Information Transmission

SHMUEL ZAMIR

The Hebrew University

AND

MORTON I. KAMIEN

Northwestern University

AND

YAIR TAUMAN

Tel Aviv University

1. INTRODUCTION

Information transmissions in strategic games were discussed by Forges (1986) and by Kamien *et al.* (1988). The latter deals with games in extensive form where an outside player, an information holder, has some information unknown to the players in the game. The information holder's strategies are various information transmissions available to him. Each one of these strategies induces a new game with the same set of players. The set of all Nash equilibrium points obtained as the unique perfect equilibrium of a game induced by an information transmission is called the inducible set. This set measures the ability of the information holder to change the game. The inducible set of any two-person zero-sum game in extensive form is characterized in Kamien *et al.* (1988). However, the characterization of the inducible sets of a non-zero sum game is, in general, far from trivial. An illustration of the characterization of an inducible set of a two-person non-zero game is given in Kamien *et al.* (1988). In this paper we focus on a different phenomenon. We provide an example where basic transmissions of information induce games with multiple and even a continuum of perfect Nash equilibria payoffs. Thus, by our definition such games do not contribute any point to the inducible set (only Nash equilibrium payoffs which are the payoff of a unique perfect equilibrium of a game induced by some transmission of information are inducible). Nevertheless, it turns out that these multiple equilibrium payoffs are all inducible. Namely, each one of these equilibrium payoffs is the

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unique Nash equilibrium of a game induced by some transmission of information. We do not know, however, how general this phenomenon is.

We start with a brief overview of the basic definitions and results in Kamien et al. (1988).

2. THE BASIC DEFINITIONS AND RESULTS

Let Γ be a finite n-person game in extensive form in which the set of players is $N = \{1, 2, \ldots, n\}$. Let $M \notin N$. We refer to M as the information holder or player M (the "maven").

DEFINITION 1.   *Player M's information of player I is a partition E of the set of all nonterminal modes of Γ.*

DEFINITION 2.

(1) *Player M's signal set is a set S of any alphabet.*
(2) *The set of player M's pure strategies is $\Sigma_o = (S^N)^E$ and his set of mixed strategies is $\Sigma = \Delta(\Sigma_o)$ (the set of probability distributions on Σ_o).*

A mixed strategy of M is called a signalling strategy.

The signal set S is the set of messages; each element of S is a message that player M can communicate to each of the other players. A pure strategy is a prescription of what message to send to each of the players in each information element $e_k \in E$. We denote a pure strategy $s \in \Sigma_0$ by $s = \{s(e)\}_{e \in E}$ with the interpretation that if the node $x \in e$ belongs to player i then this player, upon reaching this node, receives the ith component of $s(e)$ before making his move.

We assume that player M's strategy choice σ, becomes common knowledge before the beginning of the game and that he is committed to carrying it out. This eliminates the possibility of player M's "cheating" by transmitting a signal which is not according to σ. Consequently, at any point in the induced game, the beliefs of the players are consistent in Harsanyi's sense, i.e., they are derived from common prior given each player's private information.

Any strategy $\sigma \in \Sigma$ of player M modifies the original game Γ to another game with the same set of players. We denote this game by Γ_σ and call it *the game induced by the strategy σ*. If σ_0 is a pure strategy then Γ_{σ_0} is obtained from Γ by refining the information sets of each player by the signals he receives from M. If σ is the mixed strategy using pure strategies $\sigma_0^1, \ldots, \sigma_0^k$ with probabilities $p_1, \ldots, p_k$ respectively, then Γ_σ is the extensive form game in Figure 1. The "information set" in Figure 1 indicates that for each of the players the same nodes in $\Gamma_{\sigma_0^j}$ and $\Gamma_{\sigma_0^k}$ are indistinguishable unless the signals received there are distinct.
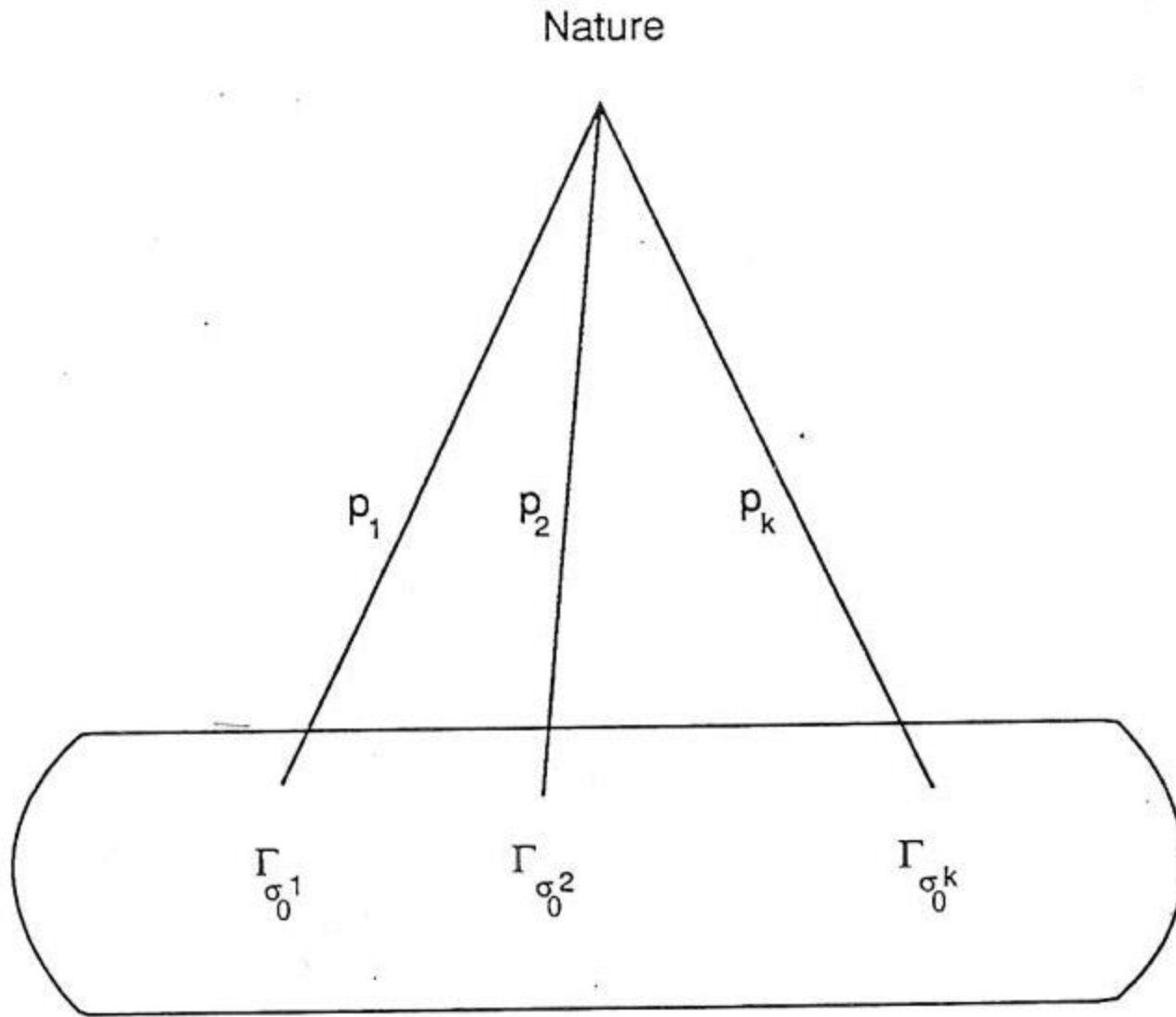
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FIG. 1. The form game Γ_σ for which σ is the mixed strategy.

DEFINITION 3.

(1) *A payoff vector $x \in R^N$ is inducible by player M in the game Γ if there exists a $\sigma \in \Sigma$ such that the induced game Γ_σ has a unique perfect Nash equilibrium point with payoff x.*

(2) *The inducible set in Γ with respect to information E is the closure of all payoffs inducible by player M with information E. (Notation: $X = X(\Gamma, E)$).*

Note that the uniqueness of σ is not required; a certain $x \in X$ may be inducible by two (or more) different strategies σ and σ'. In such a case each of Γ_σ and $\Gamma_{\sigma'}$ has a unique perfect NE with payoff x.

Remark. Definition 3 is a special case of a possibly more general definition of inducibility. Let G be a subset of the class of all games in extensive form with a given set $N = \{1, \ldots, n\}$ of players. Given a game Γ with the set of players N and given the information partition E denote

$$\Gamma_E = \{G \in G \mid \exists \sigma \in \Sigma \quad \text{s.t.} \quad G = \Gamma_\sigma\}.$$

Let Ψ be a single-valued solution concept $\Psi : G \to \mathbb{R}^n$. The inducible set x of Γ, with respect to Ψ, is defined to be the closure of the image $\overline{\Psi(\Gamma_E)}$. In Definition 3, G is the set of all games with unique perfect Nash equilibrium and Ψ associates with each game in G the payoff of its perfect Nash equilibrium.

Our first result is easily obtained.

. 1. *For any Γ and any E, the set of all payoffs inducible by M is*

OROLLARY. *The noninducible outcomes in X can be only on the boundary X.*

3. Information about Chance Moves

A class of games of special interest is that for which player M knows the outcomes of some of the chance moves in Γ. By taking the product space of all chance moves known to player M we may equivalently consider the situation in which there is *one chance move* with outcomes $O = \{o_1, \ldots, o_k\}$ ("states"), probability distribution over the states $p = (p_1, \ldots, p_k)$ and the chosen state is known to player M. We view p as a parameter of the game with range $\Delta = \{p = (p_1, \ldots, p_k) \in R^k \mid p_i \geq 0, \Sigma p_i = 1\}$, and write $\Gamma(p)$, $\Gamma_\sigma(p)$ etc.

LEMMA 2. (Splitting Strategy) *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_j \geq 0$, $\Sigma \lambda_j = 1$ and let p, $(p^j)_{j=1}^m$ be $m + 1$ points in Δ such that $p = \Sigma_{j=1}^m \lambda_j p^j$. Then in $\Gamma(p)$ player M can induce a game which is equivalent to the following: a chance move chooses a point in $\{p^1, \ldots, p^m\}$ according to the probabilities $\lambda_1, \ldots, \lambda_m$, all players are informed of the outcome p^j and then $\Gamma(p^j)$ is played.*

This is a well known result from the theory of games with incomplete information (see Mertens and Zamir, 1971, Lemma 2, p. 46). A proof is also provided in Kamien *et al.* (1988). By Lemma 2 we can now derive the following. Let $D = \{p \in \Delta \mid X(p) \neq 0\}$.

PROPOSITION 3. *The graph of the set valued function $X(\cdot)$ defined on D is a convex set in $\Delta \times R^n$.*

For a proof see Kamien et al. (1988).

EXAMPLE. *Consider the following two-person-game Γ (described in Figure 2). If player M, knowing which game is actually being played, Γ_L or Γ_R, restricts himself to the actions "inform" (I) and "do not inform" (U), he can induce the following four games:*

(I, I): Informing both players' results in the only Nash equilibrium payoff (0.5, 0.5).

(I, U): Informing Player 1 only. The informed player then uses his dominant strategies (T in Γ_L and B in Γ_R) leaving the uninformed player with the choice L, with expected payoff $(-0.5, 0)$, or R, with expected payoff $(2, 0)$. Any mixture $(y, 1 - y)$, $0 \leq y \leq 1$, of L and R by the uninformed player yields a perfect Nash equilibrium with expected payoffs $(2 - 2.5y, 0)$. Therefore, in the game induced by informing the row player only, there is a continuum of perfect Nash
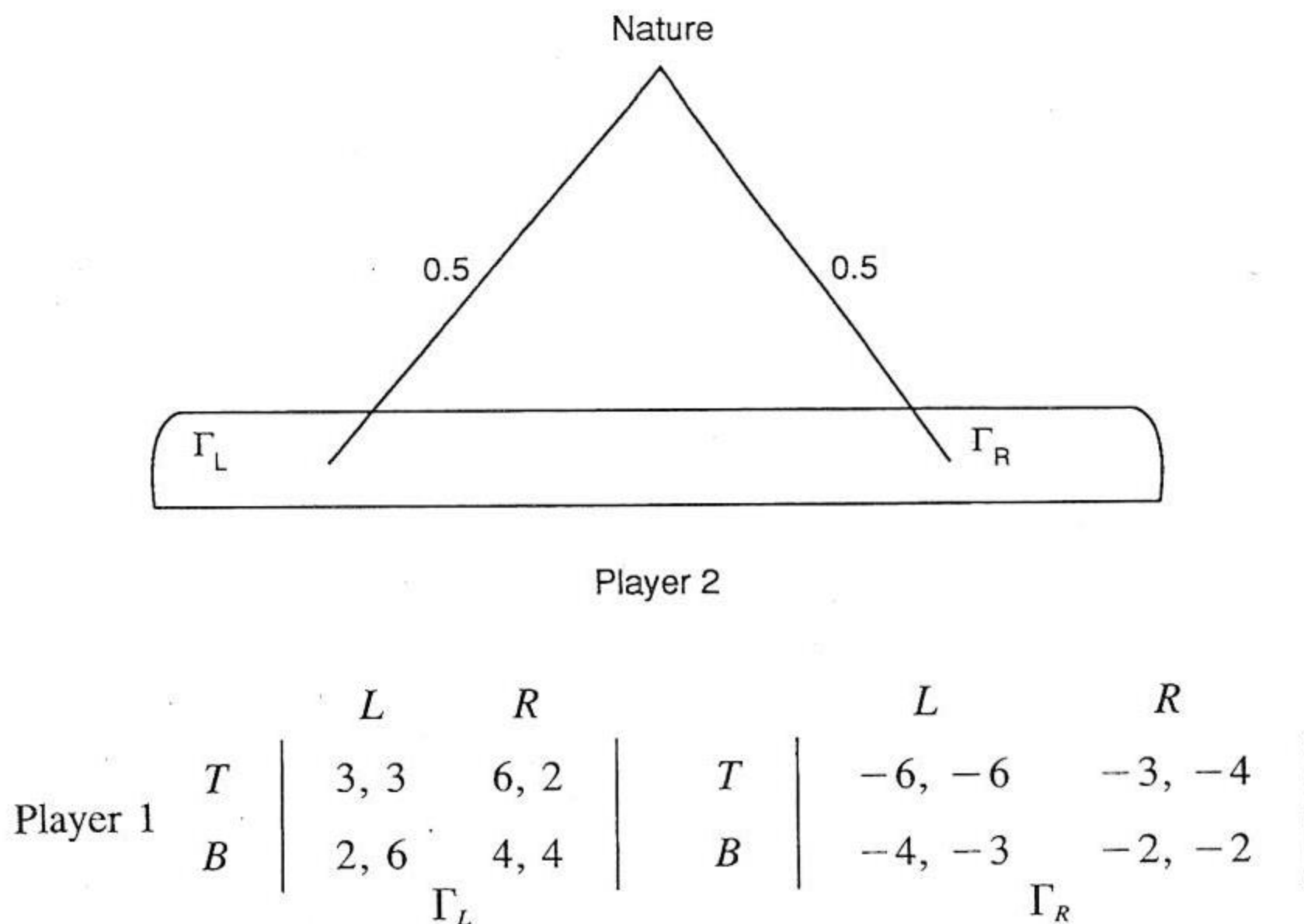
FIG. 2. Two person game Γ.

equilibria with payoffs consisting of the line segment $[(-0.5, 0), (2, 0)]$. Similarly, the action (U, I) induces a game in which each point of the line segment $[(0, -0.5), (0, 2)]$ is a perfect Nash equilibrium payoff.

(U, U): If neither player is informed, they play the original game which is equivalent to

$$
\begin{array}{c|cc}
 & L & R \\
\hline
T & -1.5, -1.5 & 1.5, -1 \\
B & -1, 1.5 & 1, 1
\end{array}
$$

This game has two pure Nash equilibria with payoffs $(1.5, -1)$ and $(-1, 1.5)$ and a mixed Nash equilibrium in which each player plays the pure strategies with equal probability, yielding the expected payoff $(0, 0)$.

The I–U "matrix" summarizing the outcome of these four actions is:

$$
\begin{array}{c|cc}
 & I & U \\
\hline
I & (0.5, 0.5) & [(-0.5, 0), (2, 0)] \\
U & [(0, -0.5), (0, 2)] & [(1.5, -1), (-1, 1.5), (0, 0)]
\end{array}
$$

Which payoffs of the I–U matrix are in the inducible set? According to our definition, only the outcome $(0.5, 0.5)$ is inducible, by (I, I). Any other action leads to a game with a multiplicity of perfect Nash equilibrium payoffs. Never-

theless, we will show below that there are modes of information disclosure that enable Player M to induce any point in a rather larger set X which contains the convex hull of all the Nash equilibrium payoffs described in the I–U matrix above, and more.

PROPOSITION 4. *Any point in the set X which is the open convex hull of* $\{(2, 0), (0, 2), (1.5, -1), (-1, 1.5), (-1, 0), (0, -1)\}$, *is inducible.*

In Fig. 3 the polyhedron *ABCDEFG* is the set of all payoffs which are of the form $x = 0.5x_L + 0.5x_R$ where x_L is a point in the convex hull of $\{(3, 3), (6, 2), (2, 6), (4, 4)\}$ and x_R is a point in the convex hull of $\{(-6, -6), (-3, -4), (-4, -3), (-2, -2)\}$. The polyhedron *ABCRSG* is the convex hull of all Nash equilibrium payoffs in the *I–U* matrix and strictly larger than it, is the shaded open polyhedron *ABCMNG*, each point of which is inducible by the information holder.

Proof.   Let $\Gamma(p)$ be the game starting with a chance move which selects one of the two games Γ_L and Γ_R with probabilities p and $1 - p$, respectively. Thus the game under consideration is $\Gamma(0.5)$ and we are interested in the inducible set $X(0.5)$. The games $\Gamma(1)$ and $\Gamma(0)$ are Γ_L and Γ_R, respectively. These have perfect Nash equilibrium payoffs $(3, 3)$ and $(-2, -2)$, respectively, and Player M has no role there. Therefore, we have:

(1). $X(0) = \{(-2, -2)\}$ and $X(1) = \{(3, 3)\}$. Consider now $\Gamma(p)$ and the pure signalling strategy:

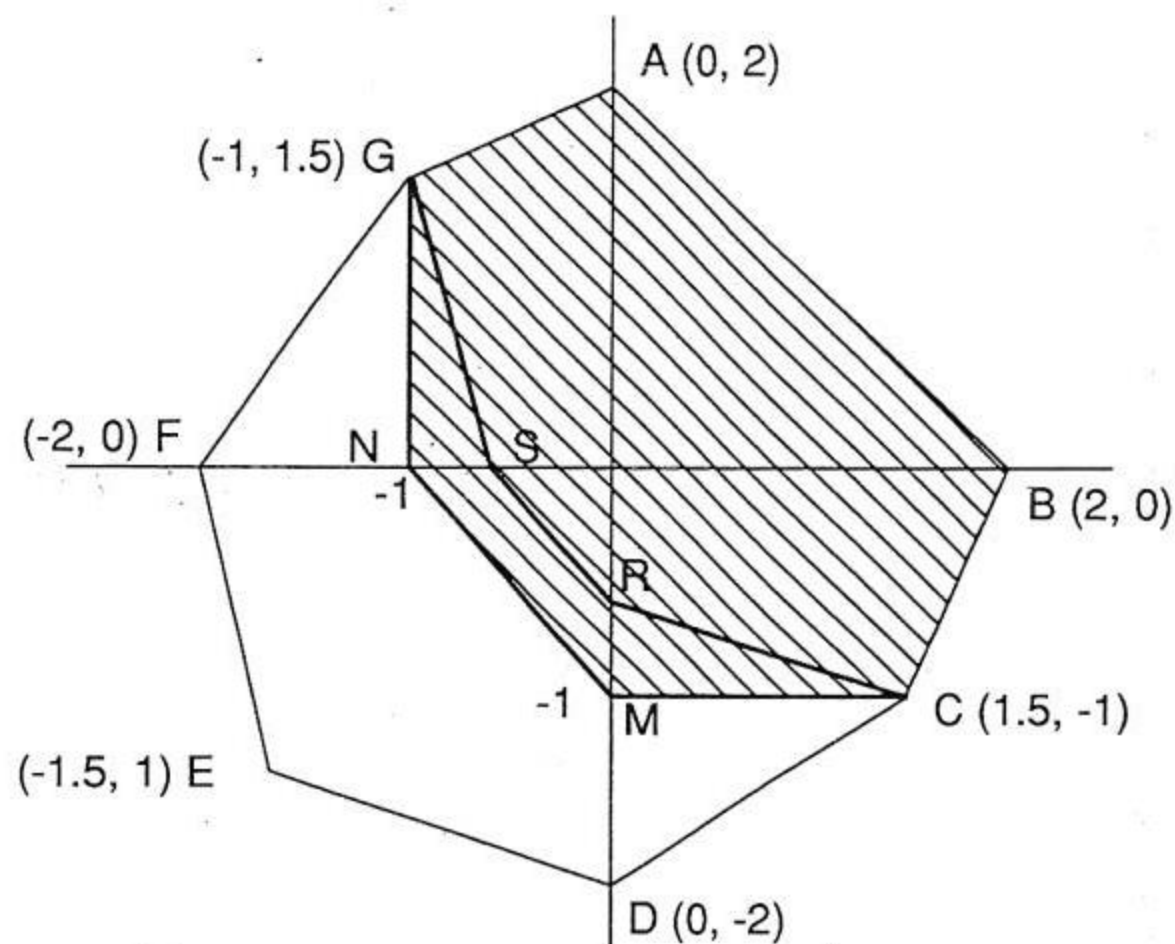$$\sigma_1 : kk \text{ at } \Gamma_L \quad \text{and} \quad rk \text{ at } \Gamma_R.$$

FIG. 3.   The inducible set $x(0.5)$.

This strategy tells Player 1 the true game and provides Player 2 no information (i.e., always k). Since for Player 1, T is a dominant strategy in Γ_L and B is dominant in Γ_R, Player 2 faces the choice between:

L with payoffs $p(3, 3) + (1 - p)(-4, -3) = (7p - 4, 6p - 3)$

and

R with payoffs $p(6, 2) + (1 - p)(-2, -2) = (8p - 2, 4p - 2)$

Therefore, for $p < 0.5$, the best reply is R, yielding a unique Nash equilibrium with payoffs $(8p - 2, 4p - 2)$. For $p > 0.5$, Player 2's best reply is L, yielding a unique Nash equilibrium with payoffs $(7p - 4, 6p - 3)$. We conclude that:

(2). $\quad \begin{cases} (8p - 2, 4p - 2) \in X(p) & \text{for} \quad 0 \le p < 0.5 \\ (7p - 4, 6p - 3) \in X(p) & \text{for} \quad 0.5 < p \le 1. \end{cases}$

Notice that (2) implies (1).

Now for $0 < \varepsilon < 0.5$, $0.5 = (0.5 - \varepsilon)/(1 + 2\varepsilon) + 2\varepsilon/(1 + 2\varepsilon)$. By (2), $(2 - 8\varepsilon, -4\varepsilon) \in X(0.5 - \varepsilon)$ and $(3, 3) \in X(1)$. So by Proposition 3, $(2 - 8\varepsilon, -4\varepsilon)/(1 + 2\varepsilon) + 2\varepsilon(3, 3)/(1 + 2\varepsilon) \in X(0.5)$, i.e., $(2 - 2\varepsilon, 2\varepsilon)/(1 + 2\varepsilon) \in X(0.5)$.

Taking $\varepsilon \to 0$ one obtains that the point $A = (2, 0)$ in Figure 3 is in $\bar{X}(0.5)$ (the closure of $X(0.5)$). Similarly, switching the roles of Players 1 and 2 we have that $(0, 2) \in \bar{X}(0.5)$. Although similar use of Proposition 3 may be employed to prove that $(-1, 0)$ and $(-1, 1.5)$ are in $\bar{X}(0.5)$, it may be instructive to exhibit directly signalling strategies which induce these outcomes in $\Gamma(0.5)$.

Inducing $(-1 + 1.5\varepsilon, 0)$. Consider the following (behavioral) signalling strategy σ by Player M:

If Γ_L : $(1/3 - \varepsilon)kk + 2\varepsilon rk + (2/3 - \varepsilon)rr.$

If Γ_R : $(2/3 + \varepsilon)kk + (1/3 - \varepsilon)rr.$

Interpretation: If the game is Γ_L, with probability $(1/3 - \varepsilon)$ communicate k to both players, with probability 2ε communicate r to Player 1 and k to Player 2, etc.

We claim that the only Nash equilibrium in the game Γ_σ, induced by this strategy is for Player 1 to play B if he obtains the signal k and play T if he obtains the signal r, and for Player 2 to always play L. Indeed, the posterior probabilities after receiving the signals are:

For Player 1: $p_k = P(\Gamma_L \mid k) = 1/3 - \varepsilon < 1/3$
$p_r = P(\Gamma_L \mid r) = 2/3 + \varepsilon > 2/3.$

For Player 2: $q_k = P(\Gamma_L \mid k) = (1/3 + \varepsilon)/(1 + 2\varepsilon) < 1/3$
$q_r = P(\Gamma_L \mid r) = (2/3 - \varepsilon)/(1 - 2\varepsilon) > 2/3.$

Since it is common knowledge that, in any event, neither player will know the true game after the signal, the available moves after the signalling will still be T, B for Player 1 and L, R for Player 2. Each player will therefore face an expected payoff matrix $\Gamma(p)$ given by

$$\Gamma(p) = p\Gamma_L + (1 - p)\Gamma_G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{cc} L & R \\ \begin{pmatrix} 9p - 6, 9p - 6 & 9p - 3, 6p - 4 \\ 6p - 4, 9p - 3 & 6p - 2, 6p - 2 \end{pmatrix} \end{array}$$

in which p equals his posterior probability for Γ_L. Now, since $p_k < 1/3$ and $p_r > 2/3$ it is a dominant strategy for Player 1 to play B when hearing k and T when hearing r. For the same reason it is dominant for Player 2 to play L when hearing r. Finally, when Player 2 hears k, then either Player 1 also heard k, in which case Player 1 plays B and Player 2's best response is L (since $q_r < 1/3$) or Player 1 heard r, which implies that the game must be Γ_L, in which case L is again a dominant strategy. This proves our claim that Γ_σ has a unique Nash equilibrium. The corresponding payoff is $0.5[(1/3 - \varepsilon)(2, 6) + 2\varepsilon(3, 3) + (2/3 - \varepsilon)(3, 3) + (2/3 + \varepsilon)(-4, -3) + (1/3 - \varepsilon)(-6, -6)] = (-1 + 1.5\varepsilon, 0)$. Taking $\varepsilon \rightarrow 0$ we obtain that $(-1, 0) \in \bar{X}(0.5)$ and similarly $(0, -1) \in \bar{X}(0.5)$.

Inducing $(-1 + 4.5\varepsilon, 1.5 - 3\varepsilon)$. With the same notation as above, consider the following (behavioral) signalling strategy:

If Γ_L : $(1/3 - \varepsilon)kk + 2\varepsilon gk + \varepsilon gr + (2/3 - 2\varepsilon)rr$.

If Γ_R : $(2/3 - 2\varepsilon)kk + \varepsilon ks + 2\varepsilon rs + (1/3 - \varepsilon)rr$.

It is readily verified that the posteriors after receiving the signals satisfy:

For Player 1: $p_k < 1/3$; $p_r < 2/3$; $p_g = 1$.

For Player 2: $q_k > 1/3$; $q_r > 2/3$; $q_s = 0$.

A discussion similar to the one for the previous case leads to the conclusion that in the game induced by this strategy there is a unique Nash equilibrium in which Player 1 plays B when receiving k or r and T when receiving g. Player 2 plays L when receiving k or r and R when receiving s. The corresponding payoff is $0.5[(1 - 3\varepsilon)(2, 6) + 3\varepsilon(3, 3) + (1 - 3\varepsilon)(-4, -3) + 3\varepsilon(-2, -2)] = (-1 + 4.5\varepsilon, 1.5 - 3\varepsilon)$, proving that $(-1, 1.5) \in \bar{X}(0.5)$ and similarly $(1.5, -1) \in \bar{X}(0.5)$. This concludes the proof of Proposition 4.

Notice that Proposition 4 does not fully determine $X(0.5)$. We conjecture, however, that the set we found is in fact the whole of $X(0.5)$.

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