})$ that satisfy the CJT.*

Proof The statements of the theorems are trivially true for the point $(1, 0)$, as it corresponds to the unique distribution: $Pr(X^1 = \ldots = X^n \ldots) = 1$, which is both exchangeable and satisfies the *CJT*. For all other points in A:

- Part 1. follows de Finetti's Theorem 3, Theorem 5, and Proposition 1.
- Part 2. follows de Finetti's Theorem 3, Theorem 5, and Proposition 2. □

3.2 Application to symmetric juries

A jury game G_n, as defined in Sect. 2, is said to be *symmetric* if

- $T^1 = T^2 = \ldots = T^n$
- The probability distribution $p^{(n)}$ is symmetric in the variables $t^1, \ldots, t^n$.

We consider a sequence of increasing juries $(G_n)_{n=1}^{\infty}$ such that G_n is symmetric for all n. In such a sequence, $\sum_n^i$ is the same for all i and all n and is denoted by $\sum$. A strategy vector $\sigma_n = (\sigma_n^1, \ldots, \sigma_n^n) \in \sum_n$ is said to be *symmetric*, if $\sigma_n^1 = \sigma_n^2 = \ldots = \sigma_n^n$.

Corollary 8 *Let $\underset{\sim}{\sigma} = (\sigma, \sigma, \ldots, \sigma, \ldots) \in \sum^{\infty}$ and let $\underset{\sim}{X} = (X^1, X^2, \ldots, X^n, \ldots)$ be the sequence of binary random variables derived from $\underset{\sim}{\sigma}$ by (1); then $\underset{\sim}{X}$ is exchangeable. If $\underset{\sim}{X}$ satisfies (7), then there exists a sequence of BNE, $\sigma_*^n = (\sigma_n^*, \ldots, \sigma_n^*)$ of G_n for $n = 1, 2, \ldots$, that satisfies the CJT.*

Proof Follows from Theorems 5 and 2 of McLennan (1998). □

4 Sufficient conditions

Having characterized the *CJT* conditions for exchangeable variables, we proceed now
to the general case and we start with sufficient conditions.

Let $X = (X^1, X^2, \ldots, X^n, \ldots)$ be a sequence of binary random variables with
range in $\{0, 1\}$ and with joint probability distribution P. The sequence X is said to
satisfy the *CJT* if

$$\lim_{n \to \infty} P\left(\sum_{i=1}^{n} X^i > \frac{n}{2}\right) = 1 \tag{29}$$

Observe that in our model the vector X results from strategic voting and (29) is the
condition corresponding to condition (2) when $X_i = X_i(\sigma_i)$ for an infinite sequence
of constant strategies $(\sigma^i)_{i=1}^{\infty}$ that satisfy CJT.

In this section, we provide sufficient conditions for a pair (X, P) to satisfy the
CJT. Recall our notation: $\overline{X}_n = (X^1 + X^2 + \cdots + X^n)/n$, $p^i = E(X^i)$, and $\overline{p}_n = (p^1 + p^2 + \cdots + p^n)/n$.

Theorem 9 *Assume that $\overline{p}_n > \frac{1}{2}$ for all $n > N_0$ and*

$$\lim_{n \to \infty} \frac{E(\overline{X}_n - \overline{p}_n)^2}{(\overline{p}_n - \frac{1}{2})^2} = 0, \tag{30}$$

or equivalently assume that

$$\lim_{n \to \infty} \frac{\overline{p}_n - \frac{1}{2}}{\sqrt{E(\overline{X}_n - \overline{p}_n)^2}} = \infty; \tag{31}$$

then the CJT is satisfied.

Proof

$$P\left(\sum_{i=1}^{n} X^i \leq \frac{n}{2}\right) = P\left(-\sum_{i=1}^{n} X^i \geq -\frac{n}{2}\right)$$

$$= P\left(\sum_{i=1}^{n} p^i - \sum_{i=1}^{n} X^i \geq \sum_{i=1}^{n} p^i - \frac{n}{2}\right)$$

$$\leq P\left(\left|\sum_{i=1}^{n} p^i - \sum_{i=1}^{n} X^i\right| \geq \sum_{i=1}^{n} p^i - \frac{n}{2}\right)$$

By Chebyshev's inequality (assuming $\sum_{i=1}^{n} p^i > \frac{n}{2}$), we have

$$P\left(\left|\sum_{i=1}^{n} p^i - \sum_{i=1}^{n} X^i\right| \geq \sum_{i=1}^{n} p^i - \frac{n}{2}\right) \leq \frac{E\left(\sum_{i=1}^{n} X^i - \sum_{i=1}^{n} p^i\right)^2}{\left(\sum_{i=1}^{n} p^i - \frac{n}{2}\right)^2}$$

$$= \frac{E\left(\overline{X}_n - \overline{p}_n\right)^2}{(\overline{p}_n - \frac{1}{2})^2}$$

As this last term tends to zero by (30), the *CJT* (29) then follows. □

Corollary 10 *If* $\sum_{i=1}^{n} \sum_{j \neq i} Cov(X^i, X^j) \leq 0$ *for* $n > N_0$ *(in particular if* $Cov(X^i, X^j) \leq 0$ *for all* $i \neq j$) *and* $\lim_{n \to \infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty$, *then the CJT is satisfied.*

Proof Since the variance of a binary random variable X with mean p is $p(1-p) \leq 1/4$ we have for $n > N_0$,

$$0 \leq E(\overline{X}_n - \overline{p}_n)^2 = \frac{1}{n^2} E\left(\sum_{i=1}^{n} (X^i - p^i)\right)^2$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} Var(X^i) + \sum_{i=1}^{n} \sum_{j \neq i} Cov(X^i, X^j)\right) \leq \frac{1}{4n}$$

Therefore, if $\lim_{n \to \infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty$, then

$$0 \leq \lim_{n \to \infty} \frac{E(\overline{X}_n - \overline{p}_n)^2}{(\overline{p}_n - \frac{1}{2})^2} \leq \lim_{n \to \infty} \frac{1}{4n(\overline{p}_n - \frac{1}{2})^2} = 0$$

□

Remark 2 It follows from Eq. 30 that any (X, P) satisfying this sufficient condition must have $\underline{y} = 0$; that is, it corresponds to a point $(\underline{p}, 0)$ in the $\mathbb{R}^2$ plane. Thus, any distribution with $\underline{y} > 0$ that satisfy the *CJT*, does not satisfy this sufficient condition. In particular, this is true for the exchangeable sequences (with $\underline{y} > 0$) we identified in Sect. 3 and the non-exchangeable sequences satisfying the *CJT* we will see in Sect. 7.

Remark 3 Note that under the condition of Corollary 10, namely, for bounded random variables with all covariances being non-positive, the (weak) Law of Large Numbers (*LLN*) holds for arbitrarily dependent variables (see, e.g., Feller (1957), vol I, exercise 9, p. 262). This is not implied by Corollary 10 since, as we show in Appendix 9.3, the *CJT*, strictly speaking, is not a Law of Large Lumbers. In particular, *CJT* does not imply *LLN* and *LLN* does not imply *CJT*.

Remark 4 When $X^1, X^2, \ldots, X^n, \ldots$ are independent, then under mild conditions $\lim_{n \to \infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty$ is a necessary and sufficient condition for *CJT* (see Berend and Paroush 1998).

5 Necessary conditions

We start this section with a simple observation and then state two necessary conditions that do not fully imply one another in either direction.

Proposition 4 *Given a sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ of binary random variables with a joint probability distribution P, if the CJT holds then $\underline{p} \geq \frac{1}{2}$.*

Proof Define a sequence of events $(B_n)_{n=1}^{\infty}$ by $B_n = \{\omega \mid \overline{X}_n(\omega) \geq 1/2\}$. Since the CJT holds, $\lim_{n \to \infty} P\left(\sum_{i=1}^{n} X^i > \frac{n}{2}\right) = 1$ and hence $\lim_{n \to \infty} P(B_n) = 1$. Since

$$\overline{p}_n - \frac{1}{2} = E\left(\overline{X}_n - \frac{1}{2}\right) \geq -\frac{1}{2} P(\Omega \backslash B_n),$$

taking the lim inf, the right-hand side tends to zero and we obtain that $\lim \inf_{n \to \infty} \overline{p}_n = \underline{p} \geq \frac{1}{2}$. $\qquad \square$

5.1 A necessary condition with respect to the L_2 norm

In this section, we provide a necessary condition with respect to the L_2 norm for a general sequence (X, P) to satisfy the CJT. That is, a condition in terms of two characteristics, $\underline{p} = \lim \inf_{n \to \infty} \overline{p}_n$ and $\underline{y} = \lim \inf_{n \to \infty} E(\overline{X}_n - \overline{p}_n)^2$.

Theorem 11 *Let $X = (X^1, X^2, \ldots, X^n, \ldots)$ be a sequence of binary random variables with joint distribution P. If (X, P) satisfy the CJT, then $\underline{y} \leq (\underline{p} - \frac{1}{2})(1 - \underline{p})$.*

Proof Recall our notation $B_n = \{\omega \in \Omega \mid \overline{X}_n(\omega) \geq \frac{1}{2}\}$; then, since (X, P) satisfy the CJT, $\lim_{n \to \infty} P(B_n) = 1$. The main part of the proof is a direct computation of $E(\overline{X}_n(\omega) - \overline{p}_n)^2$. Denote by $B_n^c := \Omega \setminus B_n$ the complement of B_n; then:

$$
\begin{aligned}
E(\overline{X}_n(\omega) - \overline{p}_n)^2 &= E\left(\overline{X}_n(\omega) - \frac{1}{2} + \frac{1}{2} - \overline{p}_n\right)^2 \\
&= E\left(\overline{X}_n(\omega) - \frac{1}{2}\right)^2 + 2\left(\frac{1}{2} - \overline{p}_n\right) E\left(\overline{X}_n(\omega) - \frac{1}{2}\right) \\
&\quad + \left(\frac{1}{2} - \overline{p}_n\right)^2 \\
&= E\left(\overline{X}_n(\omega) - \frac{1}{2}\right)^2 - \left(\frac{1}{2} - \overline{p}_n\right)^2 \\
&= \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right)^2 dP + \int_{B_n} \left(\overline{X}_n(\omega) - \frac{1}{2}\right)\left(\overline{X}_n(\omega) - \frac{1}{2}\right) dP \\
&\quad - \left(\frac{1}{2} - \overline{p}_n\right)^2
\end{aligned}
$$

$$\leq \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right)^2 dP + \frac{1}{2} \int_{B_n} \left(\overline{X}_n(\omega) - \frac{1}{2}\right) dP - \left(\frac{1}{2} - \overline{p}_n\right)^2.$$

Thus,

$$E(\overline{X}_n(\omega) - \overline{p}_n)^2 \leq \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right)^2 dP - \frac{1}{2} \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right) dP$$

$$+ \frac{1}{2} E\left(\overline{X}_n(\omega) - \frac{1}{2}\right) - \left(\frac{1}{2} - \overline{p}_n\right)^2$$

$$= \frac{1}{2}\left(\overline{p}_n - \frac{1}{2}\right) - \left(\frac{1}{2} - \overline{p}_n\right)^2 + \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right)^2 dP$$

$$- \frac{1}{2} \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right) dP$$

$$= \left(\overline{p}_n - \frac{1}{2}\right)(1 - \overline{p}_n) + \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right)^2 dP$$

$$- \frac{1}{2} \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right) dP$$

For any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for $n > N(\varepsilon)$,

$$\int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right)^2 dP < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \frac{1}{2} \int_{B_n^c} \left(\overline{X}_n(\omega) - \frac{1}{2}\right) dP \right| < \frac{\varepsilon}{2}.$$

Hence for $n > N(\varepsilon)$,

$$E(\overline{X}_n(\omega) - \overline{p}_n)^2 \leq \left(\overline{p}_n - \frac{1}{2}\right)(1 - \overline{p}_n) + \varepsilon. \tag{32}$$

We conclude that

$$\underline{y} = \liminf_{n \to \infty} E(\overline{X}_n - \overline{p}_n)^2 \leq \liminf_{n \to \infty} \left(\overline{p}_n - \frac{1}{2}\right)(1 - \overline{p}_n) + \varepsilon,$$

for every $\varepsilon > 0$. Hence,

$$\underline{y} = \liminf_{n \to \infty} E(\overline{X}_n - \overline{p}_n)^2 \leq \liminf_{n \to \infty} \left(\overline{p}_n - \frac{1}{2}\right)(1 - \overline{p}_n).$$
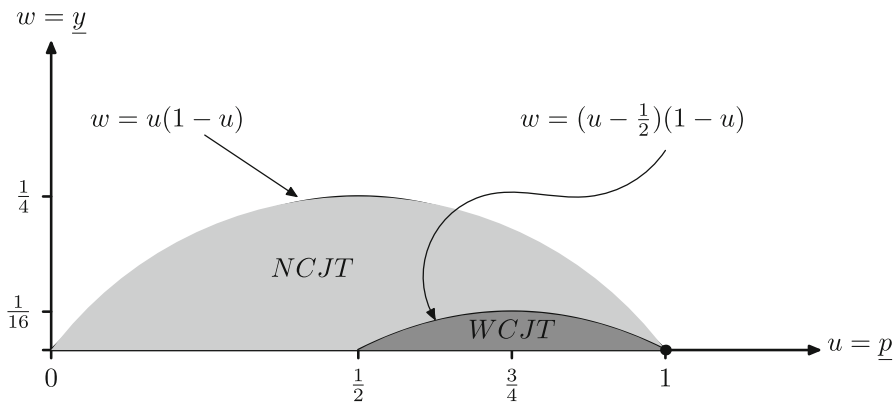
Fig. 4 The *CJT* region of validity for general distributions

Choose a sequence $(n_k)_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \overline{p}_{n_k} = \underline{p}$; then

$$\underline{y} \le \liminf_{k\to\infty} \left(\overline{p}_{n_k} - \frac{1}{2}\right)(1 - \overline{p}_{n_k}) = \left(\underline{p} - \frac{1}{2}\right)(1 - \underline{p}).^6$$

$\square$

Figure 4 depicts the regions of validity of the *CJT* in the $\mathbb{R}^2$ plane: any distribution for which the parameters $(\underline{p}, \underline{y})$ lie in lightly colored region denoted by *NCJT*, does not satisfy the *CJT*. In particular, if a sequence of strategy vectors $(\sigma_n)_{n=1}^{\infty}$ in McLennan's theorem (i.e. maximizers in Equation (3)) does not satisfy the necessary condition (i.e., the corresponding $(\underline{p}, \underline{y})$ lies in the region *NCJT*) then there is no sequence of strategies $(\sigma_n)_{n=1}^{\infty}$, whether constant or not, that satisfy the *CJT*. The dark region, denoted by *WCJT* (for *weak CJT*), is the closed area below the small parabola. Any distribution that satisfies the *CJT* must have parameters $(\underline{p}, \underline{y})$ in this region. However, for general distributions this is not a sufficient condition; as we shall see later, for any $(\underline{p}, \underline{y})$ in this region, excluding $(1, 0)$, there is a sequence with these parameters that does not satisfy the *CJT*.

5.2 A necessary condition with respect to the L_1 norm

In this section, we provide a necessary condition with respect to the L_1 norm for a general sequence (X, P) to satisfy the *CJT*. That is, a condition in terms of two characteristics, $\underline{p} = \liminf_{n\to\infty} \overline{p}_n$ and $\underline{y}^* = \liminf_{n\to\infty} E\left|\overline{X}_n - \overline{p}_n\right|$.

---

[6] Since for any $\varepsilon > 0$ inequality (32) holds for all $n > N(\varepsilon)$, then for a subsequence $(n_k)_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \overline{p}_{n_k} = \tilde{p}$ and $\lim_{k\to\infty} E(\overline{X}_{n_k} - \overline{p}_{n_k})^2 = \tilde{y}$, we get $\tilde{y} \le (\tilde{p} - \frac{1}{2})(1 - \tilde{p})$. It follows that if (X, P) satisfies the *CJT*, then any limit point of $(\overline{p}_n, E(\overline{X}_n - \overline{p}_n)^2)$ is in the region A of Fig. 3 (or region *WCJT* in Fig. 4). We are indebted to A. Neyman for a discussion concerning this observation.
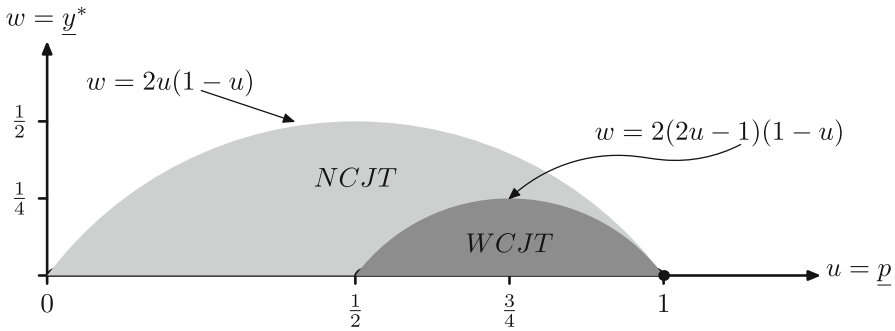
Fig. 5 The *CJT* region of validity with respect to the L_1 norm

Theorem 12 *Let* $X = (X^1, X^2, \ldots, X^n, \ldots)$ *be a sequence of binary random variables with joint distribution* P. *If* (X, P) *satisfy the CJT, then* $\underline{y}^* \leq 2(2\underline{p} - 1)(1 - \underline{p})$.

Proof See Appendix 9.2   □

Figure 5 depicts the regions of validity of the *CJT* in the $\mathbb{R}^2$ plane with respect to the L_1 norm; the analog of Fig. 4.

Strangely enough, Theorem 12 and Theorem 11 do not imply each other in either direction. Furthermore, the techniques of the proofs for the L_1 norm and for the L_2 norm are very different. We could derive only a weak implication in one direction which stems from the following lemma:

Lemma 1 *One always has:* $\underline{y}^* \geq 2\underline{y}$.

Proof Denoting $A_n = \{\omega \in \Omega \mid \overline{p}_n - \overline{X}_n(\omega) \geq 0\}$, we have:

$$\int_{A_n} (\overline{p}_n - \overline{X}_n)^2 \, dP = \int_{A_n} (\overline{p}_n - \overline{X}_n)(\overline{p}_n - \overline{X}_n) \, dP$$

$$\leq \overline{p}_n \int_{A_n} (\overline{p}_n - \overline{X}_n) \, dP = \overline{p}_n \frac{y_n^*}{2}.$$

Similarly,

$$\int_{A_n^c} (\overline{X}_n - \overline{p}_n)^2 \, dP \leq (1 - \overline{p}_n) \frac{y_n^*}{2}.$$

Hence, for all n we have:

$$y_n := E(\overline{X}_n - \overline{p}_n)^2 = \int_{\Omega} (\overline{X}_n - \overline{p}_n)^2 \, dP \leq \overline{p}_n \frac{y_n^*}{2} + (1 - \overline{p}_n) \frac{y_n^*}{2} = \frac{y_n^*}{2}.$$
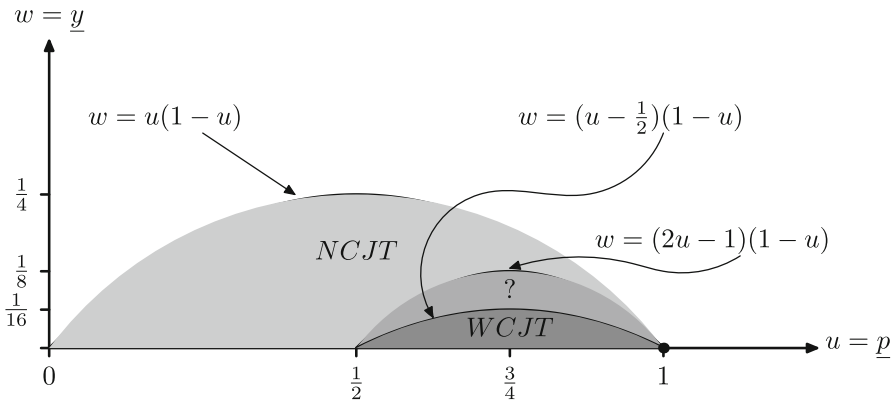
Fig. 6 The *CJT* validity region with respect to the L_2 norm as implied by the condition with respect to the L_1 norm

Taking a subsequence $(n_k)_{k=1}^\infty$ such that $\lim_{k\to\infty} y_{n_k}^* = \underline{y}^*$, we conclude that

$$\underline{y}^* \geq 2 \liminf_{k\to\infty} y_{n_k} \geq 2\underline{y}. \qquad \square$$

Combining Lemma 1 with Theorem 12 yields,

Corollary 13 *Let $X = (X^1, X^2, \ldots, X^n, \ldots)$ be a sequence of binary random variables with joint distribution P. If $\underline{y} > (2\underline{p} - 1)(1 - \underline{p})$, then (X, P) does not satisfy the CJT.*

Figure 6 depicts the conclusion of the last corollary: the region with the lightest color, denoted by *NCJT*, is the region in which the *CJT* is not satisfied for any (X, P) with these values of $(\underline{p}, \underline{y})$. The darkest region, denoted by *WCJT*, is the region of $(\underline{p}, \underline{y})$ for which there exist (X, P) with these parameters that satisfy the *CJT*. Clearly, this is a weaker result than Theorem 11 that we obtained directly for the L_2 norm and is described in Figure 4 according to which, the crescent in Figure 6, denoted by "?", belongs to the *NCJT* region.

6 Distributions in *WCJT* that do not satisfy the *CJT*

In this section, we prove that the necessary conditions stated in Theorems 11 and 12 are not sufficient. In fact, we prove a stronger result, namely: to any pair of parameters in the closure of the dark *WCJT* region (either in Fig. 4 for the L_2 norm or in Fig. 5 for the L_1 norm), excluding the point $(1, 0)$, there is a distribution that does not satisfy the *CJT*. We shall prove this only for the L_2 norm (the proof for the L_1 norm is similar). This is established by the following:

Theorem 14 *For any $(u, w) \in \{(u, w) \mid 0 < u < 1; \ 0 \leq w \leq u(1 - u)\}$, there is a sequence of binary random variables Z with joint distribution H such that:*

(i) $E(Z^i) = u, \forall i$.

(ii) $\liminf_{n\to\infty} E(\overline{Z}_n - u)^2 = w$.
(iii) The distribution H does not satisfy the CJT.

Proof For $0 < u < 1$,

- let (X, F_0) be given by $X^1 = X^2 = \ldots = X^n = \ldots$ and $E(X^i) = u$;
- let (Y, F_1) be a sequence of of *i.i.d.* random variables $(Y^i)_{i=1}^\infty$ with expectation u.
- For $0 < t \leq 1$ let (Z_t, H_t) be the pair in which $Z_t^i = tX^i + (1 - t)Y^i$ for $i = 1, 2, \ldots$ and H_t is the product distribution $H_t = F_0 \times F_1$ (that is, the X and the Y sequences are independent).

Note first that $E(Z_t^i) = u$ for all i and

$$\lim_{n\to\infty} E(\overline{Z}_{t,n} - u)^2 = \lim_{n\to\infty} \left((1 - t)\frac{u(1 - u)}{n} + tu(1 - u) \right) = tu(1 - u),$$

and therefore, the pair (Z_t, H_t) corresponds to the point (u, w) in the L_2 space, where $w = tu(1 - u)$ ranges in $(0, u(1 - u))$ as $0 < t \leq 1$.

Finally, (Z_t, H_t) does not satisfy the *CJT* since for all n,

$$Pr\left(\overline{Z}_{t,n} > \frac{1}{2} \right) \leq 1 - Pr(Z_t^1 = Z_t^2 = \ldots = 0) = 1 - t(1 - u) < 1.$$

As this argument does not apply for $t = 0$ it remains to prove that, except for $(1, 0)$, to any point $(u, 0)$ on the x axis corresponds a distribution that does not satisfy the *CJT*. For $0 \leq u \leq 1/2$, the sequence (Y, F_1) of *i.i.d.* random variables $(Y^i)_{i=1}^\infty$ with expectation u does not satisfy the *CJT*, as follows from the result of Berend and Paroush (1998). For $1/2 < u < 1$ such a sequence of *i.i.d.* random variables does satisfy the *CJT* and we need the following more subtle construction.

Given the two sequences (X, F_0) and (Y, F_1) defined above, we construct a sequence $Z = (Z^i)_{i=1}^\infty$ consisting of alternating blocks of X^is and Y^is, with the probability distribution on Z being that induced by the product probability $H = F_0 \times F_1$. Clearly $E(Z^i) = u$ for all i, in particular $\overline{p}_n = u$ for all n and $\underline{p} = u$. We denote by B_ℓ the set of indices of the ℓth block and its cardinality by b_ℓ. Thus, $n(\ell) = \sum_{j=1}^\ell b_j$ is the index of Z^i at the end of the ℓth block. Therefore,

$$B_{\ell+1} = \{n(\ell) + 1, \ldots, n(\ell) + b_{\ell+1})\} \quad \text{and} \quad n(\ell + 1) = n(\ell) + b_{\ell+1}.$$

Define the block size b_ℓ inductively by:

1. $b_1 = 1$, and for $k = 1, 2, \ldots$;
2. $b_{2k} = k \sum_{j=1}^k b_{2j-1}$ and $b_{2k+1} = b_{2k}$.

Finally, we define the sequence $Z = (Z^i)_{i=1}^\infty$ to consist of X^is in the odd blocks and Y^is in the even blocks, that is,

$$Z^i = \begin{cases} X^i & \text{if } i \in B_{2k-1} & \text{for some } k = 1, 2, \ldots \\ Y^i & \text{if } i \in B_{2k} & \text{for some } k = 1, 2, \ldots \end{cases}$$

Denote by $n_x(\ell)$ and $n_y(\ell)$ the number of X coordinates and Y coordinates respectively in the sequence Z at the end of the ℓth block and by $n(\ell) = n_x(\ell) + n_y(\ell)$ the number of coordinates at the end of the ℓth block of Z. It follows from 1 and 2 (in the definition of b_ℓ) that for $k = 1, 2, \ldots$,

$$n_x(2k-1) = n_y(2k-1) + 1 \tag{33}$$

$$\frac{n_x(2k)}{n_y(2k)} \leq \frac{1}{k} \quad \text{and hence also} \quad \frac{n_x(2k)}{n(2k)} \leq \frac{1}{k} \tag{34}$$

It follows from (33) that at the end of each odd-numbered block $2k-1$, there is a majority of X_i coordinates that with probability $(1-u)$ will all have the value 0. Therefore,

$$Pr\left(\overline{Z}_{n(2k-1)} < \frac{1}{2}\right) \geq (1-u) \quad \text{for } k = 1, 2, \ldots,$$

and hence

$$\liminf_{n \to \infty} Pr\left(\overline{Z}_n > \frac{1}{2}\right) \leq u < 1;$$

that is, (Z, H) does not satisfy the *CJT*.

It remains to show that

$$\underline{y} = \liminf_{n \to \infty} E(\overline{Z}_n - \overline{p}_n)^2 = 0.$$

To do so, we show that the subsequence of $\{E((\overline{Z}_n - \overline{p}_n)^2)\}_{n=1}^{\infty}$ corresponding to the end of the even-numbered blocks converges to 0, namely,

$$\lim_{k \to \infty} E(\overline{Z}_{n(2k)} - \overline{p}_{n(2k)})^2 = 0.$$

Indeed,

$$E(\overline{Z}_{n(2k)} - \overline{p}_{n(2k)})^2 = E\left(\frac{n_x(2k)}{n(2k)}(X^1 - u) + \frac{1}{n(2k)}\sum_{i=1}^{n_y(2k)}(Y^i - u)\right)^2.$$

Since the Y^is are *i.i.d.* and independent of X^1 we have

$$E(\overline{Z}_{n(2k)} - \overline{p}_{n(2k)})^2 = \frac{n_x^2(2k)}{n^2(2k)}u(1-u) + \frac{n_y(2k)}{n^2(2k)}u(1-u),$$

and by property (34) we get finally:

$$\lim_{k \to \infty} E(\overline{Z}_{n(2k)} - \overline{p}_{n(2k)})^2 \leq \lim_{k \to \infty} \left(\frac{1}{k^2}u(1-u) + \frac{1}{n(2k)}u(1-u)\right) = 0,$$

concluding the proof of the theorem. $\square$

An immediate implication of Theorem 14 is the following:

Corollary 15 *For any pair of parameters* $(\underline{p}, \underline{y})$ *satisfying* $1/2 \leq \underline{p} < 1$ *and* $0 \leq \underline{y} \leq (\underline{p} - 1/2)(1 - \underline{p})$ *(that is, the point* $(\underline{p}, \underline{y})$ *is in the closure of the region WCJT in Fig. 4, excluding* $(1, 0)$*), there is a distribution with these parameters that does not satisfy the CJT.*

7 Non-exchangeable sequences satisfying the *CJT*

In this section, we prove the existence of sequences (X, P) of dependent random variables, sequences that are non-exchangeable and satisfy the *CJT*. By Theorem 11, such distributions must have their parameter in the closure of the dark *WCJT* region (either in Fig. 4 in L_2 or in Fig. 5 in L_1). In fact, we shall prove that for any point in this region there is a distribution that satisfies the *CJT*, and is not exchangeable. We shall prove that only in the L_2 plane. The proof for the L_1 plane is similar. The construction of these sequences uses the idea of the *interlacing* of two sequences, which can be generalized and proves to be useful.

Theorem 16 *Let* $t \in [0, \frac{1}{2}]$. *If F is a distribution with parameters* $(\underline{p}, \underline{y})$, *then there exists a distribution H with parameters* $\tilde{p} = 1 - t + t\underline{p}$ *and* $\tilde{y} = t^2\underline{y}$ *that satisfy the CJT.*

Proof To illustrate the idea of the proof we first prove (somewhat informally) the case $t = 1/2$. Let $X = (X^1, X^2, \ldots, X^n, \ldots)$ be a sequence of binary random variables with a joint probability distribution F. Let G be the distribution of the sequence $Y = (Y^1, Y^2, \ldots, Y^n, \ldots)$, where $EY^n = 1$ for all n (that is, $Y^1 = Y^2 = \ldots Y^n = \ldots$ and $P(Y^i = 1) = 1 \; \forall i$). Consider now the following *interlacing* of the two sequences X and Y:

$$Z = (Y^1, Y^2, X^1, Y^3, X^2, Y^4, X^3, \ldots, Y^n, X^{n-1}, Y^{n+1}, X^n \ldots),$$

and let the probability distribution H of Z be the product distribution $H = F \times G$. It is verified by straightforward computation that the parameters of the distribution H are in accordance with the theorem for $t = \frac{1}{2}$, namely, $\tilde{p} = \frac{1}{2} + \frac{1}{2}\underline{p}$ and $\tilde{y} = \frac{1}{4}\underline{y}$. Finally, as each initial segment of voters in Z contains a majority of Y^is (thus with all values 1), the distribution H satisfies the *CJT*, completing the proof for $t = \frac{1}{2}$.

The proof for a general $t \in [0, 1/2)$ follows the same lines: we construct the sequence Z so that any finite initial segment of n variables, includes "about, but not more than" the initial tn segment of the X sequence, and the rest is filled with the constant Y_i variables. This will imply that the *CJT* is satisfied.

Formally, for any real $x \geq 0$ let $\lfloor x \rfloor$ be the largest integer less than or equal to x and let $\lceil x \rceil$ be smallest integer greater than or equal to x. Note that for any n and any $0 \leq t \leq 1$ we have $\lfloor tn \rfloor + \lceil (1 - t)n \rceil = n$; thus, one and only one of the following holds:
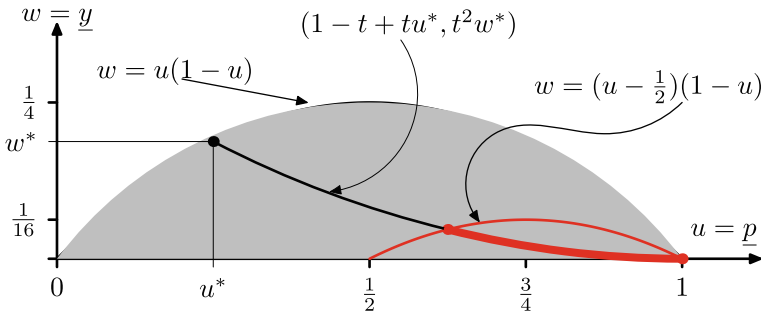
Fig. 7 Interlacing with $(1, 0)$ in L_2

(i) $\lfloor tn \rfloor < \lfloor t(n+1) \rfloor$ or

(ii) $\lceil (1-t)n \rceil < \lceil (1-t)(n+1) \rceil$

From the given sequence X and the above-defined sequence Y (of constant 1 variables) we define now the sequence $Z = (Z^1, Z^2, \ldots, Z^n, \ldots)$ as follows: $Z^1 = Y^1$ and for any $n \geq 2$, let $Z^n = X^{\lfloor t(n+1) \rfloor}$ if (i) holds and $Z_n = Y^{\lceil (1-t)(n+1) \rceil}$ if (ii) holds. This inductive construction guarantees that for all n, the sequence contains $\lfloor tn \rfloor$ X^i coordinates and $\lceil (1-t)n \rceil$ Y^i coordinates. The probability distribution H is the product distribution $F \times G$. The fact that (Z, H) satisfies the *CJT* follows from:

$$\lceil (1-t)n \rceil \geq (1-t)n > tn \geq \lfloor tn \rfloor,$$

and finally $\tilde{p} = 1 - t + t\underline{p}$ and $\tilde{y} = t^2 \underline{y}$ is verified by straightforward computation. $\square$

Remark 5 • Note that the sequence Z is clearly not exchangeable (except for the case $t = 0$ which corresponds to $(1, 0)$).

• The interlacing of the two sequences X and Y described in the proof of Theorem 16 may be defined for any $t \in [0, 1]$. We were specifically interested in $t \in [0, 1/2]$ since this guarantees the *CJT*.

Figure 7 depicts the interlacing procedure: the parabolic line joining (u^*, w^*) to the point $(1, 0)$, corresponds to all interlacing with $t \in [0, 1]$. The lower part, described as a thick line, corresponds to interlacing when $t \in [0, 1/2]$. For these values of t, the interlacing yields distributions satisfying the *CJT*. The small parabola is the locus of points corresponding to $t = 1/2$ when (u^*, w^*) ranges over the parabola $w = u(1-u)$.

Corollary 17 *For any* $(\underline{p}, \underline{y})$ *in the set*

$$\overline{A} = \{(\underline{p}, \underline{y}) \mid 0 \leq \underline{y} \leq (\underline{p} - 1/2)(1 - \underline{p}) ; \quad 1/2 \leq \underline{p} \leq 1\}$$

(this is the closure of the region *WCJT* in Fig. 4), *there is a sequence of non-exchangeable random variables, with these parameters, that satisfy the CJT.*

Proof By straightforward verification we observe that the set $\overline{A}$ is obtained from Theorem 16 by letting $(\underline{p}, \underline{y})$ range over the points of parabola $w = u(1-u)$ defining the

feasible set FE_2. In other words, $\overline{A}$ can also be written as:

$$\overline{A} = \{(\underline{p}, \underline{y}) \mid \underline{p} = 1 - t + tu \;;\;\; \underline{y} = t^2 u(1-u) \;;\;\; 0 \le t \le 1/2, \;\; 0 \le u \le 1\} \quad \square$$

Note that $\overline{A}$ is the closure of the set A defined in Eq. 28 for exchangeable variables, but $\overline{A} \ne A$. More specifically, the points $(\underline{p}, \underline{y})$ on the parabola $\underline{y} = (\underline{p} - 1/2)(1 - \underline{p})$, excluding $(1, 0)$, are in $\overline{A}$ but not in A. For each of these points there is a corresponding sequence satisfying the *CJT* but this sequence cannot be exchangeable.

Finally, combining Corollary 17 and Theorem 14 yields:

Corollary 18 *For any point $(\underline{p}, \underline{y})$ in $\overline{A} \setminus \{(1, 0)\}$ there is a corresponding sequence satisfying the CJT and a corresponding sequences that does not satisfy the CJT.*

7.1 Other distributions satisfying the *CJT*: general interlacing

So far, we have identified three types of distributions that satisfy the *CJT*; all correspond to parameters $(\underline{p}, \underline{y})$ in the set $\overline{A}$, the closure of the region *WCJT* in Fig. 4.

1. Distributions satisfying the sufficient condition (Theorem 9).
2. Exchangeable distributions characterized in Theorem 5.
3. Non-exchangeable distributions obtained by interlacing with constant sequence $Y = (1, 1, \ldots)$ (Theorem 16).

In this section, we construct more distributions satisfying the *CJT* that are not in either of the three families mentioned above. We do so by generalizing the notion of the "interlacing" of two distributions that we introduced in Sect. 7.

Definition 2 Let $X = (X^1, X^2, \ldots, X^n, \ldots)$ be a sequence of binary random variables with joint probability distribution F and let $Y = (Y^1, Y^2, \ldots, Y^n, \ldots)$ be another sequence of binary random variables with joint distribution G. For $t \in [0, 1]$, the *t-interlacing* of (X, F) and (Y, G) is the pair $(Z, H) := (X, F) *_t (Y, G)$ where for $n = 1, 2, \ldots,$

$$Z^n = \begin{cases} X^{\lfloor tn \rfloor} & \text{if } \lfloor tn \rfloor > \lfloor t(n-1) \rfloor \\ Y^{\lceil (1-t)n \rceil} & \text{if } \lceil (1-t)n \rceil > \lceil (1-t)(n-1) \rceil \end{cases}, \tag{35}$$

and $H = F \times G$ is the product probability distribution of F and G.

The following lemma is a direct consequence of Definition 2.

Lemma 2 *If (X, F) and (Y, G) satisfy the CJT, then for any $t \in [0, 1]$ the pair $(Z, H) = (X, F) *_t (Y, G)$ also satisfies the CJT.*

Proof We may assume that $t \in (0, 1)$. Note that

$$\left\{ \omega \mid \overline{Z}_n(\omega) > \frac{1}{2} \right\} \supseteq \left\{ \omega \mid \overline{X}_{\lfloor tn \rfloor}(\omega) > \frac{1}{2} \right\} \cap \left\{ \omega \mid \overline{Y}_{\lceil (1-t)n \rceil}(\omega) > \frac{1}{2} \right\}$$

By our construction and the fact that both (X, F) and (Y, G) satisfy the *CJT*,

$$\lim_{n \to \infty} F\left(\overline{X}_{\lfloor tn \rfloor} > \frac{1}{2}\right) = 1 \ \text{ and } \ \lim_{n \to \infty} G\left(\overline{Y}_{\lceil (1-t)n \rceil} > \frac{1}{2}\right) = 1.$$

As

$$H\left(\overline{Z}_n > \frac{1}{2}\right) \ge F\left(\overline{X}_{\lfloor tn \rfloor} > \frac{1}{2}\right) \cdot G\left(\overline{Y}_{\lceil (1-t)n \rceil} > \frac{1}{2}\right),$$

the proof follows. □

Thus, from any two distributions satisfying the *CJT* we can construct a continuum of distributions satisfying the *CJT*. These distributions will generally be outside the union of the three families listed above.

8 Conclusions

We have analyzed the Condorcet jury problem in a detailed manner as a strategic game with incomplete information (Sect. 2). This framework has the following advantages:

(I)   It is in line with the modern approach of Austen-Smith and Banks (1996);
(II) It enables us to focus on a natural candidate BNE for satisfying the *CJT*, namely, McLennan's BNE (see Theorem 2);
(III) It explains, in a transparent way, Condorcet's own model which was originally restricted to two types of voters and informative voting;
(IV) It enables us to deal with "reverse voting" and "random voting" (and other strategies) without altering our model, since we consider *all possible strategies* and not only informative voting (see Examples 3 and 2);
(V)   Using our model, we find (sharp) necessary conditions for the existence of a sequence of BNE that satisfies the CJT. Indeed, as we stated on page (22), if a McLennan sequence of BNE does not satisfy one of our necessary conditions, then *no other sequence of strategies* satisfies the *CJT*.
(VI) Informative voting may not be a *Bayesian-Nash Equilibrium* (See Austen-Smith and Banks (1996)).

Technically, we deal, in most of the paper, with distributions of sequences of binary random variables that are derived from sequences of strategies of the players. This is mainly implied by the fact that the *CJT* is a probabilistic property. In Sect. 3 we find necessary and sufficient conditions for a sequence of exchangeable variables to satisfy the *CJT* (see Theorem 5). We then go on to find a purely geometrical characterization for our result. For exchangeable variables, the geometric condition is fully determined by the expectation of a variable in the sequence and the covariance of the two variables (see Theorem 7). Our sufficient conditions are standard and are derived by Chebyshev's inequality. On the other hand, the derivation of the necessary conditions makes use of special techniques. The necessary condition with respect to the L_2 norm, quite surprisingly, coincides with the necessary condition for exchangeable

variables. However, of course, it is not a sufficient condition for general sequences of random variables. Indeed in Sect. 6, we provide sequences of random variables satisfying this necessary condition but do not satisfy the *CJT* (see Theorem 14 and Corollary 15). The necessary condition with respect to the L_1 norm is difficult to derive and may be useful in special cases. Finally, we introduce in Sect. 7 the operation of interlacing of two sequences of random variables. This enables to generate many new dependant sequences of binary random variables that satisfy the *CJT*.

9 Appendix

9.1 Every sequence of of binary random variables is attainable

In this section, we prove what we claimed towards the end of Sect. 2, namely, that for any infinite sequence of binary random variables X there is a sequence of games $(G_n)_{n=1}^{\infty}$ and an infinite sequence of constant strategies $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n, \ldots)$ that yield this X as the infinite sequence of the indicators of correct voting.

Let $X = (X^1, X^2, \ldots, X^n, \ldots)$ be a sequence of binary random variables on some probability space $(\tilde{\Omega}, \mathscr{B}, \mathscr{P})$. Let P also denote the distribution of X. In our model let $T^i = \{t_0^i, t_1^i\}$ be the type set of juror i and let the type of juror i, $t^i = t^i(\theta, X^i(\omega))$ be defined by: $t^i(g, 0) = t^i(z, 1) = t_0^i$ and $t^i(g, 1) = t^i(z, 0) = t_1^i$. We define the probability distribution $p^{(n)}$ on $\Omega_n = \Theta \times T^1 \times \cdots \times T^n$ as follows: let $p^{(n)}(z) = p^{(n)}(g) = 1/2$; for $\varepsilon_k \in \{0, 1\}$; $k = 1, \ldots, n$ let

$$\tilde{p}(g, X^1 = \varepsilon_1, \ldots, X^n = \varepsilon_n) = \tilde{p}(z, X^1 = 1 - \varepsilon_1, \ldots, X^n = 1 - \varepsilon_n)$$
$$= \frac{1}{2} P(X^1 = \varepsilon_1, \ldots, X^n = \varepsilon_n)$$

and define

$$p^{(n)}(\theta, t^1, t^2, \ldots, t^n) = \tilde{p}(\theta, X^1, X^2, \ldots, X^n).$$

The sequence $(p^{(n)})_{n=1}^{\infty}$ clearly satisfies the projective structure required for the Kolmogorov's extension theorem (that is, the marginal distribution of $p^{(n+1)}$ on Ω_n is equal to $p^{(n)}$). It defines therefore a probability distribution p on $\Omega = \lim_{\infty \leftarrow n} \Omega_n$.

Define now the (informative voting) strategies σ^i by: $\sigma^i(t_0^i) = a$ and $\sigma^i(t_1^i) = c$, and let $\tilde{X}^1, \ldots, \tilde{X}^n \ldots$ be the indicators of correct voting (w.r.t. this σ), then

$$\tilde{X}^i(g, t^i(g, 1)) = \tilde{X}^i(z, t^i(z, 1)) = 1 \quad \text{and} \quad \tilde{X}^i = 0 \text{ otherwise.}$$

Thus,

$$\tilde{X}^i(\theta, \omega) = 1 \quad \Leftrightarrow \quad X^i(\omega) = 1,$$

which means that we obtained the original given sequence.

9.2 Proof of Theorem 12

In this section, we provide a necessary condition for a general sequence of binary random variables $X = (X^1, X^2, \ldots, X^n, \ldots)$ with joint distribution P, in terms of two of its characteristics namely, $\underline{p} = \liminf_{n \to \infty} \overline{p}_n$ and $\underline{y}^* = \liminf_{n \to \infty} E|\overline{X}_n - \overline{p}_n|$.

Let $y_n^* = E|\overline{X}_n - \overline{p}_n|$; then $\underline{y}^* = \liminf_{n \to \infty} y_n^*$. For $n = 1, 2, \ldots$, let

$$A_n = \{\omega \in \Omega \mid \overline{p}_n - \overline{X}_n(\omega) \geq 0\} \quad \text{and} \quad A_n^c = \Omega \setminus A_n.$$

Then, since $E(\overline{X}_n - \overline{p}_n) = 0$,

$$\int_{A_n^c} (\overline{X}_n - \overline{p}_n) \, dP = \frac{y_n^*}{2} \quad \text{and} \quad \int_{A_n^c} (1 - \overline{p}_n) \, dP = (1 - \overline{p}_n)P(A_n^c) \geq \frac{y_n^*}{2}. \quad \text{Hence,}$$

$$P(A_n) = 1 - P(A_n^c) \leq 1 - \frac{y_n^*}{2(1 - \overline{p}_n)}. \tag{36}$$

Also,

$$\int_{A_n} (\overline{p}_n - \overline{X}_n) \, dP = \overline{p}_n P(A_n) - \int_{A_n} \overline{X}_n \, dP = \frac{y_n^*}{2}. \tag{37}$$

Hence, since $\overline{X}_n \geq 0$,

$$P(A_n) \geq \frac{y_n^*}{2\overline{p}_n}. \tag{38}$$

Assuming $\underline{y}^* > 0$ and $\underline{p} < 1$, it follows from (38) and (36) that there is a subsequence $(n_k)_{k=1}^{\infty}$ such that $(P(A_{n_k}))_{k=1}^{\infty}$ is uniformly bounded away from 0 and 1,

$$\lim_{k \to \infty} \overline{p}_{n_k} = \underline{p} \quad \text{and} \quad \lim_{k \to \infty} P(A_{n_k}) = \ell \quad \text{where} \quad 0 < \ell < 1. \tag{39}$$

Lemma 3 *Let $t > 0$; then*

$$\liminf_{k \to \infty} P\left(\left\{\omega \in A_{n_k} \;\middle|\; \overline{p}_{n_k} - \overline{X}_{n_k}(\omega) \leq \frac{y_{n_k}^*}{2P(A_{n_k})} - t\right\}\right) < \ell. \tag{40}$$

Proof Assume by contradiction that (40) does not hold; then, since the sets on the left-hand side are subsets of A_{n_k}, it follows from (39) that:

$$\lim_{k \to \infty} P\left(\left\{\omega \in A_{n_k} \;\middle|\; \overline{P}_{n_k} - \overline{X}_{n_k}(\omega) \le \frac{y_{n_k}^*}{2P(A_{n_k})} - t\right\}\right) = \ell. \tag{41}$$

Denote: $\tilde{A}_{n_k} = \left\{\omega \in A_{n_k} \;\middle|\; \overline{P}_{n_k} - \overline{X}_{n_k}(\omega) \le \frac{y_{n_k}^*}{2P(\tilde{A}_{n_k})} - t\right\}$. Then,

$$\overline{P}_{n_k} P(\tilde{A}_{n_k}) - \int_{\tilde{A}_{n_k}} \overline{X}_{n_k}(\omega) dP \le \frac{y_{n_k}^* P(\tilde{A}_{n_k})}{2P(A_{n_k})} - t P(\tilde{A}_{n_k}). \tag{42}$$

Clearly, $\lim_{k \to \infty} P(\tilde{A}_{n_k}) = \ell = \lim_{k \to \infty} P(A_{n_k})$ and since $\tilde{A}_{n_k} \subseteq A_{n_k}$, we have

$$\lim_{k \to \infty} \left| \int_{\tilde{A}_{n_k}} \overline{X}_{n_k}(\omega) dP - \int_{A_{n_k}} \overline{X}_{n_k}(\omega) dP \right| = 0.$$

Thus for k_0 sufficiently large, the inequality (42) contradicts the last equality in (37) for $n = n_{k_0}$. $\qquad\square$

Let

$$B_n = A_n \setminus \tilde{A}_n = \left\{\omega \in A_n \;\middle|\; \overline{P}_n - \overline{X}_n(\omega) > \frac{y_n^*}{2P(A_n)} - t\right\};$$

then, by Lemma 3, there is a subsequence $(B_{n_k})_{k=1}^{\infty}$ and $q > 0$, such that $P(B_{n_k}) > q > 0$ for all k, that is

$$\overline{X}_{n_k}(\omega) < \overline{P}_{n_k} - \frac{y_{n_k}^*}{2P(A_{n_k})} + t; \quad \forall \omega \in B_{n_k}; \quad \forall k. \tag{43}$$

Example 4 Let $\frac{1}{2} \le \underline{p} < 1$ and $\underline{y}^* = 2\underline{p}(1 - \underline{p})$; then, by (36) and (43) we have

$$\overline{X}_{n_k}(\omega) < \overline{P}_{n_k} - \frac{y_{n_k}^*}{2\left(1 - \frac{y_{n_k}^*}{2(1 - \overline{P}_{n_k})}\right)} + t; \quad \forall \omega \in B_{n_k}; \quad \forall k. \tag{44}$$

By taking subsequences of $(n_k)_{k=1}^{\infty}$ (to make $y_{n_k}^*$ converge) we may assume w.l.o.g. that:

$$\lim_{k \to \infty} \left(\overline{P}_{n_k} - \frac{y_{n_k}^*}{2\left(1 - \frac{y_{n_k}^*}{2(1 - \overline{P}_{n_k})}\right)} + t\right) = \underline{p} - \frac{\underline{y}^* + \varepsilon}{2\left(1 - \frac{\underline{y}^* + \varepsilon}{2(1 - \underline{p})}\right)} + t, \quad \text{for some } \varepsilon \ge 0.$$

Thus, for some k_0 we have

$$\overline{X}_{n_k}(\omega) < \underline{p} - \frac{\underline{y}^* + \varepsilon}{2\left(1 - \frac{\underline{y}^* + \varepsilon}{2(1 - \underline{p})}\right)} + 2t; \quad \forall \omega \in B_{n_k}; \quad \forall k > k_0. \tag{45}$$

Inserting $\underline{y}^* = 2\underline{p}(1 - \underline{p})$ we have:

$$\underline{p} - \frac{\underline{y}^* + \varepsilon}{2\left(1 - \frac{\underline{y}^* + \varepsilon}{2(1 - \underline{p})}\right)} + 2t \leq \underline{p} - \frac{\underline{y}^*}{2\left(1 - \frac{\underline{y}^*}{2(1 - \underline{p})}\right)} + 2t$$

$$= \underline{p} - \frac{2\underline{p}(1 - \underline{p})}{2(1 - \frac{2\underline{p}(1-\underline{p})}{2(1-\underline{p})})} + 2t = 2t,$$

implying that

$$\overline{X}_{n_k}(\omega) < 2t; \quad \forall \omega \in B_{n_k}; \quad \forall k > k_0. \tag{46}$$

As $t > 0$ is arbitrary, in particular, if $2t < 1/2$; since $P(B_{n_k}) > q > 0$ for all k, inequalities (46) imply that (X, P) does not satisfy the *CJT*.

We conclude: *No distribution with $\frac{1}{2} \leq \underline{p} < 1$ and $\underline{y}^* = 2\underline{p}(1 - \underline{p})$ satisfies the CJT*.

Inspired by the previous example we move now to the proof of Theorem 12 stating the general necessary condition for the *CJT* in L_1.

Theorem 19 *Let $X = (X^1, X^2, \ldots, X^n, \ldots)$ be sequence of binary random variables with joint distribution P. If $\underline{y}^* > 2(2\underline{p} - 1)(1 - \underline{p})$, then (X, P) does not satisfy the CJT*.

Proof Let $\tilde{x} = (2\underline{p} - 1)(1 - \underline{p})$ and notice that $x/(1 - x/(1 - \underline{p}))$ is an increasing function for $x < 1 - \underline{p}$. Since $\underline{y}^*/2 > \tilde{x}$, let t be such that

$$0 < t < \frac{1}{2}\left(\frac{\underline{y}^*}{2\left(1 - \frac{\underline{y}^*}{2(1 - \underline{p})}\right)} - \frac{\tilde{x}}{1 - \frac{\tilde{x}}{1 - \underline{p}}}\right)$$

By Lemma 3, there exists a sequence of events $(B_{n_k})_{k=1}^{\infty}$ and $q > 0$, such that $P(B_{n_k}) > q > 0$ for all k, and (43) and, (by choosing an appropriate subsequence), (45) are satisfied. Thus, on these events we have,

$$\overline{X}_{n_k}(\omega) < \underline{p} - \frac{\underline{y}^* + \varepsilon}{2\left(1 - \frac{\underline{y}^* + \varepsilon}{2(1 - \underline{p})}\right)} + 2t \leq \underline{p} - \frac{\underline{y}^*}{2\left(1 - \frac{\underline{y}^*}{2(1 - \underline{p})}\right)} + 2t$$

$$< \underline{p} - \frac{\tilde{x}}{1 - \frac{\tilde{x}}{1 - \underline{p}}} + 2t - 2t.$$

Substituting $\tilde{x} = (2\underline{p} - 1)(1 - \underline{p})$, we have

$$\overline{X}_{n_k}(\omega) < \underline{p} - \frac{(2\underline{p} - 1)(1 - \underline{p})}{1 - (2\underline{p} - 1)} = \frac{1}{2}.$$

We conclude that $\overline{X}_{n_k}(\omega) < \frac{1}{2}$, for all $\omega \in B_{n_k}$ and for all $k > k_0$, implying that (X, P) does not satisfy the *CJT*. $\square$

9.3 The *CJT* and the law of large numbers

At first sight, the asymptotic *CJT* condition may look rather similar to the well-known *Law of Large Numbers* (*LLN*). It is the purpose of this section to clarify and state precisely the relationship between these two concepts.

Recall that an infinite sequence of binary random variables $X = (X^1, X^2, \ldots, X^n, \ldots)$ with a joint probability distribution P satisfies the (weak) Law of Large Numbers (*LLN*) if (in our notations):

$$\forall \varepsilon > 0, \quad \lim_{n \to \infty} P\left(|\overline{X}_n - \overline{p}_n| < \varepsilon\right) = 1 \tag{47}$$

while it satisfies the *CJT* if:

$$\lim_{n \to \infty} P\left(\overline{X}_n > \frac{1}{2}\right) = 1 \tag{48}$$

Since by Proposition 4, the condition $\underline{p} \geq \frac{1}{2}$ is necessary for the validity of the *CJT*, let us check the relationship between the *LLN* and the *CJT* in this region. Our first observation is:

Proposition 5 *For a sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ with probability distribution P satisfying $\underline{p} > \frac{1}{2}$, if the LLN holds then the CJT also holds.*

Proof Let $\underline{p} = 1/2 + 3\delta$ for some $\delta > 0$ and let N_0 be such that $\overline{p}_n > 1/2 + 2\delta$ for all $n > N_0$; then for all $n > N_0$ we have

$$P\left(\overline{X}_n > \frac{1}{2}\right) \geq P\left(\overline{X}_n \geq \frac{1}{2} + \delta\right) \geq P\left(|\overline{X}_n - \overline{p}_n| < \delta\right)$$

Since the last expression tends to 1 as $n \to \infty$, the first expression does too, and hence the *CJT* holds. $\square$

Remark 6 The statement of Proposition 5 does not hold for $\underline{p} = \frac{1}{2}$. Indeed, the sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ of *i.i.d.* variables with $P(X^i = 1) = P(X^i = 0) = 1/2$ satisfies the *LLN* but does not satisfy the *CJT* since it does not satisfy $\lim_{n \to \infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty$ which is a necessary and sufficient condition for *CJT* (see Berend and Paroush 1998).

Unfortunately, Proposition 5 is of little use to us. This is due to the following fact:

Proposition 6 *If the random variables of the sequence $X = (X^1, X^2, \ldots, X^n, \ldots)$ are uniformly bounded then the condition*

$$\lim_{n \to \infty} E \left(\overline{X}_n - \overline{p}_n \right)^2 = 0$$

is a necessary condition for LLN to hold.

The proof is elementary and can be found, e.g., in Uspensky (1937, p. 185).

It follows thus from Proposition 6 that LLN cannot hold when $y > 0$ and thus we cannot use Proposition 5 to establish distributions in this region that satisfy the CJT.

Summing up, The LLN and the CJT are substantially two different properties that do not imply each other. The partial implication $LLN \Rightarrow CJT$ applies only for the horizontal line in L_2; $(\underline{p}, 0)$, for $\underline{p} > 1/2$, where the CJT is easily established directly. Furthermore, all distributions with $y > 0$ for which we established the validity of the CJT do not satisfy the LLN.

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