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RESEARCH ARTICLE

Multiple equilibria in asymmetric first-price auctions

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Abstract Maskin and Riley (Games Econ Behav 45:395–409, 2003) and Lebrun (Games Econ Behav 55:131–151, 2006) prove that the Bayes–Nash equilibrium of first-price auctions is unique. This uniqueness requires the assumption that a buyer never bids above his value (which amounts to the elimination of weakly dominated strategies). We demonstrate that, in asymmetric first-price auctions (with or without a minimum bid), the relaxation of this assumption results in additional equilibria that are substantial. Although in each of these additional equilibria no buyer wins with a bids above his value, the allocation of the object and the selling price may vary among the equilibria. In particular, we show that these yield higher revenue. We show that such phenomena can only occur under certain types of asymmetry in the distributions of values.

Keywords Asymmetric auctions · First-price auctions · Multiple equilibria

JEL Classification C72 · D44

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1 Introduction

In symmetric auctions, there is a unique Bayes–Nash equilibrium (see Vickrey 1961; McAdams 2007).[1] This uniqueness also applies to asymmetric auctions; however, with the additional assumption that a buyer never bids above his value (see Lebrun 2006; Maskin and Riley 2003). This assumption amounts to the elimination of weakly dominated strategies which, in other contexts of game theory, may be questionable. [2] We demonstrate that, in asymmetric first-price auctions (with or without a minimum bid), the relaxation of this assumption may result in additional equilibria that are substantially different from each other. Although in each of these additional equilibria no buyer wins with a bid above his value, the allocation of the object and the selling price vary among the equilibria. These additional equilibria are closely related to equilibria in an environment with a minimum bid where buyers do not bid above their values.

To present our main observation, consider the following example.

Example 1 Buyer 1 has values drawn uniformly from [0,5]. Buyer 2 has values drawn uniformly from [6,7]. There is no minimum bid.

Claim *(Equilibrium 1): The following pair of inverse bid functions form an equilibrium, buyer 1 bids his value if* $v_1 \leq 3$ *(i.e.,* $v_1(b) = b$ *if* $b < 3$*), and, otherwise,*

$$v_1(b) = \frac{36}{(2b-6)\left(\frac{1}{5}\right)e^{\frac{9}{4}+\frac{6}{6-2b}} + 24 - 4b}, \tag{1}$$

$$v_2(b) = 6 + \frac{36}{(2b-6)\,(20)\,e^{-\frac{9}{4}-\frac{6}{6-2b}} - 4b}. \tag{2}$$

for $3 \leq b \leq 4\frac{1}{3}$ *(see Fig. 1 for a graph of the bid functions).*

Proof It follows from (Kaplan and Zamir 2012) that this is the unique equilibrium under the assumption that no buyer bids more than his value. □

Now, by allowing buyers to bid more than their values, we are able present two other equilibria in which such bidding occurs off the equilibrium path.

Claim *(Equilibrium 2): The following vector of bid functions* $\widetilde{\boldsymbol{b}}$ *form an equilibrium. Buyer 1 bids* $\widetilde{b}_1(v_1) = \frac{v_1}{2} + 2$ *if* $v_1 > 4$ *and* $\widetilde{b}_1(v_1) = v_1/4 + 3$, *otherwise. Buyer 2 bids* $\widetilde{b}_2(v_2) = \frac{v_2}{2} + 1$. *(See Fig. 2.)*

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[1] This uniqueness requires a low bound for the bids (such as 0). See (Baye and Morgan 1999) and (Kaplan and Wettstein 2000) for details.

[2] For example, in strategic form games, the iterated elimination of weakly dominated strategies may eliminate a Nash equilibrium that strictly Pareto dominates all other Nash equilibria (see Maschler et al. 2013, page 108, Example 4.34). Furthermore, different orders of the elimination of the weakly dominated strategies may result in different outcomes (see Maschler et al. (2013), page 95, Example 4.15).

Proof To prove that this is indeed an equilibrium we argue as follows.

- Since buyer 2 is always bidding 4 or more, buyer 1 with $v_1 < 4$, has no profitable deviation since any deviation to $b_1 < 4$ is irrelevant and any deviation to $b_1 > 4$ yields a negative expected payoff.
- Buyer 2 with value v_2 cannot profit by deviating to $b_2 < 4$. Indeed, the probability of winning with such a bid is $4(b_2 - 3)/5$ and hence the best bid in this region is

  $$\widetilde{b}_2 = \arg \max_{b_2 \in [3,4]} \frac{4(b_2 - 3)}{5}(v_2 - b_2) = \min \left\{ \max \left\{ \frac{v_2 + 3}{2}, 3 \right\}, 4 \right\} = 4,$$

  since $v_2 \geq 6$.
- Buyer 2 with value v_2 bidding b_2 in [4, 4.5] has probability $(2b_2 - 4)/5$ of winning. Hence, his best bid in this region is

  $$\widetilde{b}_2 = \arg \max_{b_2 \in [4,4.5]} \frac{2b_2 - 4}{5}(v_2 - b) = \min \left\{ \max \left\{ \frac{v_2}{2} + 1, 4 \right\}, 4.5 \right\} = \frac{v_2}{2} + 1,$$

  since $v_2 \in [6, 7]$.

Buyer 1 with $v_1 > 4$ cannot profit by deviating to $b_1 < 4$ (again since buyer 2 is always bidding 4 or more). For $b_1 \in [4, 4.5]$, the probability of winning is $(2b_1 - 8)$ and hence the best reply to buyer 2's bid function is

$$\widetilde{b}_1 = \arg \max_{b_1 \in [4,4.5]} (2b_1 - 8)(v_1 - b) = \frac{v_1}{2} + 2.$$

□

Note that in Equilibrium 2, buyer 1 bids strictly more than his value for $v_1 \in [0, 4)$.

Claim *Equilibrium 3: The following vector of bid functions* $\widehat{\boldsymbol{b}}$ *form an equilibrium. Buyer 1 bids* $\widehat{b}_1(v_1) = v_1/5 + 4$ *and buyer 2 bids 5. (See Fig. 3.)*

Proof Note that buyer 1 has no incentive to deviate to bidding above 5 since winning would yield a negative profit for him. There is also no incentive for buyer 1 to deviate to a bid below 5, since it would yield the same profit of zero. Given buyer 1's strategy, buyer 2 then faces the following maximization problem:

$$\widehat{b}_2(v_2) = \arg \max_{b_2 \in [0,5]} (b_2 - 4)(v_2 - b_2) = \min \left\{ \max \left\{ \frac{v_2 + 4}{2}, 4 \right\}, 5 \right\} = 5,$$

since $v_2 \geq 6$. □

Note that in Equilibrium 3, buyer 1 always bids strictly more than his value for all $v_1 \in [0, 5)$.

The revenue clearly differs among all three equilibria; yet, this is still consistent with revenue equivalence (Myerson 1981) since all three equilibria yield different allocations, and hence revenue need not be the same. See Fig. 4 for the expected revenue and probability that buyer 1 wins in each of the possible equilibria.

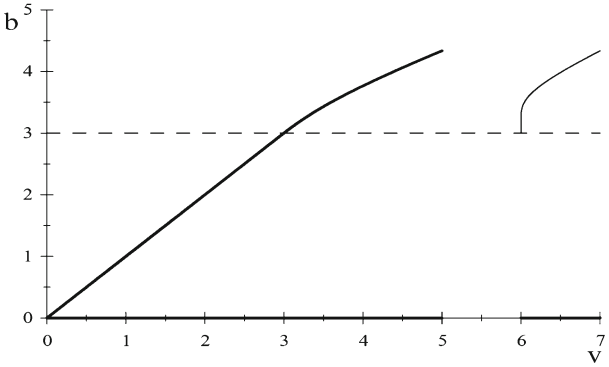

Fig. 1 Equilibrium 1. The thicker line is buyer 1’s bid function

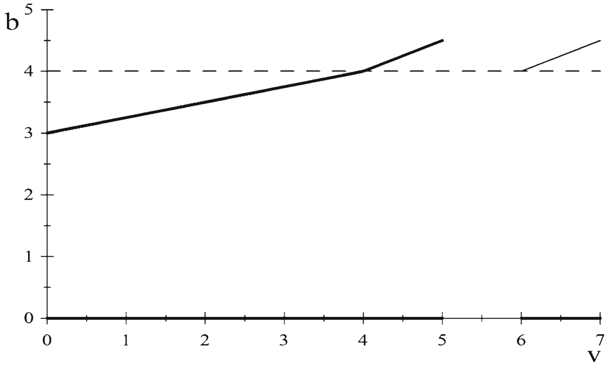

Fig. 2 Equilibrium 2. The thicker line is buyer 1’s bid function

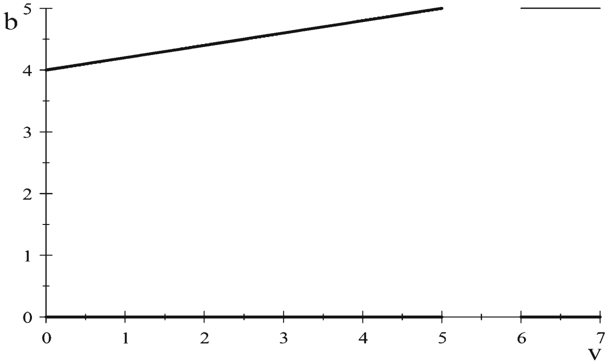

Fig. 3 Equilibrium 3. The thicker line is buyer 1’s bid function

	Probability that buyer 1 wins	Revenue
Equilibrium 1	0.13333	3.99099
Equilibrium 2	0.1	4.26666
Equilibrium 3	0	5

Fig. 4 Probability that buyer 1 wins and the expected revenue in each of the three equilibria

2 Model

Consider n buyers bidding for an indivisible good. Each buyer i has his value drawn from a distribution F_i with support $[\underline{v}_i, \overline{v}_i]$ (where $0 \leq \underline{v}_i < \overline{v}_i$). This environment will be fixed throughout the paper. The selling mechanism that we will consider is a first-price auction with a minimum bid m, which we denote by A^m. Let $b_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}$ denote a bid function (pure strategy) and $\boldsymbol{b} = (b_i)_{i=1}^n$ denote a vector of bid functions.

Definition 1 An equilibrium $\boldsymbol{b}^m$ of auction A^m is said to be *standard* if $P(\{b_i^m(v_i) > v_i \text{ and } b_i^m(v_i) \geq m\}) = 0$ for all i; otherwise, it is called *non-standard*.

To elaborate on the definition, let us call an acceptable bid, a bid greater than or equal to the minimum bid. The above definition says that in a standard equilibrium no buyer makes an acceptable bid that is (strictly) above his value, even if such bids that never win in equilibrium. In contrast, in a non-standard equilibrium there is a least one buyer that with positive probability makes an acceptable bid that is strictly above his value (but still never wins in the equilibrium).

As mentioned above, in our model, there is a unique standard equilibrium (see Maskin and Riley 2003; Lebrun 2006). Also, in a non-standard equilibrium, although some buyers may bid above their values, no buyer that bids above his value wins with positive probability. Such bidding cannot occur on the equilibrium path since such a buyer would have a profitable deviation (for example, bidding his value). Nevertheless, as we already saw in our example, the ability to bid above one's value may substantially affect the allocation of the object and the selling price. In the above example, the first equilibrium is standard, while the other two are non-standard.

In the following, we compare equilibria of two different mechanisms, namely, two first-price auctions with different minimum bids.

Definition 2 A vector of bid functions $\boldsymbol{b}^m$ in auction A^m is said to be equivalent to a vector of bid functions $\boldsymbol{b}^{\widetilde{m}}$ in auction $A^{\widetilde{m}}$ if for any realization of values of the buyers, both vectors yield the same ex-post payoffs for the buyers and the seller and the same allocation of the good. We denote equivalence between $\boldsymbol{b}^m$ and $\boldsymbol{b}^{\widetilde{m}}$ by $\boldsymbol{b}^m \approx \boldsymbol{b}^{\widetilde{m}}$.

Remark 1 The equivalence $\boldsymbol{b}^m \approx \boldsymbol{b}^{\widetilde{m}}$ does not imply the equality of the bid functions, $\boldsymbol{b}^m = \boldsymbol{b}^{\widetilde{m}}$. In the opposite direction, $\boldsymbol{b}^m = \boldsymbol{b}^{\widetilde{m}}$ does not imply $\boldsymbol{b}^m \approx \boldsymbol{b}^{\widetilde{m}}$.

As an example of the first claim in the remark, consider the environment where buyer 1 has a value uniformly drawn from [0, 1] and buyer 2 has a value uniformly drawn from [4, 5]. Consider two equilibria for two different minimum bids. With a minimum bid of 0, buyer 1 bids his value and buyer 2 bids 1. With a minimum bid of 1, buyer 1 bids 0 and buyer 2 bids 1. These two equilibria are equivalent but have different bid functions.

For an example in the opposite direction, consider the symmetric auction where both buyers have values drawn uniformly from [0, 1] and bid half their value. This is an equilibrium when the minimum bid is 0 and when the minimum bid is 1. However, in the former case, buyers always receive the object and in the latter case they never do. Hence, the two equilibria are not equivalent.

Generalizing the insight from the above example that proves the first part of the remark, consider two buyers where buyer 1's value distribution is on [0, 1] and buyer 2's value distribution is on $[\alpha, \beta]$ where $0 < \alpha < \beta$. Assume that there is a minimum bid m where $0 < m < \alpha$. (For now, assume that whenever there is a tie, the winner is buyer 2.) Consider a standard equilibrium with this minimum bid m. This is still an equilibrium if it is modified with the only change that whenever buyer 1's value is below m, he bids m. Furthermore, if the minimum bid is now lowered to $\widetilde{m} < m$, this modified vector of bid functions is still an equilibrium, but it is clearly non-standard. The following proposition captures this intuition (without the assumption regarding the case of a tie).

Proposition 1 *For any standard equilibrium* $\boldsymbol{b}^m$ *of auction* A^m *where* $\min_i \underline{v}_i < m < \max_i \underline{v}_i$ *and for any* $\widetilde{m} < m$*, there exists a non-standard equilibrium* $\boldsymbol{b}^{\widetilde{m}}$ *of auction* $A^{\widetilde{m}}$ *that is equivalent to* $\boldsymbol{b}^m$ ($\boldsymbol{b}^{\widetilde{m}} \approx \boldsymbol{b}^m$).

Proof It is enough to prove this for $\boldsymbol{b}^m$ in which $b_i^m(v_i) \geq m$ if and only if $v_i \geq m$. If instead $\boldsymbol{b}^m$ does not satisfy this condition, we can construct an equivalent standard equilibrium $\widehat{\boldsymbol{b}}^m$ of A^m that does satisfy this property. Since the equivalence relationship is clearly transitive, any non-standard $\boldsymbol{b}^{\widetilde{m}}$ equivalent to $\widehat{\boldsymbol{b}}^m$ would also be equivalent to $\boldsymbol{b}^m$. More specifically, define $\widehat{\boldsymbol{b}}^m$ as follows: $\widehat{b}_i^m(v_i) \stackrel{\text{def}}{=} b_i^m(v_i)$ for all i, and for all $v_i \geq m$ where $b_i^m(v_i) \geq m$ and for all $v_i < m$ (and hence $b_i^m(v_i) < m$ since $\boldsymbol{b}^m$ is a standard equilibrium). Also, define $\widehat{b}_i^m(v_i) \stackrel{\text{def}}{=} \frac{m+v_i}{2}$ for all i, for all $v_i \geq m$ where $b_i^m(v_i) < m$. Notice that when $v_i \geq m$ and $b_i^m(v_i) < m$, buyer i is not winning (with positive probability) in $\boldsymbol{b}^m$ and hence $\widehat{b}_i^m(v_i)$ is also not winning against $\boldsymbol{b}_{-i}^m$, since this would be a profitable deviation which is impossible since $\boldsymbol{b}^m$ is an equilibrium of A^m. Since by construction, $\widehat{b}_j^m(v_j) \geq b_j^m(v_j)$ for all $j \neq i$, $\widehat{b}_i^m(v_i)$ is also not winning against $\widehat{\boldsymbol{b}}_{-i}^m$. Thus, $\widehat{\boldsymbol{b}}^m \approx \boldsymbol{b}^m$. Finally, $\widehat{\boldsymbol{b}}^m$ is an equilibrium of auction A^m since the winning bids are the same as in $\boldsymbol{b}^m$ and the losing bids are weakly higher.

Given an equilibrium $\boldsymbol{b}^m$ of auction A^m where $\min_i \underline{v}_i < m < \max_i \underline{v}_i$ and $\widetilde{m} < m$, let ε be such that $0 < \varepsilon < \min\{\max_i \underline{v}_i - m, m - \widetilde{m}, m\}$. Define $\boldsymbol{b}^{\widetilde{m}}$ as follows: $b_i^{\widetilde{m}}(v_i) \stackrel{\text{def}}{=} b_i^m(v_i)$ for all i and for all $v_i \geq m$ and $b_i^{\widetilde{m}}(v_i) \stackrel{\text{def}}{=} m - \varepsilon + \frac{F_i(v_i)}{F_i(m)}\varepsilon$ for all v_i s.t. $v_i < m$. We created $\boldsymbol{b}^{\widetilde{m}}$ from $\boldsymbol{b}^m$ by keeping the bids the same for values weakly above m and distributing all bids below m uniformly on the interval $[m - \varepsilon, m]$. We now proceed to show that $\boldsymbol{b}^{\widetilde{m}}$ is a non-standard equilibrium of auction $A^{\widetilde{m}}$ and it is equivalent to $\boldsymbol{b}^m$ under the assumption that $b_i^m(v_i) \geq m$ if and only if $v_i \geq m$.

Step 1. $\boldsymbol{b}^{\widetilde{m}} \approx \boldsymbol{b}^m$.

Any buyer i with $\underline{v}_i > m$ always bids higher than m in $\boldsymbol{b}^m$ and hence by definition any buyer i with $\underline{v}_i > m$ always bids higher than m in $\boldsymbol{b}^{\widetilde{m}}$. Since $\max_i \underline{v}_i > m$, both in $\boldsymbol{b}^{\widetilde{m}}$ and in $\boldsymbol{b}^{m}$, there is at least one buyer whose bid is greater than or equal to m for all his values (i.e., with probability 1). Hence, no buyer wins with positive probability with a bid strictly below m . Since $b_i^{\widetilde{m}}(v_i) = b_i^m(v_i)$ for values $v_i \geq m$, the allocation and the price paid for the object in $\boldsymbol{b}^{\widetilde{m}}$ of auction $A^{\widetilde{m}}$ is the same as in the equilibrium $\boldsymbol{b}^{m}$ of auction A^m. Thus, $\boldsymbol{b}^{\widetilde{m}} \approx \boldsymbol{b}^{m}$.

Step 2. We next show that $\boldsymbol{b}^{\widetilde{m}}$ is an equilibrium of $A^{\widetilde{m}}$.

Step 2a. In $\boldsymbol{b}^{\widetilde{m}}$ of auction $A^{\widetilde{m}}$, no buyer with value less than m has an incentive to deviate.

In fact, since the selling price is always greater than or equal to m, no buyer with a value less than m would have an incentive to deviate; any deviation to a bid less than m would not affect the payoff (as it would never win) and any deviation to a bid greater than or equal to m would result in a non-positive payoff.

Step 2b. In $\boldsymbol{b}^{\widetilde{m}}$ of auction $A^{\widetilde{m}}$, no buyer i with value $v_i \geq m$ has an incentive to deviate to $b_i \geq m$.

Recall that by definition of $\boldsymbol{b}^{\widetilde{m}}$, we have $b_i^{\widetilde{m}}(v_i) = b_i^m(v_i)$ for all $v_i \geq m$ (and hence by our assumption on $\boldsymbol{b}^{m}$, for all $b_i^m(v_i) \geq m$). Since the distribution of bids above m is the same for both $\boldsymbol{b}^{\widetilde{m}}$ and $\boldsymbol{b}^{m}$ and in $\boldsymbol{b}^{m}$ there is no incentive to deviate to a bid above m, in $\boldsymbol{b}^{\widetilde{m}}$ there is also no incentive to deviate to a bid above m. Formally, let $W_i^m(b_i; \boldsymbol{b}^m_{-i})$ be the winning probability of buyer i when bidding b_i against $\boldsymbol{b}^m_{-i}$ in auction A^m and let $W_i^{\widetilde{m}}(b_i; \boldsymbol{b}^{\widetilde{m}}_{-i})$ be his winning probability when bidding b_i against $\boldsymbol{b}^{\widetilde{m}}_{-i}$ in auction $A^{\widetilde{m}}$. Let $\pi_i^m(v_i, b_i; \boldsymbol{b}^m_{-i}) \overset{\text{def}}{=} W_i^m(b_i; \boldsymbol{b}^m_{-i})(v_i - b_i)$ and $\pi_i^{\widetilde{m}}(v_i, b_i; \boldsymbol{b}^{\widetilde{m}}_{-i}) \overset{\text{def}}{=} W_i^{\widetilde{m}}(b_i; \boldsymbol{b}^{\widetilde{m}}_{-i})(v_i - b_i)$ be the expected profit of buyer i when bidding b_i against $\boldsymbol{b}^m_{-i}$ in auction A^m and against $\boldsymbol{b}^{\widetilde{m}}_{-i}$ in auction $A^{\widetilde{m}}$, respectively. Observe that for all $b_i \geq m$, it follows from the definition of $\boldsymbol{b}^{\widetilde{m}}$ that $W_i^{\widetilde{m}}(b_i; \boldsymbol{b}^m_{-i}) = W_i^m(b_i; \boldsymbol{b}^{\widetilde{m}}_{-i})$ and hence $\pi_i^m(v_i, b_i; \boldsymbol{b}^m_{-i}) = \pi_i^{\widetilde{m}}(v_i, b_i; \boldsymbol{b}^{\widetilde{m}}_{-i})$. Since $\pi_i^m(v_i, b_i^m(v_i); \boldsymbol{b}^m_{-i}) \geq \pi_i^m(v_i, b_i; \boldsymbol{b}^m_{-i})$ for all $b_i \geq m$ (since $\boldsymbol{b}^{m}$ is an equilibrium), we have $\pi_i^{\widetilde{m}}(v_i, b_i^{\widetilde{m}}(v_i); \boldsymbol{b}^{\widetilde{m}}_{-i}) = \pi_i^{\widetilde{m}}(v_i, b_i^m(v_i); \boldsymbol{b}^{\widetilde{m}}_{-i}) = \pi_i^m(v_i, b_i^m(v_i); \boldsymbol{b}^m_{-i}) \geq \pi_i^m(v_i, b_i; \boldsymbol{b}^m_{-i}) = \pi_i^{\widetilde{m}}(v_i, b_i; \boldsymbol{b}^{\widetilde{m}}_{-i})$ for all $b_i \geq m$.

Step 2c. In $\boldsymbol{b}^{\widetilde{m}}$ of auction $A^{\widetilde{m}}$, no buyer i with value $v_i \geq m$ has an incentive to deviate to $b_i < m$.

Since, as we saw in the proof of Step 1, there is a buyer j that in $\boldsymbol{b}^{\widetilde{m}}$ bids m or more with probability one (namely, any buyer j with $\underline{v}_j \geq m$), no buyer $i \neq j$ has a profitable deviation to bid $b_i < m$ since such a bid would not win. It remains to show that j also has no incentive to deviate to $b_j < m$. Now if there are two or more buyers with $\underline{v}_j \geq m$, then there is no incentive for any such buyer to bid $b_j < m$ since another buyer j' with $\underline{v}_{j'} \geq m$ is bidding m or more with probability one. Hence, we are only left with the case of one buyer j with $\underline{v}_j \geq m$ and all other $n - 1$ buyers with $\underline{v}_i < m$.

By the definition of $\boldsymbol{b}^{\widetilde{m}}$, there are no bids below $m - \varepsilon$; hence clearly any $b_j < m - \varepsilon$ is not profitable. Thus, it is enough to show that for buyer j and for any possible value v_j, the expected profit of bidding $b_j \in [m - \varepsilon, m)$ is increasing in b_j. If this is so and if there is a profitable deviation to $b_j \in [m - \varepsilon, m)$, then there is a profitable deviation to m (the expected profit is upper semi-continuous). This would be in contradiction to what we proved in Step 2b.

To prove this monotonicity, note that $\frac{\partial \pi_j^{\widetilde{m}}(v_j, b; \boldsymbol{b}_{-j}^{\widetilde{m}})}{\partial b} = W^{\widetilde{m}\prime}(b; \boldsymbol{b}_{-j}^{\widetilde{m}})(v_j - b) - W^{\widetilde{m}}(b; \boldsymbol{b}_{-j}^{\widetilde{m}})$ and also note that by the definition of $b_i^{\widetilde{m}}(v_i)$ the distribution of bids for values below m of player i is uniform on $[m - \varepsilon, m)$. Since all other $n - 1$ buyers bid $[m - \varepsilon, m)$ with positive probability, we have $W^{\widetilde{m}}(b; \boldsymbol{b}_{-j}^{\widetilde{m}}) = (b - (m - \varepsilon))^{n-1}/c$ (for $b \in [m - \varepsilon, m)$), and $W^{\widetilde{m}\prime}(b; \boldsymbol{b}_{-j}^{\widetilde{m}}) = (n - 1)(b - (m - \varepsilon))^{n-2}/c$ where c is a constant. Hence, since $n \geq 2$, we have

$$\begin{aligned}\frac{\partial \pi_j^{\widetilde{m}}(v_j, b; \boldsymbol{b}_{-j}^{\widetilde{m}})}{\partial b} &= \frac{(b - (m - \varepsilon))^{n-2}}{c}\left[(n - 1)(v_j - b) - (b - (m - \varepsilon))\right] \\ &\geq \frac{(b - (m - \varepsilon))^{n-2}}{c}\left[v_j - 2b + m - \varepsilon\right].\end{aligned}$$

The expression $\frac{(b-(m-\varepsilon))^{n-2}}{c} \geq 0$ for all $b \in [m - \varepsilon, m]$. Since $\varepsilon < \underline{v}_j - m$, when $b = m$, the expression $v_j - 2b + m - \varepsilon = v_j - m - \varepsilon \geq \underline{v}_j - m - \varepsilon \geq 0$. Since this last expression is strictly decreasing in b, it is positive for $b < m$. Hence, $\frac{\partial \pi_j^{\widetilde{m}}(v_j, b; \boldsymbol{b}_{-j}^{\widetilde{m}})}{\partial b} \geq 0$ for all $b \in [m - \varepsilon, m)$. This together with the condition that $W^{\widetilde{m}}(m - \varepsilon; \boldsymbol{b}_{-j}^{\widetilde{m}}) = 0$ prove that $\pi_j^{\widetilde{m}}(v_j, m; \boldsymbol{b}_{-j}^{\widetilde{m}}) \geq \pi_j^{\widetilde{m}}(v_j, b; \boldsymbol{b}_{-j}^{\widetilde{m}})$ for all $b \leq m$.

This concludes the proof that $\boldsymbol{b}^{\widetilde{m}}$ is an equilibrium of auction $A^{\widetilde{m}}$.

Step 3. The equilibrium $\boldsymbol{b}^{\widetilde{m}}$ is non-standard.

Since there exists a buyer k with $\underline{v}_k < m$, there exists a $\mu > 0$ such that for values between $\underline{v}_k$ and $\underline{v}_k + \mu$, buyer k bids in the interval $[m - \varepsilon, m]$. For small enough ε and μ, these bids are strictly greater than his values and greater than $\widetilde{m}$ (since $\varepsilon < m - \widetilde{m}$); hence, $\boldsymbol{b}^{\widetilde{m}}$ is non-standard. □

We now show that while the condition $m < \max_i \underline{v}_i$ appearing in Proposition 1 is not necessary for the existence of a non-standard equilibrium, the weaker condition, $m \leq \max_i \underline{v}_i$, is a necessary condition.

Proposition 2 *If $m > \max_i \underline{v}_i$, then there does not exist a non-standard equilibrium $\boldsymbol{b}^m$ of auction A^m.*

Proof Assume by way of contradiction that there is a non-standard equilibrium $\boldsymbol{b}^m$ of auction A^m. Then, there is a buyer j with value v_j that bids with positive probability $b_j(v_j) > v_j$ where $b_j(v_j) \geq m$. There is also a positive probability that $\max_{i \neq j} v_i < b_j(v_j)$ (since $b_j(v_j) \geq m > \max_i \underline{v}_i$). These two events are independent; hence, their intersection has positive probability. In this event, buyer j will win the auction and pay more than his value, which cannot be the case in equilibrium. Therefore, there cannot be a non-standard equilibrium $\boldsymbol{b}^m$ of auction A^m. □

Remark 2 The condition of Proposition 2 cannot be weakened to $m \geq \max_i \underline{v}_i$. In other words, there may indeed be a non-standard equilibrium when $m = \max_i \underline{v}_i$, as demonstrated by the following example. There are two buyers with values uniformly distributed on [1, 2] and one buyer with a value uniformly distributed on [0, 1/2] and a minimum bid $m = 1$. In this case, $m = \max_i \underline{v}_i$. There exists an equilibrium in which the first two buyers bid $b_i(v_i) = (v_i + 1)/2$ for $i = 1, 2$ and $v_i \in [1, 2]$ and buyer 3 bids $b_3(v_3) = 1$ for all $v_3 \in [0, 1/2]$. This equilibrium is non-standard since the buyer 3 bids is strictly more than his value and weakly more than m.

We make use of Proposition 2 to prove a stronger result, namely:

Proposition 3 *If $m > \max_i \underline{v}_i$ and $\boldsymbol{b}^m$ is an equilibrium of auction A^m, then there does not exist an $\widetilde{m}$ and non-standard equilibrium $\boldsymbol{b}^{\widetilde{m}}$ of auction $A^{\widetilde{m}}$ such that $\boldsymbol{b}^{\widetilde{m}} \approx \boldsymbol{b}^m$.*

Proof By Proposition 2, $\boldsymbol{b}^m$ must be a standard equilibrium of auction A^m. If $\widetilde{m} > \max_i \underline{v}_i$, then by Proposition 2, there is no non-standard equilibrium of $A^{\widetilde{m}}$. A fortiori, there is no non-standard equilibrium that is equivalent to $\boldsymbol{b}^m$. So, now we need to examine the case where $\widetilde{m} \leq \max_i \underline{v}_i < m$. Assume that $\boldsymbol{b}^{\widetilde{m}}$ is a non-standard equilibrium of $A^{\widetilde{m}}$. We will show that in this case, there is a positive probability event in which the object is allocated in $\boldsymbol{b}^m$ of auction A^m but not allocated in $\boldsymbol{b}^{\widetilde{m}}$ of auction $A^{\widetilde{m}}$ (and hence $\boldsymbol{b}^{\widetilde{m}} \not\approx \boldsymbol{b}^m$). Indeed, this happens when there exists a buyer j such that $\widetilde{m} < v_j < m$ and $\max_{i \neq j} v_i < v_j$. Note that, this happens with positive probability since $\widetilde{m} \leq \max_i \underline{v}_i$ implies that $\widetilde{m} < \max_i \overline{v}_i$ by the assumption that $\underline{v}_i < \overline{v}_i$ for all i . In this event, the object is not allocated in (the standard) equilibrium $\boldsymbol{b}^m$ since all values are below m. On the other hand, the object must be allocated in equilibrium $\boldsymbol{b}^{\widetilde{m}}$. Otherwise, all buyers with such values must bid below $\widetilde{m}$. However, if this were the case, then buyer j could earn strictly positive profit by bidding $v_j - \varepsilon$ for small enough $\varepsilon > 0$ (s.t. $v_j - \varepsilon > \widetilde{m}$). ☐

We next show that any non-standard equilibrium has an equivalent standard equilibrium with a higher minimum bid. This is evident from our example in Equilibrium 3 where if the seller sets a minimum bid of $m = 5$, then the Equilibrium 3 bid functions (above the minimum bid) form the unique standard equilibrium.

Proposition 4 *For any non-standard equilibrium $\boldsymbol{b}^{\widetilde{m}}$ of $A^{\widetilde{m}}$, there exists an $m > \widetilde{m}$ and a $\boldsymbol{b}^m$ such that $\boldsymbol{b}^m$ is a standard equilibrium of A^m and $\boldsymbol{b}^m \approx \boldsymbol{b}^{\widetilde{m}}$.*

Proof Consider $m = \sup_{i, v_i} \{b_i^{\widetilde{m}}(v_i) : b_i^{\widetilde{m}}(v_i) > v_i \text{ and } b_i^{\widetilde{m}}(v_i) \geq \widetilde{m}\}$. Clearly, $m \geq \widetilde{m}$. Define $b_i^m(v_i) \overset{\text{def}}{=} \min\{v_i, b_i^{\widetilde{m}}(v_i)\}$. By definition of m, $b_i^{\widetilde{m}}(v_i) \geq m$ implies that $b_i^{\widetilde{m}}(v_i) \leq v_i$. Hence, $b_i^m(v_i) = b_i^{\widetilde{m}}(v_i)$ for all $b_i^{\widetilde{m}}(v_i) \geq m$. In $\boldsymbol{b}^{\widetilde{m}}$ of auction $A^{\widetilde{m}}$, the probability that the winning bid is strictly below m is zero. Otherwise, there is a positive probability that there is a buyer j who wins while bidding $b_j^{\widetilde{m}}(v_j) \leq b^* < m$. Then, any bid greater than b^* (by any buyer) must win with a positive probability. However, by definition of m, there is a buyer k bidding $b_k^{\widetilde{m}}(v_k)$ where $m \geq b_k^{\widetilde{m}}(v_k) > v_k$ and $b_k^{\widetilde{m}}(v_k) > b^*$. This buyer k will be winning with positive probability in $A^{\widetilde{m}}$ while bidding above his value which cannot happen in equilibrium. Hence, $\boldsymbol{b}^m \approx \boldsymbol{b}^{\widetilde{m}}$ since in both auctions all winning bids are (weakly) above m and in that region, $\boldsymbol{b}^m$ and $\boldsymbol{b}^{\widetilde{m}}$ coincide.

To see that $\boldsymbol{b}^m$ is indeed an equilibrium of A^m, observe that any buyer i with value v_i not winning in $\boldsymbol{b}^{\widetilde{m}}$ is still not winning in $\boldsymbol{b}^m$ and has no incentive to change his bid

$\boldsymbol{b}_i^m(v_i)$ (since $\boldsymbol{b}_i^m(v_i) \le b_i^{\widetilde{m}}(v_i)$ and winning bids are the same). If $b_i^{\widetilde{m}}(v_i)$ is winning in $\boldsymbol{b}^{\widetilde{m}}$ of $A^{\widetilde{m}}$, then $b_i^m(v_i) = b_i^{\widetilde{m}}(v_i) \ge m$. Since winning bids are the same, there are no profitable deviations from b_i^m, since there are no profitable deviations from $b_i^{\widetilde{m}}$. We conclude that $\boldsymbol{b}^m$ is an equilibrium in A^m, and as we proved before this implies that $\boldsymbol{b}^m \approx \boldsymbol{b}^{\widetilde{m}}$. Finally, since if $m = \widetilde{m}$, then the same equilibrium would be standard and non-standard—a contradiction. Thus, $m > \widetilde{m}$. □

Given a non-standard equilibrium $\boldsymbol{b}^{\widetilde{m}}$ of $A^{\widetilde{m}}$, let $m(\boldsymbol{b}^{\widetilde{m}})$ be the minimum bid defined in the proof of Proposition 4. Namely, $m(\boldsymbol{b}^{\widetilde{m}}) = \sup_{i,v_i}\{b_i^{\widetilde{m}}(v_i) : b_i^{\widetilde{m}}(v_i) > v_i$ and $b_i^{\widetilde{m}}(v_i) \ge \widetilde{m}\}$. The following corollary states that $b_i^{\widetilde{m}}$ is still a non-standard equilibrium in any $A^{\widehat{m}}$ with $\widehat{m} < m(\boldsymbol{b}^{\widetilde{m}})$.

Corollary 1 *For any non-standard equilibrium* $\boldsymbol{b}^{\widetilde{m}}$ *of* $A^{\widetilde{m}}$, *for any* $\widehat{m} < m(\boldsymbol{b}^{\widetilde{m}})$, $\boldsymbol{b}^{\widehat{m}} \overset{\text{def}}{=} \boldsymbol{b}^{\widetilde{m}}$ *is a non-standard equilibrium of* $A^{\widehat{m}}$ *and* $\boldsymbol{b}^{\widehat{m}} \approx \boldsymbol{b}^{\widetilde{m}}$.

Proof The arguments for why $\boldsymbol{b}^{\widehat{m}}$ is an equilibrium of $A^{\widehat{m}}$ are similar to that of the previous proposition. The equilibrium is non-standard since $\widehat{m} < m(\boldsymbol{b}^{\widetilde{m}})$ and by the definition of $m(\boldsymbol{b}^{\widetilde{m}})$ there is a buyer i with value v_i where $b_i^{\widehat{m}}(v_i) > v_i$ and $b_i^{\widehat{m}}(v_i) > \widehat{m}\}$. □

In the examples in the Introduction, the second equilibrium $\widetilde{\boldsymbol{b}}$ which is a non-standard equilibrium of A^0 (first-price auction with no minimum bid) is such that $\boldsymbol{b}^4 \overset{\text{def}}{=} \widetilde{\boldsymbol{b}}$ is a standard equilibrium of A^4. Furthermore, these two equilibria are equivalent: $\widetilde{\boldsymbol{b}} \approx \boldsymbol{b}^4$. Similarly, the third equilibrium $\widehat{\boldsymbol{b}}$ is a non-standard equilibrium of A^0, $\widehat{\boldsymbol{b}}^4 \overset{\text{def}}{=} \widehat{\boldsymbol{b}}$ is a non-standard equilibrium of A^4, while $\boldsymbol{b}^5 \overset{\text{def}}{=} \widehat{\boldsymbol{b}}$ is a standard equilibrium in A^5, and all three equilibria are equivalent: $\widehat{\boldsymbol{b}} \approx \widehat{\boldsymbol{b}}^4 \approx \boldsymbol{b}^5$.

We can use the above results to confirm the result in the literature that there are no non-standard equilibria in symmetric auctions. In fact, we have the following stronger result.

Corollary 2 *In an asymmetric auction, with a common lower bound of the support, that is, there exists a* $\underline{v}$ *such that* $\underline{v} = \underline{v}_i$ *for all i, there does not exist a non-standard equilibrium. Consequently, in a symmetric auction, there does not exist a non-standard equilibrium.*

Proof By Proposition 2, there cannot be a non-standard equilibrium when the minimum bid $\widetilde{m} > \underline{v}$. However, if $\widetilde{m} = \underline{v}$, then there also cannot be a non-standard equilibrium since then there would be a buyer bidding strictly above his value (and thereby strictly above $\widetilde{m}$). We would then have $m(\boldsymbol{b}^{\widetilde{m}}) > \underline{v}$. But by Corollary 1, there would exist a non-standard equilibrium for $\underline{v} < \widehat{m} < m(\boldsymbol{b}^{\widetilde{m}})$, in contradiction to Proposition 2. If there are no non-standard equilibria for $\widetilde{m} = \underline{v}$, then there are no non-standard equilibria for $\widetilde{m} < \underline{v}$. Since if $\widetilde{m} < \underline{v}$, in equilibrium the infimum of the winning bids must not be strictly less than $\underline{v}$. (Otherwise, any buyer bidding close to this infimum would have a profitable deviation.) Hence, any non-standard equilibrium for $m < \underline{v}$ would also be a non-standard equilibrium for $m = \underline{v}$. Under symmetry, we have a $\underline{v} \overset{\text{def}}{=} \underline{v}_i$ for all i. □

Lebrun (2004) shows in Lemma A5.2-2 that in a first-price auction, in the region where the minimum bid is binding, raising the minimum bid from $\widetilde{m}$ to m, the bid functions of the buyers in the standard Bayes–Nash equilibrium increase (pointwise for $v \geq m$). Combined with Proposition 4, we obtain the following Corollary.

Corollary 3 *In a first-price auctions, if* $\boldsymbol{b}^{\widetilde{m}}$ *is a standard equilibrium of* $A^{\widetilde{m}}$ *and* $\boldsymbol{b}'^{\widetilde{m}}$ *is a non-standard equilibrium of* $A^{\widetilde{m}}$*, then the revenue from* $\boldsymbol{b}'^{\widetilde{m}}$ *is strictly higher than the revenue from* $\boldsymbol{b}^{\widetilde{m}}$*.*

Proof By Proposition 4, $\boldsymbol{b}'^{\widetilde{m}}$ is equivalent to a standard equilibrium $\boldsymbol{b}^m$ in A^m where $m > \widetilde{m}$ (and therefore yields the same revenue). Denote by $Rev(\boldsymbol{b}^m)$ the expected selling price with bid functions $\boldsymbol{b}^m$. It follows from (Lebrun 2004) that $Rev(\boldsymbol{b}'^{\widetilde{m}}) = Rev(\boldsymbol{b}^m) > Rev(\boldsymbol{b}^{\widetilde{m}})$. □

This means that from the perspective of the seller, the existence of a non-standard equilibrium is in a way an indication that the minimum bid is not optimally set; it can be raised to yield higher revenue. (Although the seller may be obligated to keep a low minimum bid or the buyers may know each others' values better than the seller knows.)

3 Concluding remarks

Our main observation is that in an asymmetric first-price auction A^m with a minimum bid m, besides the unique standard equilibrium, there may be additional non-standard equilibria where some buyers make acceptable bids above their values. In game theoretic terminology, this means that buyers are using weakly dominated strategies. This is the issue that we focus on in this paper in the context of a first-price auction. We find that this multiplicity is non-trivial in the sense that both revenue and the allocation of the good can be different from those in the standard equilibrium. However, this can only occur in asymmetric auctions, more precisely, only when $\min_i \underline{v}_i < m \leq \max_i \underline{v}_i$. We characterize non-standard equilibria in four propositions. Our main result, Proposition 1, shows how a standard equilibrium can form the basis for a non-standard equilibrium with a different (smaller) minimum bid. Propositions 2 and 3 provide sufficient conditions under which there does not exist a non-standard equilibrium. Finally, Proposition 4 shows how a non-standard equilibrium can form the basis of a standard equilibrium with a different (larger) minimum bid.

This paper fits into the small body of literature on multiple equilibria in auctions. For first-price auctions (equivalent to Bertrand price competition with a unit demand), (Baye and Morgan 1999) and (Kaplan and Wettstein 2000) show that there can be additional equilibria with mixed strategies. These require that there is no lower bound on the bids submitted. Also, in Bertrand price competition with asymmetric costs and complete information, (Erlei 2002) finds additional equilibria. For the second-price auction, (Blume and Heidhues 2004) and (Blume et al. 2009) show that there can be additional equilibria if one relaxes the assumption that buyers never bid above their values. We follow them by also allowing buyers in a first-price auction to bid above their values and (weakly) above the minimum bid. This seems worth considering given the large body of experimental literature that shows that weakly dominated strategies are not always eliminated (see Binmore 1999, for a discussion). We find that these additional equilibria are particularly important in first-price asymmetric auctions since they may be substantially different in both revenue and outcome.

Kagel and Levin (1993) find that in second-price auctions buyers bid strictly above their values 67.5 % of the time, while (Garratt et al. 2012) find that seasoned e-Bay participants still bid above their value in laboratory second-price auctions 37.5 % of the time. However, experimental results show that in first-price auctions buyers seldom bid above their values. (Kagel and Levin 1993) report 0.4 %. [3] Nevertheless, they sometimes do. Buyers have simply to believe that other buyers may sometimes bid about their value. Given the overbidding frequency in second-price auctions, this may be reasonable particularly in that this belief may not be falsified. From the Seller's viewpoint, it may be worthwhile to try to maintain this belief by not to reveal losing bids. We conclude that the widely used assumption that buyers do not bid above their values, or more generally, do not use weakly dominated strategies should be taken more carefully in asymmetric auctions.

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[3] Kagel and Levine examine a symmetric environment where according to Corollary 2, there should be no overbidding. In preliminary data (of fifty thousand bids) on asymmetric auctions, (Kaplan and Turocy 2014) find 1.1 % of buyers bid above their value. This can be broken down to 3.9 % for those with values lower than the minimum bid and 0.48 % of those with values greater than the minimum bid.

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