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Type indeterminacy: A model of the KT(Kahneman–Tversky)-man[☆]

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ABSTRACT

In this paper, we propose to use elements of the mathematical formalism of Quantum Mechanics to capture the idea that agents' preferences, in addition to being typically uncertain, can also be *indeterminate*. They are determined (i.e., realized, and not merely revealed) only when the action takes place. An agent is described by a *state* that is a superposition of potential types (or preferences or behaviors). This superposed state is projected (or "collapses") onto one of the possible behaviors at the time of the interaction. In addition to the main goal of modeling uncertainty of preferences that is not due to lack of information, this formalism seems to be adequate to describe widely observed phenomena of non-commutativity in patterns of behavior. We explore some implications of our approach in a comparison between classical and type indeterminate rational choice behavior. The potential of the approach is illustrated in two examples.

1. Introduction

It has recently been proposed that models of quantum games can be used to study how the extension of classical moves to quantum ones (i.e., complex linear combinations of classical moves) can affect the analysis of a game. For example, Eisert, Wilkens, and Lewenstein (1999) show that allowing the players to use quantum strategies in the Prisoner's Dilemma is a way of escaping the well-known "bad feature" of this game.[1] From a game-theoretical point of view the approach consists in changing the strategy spaces, and thus the interest of the results lies in the appeal of these changes.[2]

This paper also proposes to use elements of the mathematical formalism of Quantum Mechanics but with a different intention: to model uncertain preferences.[3] The basic idea is that the Hilbert space model of Quantum Mechanics can be thought of as a very general contextual predictive tool, particularly well-suited to describing experiments in psychology or in "revealing" preferences.

The well-established Bayesian approach suggested by Harsanyi (1967) to model incomplete information consists of a chance move that selects the types of the players and informs each player of his own type. For the purposes of this paper, we underline the following essential implication of this approach: all uncertainty about a player's type exclusively reflects the other players' incomplete knowledge of it. This follows from the fact that a Harsanyi type is fully determined. It is a complete, well-defined description of the characteristics of a player that is known to her. Consequently, from the point of view of the other players, uncertainty as to the type can only be due to lack of *information*. Each player has a probability distribution over the type of the other players, but her own type is fully determined and is known to her.

This brings us to the first important point at which we depart from the classical approach: we propose that in addition to informational reasons, the uncertainty about preferences is due to *indeterminacy*: prior to the moment a player acts, her (behavior) type is indeterminate. The *state* representing the player is a *superposition* of potential types. It is only at the moment when

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[1] In the classical version of the dilemma, the dominant strategy for both players is to defect and thereby to do worse than if they had both decided to cooperate. In the quantum version, there are a couple of quantum strategies that are both a Nash equilibrium and Pareto optimal and whose payoff is the one of the joint cooperation.

[2] This approach is closely related to quantum computing. It relies on the use of a sophisticated apparatus to exploit q-bits' property of entanglement in mixed strategies.

[3] In this work we borrow the elements of the quantum formalism that concern the measurement process. We will not use the part of the theory concerned with the evolution of systems over time.

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the player selects an action that a specific type is actualized.[4] It is not merely revealed but rather determined in the sense that prior to the choice, there is an irreducible multiplicity of potential types. Thus we suggest that in modeling a decision situation, we do not assume that the preference characteristics can always be fully known with certainty (neither to the decision-maker nor even to the analyst). Instead, what can be known is the state of the agent: a vector in a Hilbert space which encapsulates all existing information to predict how the agent is expected to behave in different decision situations.

This idea, daringly imported from Quantum Mechanics to the context of decision and game theory, is very much in line with Tversky and Simonson (Kahneman & Tversky, 2000), according to whom "*There is a growing body of evidence that supports an alternative conception according to which preferences are often constructed – not merely revealed – in the elicitation process. These constructions are contingent on the framing of the problem, the method of elicitation, and the context of the choice".* In Ariely, Prelec, and Lowenstein (2003), the authors show in a series of experiments that *"valuations are initially malleable but become "imprinted" after the agent is called upon to make an initial decision"* (p. 74). This view is also consistent with that of cognitive psychology, which teaches one to distinguish between objective reality and the proximal stimulus to which the observer is exposed, and to further distinguish between those and the mental representation of the situation that the observer eventually constructs. More generally, this view fits in with the observation that players (even highly rational ones) may act differently in game theoretically equivalent situations that differ only in seemingly irrelevant aspects (framing, prior unrelated events, etc.). Our theory as to why agents act differently in game theoretically equivalent situations is that they are not in the same state; (revealed) preferences are contextual because of (intrinsic) indeterminacy.

The basic analogy with Physics, which makes it appealing to adopt the mathematical formalism of Quantum Mechanics in the social sciences, is the following: we view decisions and choices as something similar to the result of a *measurement* (of the player's type). A situation of decision is then similar to an experimental setup to measure the player's type. It is modeled by an operator (called *observable*), and the resulting behavior is an eigenvalue of that operator. The non-commutativity of observables and its consequences (very central features of Quantum Mechanics) are reminiscent of many empirical phenomena like the following one exhibited in a well-known experiment conducted by Leon Festinger.[5] In this experiment, people were asked to sort a batch of spools into lots of twelve and give a square peg a quarter turn to the left. They all agreed that the task was very boring. Then, they were told that one subject was missing for the experiment and asked to convince a potential female subject in the waiting room to participate. They were offered $1 for expressing their enthusiasm for the task.[6] Some refused, but others accepted. Those who accepted maintained afterwards that the task was enjoyable. This experiment aimed at showing that attitudes change in response to cognitive dissonance. The dissonance faced by those who accepted to fake enthusiasm for $1 was due to the contradiction between the self-image of being "a good guy" and that of "being ready to lie for a dollar". Changing one's attitude and persuading oneself that the task really was interesting was a way to resolve the dissonance. Similar phenomena have been documented in hazardous industries, with employees showing very little caution in the face of a danger. Here too, experimental and empirical studies (e.g., Ben-Horin (1979)) showed how employees change their attitude after they have decided to work in a hazardous industry. More generally, suppose that an agent is presented with the same situation of decision in two different contexts. The contexts may vary with respect to the situation of decision that precedes the investigated one. In Festinger's experiment the two measurements of the attitude toward the task differ in that the second one was preceded by a question about willingness to lie about the task for a dollar. Two contexts may also vary with respect to the framing of the situation of decision (cf. Selten (1998)). If we do not observe the same decision in the two contexts, then the classical approach considers that the two situations are not identical and hence that they should be modeled so as to incorporate the context. In many cases, however, such an assumption (i.e., that the situations are different) is difficult to justify.

In contrast, we propose that the difference between the two decisions comes from the fact that the agent is not in the same state. The context (e.g., a past situation of decision to which the agent has been exposed) is represented by an operator that does not commute with the operator associated with the situation of decision currently considered. The consequence is that the initial state of the agent has changed and that the agent is therefore expected to behave differently from what she would have done if confronted directly with the situation. As in Quantum Mechanics, the non-commutativity of certain situations of decision (measurements) leads us to conjecture that the preferences of an agent are represented by a state that is indeterminate and gets determined (with respect to any particular type characteristics) in the course of interaction with the environment. In Section 3, we show how this approach can explain the reversal of preferences in a model of rational choice and that it provides a framework for explaining cognitive dissonance and framing effects.

The objective of this paper is to propose a theoretical framework for modeling the KT(Kahneman–Tversky)-man, i.e., for the "constructive preference perspective" . Our approach amounts to extending the classical representation of uncertainty in Harsanyi's style to non-classical indeterminacy. This work is a contribution to Behavioral Economics. A distinguishing feature of behavioral theories is that they often focus on rather specific anomalies (e.g., "trade-off contrast" or "extremeness aversion" (Kahneman & Tversky, 2000). Important insights have been obtained by systematically investigating the consequences on utility maximization of "fairness concerns" (Rabin, 1993), "temptation and costly self-control" (Gul & Pesendorfer, 2001) or "concerns for self-image" (Benabou & Tirole, 2002). Yet, other explanations appeal to bounded rationality, e.g., "superficial reasoning" or "choice of beliefs" (Akerlof & Dickens, 1982; Selten, 1998). In contrast, the type indeterminacy model is a framework model that addresses structural properties of preferences, i.e., their intrinsic indeterminacy. A value of our approach is in providing a unified explanation for a wide variety of behavioral phenomena.

In Section 2, we present the framework and some basic notions of quantum theory. In Section 3, we develop applications of the theory to social sciences. In Section 4, we discuss some basic assumptions of the model. The Appendix provides a brief exposition of some basic concepts of Quantum Mechanics.

2. The basic framework

In this section we present the basic notions of our framework. They are heavily inspired by the mathematical formalism of Quantum Mechanics (see, e.g., Cohen-Tannoudji, Diu, and Laloe (1973) and Cohen (1989)) from which we also borrow the notation.

4 The associated concept of irreducible uncertainty, which is the essence of indeterminacy, is formally defined in Section 2 of the paper.

5 Leon Festinger is the father of the theory of cognitive dissonance (Festinger, 1957).

6 The experience was richer. People were divided into two groups offered different amounts of money. For our purpose it is sufficient to focus on a single result.

2.1. The notions of state and superposition

The object of our investigation is individual choice behavior, which we interpret as the revelation of an agent's preferences in a situation of decision that we shall call a *DS* (Decision Situation). In this paper, we focus on decision situations that do not involve any strategic thinking. Examples of such *DS* include the choice between buying a Toshiba or a Compaq laptop, the choice between investing in a project or not, the choice between a sure gain of \$100 or a bet with probability 0.5 to win \$250 and 0.5 to win \$0, etc. When considering games, we view them as decision situations from the perspective of a single player who plays once.[7]

An agent is represented by a state which encapsulates all the information on the agent's expected choice behavior. The formalism that we present below allows for a variety of underlying models. For instance, if the choice set is $\{a, b, c\}$, we may identify three possible states corresponding to the respective choices a, b, and c. Alternatively, a state may correspond to an ordering of the 3 items, in which case we have six possible states (e.g., a state could be (a,b,c) in this order). We may also consider other models. In Section 3.1 we examine some consequences of the second case. For ease of exposition, the basic framework is presented in terms of the first case where states are identified with choices (but the reader is invited to keep in mind that other interpretations of the model are possible).

Mathematically, a state $|\psi\rangle$ is a vector in a Hilbert space $\mathcal{H}$ of finite or countably infinite dimensions, over the field of the real numbers $\mathbb{R}$.[8] The relationship between $\mathcal{H}$ and a decision situation will be specified later. For technical reasons related to the probabilistic content of the state, each state vector has to be of length one, that is, $\langle\psi|\psi\rangle^2 = 1$ (where $\langle\cdot|\cdot\rangle$ denotes the inner product in $\mathcal{H}$). So all vectors of the form $\lambda\,|\psi\rangle$, where $\lambda \in \mathbb{R}$, correspond to the same state, which we represent by a vector of length one.

A key ingredient in the formalism of indeterminacy is *the principle of superposition.* This principle states that the linear combination of any two states is itself a possible state.[9] Consider two states $|\varphi_1\rangle, |\varphi_2\rangle \in \mathcal{H}$. If $|\psi\rangle = \lambda_1|\varphi_1\rangle + \lambda_2|\varphi_2\rangle$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ then $|\psi\rangle \in \mathcal{H}$. The principle of superposition implies that, unlike the Harsanyi type space, the state space is non-Boolean.[10]

2.2. The notion of measurement and of observable

Measurement is a central notion in our framework. A measurement is an operation (or an experiment) performed on a system. It yields a result, the outcome of the measurement. A defining feature of a measurement is the so-called first-kindness property.[11] It refers to the fact that if one performs a measurement on a system and obtain**s** a result, then one will get the same result if one performs again that measurement on the same system immediately afterwards. Thus, the outcome of a first-kind measurement is reproducible but only in a next subsequent measurement. First-kindness does not entail that the first outcome is obtained when repeating a measurement if other measurements were performed on the system in between. Whether an operation is a measurement (i.e., is endowed with the property of first-kindness) is an empirical issue. When it comes to decision theory, it means that we do not, a priori, assume that any choice set can be used to measure preferences. In particular, the product set of two sets, each of which is associated with a first-kind measurement, is not in general associated with a first-kind measurement. The set of decision problems that we consider consists exclusively of decision problems that can be associated with first-kind measurements. We call these decision problems Decision Situations (*DS*).[12]

A Decision Situation A can be thought of as an experimental setup where the agent is invited to choose a particular action among all the possible actions allowed by this Decision Situation. In this paper, we will consider only the case of finitely many possible outcomes. They will be labeled from 1 to n by convention. When an agent selects an action, we say that she "plays" the *DS*. To every Decision Situation A, we will associate an *observable,* namely, a specific symmetric operator on $\mathcal{H}$, which, for notational simplicity, we also denote by A.[13] If we consider only one Decision Situation A with n possible outcomes, we can assume that the associated Hilbert space is n-dimensional and that the eigenvectors of the corresponding observable, which we denote by $|1_A\rangle, |2_A\rangle, \ldots, |n_A\rangle$, all correspond to different eigenvalues, denoted by $1_A, 2_A, \ldots, n_A$ respectively. By convention, the eigenvalue i_A will be associated with choice i.

$$A\,|k_A\rangle = k_A\,|k_A\rangle\,, \quad k = 1, \ldots, n.$$

As A is symmetric, there is a unique orthonormal basis of the relevant Hilbert space $\mathcal{H}$ formed with its eigenvectors. The basis $\{|1_A\rangle, |2_A\rangle, \ldots, |n_A\rangle\}$ is the unique orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. It is thus possible to represent the agent's state as a superposition of vectors of this basis:

$$|\psi\rangle = \sum_{k=1}^{n} \lambda_k\,|k_A\rangle\,, \tag{1}$$

where $\lambda_k \in \mathbb{R}, \forall k \in \{1, \ldots, n\}$ and $\sum_{k=1}^{n} \lambda_k^2 = 1$.

The Hilbert space can be decomposed as follows:

$$\mathcal{H} = \mathcal{H}_{1_A} \oplus, \ldots, \oplus \mathcal{H}_{n_A}, \qquad \mathcal{H}_{i_A} \perp \mathcal{H}_{j_A}, \quad i \neq j, \tag{2}$$

where $\oplus$ denotes the direct sum of the subspaces $\mathcal{H}_{1_A}, \ldots, \mathcal{H}_{n_A}$ spanned by $|1_A\rangle, \ldots, |n_A\rangle$ respectively.[14] Or, equivalently, we can write $I_{\mathcal{H}} = P_{1_A} +, \ldots, + P_{n_A}$ where P_{i_A} is the projection operator on $\mathcal{H}_{i_A}$ and $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$.

A Decision Situation A is an experimental setup and the actual implementation of the experiment is represented by a *measurement* of the associated observable A. According to the so-called Reduction Principle (see Appendix), the result of such a measurement can only be one of the n eigenvalues of A. If the

[7] All information (beliefs) and strategic considerations are embedded in the definition of the choices. Thus an agent's play of cooperation in a Prisoner's Dilemma is a play of cooperation given his information (knowledge) about the opponent.

[8] In Quantum Mechanics the field that is used is the complex numbers field. However, for our purposes the field of real numbers provides the structure and the properties needed (see e.g. Beltrametti and Cassinelli (1981) and Holland (1995)). Everything we present in the Appendix (Elements of quantum mechanics) remains true when we replace Hermitian operators with real symmetric operators.

[9] We use the term state to refer to "pure state". Some people use the term state to refer to mixture of pure states. A mixture of pure states combines indeterminacy with elements of incomplete information. They are represented by the so-called density operators.

[10] The distributivity condition defining a Boolean space is dropped for a weaker condition called orthomodularity. The basic structure of the state space is that of a logic, i.e., an orthomodular lattice. For a good presentation of Quantum Logic, a concept was introduced by Birkhoff and von Neuman (1936), and further developed by Mackey (2004), see Cohen (1989).

[11] The term first-kind measurement was introduced by Pauli.

[12] Even standard decision theory implicitly restricts its application to decision problems that satisfy the first-kindness property (or that can be derived from such decision problems). In contrast, random utility models do not require choice behavior to satisfy the first-kindness property in the formulation used in this paper.

[13] Observables in Physics are represented by Hermitian operators because QM is defined over the field of complex numbers. Here, we confine ourselves to the field of real numbers, which is why observables are represented by symmetric operators.

[14] That is, for $i \neq j$ any vector in $\mathcal{H}_{i_A}$ is orthogonal to any vector in $\mathcal{H}_{j_A}$ and any vector in $\mathcal{H}$ is a sum of n vectors, one in each component space.

result is m_A (i.e., the player selects action m) the superposition $\sum \lambda_i |i_A\rangle$ "collapses" onto the eigenvector associated with the eigenvalue m_A. The initial state $|\psi\rangle$ is projected into the subspace $\mathcal{H}_{m_A}$ (of eigenvectors of A with eigenvalue m_A). The probability that the measurement yields the result m_A is equal to $\langle m_A|\psi\rangle^2 = \lambda_m^2$, i.e., the square of the corresponding coefficient in the superposition. The coefficients themselves are called 'amplitudes of probability'. They play a key role when studying sequences of measurements (see Section 2.3). As usual, we interpret the probability of m_A either as the probability that one agent in state $|\psi\rangle$ selects action m_A or as the proportion of the agents who will make the choice m_A in a population of many agents, all in the state $|\psi\rangle$.

In our theory, an agent is represented by a state. We shall also use the term type (and eigentype) to denote a state corresponding to one eigenvector, say $|m_A\rangle$. An agent in this state is said to be of type m_A. An agent in a general state $|\psi\rangle$ can be expressed as a superposition of all eigentypes of the *DS* under consideration. Our notion of type is closely related to the notion introduced by Harsanyi. Consider a simple choice situation e.g., when an employee faces a menu of contracts. The type captures all the agent's characteristics (taste, subjective beliefs, private information) of relevance for uniquely predicting the agent's behavior. In contrast to Harsanyi, we shall not assume that there exists an exhaustive description of the agent that enables us to determine the agent's choice uniquely and *simultaneously* in all possible Decision Situations. Instead, our types are characterized by an irreducible uncertainty that is revealed when the agent is confronted with a sequence of *DS* (see Section 2.3.2 for a formal characterization of irreducible uncertainty).

Remark. Clearly, when only one *DS* is considered, the above description is equivalent to the traditional probabilistic representation of an agent by a probability vector $(\alpha_{1,\ldots,}\alpha_n)$ in which α_k is the probability that the agent will choose action k_A and $\alpha_k = \lambda_k^2$ for $k = 1, \ldots, n$. The advantage of the proposed formalism consists in enabling us to study several decision situations and the interaction between them.

2.3. More than one Decision Situation

When studying more than one *DS*, say A and B, the key question is whether the corresponding observables are commuting operators in $\mathcal{H}$, i.e., whether $AB = BA$. Whether two *DS* can be represented by two commuting operators or not is an empirical issue. We next study its mathematical implications.

2.3.1. Commuting Decision Situations

Let A and B be two *DS*. If the corresponding observables commute then there is an orthonormal basis of the relevant Hilbert space $\mathcal{H}$ formed by eigenvectors common to both A and B. Denote by $|i\rangle$ (for $i = 1, \ldots, n$) these basis vectors. We have

$$A|i\rangle = i_A|i\rangle \quad \text{and} \quad B|i\rangle = i_B|i\rangle\,.$$

In general, the eigenvalues can be degenerated (i.e., for some i and j, $i_A = j_A$ or $i_B = j_B$).[15] Any normalized vector $|\psi\rangle$ of $\mathcal{H}$ can be written in this basis:

$$|\psi\rangle = \sum_i \lambda_i |i\rangle\,,$$

where $\lambda_i \in \mathbb{R}$, and $\sum_i \lambda_i^2 = 1$. If we measure A first, we observe eigenvalue i_A with probability

$$p_A(i_A) = \sum_{j;j_A=i_A} \lambda_j^2. \tag{3}$$

If we measure B first, we observe eigenvalue j_B with probability $p_B(j_B) = \sum_{k;k_B=j_B} \lambda_k^2$. After B is measured and the result j_B is obtained, the state $|\psi\rangle$ is projected into the eigensubspace $\mathcal{E}_{j_B}$ spanned by the eigenvectors of B associated with j_B. More specifically, it collapses onto the state:

$$|\psi_{j_B}\rangle = \frac{1}{\sqrt{\sum\limits_{k;k_B=j_B} \lambda_k^2}} \sum_{k;k_B=j_B} \lambda_k |k\rangle$$

(the factor $\frac{1}{\sqrt{\sum_{k;k_B=j_B}\lambda_k^2}}$ is necessary to make $|\psi_{j_B}\rangle$ a unit vector).

When we measure A on the agent in the state $|\psi_{j_B}\rangle$, we obtain i_A with probability

$$p_A(i_A|j_B) = \frac{1}{\sum\limits_{k;k_B=j_B} \lambda_k^2} \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} \lambda_k^2.$$

So when we measure first B and then A, the probability of observing the eigenvalue i_A is $p_{AB}(i_A) = \sum_j p_B(j_B)\, p_A(i_A|j_B)$:

$$\begin{aligned} p_{AB}(i_A) &= \sum_j \frac{1}{\sum\limits_{k;k_B=j_B} |\lambda_k|^2} \sum_{k;k_B=j_B} \lambda_k^2 \sum_{\substack{l;l_B=j_B \\ \text{and } l_A=i_A}} \lambda_l^2 \\ &= \sum_{j_B} \sum_{\substack{l;l_B=j_B \\ \text{and } l_A=i_A}} \lambda_l^2 = \sum_{l;l_A=i_A} \lambda_l^2. \end{aligned}$$

Hence, $p_{AB}(i_A) = p_A(i_A)$, $\forall i$, and similarly $p_{BA}(j_B) = p_B(j_B)$, $\forall j$.

When dealing with commuting observables, it is meaningful to speak of measuring them simultaneously. Whether we measure first A and then B or first B and then A, the probability distribution on the joint outcome is $p(i_A \wedge j_B) = \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} \lambda_k^2$, so (i_A, j_B) is a well-defined event. Formally, this implies that the two *DS* can be merged into a single *DS*. When we measure it, we obtain a vector as the outcome, i.e., a value in A and a value in B. To each eigenvalue of the merged observable, we associate a type that captures all the characteristics of the agent relevant to her choices (one in each *DS*).

Remark. Note that, as in the case of a single *DS*, for two such commuting *DS* our model is equivalent to a standard (discrete) probability space in which the elementary events are $\{(i_A, j_B)\}$ and $p(i_A \wedge j_B) = \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} \lambda_k^2$. In particular, in accordance with the calculus of probability we see that the conditional probability formula holds:

$$p_{AB}(i_A \wedge j_B) = p_A(i_A) p_B(j_B|i_A).$$

This also means that the type space associated with type characteristics represented by commuting observables is equivalent to the Harsanyi type space. When all *DS* commute, a Type-Indeterminate (TI) agent cannot be distinguished from a classical agent. In particular, if the Decision Situations A and B together provide a full characterization of the agent, then all types $i_A j_B$ are mutually exclusive: knowing that the agent is of type $1_A 2_B$ it is certain that she is *not* of type $i_A j_B$ for $i \neq 1$ and/or $j \neq 2$. All uncertainty about the agent's choice behavior is due to our incomplete knowledge about her type, and it can be fully resolved by making a series of suitable measurements.

[15] In the argument that we develop in Section 3.1, the pure states are linear orders — therefore choice experiments are observables with degenerated eigenvalues.

As an example, consider the following two Decision Situations. Let A be the Decision Situation presenting a choice between a week's vacation in Tunisia and a week's vacation in Italy. And let B be the choice between buying 1000 euros of shares of Bouygues Telecom or of Deutsche Telecom. It is quite plausible that A and B commute, but whether or not this is the case is, of course, an empirical question. If A and B commute, we expect that a decision on portfolio (B) will not affect the decision-making concerning the vacation (A). And thus the order in which the decisions are made does not matter, as in the classical model.

Note, finally, that the commutativity of the observables does not exclude statistical correlations between observations. To see this, consider the following example in which A and B each have two degenerated eigenvalues in a four-dimensional Hilbert space. Denote by $|i_A j_B\rangle$ $(i = 1, 2, j = 1, 2)$ the eigenvector associated with eigenvalues i_A of A and j_B of B, and let the state $|\psi\rangle$ be given by

$$|\psi\rangle = \sqrt{\frac{3}{8}}\,|1_A 1_B\rangle + \sqrt{\frac{1}{8}}\,|1_A 2_B\rangle + \sqrt{\frac{1}{8}}\,|2_A 1_B\rangle + \sqrt{\frac{3}{8}}\,|2_A 2_B\rangle$$

Then, $p_A(1_A|1_B) = \frac{\frac{3}{8}}{\frac{3}{8}+\frac{1}{8}} = \frac{3}{4}$, $p_A(2_A|1_B) = \frac{\frac{1}{8}}{\frac{3}{8}+\frac{1}{8}} = \frac{1}{4}$.

So if we first measure B and find, say, 1_B, it is much more likely (with probability $\frac{3}{4}$) that when measuring A we will find 1_A rather than 2_A (with probability $\frac{1}{4}$). But the two *interactions* (measurements) do not affect each other; i.e., the distribution of the outcomes of the measurement of A is the same whether or not we measure B first.

2.3.2. Non-commuting Decision Situations

It is when we consider Decision Situations associated with observables that do not commute that the predictions of our model differ from those of the probabilistic one. In such a context, the quantum probability calculus ($p(i_A|\psi) = \langle i_A|\psi\rangle^2$) generates cross-terms, also called interference terms. These cross-terms are the signature of indeterminacy. In the next section, we demonstrate how this feature can capture the phenomenon of cognitive dissonance, as well as that of framing.

Consider two Decision Situations A and B with the same number n of possible choices. We shall assume for simplicity that the corresponding observables A and B have non-degenerated eigenvalues $1_A, 2_A, \ldots, n_A$ and $1_B, 2_B, \ldots, n_B$ respectively. Each set of eigenvectors $\{|1_A\rangle, |2_A\rangle, \ldots, |n_A\rangle\}$ and $\{|1_B\rangle, |2_B\rangle, \ldots, |n_B\rangle\}$ is an orthonormal basis of the relevant Hilbert space. Let $|\psi\rangle$ be the initial state of the agent

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |i_A\rangle = \sum_{j=1}^{n} \nu_j\, |j_B\rangle\,. \tag{4}$$

We note that each set of eigenvectors of the respective observables forms a basis of the state space. The multiplicity of alternative bases is a distinguishing feature of this formalism. It implies that there is no single or privileged way to describe (express) the type of the agent. Instead, there exists a multiplicity of equally informative alternative ways to characterize the agent. We shall now compute the probability for type i_A under two different scenarios. In the first scenario, we measure A on the agent in state $|\psi\rangle$. In the second scenario, we first measure B on the agent in state $|\psi\rangle$ and thereafter measure A on the agent in the state resulting from the first measurement. We can write observable B's eigenvector $|j_B\rangle$ in the basis made of A's eigenvectors:

$$|j_B\rangle = \sum_{i=1}^{n} \mu_{ij} |i_A\rangle. \tag{5}$$

Using the expression above we write the last term in Eq. (4)

$$|\psi\rangle = \sum_{j=1}^{n}\sum_{i=1}^{n} \nu_j \mu_{ij} |i_A\rangle. \tag{6}$$

From expression (6), we derive the probability $p_A(i_A)$ that the agent chooses i_A in the first scenario: $p_A(i_A) = \left(\sum_{j=1}^{n} \nu_j \mu_{ij}\right)^2$. In the second scenario, she first plays B. By (4) we see that she selects action j_B with probability ν_j^2. The state $|\psi\rangle$ is then projected onto $|j_B\rangle$. When the state is $|j_B\rangle$, the probability for i_A is given by (5), namely, μ_{ij}^2. Summing up the conditionals, we obtain the (ex-ante) probability for i_A when the agent first plays B and then A : $p_{AB}(i_A) = \sum_{j=1}^{n} \nu_j^2 \mu_{ij}^2$, which is, in general, different from $p_A(i_A) = \left(\sum_{j=1}^{n} \nu_j \mu_{ij}\right)^2$. Playing first B changes the way A is played. The difference stems from the so-called interference terms

$$p_A(i_A) = \left(\sum_{j=1}^{n} \nu_j \mu_{ij}\right)^2 = \sum_{j=1}^{n} \nu_j^2 \mu_{ij}^2 + \underbrace{2\sum_{j\neq j'} \left[\left(\nu_{j'}\mu_{ij}\right)\left(\nu_j \mu_{ij'}\right)\right]}_{\text{Interference term}}$$

$$= p_{AB}(i_A) + \text{interference term}$$

The interference term is the sum of cross-terms involving the amplitudes of probability (the Appendix provides a description of interference effects in Physics).

Some intuition about interference effects may be provided using the concept of "propensity" due to Popper (1992). Imagine an agent's mind as a system of propensities to act (corresponding to different possible actions). As long as the agent is not required to choose an action in a given *DS*, the corresponding propensities coexist in her mind; the agent has not "made up her mind". A decision situation operates on this state of "hesitation" to trigger the emergence of a single type (of behavior). But as long as alternative propensities are present in the agent's mind, they affect choice behavior by increasing or decreasing the probability of the choices in the *DS* under investigation.

An illustration of this kind of situation may be supplied by the experiment reported in Knetz and Camerer (2000). The two *DS* they study are the Weak Link (WL) game and the Prisoner's Dilemma (PD) game.[16] They compare the distribution of choices in the Prisoner's Dilemma (PD) game when it is preceded by a Weak Link (WL) game and when only the PD game is being played. Their results show that playing the WL game affects the play of individuals in the PD game. The authors appeal to an informational argument, which they call the "precedent effect".[17] However, they cannot explain the high rate of cooperation (37.5%) in the last round of the PD game (Table 5, p. 206). Instead, we propose that the WL and the PD are two *DS* that do not commute. In such a case, we expect a difference in the distributions of choices in the (last round of the) PD, depending on whether or not it was preceded by a play of the WL or another PD game. This is because the type of the agent is being modified by the play of the WL game.

Remark. In the case where A and B do not commute, they cannot have simultaneously defined values: the state of the agent is characterized by an *irreducible uncertainty*. Therefore, and in contrast with the commuting case, two non-commuting

16 The Weak Link game is a type of coordination game where each player picks an action from a set of integers. The payoffs are defined in such a manner that each player wants to select the minimum of the other players but everyone wants that minimum to be as high as possible.

17 The precedent effect hypothesis is as follows: "The shared experience of playing the efficient equilibrium in the WL game creates a precedent of efficient play strong enough to (...) lead to cooperation in a finitely repeated PD game"; see Knetz and Camerer (2000, p. 206).

observables cannot be merged into one single observable. There is no probability distribution on the events of the type "to have the value i_A for A and the value j_B for B". The conditional probability formula, derived from the law of total probability, does not hold:

$$p_A(i_A) \neq \sum_{j=1}^{n} p_B(j_B)p(i_A|j_B).$$

Indeed, $\left(\sum_{j=1}^{n} \nu_j \mu_{ij}\right)^2 = p_A(i_A) \neq \sum_{j=1}^{n} p_B(j_B)p(i_A|j_B) = \sum_{j=1}^{n} \nu_j^2 \mu_{ij}^2$.

Consequently, the choice experiment consisting of asking the agent to select a pair (i_A, j_B) out of the set of alternatives $A \times B$ is NOT a *DS*. The agent cannot simultaneously choose i_A and j_B. Of course, he can make the two choices in some ordered sequence, but such an experiment cannot be represented by a single *DS*.

We must here acknowledge a fundamental distinction between the type space of our TI-model (Type Indeterminacy model) and that of Harsanyi. In the Harsanyi type space the (pure) types are all mutually exclusive: the agent is either of type θ_i *or* of type θ_j, but she cannot be both. In the TI-model, this is not always the case. When dealing with (complete) non-commuting *DS*,[18] the types associated with respective *DS* are not mutually exclusive: knowing that the agent is of type 1_A, which is a full description of her type, it cannot be said that she is *not* of type 1_B. The eigentypes of non-commuting *DS* are "connected" in the sense that the agent can transit from one type to another under the impact of a measurement. When making a measurement of B on the agent of type 1_A, she is projected onto one of the eigenvectors of *DSB*. Her type changes from being 1_A to being some j_B.

3. The type indeterminacy model and social sciences

The theory of choice presented in this paper does not allow for a straightforward comparison with standard choice theory. Significant further elaboration is required. Yet, some implications of the type indeterminacy approach can be explored. First, we shall be interested in comparing the behavior of an agent of indeterminate type with that of a classical agent in the case where both behaviors satisfy the Weak Axiom of Revealed Preference (WARP)[19] which is a basic axiom of rational choice. Then, we will show that this framework can be used to explain two instances of behavioral anomalies that have been extensively studied in the literature.

3.1. *The TI-model and the classical rational man*

In standard decision theory, it is assumed that an individual has preferences (i.e., a complete ranking or a complete linear ordering) on the universal set of alternatives X. The individual knows her preferences, while the outsiders may not know them. But it is also possible that the individual only knows what she would choose from some limited sets of alternatives and not from the whole set X. Thus, a less demanding point of view consists in representing the choice behavior by a choice structure (i.e., a family $\mathcal{B}$ of subsets of the universal set of alternatives X and a choice rule C that assigns a nonempty set of chosen elements $C(A)$ for all $A \in \mathcal{B}$). The link between the two points of view is well known. From preferences, it is always possible to build a choice structure, but the reverse is not always true. For it to be true, the choice structure must display a certain amount of consistency (satisfying the Weak Axiom of Revealed Preference) and the family $\mathcal{B}$ must include all subsets of X of up to three elements.[20] So, preferences can be revealed by asking the individual to make several choices from subsets of X. How does this simple scheme change when we are dealing with an individual whose type is indeterminate?

Consider a situation where an individual is invited to make a choice of one item out of a set A, $A \subseteq X$. If this experiment satisfies the first-kindness property, we can consider it to be a measurement represented by an observable. The set of possible outcomes of this experiment is the set A. We also denote the observable by A.[21]

We make two key assumptions on the individual choice behavior:

A1. Choices out of a "small" subset satisfy the first-kindness property (the meaning of "small" will be made precise).

A2. Choices out of a "small" subset respect our Weak Axiom of Revealed Preference (WARP'; see below).

Assumption A1 means that "small"subsets are associated with *DS*, i.e., the experiment consisting of letting an individual choose an item out of a "small" subset of items is a measurement. Assumption A2 means that choices from "small" subsets are rational. The idea behind these assumptions is that an individual can, in her mind, structure any "small" set of alternatives, i.e., simultaneously compare those alternatives. She may not be able to do that within a "big" set, though. But this does not mean that our individual cannot make a choice from a "big" set. For example, she might use an appropriate sequence of binary comparisons and select the last winning alternative. However, such a compound operator would not in general satisfy the first-kindness property, i.e., there may not exist any *DS* representing such an operation.

A standard formulation of the WARP can be found in Mas-Colell, Whinston, and Green (1995).[22] We shall use a stronger version by assuming that $C(B)$, for any $B \in \mathcal{B}$, is a singleton.[23] For the purpose of discussing choice experiments that may not commute, it is useful to formulate the axiom in two parts:

Consider two subsets $B, B' \in \mathcal{B}$ such that $B \subset B'$.

(a) Let $x, y \in B$, $x \neq y$ with $x \in C(B)$ then $y \notin C\left(B'\right)$.

(b) Let $C\left(B'\right) \cap B \neq \varnothing$ then $C(B) = C\left(B'\right)$.

The intuition for (a) is that as we enlarge the set of items from B to B' a rational decision-maker never chooses from B' an item that is available in B but was not chosen. The intuition for (b) is that as we reduce the set of alternatives, an item chosen in the large set is also chosen in the smaller set containing that item, and no item previously not chosen becomes chosen.[24]

It can be shown that, in the classical context, (b) implies (a) (see Arrow (1959)). In our context where choice experiments can be non-commuting, we may not have such an implication. Moreover, the notion of choice function is not appropriate because it implicitly assumes the commutativity of choice (see below). Since our purpose is to investigate this issue explicitly, we express the axiom more immediately in terms of (observable) choice behavior in the following way, which we call WARP':

Consider two subsets $B, B' \in \mathcal{B}$ with $B \subset B'$.

(1) Suppose the agent chooses from B some element x. In a next subsequent measurement of B' the outcome of the choice is not in $B \setminus \{x\}$.

18 We say that a DS is complete when its outcome provides a complete characterization of the agent.

19 Samuelson (1947).

20 See for example Mas-Colell, Whinston, and Green, Microeconomic Theory, p.13.

21 The use of the same symbol for sets of items and observables should not confuse the reader. Either the context unambiguously points to the right interpretation, or we make it precise.

22 "If for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C\left(B'\right)$, we must also have $x \in C\left(B'\right)$". (Mas-Colell et al., 1995, p.10).

23 At this stage of the research we do not want to deal with indifference relations.

24 Conditions (a) and (b) are equivalent to C2 and C4 in Arrow (1959).

(2) Suppose the agent chooses from B' some element x that belongs to B. In a next subsequent measurement of B the agent's choice is x.

Points (1) and (2) capture the classical intuitions about rational choice behavior associated with (a) and (b) above. A distinction with the standard formulations of WARP is that we do not refer to a choice function but to choice behavior, and that our axiom only applies to two subsequent choices. We show below that in the context where choices commute WARP' is equivalent to WARP. But, as we shall see, the equivalence does not hold if we allow for some choices not to commute.

We now investigate the choice behavior of a type indeterminate agent under assumptions A1 and A2 above, i.e., under the assumption that choices from small subsets satisfy the property of first-kindness and that they respect WARP'. We consider, in turn, two cases. In the first one, we assume that all the *DS* considered pairwise commute. In the second one, we allow for non-commuting *DS*. In the following, we define "small" subsets as subsets of size 3 or less.

Case 1

The assumption here is that all experiments of choice from small subsets are compatible with each other, i.e., the corresponding observables commute. A first implication of the commutativity of choice experiments is that for any $B \subseteq X$, the outcome of the choice experiment B only depends on B (whatever was done before the choice in B will always be the same and, in particular, the outcome does not depend on other choice experiments that were performed before). Consider now a situation where the agent has made a choice in each one of the possible subsets. Then we know, appealing to the first-kindness property, that all subsequent choices are deterministic.[25] Therefore, we can express the outcome in terms of *a choice function* $C(X) : B \to x$ that associates to each $B \in \mathcal{B}$ a chosen item $x \in B$. A second consequence of commutativity is that the restriction of WARP' to two subsequent choices is inconsequential. This is because the outcome of any given choice experiment must be the same in any series of two consecutive experiments. In particular, WARP' implies the transitivity of choices. This can easily be seen when we take an example on a subset of three items and perform choice experiments on pairs in different orders.[26] So for the case where choice experiments commute, WARP' is equivalent to WARP and we recover the standard results. We know that if $(\mathcal{B}, C(.))$ is a choice structure satisfying WARP' and defined for $\mathcal{B}$ including all subsets of X of up to three elements, then there exists a rational preference relation that rationalizes choice behavior.[27] It is therefore natural to identify the states (types) of the individual with the preference orders that rank all the elements of X. The type space $\mathcal{H}$ has dimension $|X|!$. We obtain the well-known classical model. Thus, we make the following statement.

Statement 1. When all *DS* associated with subsets of up to 3 items commute, WARP' is equivalent to the standard Weak Axiom of Revealed Preferences. The type indeterminate agent's choice behavior is not distinguishable from that of a classical agent's.

This should not surprise us because we know from Section 2.3.1 that when all *DS* commute, a quantum (non-classical) system cannot be distinguished from a classical system (see Danilov and Lambert-Mogiliansky (2008), for a general equivalence result in a similar context). Here it means that the agent's preferences induce a deterministic choice behavior. For the case where all choice experiments on small subsets commute, a probabilistic representation of the TI-agent's choice behavior reflects our incomplete information, as opposed to an intrinsic indeterminacy or randomness.

In this model, all *DS* are coarse measurements of the type (the preference order), i.e., their outcomes are degenerated eigenvalues (see Section 2.3.1). When all *DS* commute and satisfy WARP', the type indeterminate rational agent behaves, in all respects, as like a classical rational agent. This should not surprise us since we know that when observables commute they can be merged into a single observable, and the TI-model is equivalent to the classical model.

Case 2

We now consider a case where some *DS* do not commute. We need to emphasize that, in contrast with the commuting case, a large number of non-commuting models are possible. The models differ from each other according to which *DS* commute and which do not. We investigate here a simple example that illustrates interesting issues.

Assume the set X consists of four items: a, b, c, and d. As before, any subset $A \subset X$ consisting of three elements or less is associated with a *DS* and we assume that any two consecutive choices made from small subsets respect WARP'. There exists a multiplicity of non-commuting models of this example. In order to formulate the two properties that characterize our non-commuting model, we define A_C a "contextual" choice experiment with $A \subseteq C \subseteq X$. The set C is the context in the sense of being the set of all the alternatives that are present. And A is the set from which the choice can be made. For instance, $ab_{\{a,b,c\}}$ (for ease of presentation a choice out of $\{a, b\}$ is denoted ab) is the choice experiment where the agent can choose between a and b in a context where the alternative c is present but is not available for choice. Our non-commuting model is characterized by the two following properties:

(nc) i. Let A_C and B_C be two *DS* with $|C| = 3$; then A_C and B_C commute.

(nc) ii. Let A_C and $B_{C'}$ be two *DS* with $|C| = \left|C'\right| = 3$ and $C \neq C'$; then A_C and $B_{C'}$ do not commute.[28]

Property i means that we may perform choice experiments in pairs and in triples, defined on the same context so as to elicit a preference relation on each triple taken separately. Property ii means that *DS* associated with different triples (e.g., $abc_{\{a,b,c\}}$ and $bcd_{\{b,c,d\}}$) are represented by observables that do not commute with each other. This implies that the agent does not have a preference order over the whole X. Instead, the types can be identified with preference orders on triples. Observables representing choice experiments on different triples are alternative ways to measure the individual's preferences. The type space representing the individual is a six-dimensional Hilbert space corresponding to the six ways to rank three items. There are four alternative bases spanned by the eigenvectors corresponding to the six possible rankings in each one of the four triples.

Requiring the satisfaction of WARP', *which is not formulated contextually*, imposes a certain amount of consistency in behavior

25 Before these choice experiments are conducted, the agent will generally be in a state of superposition, i.e., with no well-defined choice function with respect to the three items under consideration.

26 Consider the choice set $\{a, b, c\}$. We conduct the following two series of experiments with the same individual. In the first series, we let the agent choose from $\{a, b\}$, then from $\{b, c\}$, and last from $\{a, b, c\}$. In the second series, the agent first chooses from $\{b, c\}$, then from $\{a, b\}$, and last from $\{a, b, c\}$. Assume the outcomes of the two first experiments are a and b, then by WRAP' the third choice may be either a or b. The first two choices are not deterministic but thereafter, because of commutativity, all the other choices are deterministic. In the second series, the outcomes must be b and a respectively. So the third choice may be either a or c. By commutativity, the outcome of $\{a, b, c\}$ must be the same in the two series, which uniquely selects a. If then confronted with $\{a, c\}$ by WRAP', the agent chooses a. So a choice behavior respecting WARP' is transitive when choices commute.

27 See, e.g., Mas-Colell et al. (1995, p.13).

28 It is possible to show that properties (i) and (ii) can be combined in a model (see Zwirn (in press)).

across triples.[29] So, for instance, consider point 1 in the definition of WARP' above and let $B = \{a, b\}$ and $B' = \{a, b, c\}$ or $\{a, b, d\}$. Then WARP' says that if a is selected in the choice experiment $ab_{\{a,b,c\}}$ then, in a consecutive choice experiment $abc_{\{a,b,c\}}$, b is not chosen. Note that this already follows from nc(i), i.e, from the commutativity of choice experiments associated with one and the same context. But WARP' also says that the element b cannot be chosen in a consecutive experiment $abd_{\{a,b,d\}}$, which is an additional requirement that links two experiments associated with different contexts, namely, $ab_{\{a,b,c\}}$ and $abd_{\{a,b,d\}}$. The same must hold for any other pairs and associated triples.

Nevertheless, in our model the agent may violate transitivity. Assume the agent first chooses a in $ab_{\{a,b,c\}}$ and b in $bc_{\{a,b,c\}}$. We now ask her to make a choice in $bcd_{\{b,c,d\}}$. She picks d. Finally, we let her make a choice in $abc_{\{a,b,c\}}$. She picks c. This may happen because the type resulting from the first two choice experiments is modified by the performance of the $bcd_{\{b,c,d\}}$ choice experiment. Recall that the satisfaction of WARP' is only required for two consecutive measurements.

Another interesting feature that may arise in our model is the violation of the principle of Independence of Irrelevant Alternatives (IIA). [30] This happens because WARP' is formulated for small subsets. While this is sufficient in a world of classical decision-makers where WARP' implies IIA (see Case 1), it is no longer so in a world of type indeterminate decision-makers, as we next illustrate.

In our model, the agent only has preferences over triples of items. When invited to make a choice out of the whole set $\{a, b, c, d\}$, the individual must proceed by making a series of measurements (no single DS exists corresponding to $abcd$). For instance, she first selects from a pair, followed by a choice from a triple consisting of the first selected item and the two remaining ones. We can call such a behavior "procedural rational" because she acts as if she had preferences over the four items.

Consider the following possible line of events. Suppose the individual just made a choice in $acd_{\{a,c,d\}}$ and picked d. Thus her initial state is some superposition of type $[d > a > c]$ and type $[d > c > a]$. We now ask her to choose from $\{a, b, c, d\}$. Assume the individual first plays $ab_{\{a,b,c\}}$ then $acd_{\{b,c,d\}}$ (which by assumption of procedural rationality means that the outcome of $ab_{\{a,b,c\}}$ is a).[31] The choice of a from $ab_{\{a,b,c\}}$ changes the type of the individual. The new type, expressed in terms of (the eigenvectors representing) the preference orders on $\{a, b, c\}$, is a superposition of types $[a > b > c]$, $[a > c > b]$, and $[c > a > b]$. In order to complete her choice out of $\{a, b, c, d\}$, she now plays acd. The result of that last measurement (performed on the individual of the type resulting from the $ab_{\{a,b,c\}}$ measurement) is c with positive probability. But this violates IIA. She effectively selects c from $\{a, b, c, d\}$ while she initially picked d in $\{a, c, d\}$ where c was available (i.e., adding the irrelevant alternative b upsets the preference order between c and d). Yet, it is easy to check that, in this example, any two consecutive choices satisfy the WARP', so the choice behavior of our type indeterminate agent is "rational".

Statement 2. Under assumption (nc) a type indeterminate agent whose choice behavior satisfies WRAP' does not have a preference order over the universal set X. But she may have well-behaved preferences over subsets of X. Generally, she does not behave as a classical rational agent.

In this example, the distinction between the classical rational and the type indeterminate rational agent is only due to the non-commutativity of DS associated with different subsets of items. This distinctive feature of the TI-model of choice (i.e., the non-commutativity of some choice experiments) delivers a formulation of bounded rationality in terms of the impossibility of comparing and ordering all items *simultaneously*. Non-commutativity also implies that, as the agent makes a choice, her type (preferences) changes. The preferences of a type indeterminate agent are shaped in the process of elicitation, as proposed by Kahneman and Tversky (2000)[32].

TI-rationality Using the examples developed above, we can contribute to the discussion on the meaning of rationality in the case of a TI-agent. The essence of rational behavior in decision theory is that the agent makes her choices in order to maximize her preferences. In this section we have "operationalized" rational behavior as a behavior that satisfies WARP'. As we saw WARP' is equivalent to the standard weak axiom of revealed preferences for the case, all DS commute. But we also found that WARP' does not guarantee the existence of a preference order over the universal set X when some DS do not commute. With non-commuting DS two things happen: first the choice is random, second there is an effect of choice-making on the preferences themselves, i.e., they change. In order to be able to talk about "TI-rationality" we must impose requirements on the choice behavior in non-commuting DS and on the changes in preferences that occur as the result of choice-making. We do not intend to formulate these requirements explicitly. Instead we suggest a way to proceed.

i. The random character of the choices between incompatible DS suggests that stochastic rationality (see McFadden (1999)) may be a suitable concept. Stochastic rationality only supposes the stationarity of preferences in the population. A similar feature arises in the TI-model, which assumes stable (probabilistic) correlations between the eigentypes (preferences) over incompatible choice sets (see Section 2.3.2). An axiom of revealed stochastic preferences in the spirit of McFadden (2004) could be formulated. Such an axiom would be rather constraining because choice behavior must be consistent with a specific correlation matrix, not any stable distribution as in McFadden.

ii. Once the change of type has occurred as a consequence of choice-making, TI-rationality can be tested on the choice behavior in the new choice set. It should satisfy WARP'. The choice with respect to the previous (incompatible) choice set must be consistent with the same correlation matrix as the one governing stochastic rationality in (i) above.

We thus want to argue that it is possible to formulate a meaningful concept of TI-rationality. This concept would combine elements of standard (classical) rationality with elements of stochastic rationality depending on the context, i.e., whether the DS commute or not. In line with Quantum Mechanics, the TI-model assumes that whether DS commute or not is a property of the DS themselves, not of the measured systems. Therefore, TI-rationality can be formulated as a well-defined property of the individual choice behavior. We are aware that TI-rationality may be rather difficult to test empirically.[33]

29 We have not formally proved that WARP' and properties (i) and (ii) are compatible. But the compatibility of WARP' and non-commutativity has been proved in a similar setting in Zwirn (in press). We also know that it holds for this example in a more general context (see Sect. 4.4 in Danilov and Lambert-Mogiliansky (2008)).

30 This principle says that if a is chosen in $\{a, b, c\}$ and in particular $a \succ b$, then adding, say, inferior alternative d cannot lead to the choice of b out of $\{a, b, c, d\}$.

31 We recall that "playing a DS" means performing the corresponding measurement (see Section 2.2).

32 This is reminiscent of Nobel laureate Sen (1997), who proposes that the act of choice has an impact on the agent's preferences.

33 This is because, if we adopt a population approach, all agents must be of the same type. Alternatively, with an individual approach the tested agent must be "prepared" (after each trial), so he is again of the same initial type.

The TI-model and the RUM (Random Utility Maximization) approach From Statement 1, we know that when all *DS* associated with subsets of the universal set commute, then the agent's choice behavior cannot be distinguished from that of a classical agent. This also means that the agent's choice behavior becomes deterministic after a suitable series of experiments. All randomness is due to our incomplete information about his true preferences. This uncertainty can be fully resolved so that the agent's choice behavior can be predicted with certainty.

When some *DS* do not commute, the agent's choice behavior cannot be predicted with certainty, and there is an irreducible randomness. Are we dealing with some random utility model? The answer is NO. There exist two interpretations of RUM models. In the first, the subject of the experiments is a population of agents (each endowed with possibly volatile preferences) and in the second it is an individual agent endowed with random preferences. In both cases, the first choice is a draw from an underlying distribution either in the population or in the mind of the agent. The second choice (made by another agent, in the population interpretation, and by the same agent in the individual interpretation) is a second draw from the same distribution, which may of course not yield the same outcome. In contrast, and as a consequence of the first-kindness property (or the Reduction Principle; see Section 2.2), when a TI-agent makes a choice in some subset, the result from the next identical choice experiment is fully predictable, and it is the same as in the first experiment. In the TI-model, the act of choosing changes the type (preferences). This also has implications for other, non-commuting, choice experiments. The distribution over the possible choices in the next incompatible choice experiment depends on the outcome of the first experiment. It is not the same distribution, as before the agent played the first *DS*. Hence, a TI-model is not a RUM model, because in the latter the act of choosing has no impact on the preferences of the agent. They are and remain random, with the same distribution with respect to all choice experiments.

3.2. Examples

In this section, we demonstrate how type indeterminacy can explain two well-documented examples of so-called behavioral anomalies. With these examples we want to suggest that one contribution of our approach is to provide a unified framework which can accommodate a variety of behavioral anomalies. These anomalies are currently explained by widely different theories.

3.2.1. Cognitive dissonance

The kind of phenomena we have in mind can be illustrated as follows. Numerous studies show that employees in risky industry (like nuclear plants) often neglect safety regulations. The puzzle is that before moving into the risky industry those employees were typically able to estimate the risks correctly. They were reasonably averse to risk and otherwise behaved in an ordinary rational manner.

Psychologists developed a theory called cognitive dissonance (CD) according to which people modify their beliefs or preferences in response to the discomfort arising from conflicting beliefs or preferences. In the example above, they identify a dissonance as follows. On the one hand, the employee holds an image of himself as "a smart person" and on the other hand he understands that he deliberately chose to endanger his health (by moving to the risky job), which is presumably "not smart". So in order to cancel the dissonance, the employee decides that there is no danger and therefore no need to follow the strict safety regulation.

We propose a formal model that is very much in line with psychologists' theory of cognitive dissonance. We shall compare two scenarios involving a sequence of two non-commuting Decision Situations, each with two options.[34] Let A be a *DS* about jobs with options a_1 and a_2 corresponding to taking a job with a dangerous task (adventurous type) and respectively staying with the safe routine (habit-prone type). Let B be a *DS* about the willingness to use safety equipment with choices b_1 (risk-averse type) and b_2 (risk-loving type) corresponding to the choice of using and respectively not using the safety equipment.

First scenario: The dangerous task is introduced in an existing context. It is *imposed* on the workers. They are *only* given the choice to use or not to use the safety equipment (B). We write the initial state of the worker in terms of the eigenvectors of observable A:

$$|\psi\rangle = \lambda_1 |a_1\rangle + \lambda_2 |a_2\rangle, \qquad \lambda_1^2 + \lambda_2^2 = 1.$$

We develop the eigenvectors of A on the eigenvectors of B:

$$|a_1\rangle = \mu_{11} |b_1\rangle + \mu_{21} |b_2\rangle,$$
$$|a_2\rangle = \mu_{21} |b_1\rangle + \mu_{22} |b_2\rangle.$$

We now write the state in terms of the eigenvectors of the B operator:

$$|\psi\rangle = [\lambda_1\mu_{11} + \lambda_2\mu_{21}] |b_1\rangle + [\lambda_1\mu_{12} + \lambda_2\mu_{22}] |b_2\rangle.$$

The probability that a worker chooses to use the safety equipment is

$$\begin{aligned} p_B(b_1) &= \langle b_1|\psi\rangle^2 = [\lambda_1\mu_{11} + \lambda_2\mu_{21}]^2 \\ &= \lambda_1^2\mu_{11}^2 + \lambda_2^2\mu_{21}^2 + 2\lambda_1\lambda_2\mu_{11}\mu_{21}. \end{aligned} \tag{7}$$

Second scenario: First A, then B. The workers choose between taking a new job with a dangerous task or staying with the current safe routine. Those who choose the new job then face the choice between using safety equipment or not. Those who turn down the new job offer are asked to answer a questionnaire about their willingness to use the safety equipment for the case where they would be working in the risky industry. The ex-ante probability for observing b_1 is

$$\begin{aligned} p_{BA}(b_1) &= p_A(a_1)\, p_B(b_1|a_1) + p_A(a_2)\, p_B(b_1|a_2) \\ &= \lambda_1^2\mu_{11}^2 + \lambda_2^2\mu_{21}^2. \end{aligned} \tag{8}$$

The phenomenon of cognitive dissonance can now be formulated as the following inequality:

$$p_{BA}(b_1) < p_B(b_1),$$

which occurs in our model when $2\lambda_1\lambda_2\mu_{11}\mu_{21} > 0$.[35] We next show that interference effects may be quantitatively significant.

Numerical example

Assume for simplicity that $|\psi\rangle = |b_1\rangle$, which means that everybody in the first scenario is willing to use the proposed safety equipment. Let $prob(a_1|\psi) = 0.25$ and $prob(a_2|\psi) = .75$, so $|\lambda_1| = \sqrt{0.25}$ and $|\lambda_2| = \sqrt{0.75}$. It is possible to show that in this case we have $|\mu_{11}| = \sqrt{0.25}$ and $|\mu_{21}| = \sqrt{0.75}$.[36] We now compute $p_B(b_1)$ using the formula in (7) and recalling that $|\psi\rangle = |b_1\rangle$ (so $p_B(b_1) = 1$):

$$\begin{aligned} 1 &= \langle b_1|\psi\rangle^2 = \lambda_1^2\mu_{11}^2 + \lambda_2^2\mu_{21}^2 + 2\lambda_1\lambda_2\mu_{11}\mu_{21} \\ &= 0.0625 + 0.5625 + 2\lambda_1\lambda_2\mu_{11}\mu_{21} \\ &= 0.625 + 2\lambda_1\lambda_2\mu_{11}\mu_{21}, \end{aligned} \tag{9}$$

[34] **We implicitly assume that the two measurements are complete, i.e., not coarse. We return to this issue in the Discussion.**

[35] $p_{BA}(b_1)$ includes the probability of a choice of safety measures *both* in the group that chose the risky job and in the group that chose the safe job. This guarantees that we properly distinguish between the CD effect (change in attitude) and the selection bias.

[36] Based on the fact that $\begin{pmatrix} \mu_{11} & \mu_{22} \\ \mu_{21} & \mu_{22} \end{pmatrix}$ is a rotation matrix.

which implies that the interference effect is positive and equal to $1 - 0.625 = 0.375$. We note that it amounts to a third of the probability.

Under the second scenario the probability for using safety equipment is given by the formula in (8), i.e., it is the same sum as in (9) without the interference term:

$p_{BA}(b_1) = 0.625.$

So we see that our TI-model "delivers" cognitive dissonance: $p_{BA}(b_1) < p_B(b_1)$.

The key assumption that drives our result is that the choice between jobs and the choice between using or not using the safety equipment are measurements of two incompatible type characteristics (represented by two non-commuting observables). A possible behavioral interpretation is as follows. The job decision appeals to an abstract perception of risk, while the decision to use the safety equipment appeals to an emotional perception of concrete risks. In this interpretation, our assumption is that the two modes of perceptions are incompatible. This is consistent with evidence that shows a gap between perceptions of one and the same issue when the agent is in a "cold" (abstract) state of mind, compared to when she is in a "hot" emotional state of mind (see for instance Lowenstein (2005)).

Comments

In their article from 1982, Akerlof and Dickens explain the behavior attributed to cognitive dissonance in terms of a rational choice of beliefs. Highly sophisticated agents choose their beliefs to fit their preferences.[37] They are fully aware of the way their subjective perception of the world is biased and yet they keep to the wrong views. This approach does explain observed behavior but raises serious questions as to what rationality means. The type indeterminacy approach is consistent with psychologists' thesis that cognitive dissonance prompts a change in preferences (or attitude). We view its contribution as follows. First, the TI-model provides a model that explains the appearance of cognitive dissonance. Indeed, if coherence is such a basic need, as proposed by L. Festinger and his followers, why does dissonance arise in the first place? In the TI-model, dissonance arises when resolving indeterminacy in the first *DS* because of the "limitations" on possible psychological types (see Section 2.3.2). Second, the TI-model features a *dynamic process* such that the propensity to use safety measures is actually altered (reduced) as a consequence of the *act of choice*. This dynamic effect is reminiscent of psychologists' "drive-like property of dissonance" that leads to a change in attitude.

3.2.2. Framing effects

When alternative descriptions of one and the same decision problem induce different decisions from agents, we talk about "framing effects". Below, we discuss a well-known experiment that showed that two alternative formulations of the Prisoner's Dilemma (the standard presentation in a 2 by 2 matrix and a presentation in a decomposed form; see below) induced dramatically different rates of cooperation.

Kahneman and Tversky (2000, p. xiv) address framing effects using a two-step (non-formal) model of the decision-making process. The first step corresponds to the construction of a representation of the decision situation. The second step corresponds to the evaluation of the choice alternatives. The crucial point is that *"the true objects of evaluation are neither objects in the real world nor verbal descriptions of those objects; they are mental representations"*. To capture this feature we suggest modeling the "process of constructing a representation" in a way similar to the process of constructing preferences, i.e., as a measurement performed on the state of the agent. This is consistent with psychology that treats attitudes, values (preferences), beliefs, and representations as mental objects of the same kind.

In line with Kahneman and Tversky, we describe the process of decision-making as a sequence of two steps. The first step consists of a measurement of the agent's mental representation of the decision situation. The second step corresponds to a measurement of the agent's preferences. Its outcome is a choice. Note that here we depart from standard decision theory. We propose that agents do not always have a *unique* representation of a decision situation. Instead, uncertainty about what the decision situation actually is about can be resolved in a variety of ways, some of which may be reciprocally incompatible. The decision situation itself is, as in standard theory, defined by the monetary payoffs associated with the choices, i.e., in a unique way. The *utility* associated with the options generally depends both on the representation and on the monetary payoffs.

To illustrate this approach, we revisit the experiment reported in Pruitt (1970) and discussed in Selten (1998). Two groups of agents are invited to play a Prisoner's Dilemma. The game is presented to the first group in the usual matrix form, with the options labeled 1_G and 2_G (instead of *C* and *D*, presumably to avoid associations with the suggestive terms "cooperate" and "defect"):

	$[C]$	$[D]$
1_G	3 | 3	0 | 4
2_G	4 | 0	1 | 1

The game is presented to the second group in a decomposed form as follows:

	For me	For him
1_G	0	3
2_G	1	0

The payoffs are computed as the sum of what you take for yourself and what you get from the other player. So for instance if player 1 plays 1_G and player 2 plays 2_G, player 1 receives 0 from his own play and 0 from the other's play, which sums to 0. Player 2 receives 1 from his own play and 3 from player 1's play, which sums to 4. So we recover the payoffs (0,4) associated with the play of Cooperate for player 1 and Defect for player 2. Game theoretically, it should make no difference whether the game is presented in a matrix form or in a decomposed form. Pruitt's main experimental result is that one observes dramatically more cooperation in the game presented in decomposed form.

We now provide a possible TI-model for this situation. Let us consider a two-dimensional state space and a sequence of two incompatible measurements.[38] The first measurement determines the mental representation of the *DS*. We call *A* the (representation) observable corresponding to the matrix presentation. It has two non-degenerated eigenvalues denoted a_1 and a_2. Similarly, *B* is the observable corresponding to the decomposed presentation with two eigenvalues b_1 and b_2. If $|\psi\rangle$ is the initial state of the agent, we can write $|\psi\rangle = \alpha_1 |a_1\rangle + \alpha_2 |a_2\rangle$ or $|\psi\rangle = \beta_1 |b_1\rangle + \beta_2 |b_2\rangle$. The

[37] Akerlof and Dickens allow workers to freely choose beliefs (about risk) so as to optimize their utility which includes psychological comfort. The workers are highly rational in the sense that when selecting beliefs, they internalize their effect on their own subsequent bounded rational behavior.

[38] We again implicitly assume that the two measurements are complete, i.e., not coarse.

second measurement (i.e., the decision observable is unique; we called it G. G) has also two eigenvalues denoted 1_G and 2_G.

For the sake of concreteness, we may think of the four alternative mental representations as follows[39]:

$|a_1\rangle$: G is perceived as an (artificial) small-stakes game;

$|a_2\rangle$: G is perceived by analogy as a real-life PD-like situation (often occurring in a repeated setting).

$|b_1\rangle$: G is perceived as a test of generosity;

$|b_2\rangle$: G is perceived as a test of smartness.

When confronted with a presentation of the DS the agent forms her mental representation of the DS which prompts a change in her state from $|\psi\rangle$ to some $|a_i\rangle$ (if the frame is A) or $|b_j\rangle$ $i, j = 1, 2$ (if the frame is B). The new state can be expressed in terms of the eigenvectors of the decision situation: $|a_i\rangle = \gamma_{1i} |1_G\rangle + \gamma_{2i} |2_G\rangle$ or $|b_j\rangle = \delta_{1j} |1_G\rangle + \delta_{2j} |2_G\rangle$.

We can now formulate the framing effect as the following difference:

$$p_{GA}(i_G) \neq p_{GB}(i_G), \quad i = 1, 2. \tag{10}$$

Using our result in Section 2.3.2 we get

$$p_{GA}(1_G) = p_G(1_G) - 2\alpha_1\gamma_{11}\alpha_2\gamma_{12} \quad \text{and}$$
$$p_{GB}(1_G) = p_G(1_G) - 2\beta_1\delta_{11}\beta_2\delta_{12}$$

where $p_G(1_G)$ is the probability of choosing 1 in a (hypothetical) unframed situation. So we have a framing effect whenever $2\alpha_1\gamma_{11}\alpha_2\gamma_{12} \neq 2\beta_1\delta_{11}\beta_2\delta_{12}$.

The central experimental result discussed in Selten (1998) is that the decomposed presentation induces more cooperation than the matrix presentation. In our model, this translates into the following inequality: $2\alpha_1\gamma_{11}\alpha_2\gamma_{12} - 2\beta_1\delta_{11}\beta_2\delta_{12} > 0$. The inequality says that the interference term for 1_G is larger in the standard matrix presentation A than in the decomposed form corresponding to the B presentation. In order to better understand the meaning of this difference we shall consider a simple numerical example.

Set $\alpha_1 = \sqrt{.8}, \alpha_2 = \sqrt{.2}, \beta_1 = \sqrt{0.4}, \beta_2 = \sqrt{0.6}$. The key variables are the correlations between the "representation types", i.e., $|a_i\rangle$ or $|b_j\rangle$, and the "game type" 1_G, i.e., the numbers γ_{11}, γ_{12} and δ_{11}, δ_{12}. We propose that $\gamma_{11} = \sqrt{.3}$, which is interpreted as when the agent views the game as a small-stakes game she plays 1_G with probability .3. Similarly $\gamma_{12} = \sqrt{.7}$, which means that when the agent perceives the game by analogy with real life, she "cooperates" with probability .7. In the alternative presentation B, we propose that $\delta_{11} = 1$, i.e., when the game is perceived as a test of generosity, the agent cooperates with probability 1. When the game is perceived as a test of smartness, $\delta_{12} = 0$ (because the agent views the play of 1_G as plain stupid). Computing these figures, we get

$$2\alpha_1\gamma_{11}\alpha_2\gamma_{12} - 2\beta_1\delta_{11}\beta_2\delta_{12} = 0.366 - 0 > 0.$$

In the A presentation the contribution of both the $|a_1\rangle$ and the $|a_2\rangle$ type is positive and significant. When the agent is indeterminate, both types positively contribute, reinforcing each other. In contrast, in the B presentation the contribution from $|b_2\rangle$ is null so there is no interference between the types. When the agent determines herself (i.e., selects a representation) the contribution from indeterminacy is lost and that loss is positive with A, while it is null with B. Therefore, the probability for 1_G when the game is presented in the matrix form is larger (here by .36) than in the game presented in the decomposed form.

Comments

Selten proposes a "bounded rationality" explanation: players make a superficial analysis and do not perceive the identity of the games presented under the two forms. Our approach is closer to Kahneman and Tversky who suggest that, prior to the choice, a representation of the decision situation must be constructed. The TI-model provides a framework for "constructing" a representation such that it delivers framing effects in choice behavior. Of course framing effects can easily be obtained when assuming that the mental images are incomplete or biased. In the TI-model we do not need to appeal to such self-explanatory arguments. In the TI-model, framing effects arise as a consequence of (initial) indeterminacy of the agent's representation of the decision situation. Since alternative (non-commuting) presentations are equally valid and their corresponding representations (eigenvalues) equally informative, a highly rational agent can exhibit framing effects.

4. Discussion

In this section, we briefly discuss some formal features of our model.[40]

Our approach to decision-making yields that the type of the agents, rather than being exogenously given, emerges as the outcome of the interaction between the agent and the decision situations. This is modeled by letting a decision situation be represented by an operator (observable). Decision-making is modeled as a measurement process. It projects the initial state of the agent into the subspace of the state (type) space associated with the eigenvalue corresponding to the choice made. Observables may either pairwise commute or not. When the observables commute, the corresponding type space has the properties of the Harsanyi type space. From a formal point of view, this reflects the fact that all (pure) types are then mutually exclusive.[41] When the observables do not commute, the associated pure types are not all mutually exclusive. Instead, an agent who is in a pure state after the measurement of an observable will be in a different pure state after the measurement of another observable that is incompatible with the first one. As a consequence, the type space cannot be associated with a classical probability space and we obtain an irreducible uncertainty in behavior. The Type Indeterminacy model provides a framework where we can deal both with commuting and with non-commuting observables. In the TI-model, any type (state) corresponds to a probability measure on the type space which allows one to make predictions about the agent's behavior. It is in this sense that the TI-model generalizes Harsanyi's approach to uncertainty.

The more controversial feature of the TI-model as a framework for describing human behavior is related to the modeling of the impact of measurement on the state, i.e., how the type of the agent changes with decision-making. The rules of change are captured in the geometry of the type space and in the projection postulate. It is more than justified to question whether this seemingly very specific process should have any relevance to the Social Sciences.

It has been shown that the crucial property that gives all its structure to the process of change can be stated as a "minimal perturbation principle". The substantial content of that principle is that we require that when a coarse DS resolves some uncertainty about the type of an agent, the remaining uncertainty is left unaffected. Recall our example in case 2 of Section 3.1. When the agent chooses an item out of a subset A of three items, this prompts a resolution of some uncertainty. The type is projected

39 This is only meant as a suggestive illustration.

40 For a systematic investigation of the mathematical foundations of the HSM in terms of their relevance to the social sciences, see Danilov and Lambert-Mogiliansky (2008).

41 We say that a type is pure when it is obtained as the result of a complete measurement, i.e., the measurement of a complete set of commuting observables (CSCO).

into the eigenspace spanned by the two orderings consistent with the choice made. The minimal perturbation principle says that uncertainty relative to the ordering of the two remaining items is the same as before. In behavioral terms, this can be expressed as follows. When confronted with the necessity to make a choice, the agent only "makes the effort" to select her preferred item, while leaving the order relationship between the other items uncertain, as initially.

It may be argued that the minimal perturbation principle is quite demanding. Returning again to our example, if the mental processes involved in the search for the preferred items fully upset the initial state, the principle is violated. It could also be argued that the mental processes involved in decision-making determine the whole ranking. That would also violate the minimal perturbation principle. This short discussion suggests that selecting a good TI-model requires careful thinking and possibly some trial and error.

We do not expect the Type Indeterminacy model to be a fully realistic description of human behavior. Rather, we propose it as an idealized model of agents characterized by the fact that their type changes with decision-making. In particular, some features of the TI-model, like the symmetry of the correlation matrix in simple examples, may seem very constraining from a behavioral point of view. Consider two *DS* with two options, e.g., the Prisoner's Dilemma and the Ultimatum game (with options "share fairly" and "share greedily"). Assuming those *DS* have non-degenerated eigenvalues, the symmetry of the correlation matrix means that the probability of playing (e.g., defecting after having played, say, greedy) is exactly the same as the probability of playing greedy after having played defect. We do not expect this kind of equality to hold in general. A failure of this equality to hold may be due to the fact that the *DS* have some degenerated eigenvalues. In fact, this is likely to be the case in the Social Sciences, since we often deal with rather coarse measurements of the agent's type. Unfortunately, a model with coarse measurements is plagued by serious limitations regarding quantitative predictions. This is because there is no single well-defined correlation matrix linking two non-commuting observables representing two incompatible coarse measurements. It is easy to understand why. Consider again our cognitive dissonance example. Assume that the basis for the *A DS* (choice of job) is three-dimensional so that we have three eigentypes: the adventurous, the habit-prone, and the reasonable type. Similarly, the basis of the *B DS* (decision to use the safety equipment) is three-dimensional with the corresponding eigentypes: risk-loving, risk-averse, and risk-neutral.[42] Now assume the choice for the new job (in *A*) can be made by the adventurous type or the reasonable type or any superposition of the two. Since the eigentypes are orthogonal, the (probabilistic) outcome of the measurement of the *B DS* is not the same, whether the worker is of the adventurous type or of the reasonable type. Therefore, knowing the first choice (e.g., for the new job), we cannot give the probabilities for the outcomes in the *B DS*. There is no unique correlation between the choice of new job and the decision to use or not to use safety equipment. It depends on the type of the worker after the decision, which is not uniquely defined by the outcome. Notwithstanding this lack of uniqueness, it remains true that the first choice changes the type of the worker, i.e., we have non-commutativity and so cognitive dissonance phenomena can be explained by a TI-model.

Nevertheless, keeping in mind some reservations as to its realism, our view is that the Type Indeterminacy model can provide a fruitful framework for analyzing, explaining, and predicting human behavior. Clearly, much additional work is needed to extend the TI-model to strategic and repeated decision-making. We are currently exploring this second stage of our research program.

As a final remark, it should be emphasized that not all instances of non-commutativity in choice behavior call for Hilbert space modeling. Theories of addiction feature effects of past choices on future preferences. And in standard consumer theory, choices do have implications for future behavior, i.e., when goods are substitutes or complements. But in those cases we *do* expect future preferences to be affected by the choices. The Type Indeterminacy model of decision-making can be useful when we expect choice behavior to be consistent with the standard probabilistic model, because nothing justifies a modification of preferences. Yet, actual behavior contradicts those expectations.

[42] Two non-commuting complete observables always have the same dimensionality for the dimensionality of the system they measure.

Appendix. Elements of Quantum Mechanics

A.1. States and observables

In Quantum Mechanics the state of a system is represented by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. According to the superposition principle, every complex linear combination of state vectors is a state vector. A Hermitian operator called an observable is associated with each physical property of the system.

Theorem 1. *A Hermitian operator A has the following properties:*

- *Its eigenvalues are real.*
- *Two eigenvectors corresponding to different eigenvalues are orthogonal.*
- *There is an orthonormal basis of the relevant Hilbert space formed with the eigenvectors of A.*

Let us call $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$ the normalized eigenvectors of A forming a basis of $\mathcal{H}$. They are associated with eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$, so $A|v_i\rangle = \alpha_i|v_i\rangle$. The eigenvalues can possibly be degenerated, i.e., for some i and j, $\alpha_i = \alpha_j$. This means that there is more than one linearly independent eigenvector associated with the same eigenvalue. The number of these eigenvectors defines the degree of degeneracy of the eigenvalue, which in turn defines the dimension of the eigensubspace spanned by these eigenvectors. In this case, the orthonormal basis of $\mathcal{H}$ is not unique because it is possible to replace the eigenvectors associated with the same eigenvalue by any complex linear combination of them to get another orthonormal basis. When an observable A has no degenerated eigenvalue, there is a unique orthonormal basis of $\mathcal{H}$ formed with its eigenvectors. In this case (see below), it is by itself a Complete Set of Commuting Observables.

Theorem 2. *If two observables A and B commute there is an orthonormal basis of $\mathcal{H}$ formed by eigenvectors common to A and B.*

Let A be an observable with at least one degenerated eigenvalue and B another observable commuting with A. There is no unique orthonormal basis formed by A eigenvectors. But there is an orthonormal basis of the relevant Hilbert space formed by eigenvectors common to A and B. By definition, $\{A, B\}$ is a Complete Set of Commuting Observables (CSCO) if this basis is unique. Generally, a set of observables $\{A, B, \ldots\}$ is said to be a CSCO if there is a unique orthonormal basis formed by eigenvectors common to all the observables of the set.

A.2. Measurements

An observable A is associated with each physical property of a system S. Let $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$ be the normalized eigenvectors of A associated respectively with eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ and forming a basis of the relevant state space. Assume the system is in the normalized state $|\psi\rangle$. A measurement of A on S obeys the following rules, collectively called the "Wave Packet Reduction Principle" (the Reduction Principle).

Reduction principle

1. When a measurement of the physical property associated with an observable A is made on a system S in a state $|\psi\rangle$, the result can only be one of the eigenvalues of A.

2. The probability of getting the non-degenerated value α_i is $P(\alpha_i) = |\langle v_i|\psi\rangle|^2$.

3. If the eigenvalue is degenerated then the probability is the sum over the eigenvectors associated with this eigenvalue: $P(\alpha_i) = \sum \left|\left\langle v_i^j|\psi\right\rangle\right|^2$.

4. If the measurement of A on a system S in the state $|\psi\rangle$ has given the result α_i then the state of the system immediately after the measurement is the normalized projection of $|\psi\rangle$ onto the eigensubspace of the relevant Hilbert space associated with α_i. If the eigenvalue is not degenerated then the state of the system after the measurement is the normalized eigenvector associated with the eigenvalue.

If two observables A and B commute then it is possible to measure both simultaneously: the measurement of A is not altered by the measurement of B. This means that measuring B after measuring A does not change the value obtained for A. If we again measure A after a measurement of B, we again get the same value for A. Both observables can have a definite value.

A.2.1. Interferences

The archetypal example of interferences in quantum mechanics is given by the famous two-slits experiment.[43] A parallel beam of photons falls on a diaphragm with two parallel slits and strikes a photographic plate. A typical interference pattern showing alternate bright and dark rays can be seen. If one slit is shut then the previous figure becomes a bright line in front of the open slit. This is perfectly understandable if we consider photons as waves, as it is the assumption in classical electromagnetism. The explanation is based on the fact that when both slits are open, one part of the beam goes through one slit and the other part through the other slit. Then, when the two beams join on the plate, they interfere constructively (giving bright rays) or destructively (giving dark ones), depending on the difference in the length of the paths they have followed. But a difficulty arises if photons are considered as particles, as can be the case in quantum mechanics. Indeed, it is possible to decrease the intensity of the beam so as to have only one photon traveling at a time. In this case, if we observe the slits in order to detect when a photon passes through (for example, by installing a photodetector in front of the slits), it is possible to see that each photon goes through only one slit. It is never the case that a photon splits to go through both slits. The photons behave like particles. Actually, the same experiment was done with electrons instead of photons, with the same result. If we do the experiment this way with electrons (observing which slit the electrons go through, i.e., sending light through each slit to "see" the electrons), we see that each electron goes through just one slit and, in this case, we get no interference. If we repeat the same experiment without observing which slit the electrons pass through then we recover the interference pattern. Thus, the simple fact that we observe which slit the electron goes through destroys the interference pattern (two single-slit patterns are observed). The quantum explanation is based on the assumption that when we don't observe through which slit the electron has gone then its state is a superposition of both states "gone through slit 1" and "gone through slit 2",[44] while when we observe it, its state collapses onto one of these states. In the first case, the position measurement is made on electrons in the superposed state and gives an interference pattern, since both states are manifested in the measurement. In the second case, the position is measured on electrons in a definite state, and no interference arises. In other words, when only slit 1 is open we get a spectrum, say S_1 (and S_2 when only slit 2 is open). We expect to get a spectrum S_{12} that sums the two previous spectra when both slits are open, but this is not the case: $S_{12} \neq S_1 + S_2$.

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43 See, e.g., Feynman (1980) for a very clear presentation.

44 This doesn't mean that the photon actually went through both slits. This state simply can't be interpreted from a classical point of view.