Repeated games with asymmetric information
modeling financial markets with two risky assets *

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Abstract. We consider multistage bidding models where two types of risky assets (shares) are traded between two agents having different information on liquidation prices of traded assets. These prices are integer random variables that are determined by the initial chance move according to a known to both players probability distribution $p$ over two-dimensional integer lattice. Player 1 is informed on the prices of both types of shares, Player 1 is not. The bids may take any integer values.

The model of $n$-stage bidding is reduced to the zero-sum repeated game with lack of information on one side. We show that, if liquidation prices of shares have finite variances, then the sequence of values of $n$-step games is bounded. This makes reasonable to consider the bidding of unlimited duration that is reduced to the infinite game $G_\infty(p)$. We give the solutions for these games.

We begin with constructing solutions for these games with distributions $p$ having two- and three-point supports. Next we build the optimal strategies of Player 1 for bidding games $G_\infty(p)$ with arbitrary distributions $p$ as convex combinations of his optimal strategies for such games with distributions having two- and three-point supports. To do this we construct the symmetric representation of probability distributions with fixed integer expectation vectors as convex combination of distributions with not more than three-point supports and with the same expectation vectors.

1. Introduction. Modelling financial markets with two risky assets by repeated games with asymmetric information

We investigate the model of multistage bidding where two types of risky assets (shares) are traded between two agents having different information on liquidation prices of traded assets. These prices are integer random variables that are determined by the initial chance move for the whole period of bidding according to a known to both players probability distribution $p \in \Delta(\mathbb{Z}^2)$ over two-dimensional integer lattice. Player 1 knows the prices of both types of shares, Player 1 does not have this information. Player 2 knows that Player 1 is an insider.

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At each step of bidding both players make simultaneously their integer bids, i.e. they post their prices for each type of shares. The player who posts the larger price for a share of given type buys one share of this type from his opponent for this price. Any integer bids are admissible. Players aim to maximize the values of their final portfolios, i.e. money plus obtained shares evaluated by their liquidation prices. Both players remember all previous bids, both their own and their opponent’s. This allows Player 2 to draw conclusions on real share prices from the actions of Player 1 and forces Player 1 to prevent such deduction.

This model with one risky asset and with arbitrary (not necessarily integer) bids was introduced by De Meyer and Moussa Saley (2002) to show that the Brownian component in the price evolution at financial markets may have a strategic origin. The same result was demonstrated in De Meyer (2010) for models with perfectly general trading mechanisms. The thesis of Gensbittel (2010) contains analogous results for the model with two risky assets and with arbitrary bids.

The described model of \( n \)-stage bidding is reduced to the zero-sum repeated game \( G_n(p) \) with lack of information on one side as introduced by Aumann and Mashler (1995) and with two-dimensional one-step actions with components corresponding to bids for each type of assets. It is easy to show that if the expectations of share prices are finite, then the values \( V_n(p) \) of \( n \)-stage bidding games \( G_n(p) \) exist. The value of such game does not exceed the sum of values of games modeling the bidding with one-type shares. The obtained result means that a simultaneous bidding of two types of risky assets is less profitable for the insider than separate bidding of one-type shares. This is explained by the fact that the simultaneous bidding leads to revealing more insider information, because the bids for shares of each type provide information on shares of other type.

In section 2 we show that if both share prices have finite variances, then the values \( V_n(p) \) of \( n \)-stage bidding games do not exceed the function \( H(p) \) that is the smallest piecewise linear function equal to the one half of the sum of share price variances for distributions with integer expectations of both share prices.

To prove this we define the set of strategies \( \tau^*(p) \) of Player 2 that ensure these upper bounds. The strategy \( \tau^*(p) \) is a direct combination of optimal strategies of Player 2 for the games with one-type risky asset. The initial bids are the integer parts of expectations of corresponding liquidation prices. At the step \( t > 1 \), the bid for a given share type depends on the result of bidding with this share type at the previous step. If the buyer was Player 1, then the next bid increases for one unit; if the buyer was Player 2, then the next bid decreases for one unit; if there was a tie, then the next bid remains the same.

This makes reasonable to consider the bidding of unlimited duration without an artificial beforehand given restriction \( n \) for number of steps. This bidding model is reduced to the infinite game \( G_\infty(p) \). We show that this game terminates naturally when the posterior expectations of both liquidation prices become close enough to their real values. Further we show that the value \( V_\infty(p) \) coincides with \( H(p) \). Observe that \( H(p) \) is the sum of values of infinite games with one-type assets studied in Domansky and Kreps (2009).
In section 3 we construct optimal strategies $\sigma^*$ of Player 1 that ensure $H(p)$ for games $G_\infty(p)$ with two states. We base on the results for games with one-type assets and with two states obtained in Domansky (2007).

The defined strategy $\sigma^*$ of Player 1 generates the asymmetric random walk of posterior probabilities by adjacent points of the lattice formed with such probabilities that at least one of price expectations takes integer value, with probabilities of jumps that provide martingale characteristics of posterior probabilities and with absorption at extreme points.

In section 4 we construct optimal strategies $\sigma^*$ of Player 1 that ensure $H(p)$ for games $G_\infty(p)$ with three states. The martingale of posterior mathematical expectations generated by the optimal strategy of Player 1 for the game with the three-point support distribution represents a symmetric random walk over points of integer lattice lying within the triangle spanned over the support points of distribution. The symmetry is broken at the moment when the walk hits the triangle boundary. Starting at this moment, the game get over one of games with distributions having two-point supports.

In section 5 we construct the symmetric representation of probability distributions $p \in \Theta(k, l)$ over two-dimensional integer lattice with a fixed integer expectation vector $(k, l)$, as a convex combination of extreme points of the set $\Theta(k, l)$, i.e. distributions with not more than three-point supports. The more general construction is given in Domansky (2011), where such symmetric representation is constructed for arbitrary distributions over the plane with a fixed expectation vector.

This representation is a straight generalization of the analogous representation for one-dimensional distributions with a fixed expectation that was exploited in Domansky and Kreps (2009) for constructing solutions of bidding games with a single risky asset. The similar disintegration of centered probability distributions on the real line into centred distributions supported at two points each was used by Skorokhod (1961) in the proof of his famous representation for a sequence of sums of independent centered random variables by means of a Brownian motion stopped at random times.

The coefficients of decomposition may be treated as probabilities of corresponding extreme distributions with not more than three-point supports. The choice of a point on the two-dimensional integer lattice in accordance with the distribution $p$ can be realized by means of the two-step lottery: the first step chooses an extreme distribution and the second step chooses a point in its support. This treatment allows to calculate the conditional probabilities of one or two complementary points given one point $(x, y) \neq (k, l)$ in the support of extreme distribution. These conditional probabilities turn to be the same for all points of any ray starting at $(k, l)$. This property is characteristic for our decomposition.

In section 6 we construct the optimal strategy of Player 1 for the bidding game for shares of two types with arbitrary distribution with an integer expectation vector $(k, l)$, as a convex combination of his optimal strategies for such games with distributions having not more than three-point supports. If the state chosen by chance move is $(k, l)$, then Player 1 stops the game. In this case he can not get any profit from his informational
advantage.

If the state chosen by chance move is \((x, y) \neq (k, l)\), then he chooses one or two complementary points by means of the lottery with the conditional probabilities of these complements. Further he plays his optimal strategy for the state \((x, y)\) in the game with distribution having two- or three-point support that is the state \((x, y)\) and the chosen complement.

We get the solutions for infinite games with arbitrary probability distributions over two-dimensional integer lattice with finite component variances. Both players have optimal strategies. The optimal strategy of Player 2 is a direct combination of his optimal strategies for the games with one-type risky asset. The value of such game is equal to the sum of values of corresponding games with one risky asset. Thus, the profit that Player 2 gets under simultaneous \(n\)-step bidding in comparison with separate bidding of each type of shares disappears in the game of unbounded duration.

2. Upper bounds for values \(V_n(p)\)

Here we consider the set \(M^2(\mathbb{Z}^2)\) of probability distributions \(p = (p(u, v))\) over the two-dimension integer lattice \(\mathbb{Z}^2\) with finite second moments

\[
m_u^2[p] = \sum_{u, v = -\infty}^{\infty} u^2 \cdot p(u, v) < \infty, \quad m_v^2[p] = \sum_{u, v = -\infty}^{\infty} v^2 \cdot p(u, v) < \infty.
\]

The set \(M^2\) is a closed convex subset of Banach space \(L^1(\mathbb{Z}^2, \{u^2 + v^2\})\) of mappings \(l : \mathbb{Z}^2 \to R^1\) with the norm

\[
||l|| = \sum_{u, v = -\infty}^{\infty} |l(u, v)|(u^2 + v^2).
\]

For \(p \in M^2(\mathbb{Z}^2)\), the random variables \(u, v\), determining the prices of shares, belong to \(L^2\) and have finite variances

\[
D_p[u] = m_u^2[p] - (m_u^1[p])^2, \quad D_p[v] = m_v^2[p] - (m_v^1[p])^2.
\]

The main result of this section is that, for \(p \in M^2(\mathbb{Z}^2)\), the sequence \(V_n(p)\) of values remains bounded as \(n \to \infty\).

To prove this we define the set of infinite strategies \(\tau^{(k,l)}\) of Player 2, suitable for the games \(G_n(p)\) with arbitrary \(n\).

**Definition 2.1:** The first move \(\tau_1^{(k,l)}\) is the action \((k, l)\). For \(t > 1\), the \(e\)-th component of the move \(\tau_t^{(k,l)}\), \(e = 1, 2\), depends on the last observed pair of \(e\)-th components of actions \((i_{t-1}^e, j_{t-1}^e)\) of both players:

\[
j_t^e = \begin{cases} 
 j_{t-1}^e - 1, & \text{if } i_{t-1}^e < j_{t-1}^e \\
 j_{t-1}^e, & \text{if } i_{t-1}^e = j_{t-1}^e \\
 j_{t-1}^e + 1, & \text{if } i_{t-1}^e > j_{t-1}^e
\end{cases}
\]
Proposition 2.2: For the state $s = (u, v) \in \mathbb{Z}^2$ the strategy $\tau^{(k,l)}$ ensures the payoff

$$\max_{\sigma} K_n^{a,b}(\sigma, \tau^{(k,l)}|(u, v)) \leq (u - k)(u - k - 1)/2 + (v - l)(v - l - 1)/2.$$  \hspace{1cm} (2.1)

Proof: The strategy $\tau^{(k,l)}$ prescribe Player 2 to operate separately with each of assets. Hence Player 1 can do the same. Therefore the assertion follows from Proposition 1 of Domansky, Kreps (2009). This proves Proposition 2.2.

Set

$$H(p) = 1/2 \cdot (D_p[u] + D_p[v] - \alpha(p)(1 - \alpha(p)) - \beta(p)(1 - \beta(p)))$$  \hspace{1cm} (2.2)

where $\alpha(p) = E_p[u] - \text{ent}[E_p[u]], \beta(p) = E_p[v] - \text{ent}[E_p[v]]$ and ent$[x], x \in \mathbb{R}$ is the integer part of $x$.

$H(p)$ is a continuous, concave, and piecewise linear function over $M^2(\mathbb{Z}^2)$. The domains of linearity of function $H(p)$ are

$$L(k,l) = \{p : E_p[u] \in [k, k + 1], E_p[v] \in [l, l + 1]\}, \ (k,l) \in \mathbb{Z}^2.$$

Its peak points are

$$\Theta(k,l) = \{p : E_p[u] = k; E_p[v] = l\}.$$

Theorem 2.3: For $p \in M^2(\mathbb{Z}^2)$, the values $V_n(p)$ are bounded from above by the function $H(p)$.

For $p \in L(k,l)$ the upper bound $H$ is ensured with the strategy $\tau^{(k,l)}$. For $p \in \Theta(k,l)$ the upper bound $H$ is ensured with the strategies $\tau^{(k,l)}$, $\tau^{(k-1,l)}$, $\tau^{(k,l-1)}$, and $\tau^{(k-1,l-1)}$.

Proof: It follows from Proposition 1 that there is the following not depending on $n$ upper bound for $V_n(p)$:

$$V_n(p) \leq \min_{(k,l)} \frac{1}{2} \sum_{u,v=-\infty}^{\infty} ((u - k)(u - k - 1) + (v - l)(v - l - 1)) \cdot p(u,v)$$  \hspace{1cm} (2.3)

Observe that, if $E_p[u] - k = \alpha, E_p[v] - l = \beta$, then

$$\frac{1}{2} \sum_{u,v=-\infty}^{\infty} ((u - k)(u - k - 1) + (v - l)(v - l - 1)) \cdot p(u,v)$$

$$= \frac{1}{2}(D_p[u] + D_p[v] - \alpha(p)(1 - \alpha(p)) - \beta(p)(1 - \beta(p))).$$
Corollary 2.3.1: The strategies $\tau^m, m = 0, 1, \ldots$ guarantee the same upper bound $H(p)$ for the upper value of the infinite game $G_\infty(p)$.

3. Solutions for games $G_\infty(z_1, z_2, p)$ with two states

In this section we show that, for games $G_\infty(p)$ with the support of distribution $p$ containing two states $z_1, z_2 \in \mathbb{Z}^2$, the value $V_\infty(p)$ is equal to $H(p)$.

A distribution with the support $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ is uniquely determined with expectations of coordinates. For any point $w = (u, v) = p_1 \cdot z_1 + p_2 \cdot z_2, p_i \in [0, 1], p_1 + p_2 = 1$, the distribution $p^{x_1,x_2}_w$ such that $E_{p^{x_1,x_2}_w}[x] = u, E_{p^{x_1,x_2}_w}[y] = v$, is given with probabilities $p^{x_1,x_2}_w(z_i) = p_i$.

Without loss of generality we assume that one of these points is $(0, 0)$. Thus there are two states $0 = (0, 0)$ and $z = (x, y)$, where $x$ and $y$ are integers and $x > 0$. The distribution $p^x_0$ can be depicted with a scalar parameter $p \in [0, 1]$ – the probability of state $z$. For definiteness set $y > 0$.

Observe that the function $H(p)$ is equal to the sum of values

$$H(p) = V^x_\infty(p) + V^y_\infty(p) \quad (3.1)$$

of one asset games $G^x_\infty(p)$ and $G^y_\infty(p)$ considered in Domansky (2007).

The function $V^m_\infty(p)$ is a piecewise linear continuous concave function of $p \in [0, 1]$. The set of its break points is the regular lattice $\{k/m, k = 0, \ldots, m\}$ with the values $V^m_\infty(k/m) = k(m - k)/2$. Therefore, for $p \in [k/m, (k + 1)/m]$,

$$V^m_\infty(p) = (pm - k)(k + 1)(m - k - 1)/2 + (1 - pm + k)k(m - k)/2$$

$$= k(m - k)/2 + (pm - k)(m - 2k - 1)/2. \quad (3.2)$$

For $p \in [(k - 1)/m, k/m]$,

$$V^m_\infty(p) = k(m - k)/2 - (k - pm)(m - 2k + 1)/2. \quad (3.3)$$

Thus the function $H(p)$ is a piecewise linear continuous concave function of $p \in [0, 1]$. The set of its break points is the irregular lattice $D(x, y) \subset [0, 1]$: 

$$D(x, y) = \{k/x, k = 0, \ldots, x\} \cup \{l/y, l = 0, \ldots, y\}.$$ 

Further we enumerate the points of the lattice $D(x, y)$ in ascending order $D(x, y) = \{p_i\}, i = 0, 1, \ldots, I, p_0 = 0, p_I = 1, p_i < p_{i+1}$. 

\[\]
By Corollary 2.3.1 the optimal strategy \( \tau^* \) guarantees to Player 2 the loss not exceeding the function \( H(p) \). Therefore it is sufficient to show that there is an optimal strategy \( \sigma^* \) of Player 1 that guarantees him this gain at the break points of function \( H(p) \), i.e. for the initial probability \( p \) belonging to the lattice \( D(x, y) \).

Now we present a definition of first moves of the strategy \( \sigma^* \) for \( p_i \in D(x, y) \).

**Definition 3.1:** For any initial probability \( p_i \) the first move of the strategy \( \sigma^* \) makes use of two actions \( a_i^- \) and \( a_i^+ \).

For \( p_i = k/x \neq 1/y \), these actions are \( a_i^- = (k-1, l) \) and \( a_i^+ = (k, l) \).

For \( p_i = 1/y \neq k/x \), these actions are \( a_i^- = (k, l-1) \) and \( a_i^+ = (k, l) \).

For \( p_i = k/x = 1/y \), these actions are \( a_i^- = (k-1, l-1) \) and \( a_i^+ = (k, l) \).

The posterior probabilities \( p(z|a_i^-) \) and \( p(z|a_i^+) \) are the left and the right adjacent points \( p_{i-1} \) and \( p_{i+1} \) of the lattice \( D(x, y) \) correspondingly.

Consequently the total probabilities of actions are

\[
q(a_i^-) = \frac{p_{i+1} - p_i}{p_{i+1} - p_{i-1}}, \quad q(a_i^+) = \frac{p_i - p_{i-1}}{p_{i+1} - p_{i-1}}.
\]

This first move is realized with the following conditional probabilities of action \( a_i^+ \):

\[
f^*(a_i^+|z) = \frac{(p_i - p_{i-1})p_{i+1}}{(p_{i+1} - p_{i-1})p_i}, \quad f^*(a_i^+|0) = \frac{(p_i - p_{i-1})(1 - p_{i+1})}{(p_{i+1} - p_{i-1})(1 - p_i)}.
\]

As the posterior probabilities also belong to the lattice \( D(x, y) \) this set of moves defines the infinite strategy \( \sigma^* \). The defined strategy \( \sigma^* \) of Player 1 generates the asymmetric random walk of posterior probabilities of state \( z \) by adjacent points of the irregular lattice \( D(x, y) \) with the probabilities of jumps that provide the martingale characteristics of posterior probabilities and with absorption at the extreme points \( p_0 = 0 \) and \( p_I = 1 \).

**Theorem 3.2:** The value \( V_\infty(p) \) of the game \( G_\infty(p) \) with two states 0 and \( z = (x, y) \), and with the probability \( p \) of the state \( z \) is equal to the function \( H(p) \). Both players have optimal strategies.

For the initial probability \( p_i \in D(x, y) \), one of optimal strategies of Player 1 is the strategy \( \sigma_i^* \) of Definition 2.

For the initial probability \( p \in (k/x, (k+1)/x) \cap (l/y, (l+1)/y) \) a unique optimal strategy of Player 2 is the strategy \( \tau^* = \tau^{kl} \), defined in Definition 1. For points of the lattice \( D(x, y) \) any optimal strategy for adjacent intervals is optimal.

**Proof:** At first we show that the one-step gain of Player 1 corresponding to the first move \( \sigma_i^* \) combined with the optimal gain \( H \) at the points of posterior probabilities generated by this move and weighted by total probabilities of actions satisfy Bellman optimality equations.
For \( p_i = k/x \neq l/y \), the one-step gain of Player 1 corresponding to the first move \( \sigma^*_1 \) in the game \( G_\infty(0, z, p) \) is equal to his gain in the one-asset game \( G_\infty^x(p) \)

\[
\min_{(k', l')} K_1^x(\sigma^*_1, (k', l')|0, z, p_i) = \min_{k'} K_1^m(\sigma^*_1, k'|p_i)
\]

\[
= \frac{x(p_{i+1} - p_i)(p_i - p_{i-1})}{p_{i+1} - p_{i-1}}, \quad (3.4)
\]

Here the minimum in the left part is attained at \((k', l') = (k - 1, l)\) and \((k, l)\), and the minimum in the right part is attained at \(k' = k - 1\) or \(k\).

For this move, taking into account (3.2), (3.3), (3.4), we get

\[
\min_{k'} K_1^x(\sigma^*_1, x, y, p) + q(k - 1)V_\infty^x(p_{i-1}) + q(k)V_\infty^x(p_{i+1})
\]

\[
= \frac{x(p_{i+1} - p_i)(p_i - p_{i-1})}{p_{i+1} - p_{i-1}}
\]

\[
+ \frac{p_{i+1} - p_i}{p_{i+1} - p_{i-1}}(k(x - k)/2 - x(p_i - p_{i-1})(x - 2k + 1)/2)
\]

\[
+ \frac{p_i - p_{i-1}}{p_{i+1} - p_{i-1}}(k(x - k)/2 + x(p_{i+1} - p_i)(x - 2k - 1)/2)
\]

\[
= k(x - k)/2 = V_\infty^x(p_i). \quad (3.5)
\]

Thus the Bellman optimality equation is fulfilled for one-asset game. On the other hand, three points \( p_{i-1}, p_i, p_{i+1} \) are situated on the same linearity interval of function \( V_\infty^x(p) \), i.e.

\[
q(k - 1)V_\infty^x(p_{i-1}) + q(k)V_\infty^x(p_{i+1}) = V_\infty^x(p_i). \quad (3.6)
\]

Summing (3.5) and (3.6), and also taking into account (3.1) we obtain

\[
\min_{(k', l')}(\sigma^*_1, (k', l')|0, z, p_i) + q(k - 1, l)H(p_{i-1}) + q(k, l)H(p_{i+1}) = H(p_i), \quad (3.7)
\]

i.e., for \( p_i = k/x \neq l/y \) and for the move \( \sigma^*_1 \) in the game \( G_\infty(0, z, p) \), function \( H \) satisfies Bellman optimality equation.

For \( p_i = l/y \neq k/x \), the proof of this fact is analogous with replacement of \( x \) and \( y \).

For \( p_i = k/x = l/y \), the Bellman optimality equations (3.5) are fulfilled for both one-asset games \( G_\infty^x(p) \) and \( G_\infty^y(p) \). Summing these optimality equations we obtain the optimality equation for the two-asset game \( G_\infty(p) \).

Thus function \( H \) satisfies the Bellman optimality equation for all initial probabilities \( p_i \in D(x, y) \). Iterating this optimality equation and taking into account that generated by the strategy \( \sigma^{*} \) random walk of posterior probabilities terminates in a finite mean number
of steps, we see that, for the initial probability \( p_i \in D(x, y) \), the strategy \( \sigma^* \) guarantees Player 1 the gain \( H(p_i) \).

\[ \Box \]

4. Solutions for games \( G_\infty(p) \) with three states

In this section we show that, for games \( G_\infty(p) \) with the support of distribution \( p \) containing three states \( z_1, z_2, z_3 \in \mathbb{Z}^2 \), the value \( V_\infty(p) \) coincides with \( H(p) \).

We assume that three points

\[ z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2), \quad z_3 = (x_3, y_3), \quad z_1, z_2, z_3 \in \mathbb{Z}^2 \]

are enumerated counterclockwise. It follows that, for \( w \in \Delta(z_1, z_2, z_3) \), \( \det[z_i - w, z_{i+1} - w] \geq 0 \), where \( \det[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1} \). Notice that arithmetical operations with subscripts are fulfilled modulo 3.

A distribution with the support \( z_1, z_2, z_3 \) is uniquely determined with expectations of coordinates. For any point \( w = (u, v) \in \Delta(z_1, z_2, z_3) \) the distribution \( p^w_{z_1, z_2, z_3} \) such that

\[ E_{p^w_{z_1, z_2, z_3}}[x] = u, \quad E_{p^w_{z_1, z_2, z_3}}[y] = v, \]

is given with probabilities

\[ p^w_{z_1, z_2, z_3}(z_i) = \frac{\det[z_{i+1} - w, z_{i+2} - w]}{\sum_{j=1}^{3} \det[z_j - w, z_{j+1} - w]}. \quad (4.1) \]

Observe that \( \sum_{j=1}^{3} \det[z_j - w, z_{j+1} - w] = \det[z_1 - z_3, z_2 - z_3] \) does not depend on \( w \).

By Corollary 2.3.1 the optimal strategy \( \tau^* \) guarantees Player 2 the loss not exceeding \( H(p) \). It follows from Theorem 3.2 that, for \( p^w_{z_1, z_2, z_3} \) with \( w = (u, v) \) belonging to the boundary of the triangle \( \Delta(z_1, z_2, z_3) \), the equality \( V_\infty(p^w_{z_1, z_2, z_3}) = H(p^w_{z_1, z_2, z_3}) \) holds. For other points \( w = (u, v) \in \Delta(z_1, z_2, z_3) \), the function \( H(p^w_{z_1, z_2, z_3}) \) is the least concave majorant of its values at the points \( p^w_{z_1, z_2, z_3} \) with \( w = (u, v) \in \mathbb{Z}^2 \) and at the boundary of \( \Delta(z_1, z_2, z_3) \). Therefore this is sufficient to show that there is a strategy \( \sigma^* \) of Player 1 that guarantees him \( H(p^w_{z_1, z_2, z_3}) \), for \( w = (u, v) \in \mathbb{Z}^2 \).

For the point \( w = (u, v) \in \mathbb{Z}^2 \) that belongs to the triangle \( \Delta(z_1, z_2, z_3) \)

\[ H(p^w_{z_1, z_2, z_3}) = \frac{1}{2}(\sum_{i=1}^{3} (x_i^2 + y_i^2)p^w_{z_1, z_2, z_3}(z_i) - (u^2 + v^2)). \quad (4.2) \]

For \( p^w_{z_1, z_2, z_3} \) with \( w = (u, v) \in \mathbb{Z}^2 \), the first step of strategy \( \sigma^* \) may efficiently use the actions \((u - 1, v - 1), (u, v - 1), (u - 1, v)\) and \((u, v)\). With the help of these actions Player 1 can perform moves such that the modulus of difference between posterior expectations of each coordinate and its initial expectation is not more than one.
There are several types of optimal first moves of Player 1, in particular, the first moves $\sigma^{NE-SW}_1$ (north-east – south-west), $\sigma^{NW-SE}_1$, and their probabilistic mixtures. Denote $e = (1, 1), \bar{e} = (1, -1)$. The first move $\sigma^{NE-SW}_1$ exploits only two actions $w - e$ and $w$ with posterior expectations $w - b \cdot e$ and $w + a \cdot e$. The first move $\sigma^{NW-SE}_1$ makes use of actions $(u - 1, v)$ and $(u, v - 1)$ with posterior expectations $w - b \bar{e}$ and $w + a \bar{e}$.

Further we define the first move $\sigma^{NE-SW}_1$ both in terms of posterior expectations and in terms of conditional probabilities of actions. We assume w.l.o.g. that $w = 0 \in \triangle(z_1, z_2, z_3)$. The span of this move is defined with a mutual disposition of the points $-e, e$ and the triangle $\triangle(z_1, z_2, z_3)$. If $z_i = k \cdot e$ for some $i = 1, 2, 3, k > 0$, then put $a = 1$. If $z_i = k \cdot -e$ for some $i = 1, 2, 3, k > 0$, then put $b = 1$.

If $z_i \neq k \cdot e, i = 1, 2, 3, k > 0$, then there is a unique $i = i^+$ such that the half-line starting at 0 and passing through $e$ crosses the side $z_{i^+}, z_{i^++1}$ of the triangle $\triangle(z_1, z_2, z_3)$. If $z_i \neq k \cdot -e, i = 1, 2, 3, k > 0$, then there is a unique $i = i^- \neq i^+$ such that the half-line starting at 0 and passing through $-e$ crosses the side $z_{i^-}, z_{i^-+1}$. Put

$$a = \min\left(\frac{\det[z_{i^+}, z_{i^++1}]}{\det[e, z_{i^++1} - z_{i^+}]}, 1\right), \quad b = \min\left(\frac{\det[z_{i^-}, z_{i^-+1}]}{\det[-e, z_{i^-+1} - z_{i^-}]}, 1\right).$$

**Definition 4.1:** The first move $\sigma^{NE-SW}_1$ for the game $G_\infty(p^0_{z_1, z_2, z_3})$ makes use of actions $-e$ and 0. The posterior expectations are

$$E_p[z - e] = -b \cdot e, \quad E_p[z | 0] = a \cdot e.$$ 

The total probabilities of actions are

$$q(e) = a/(b + a), \quad q(0) = b/(b + a).$$

This move is realized with the conditional probabilities of actions:

$$f^*(-e | z_i) = \frac{a \det[z_{i+1} + b \cdot e, z_{i+2} + b \cdot e]}{(b + a) \det[z_{i+1}, z_{i+2}]}, \quad i = 1, 2, 3;$$

$$f^*(0 | z_i) = \frac{b \det[z_{i+1} - a \cdot e, z_{i+2} - a \cdot e]}{(b + a) \det[z_{i+1}, z_{i+2}]}, \quad i = 1, 2, 3.$$

**Remark.** The martingale of posterior expectations generated by the optimal strategy of Player 1 is a symmetric random walk over the adjacent points of the lattice $\mathbb{Z}^2$ disposed inside the triangle $\triangle(z_1, z_2, z_3)$. The symmetry of this random walk is broken at the moment when it hits the triangle boundary. Beginning from this moment the game degenerates into one of two-point games with the distribution support being either $z_{i^+}, z_{i^++1}$, or $z_{i^-}, z_{i^-+1}$.
If \( a < 1 \), then after observing the action 0 the next game is \( G_\infty(p_{z_1,z_2,z_3}^{ae}) \) with the probabilities of states

\[
p(z_i^+) = \frac{\det[e, z_{i+1}]}{\det[e, z_{i+1} - z_i]}, \quad p(z_i^{-1}) = \frac{\det[z_i, e]}{\det[e, z_{i+1} - z_i]}
\]

If \( b < 1 \), then after observing the action \(-e\) the next game is \( G_\infty(p_{z_1,z_2,z_3}^{-be}) \) with the probabilities of states

\[
p(z_i^-) = \frac{\det[e, z_{i-1}]}{\det[e, z_{i-1} - z_i]}, \quad p(z_i^{-1}) = \frac{\det[z_i, e]}{\det[e, z_{i-1} - z_i]}
\]

**Theorem 4.2:** The value \( V_\infty(p_{z_1,z_2,z_3}^0) \) of the game \( G_\infty(p_{z_1,z_2,z_3}^0) \) is equal to the function \( H(p) \) given by (5). Both players have optimal strategies.

The optimal strategy of Player 2 is given by Definition 2.1.

For \( w = (u, v) \in \mathbb{Z}^2 \), one of optimal strategies of Player 1 is the strategy \( \sigma^* \) of Definition 4.1.

**Proof:** Taking into account Corollary 2.3.1 and Theorem 3.1 this is sufficient to show that the one-step gain corresponding to the first move \( \sigma_1^{NE-\text{SW}} \) of optimal strategy of Player 1 combined with the gain \( H(p) \) at the points of posterior probabilities generated by this move and weighted by total probabilities of actions satisfies Bellman optimality equations.

The best replies of Player 2 to the first move \( \sigma_1^{NE-\text{SW}} \) are actions 0, \(-e, (-1, 0)\), and \((0, -1)\). Corresponding one-step gain of Player 1 is equal to \( 2ab/(b + a) \). In fact,

\[
K_1(\sigma_1^{NE-\text{SW}}, 0|p_{z_1,z_2,z_3}^0) = -q(-e)E_p[x + y|e] = 2ab/(b + a);
\]

\[
K_1(\sigma_1^{NE-\text{SW}}, -e|p_{z_1,z_2,z_3}^0) = q(0)E_p[x + y|0] = 2ab/(b + a).
\]

For actions \((0, -1)\) and \((-1, 0)\) of Player 2 the proof is analogous.

It follows from (4.1) and (4.2) that

\[
H(p_{z_1,z_2,z_3}^0) = \frac{\sum_{i=1}^3 (x_i^2 + y_i^2) \det[z_{i+1}, z_{i+2}]}{2 \det[z_1 - z_3, z_2 - z_3]};
\]

\[
H(p_{z_1,z_2,z_3}^{ae}) = H(p_{z_1,z_2,z_3}^0) - a \frac{\sum_{i=1}^3 (x_i^2 + y_i^2) \det[z_{i+1} - z_{i+2}, e]}{2 \det[z_1 - z_3, z_2 - z_3]} - a;
\]

\[
H(p_{z_1,z_2,z_3}^{-be}) = H(p_{z_1,z_2,z_3}^0) + b \frac{\sum_{i=1}^3 (x_i^2 + y_i^2) \det[z_{i+1} - z_{i+2}, e]}{2 \det[z_1 - z_3, z_2 - z_3]} - b.
\]

We get

\[
2ab/(b + a) + q(-e)H(p_{z_1,z_2,z_3}^{-be}) + q(0)H(p_{z_1,z_2,z_3}^{ae}) = H(p_{z_1,z_2,z_3}^0),
\]
i.e., for $p^0_{z_1,z_2,z_3}$ and for the move $\sigma^{NE-SW}_i$ in the game $G_\infty(p^0_{z_1,z_2,z_3})$, function $H$ satisfies Bellman optimality equation.

5. Decomposition of distributions over the lattice $\mathbb{Z}^2$

The sets

$$\Theta(r, s) = \{p : E_p[u] = r, \ E_p[v] = s\}, \ (r, s) \in \mathbb{Z}^2,$$

are closed convex subsets of Banach space $L^1(\{|u| + |v|\})$. Here we give a representation of the set $\Theta(r, s)$ as a convex hull of its extreme points and a decomposition of linear functions over this set corresponding to this representation. The extreme points of the set $\Theta(r, s)$ are the degenerate distribution $\delta^{(r,s)}$ with the single-point support and distributions with two-point and three-point supports.

This is sufficient to give the representation for the set $\Theta(0, 0)$.

The support of distribution $p \in \Theta(0, 0)$ with two-point support is situated over a straight line passing through $(0, 0)$. Any straight line passing through $(0, 0)$ is uniquely defined with a point $w = (u, v) \in \mathbb{Z}^2, v \geq 0$. Here $w$ is the nearest to $(0, 0)$ lattice point in the upper half-plane, (but not the point $(-1, 0)$), with $(u, v)$ being relatively prime pair of integers. Let $W \subset \mathbb{Z}^2$ be the set of such points.

There is a collection of distributions $p^0_{wk,-wl} \in \Theta^2(0, 0)$ with two-point supports $\{-l \cdot w, k \cdot w\}$ over the straight line defined with a point $w$, given with probabilities

$$p^0_{wk,-wl}(-l \cdot w) = \frac{k}{k+l}, \quad p^0_{wk,-wl}(k \cdot w) = \frac{l}{k+l},$$

$k = 1, 2, \ldots, l = 1, 2, \ldots$.

For $w \in W$, denote by $p_w$ the part of distribution $p \in \Theta(0, 0)$ lying on the straight line passing through $(0, 0)$ and $w$. Put

$$m^-(p_w) = \sum_{k=1}^{\infty} k \cdot p(-k \cdot w); \quad m^+(p_w) = \sum_{k=1}^{\infty} k \cdot p(k \cdot w),$$

and let $m(p_w) = -m^-(p_w) + m^+(p_w)$ be the moment with respect to the point $(0, 0)$ of the part $p_w$ of distribution $p \in \Theta(0, 0)$.

If the moment $m(p_w) = 0$, then, by analogy with Proposition 2 of Domansky V., Kreps V. (2009) (decomposition of distributions over the lattice $\mathbb{Z}^1$), the part $p_w$ of distribution $p \in \Theta(0, 0)$ has a following symmetric representation as a convex combination of these distributions:

$$p_w = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{k+l}{m^+(p_w)} p(-l \cdot w)p(k \cdot w) \cdot p^0_{wk,-wl}.$$
**Proposition 5.1:** Any distribution $p \in \Theta(0,0)$ can be represented as a sum of three (substochastic) distributions

$$p = p^1 + p^2 + p^3,$$

where $p^1 = p(0,0)\delta^0$ is the degenerate distribution with the single-point support $0 = (0,0)$; $p^2 \in \Theta^2(0,0)$, where $\Theta^2(0,0)$ is the class of distributions $p \in \Theta(0,0)$ such that the moment of $p^2$ on any straight line passing through $(0,0)$ is equal to zero; $p^3 \in \Theta^3(0,0)$, where $\Theta^3(0,0)$ is the class of distributions $p \in \Theta(0,0)$ that have only one loaded ray for any straight line passing through $(0,0)$.

Distributions of class $\Theta^2(0,0)$ ($\Theta^3(0,0)$) are represented as convex combinations of two-point (three-point) distributions only.

**Proof:** For any $k \in \mathbb{N}, w \in W$, put

$$p^2(-k \cdot w) = \frac{p(-k \cdot w) \cdot m^+(p_w)}{m^-(p_w) \lor m^+(p_w)}, \quad p^2(k \cdot w) = \frac{p(k \cdot w) \cdot m^-(p_w)}{m^-(p_w) \lor m^+(p_w)},$$

where $m^+_w(p) \lor m^-_w(p) = \max(m^-_w(p), m^+_w(p))$.

This is easy to see that the distribution $p^2 \in \Theta^2(0,0)$ and the distribution $p^3 = p - p^1 - p^2 \in \Theta^3(0,0)$.

Consequently, we get the following result.

**Proposition 5.2:** The part $p^2$ of distribution $p \in \Theta(0,0)$ has the following representation as a convex combination of extreme points with two-point supports:

$$p^2 = \sum_{w \in W} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{k + l}{m^-_w(p) \lor m^+_w(p)} p(-l \cdot w) p(k \cdot w) p^0_{wk, -wl}.$$ 

Further we give a representation of distributions of class $\Theta^3(0,0)$ as convex combinations of distributions $p^0_{z_1, z_2, z_3}$ with three-point supports.

Let $\Delta^0$ be the set of three-point sets $(z_1, z_2, z_3)$ such that $0 \in \Delta(z_1, z_2, z_3)$, and let $\Delta^0(z)$ be the set of ordered two-point sets $(z_2, z_3)$ such that $\det[z_2, z_3] > 0$ and the three-point set $(z, z_2, z_3)$ belongs to $\Delta^0$.

**Lemma 5.3:** For any distribution $p \in \Theta^3(0,0)$ the amount

$$\sum_{(z_2, z_3) \in \Delta^0(z)} p(z_2) p(z_3) \det[z_2, z_3] = \Phi(p)$$

does not depend on $z$, i.e. this is an invariant of the distribution $p \in \Theta^3(0,0)$.

**Proof:** Assume that the distribution $p \in \Theta^3(0,0)$ has a finite support. Consider two points $z, z' \in \mathbb{Z}^2$ from support $\mathcal{p}$ such that there is no points $z''$ from support $\mathcal{p}$ with $\arg z < \arg z'' < \arg z'$. Let

$$U(z, z') = \{z'' \in \mathbb{Z}^2 : \arg z + \pi < \arg z'' < \arg z' + \pi \}.$$
Then
\[ \Phi(p, z) - \Phi(p, z') = \sum_{z_3 \in U(z, z')} \sum_{z_2 \in \mathbb{Z}^2} p(z_2) p(z_3) \det[z_2, z_3]. \]

Since, for distributions \( p \in \Theta(0, 0) \),
\[ \sum_{z_2 \in \mathbb{Z}^2} p(z_2) \det[z_2, z_3] = 0, \]
we obtain
\[ \Phi(p, z) - \Phi(p, z') = 0 \]
This proves Lemma 5.3 for distributions \( p \in \Theta^3(0, 0) \) with a finite support. For proving it in general case this is enough to pass to the limit.

**Remark.** This Lemma is a two-dimensional analog of the fact that for \( p \in \Theta(0) \subset \Delta(\mathbb{Z}) \)
\[ \sum_{t=1}^{\infty} t \cdot p(t) = \sum_{t=1}^{\infty} t \cdot p(-t). \]

**Proposition 5.4:** Any distribution \( p \in \Theta^3(0, 0) \) has the following representation as a convex combination of extreme points with three-point supports:
\[ p = \sum_{(z_1, z_2, z_3) \in \Delta^0} \alpha(p, z_1, z_2, z_3) \cdot p_{z_1, z_2, z_3}^0 \quad (5.1) \]
with coefficients
\[ \alpha(p, z_1, z_2, z_3) = \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}{\Phi(p)} p(z_1) p(z_2) p(z_3). \quad (5.2) \]

**Proof:** Take a point \( z_1 \in \mathbb{Z}^2 \). This point occurs in three point set \((z_1, z_2, z_3)\) if \((z_2, z_3) \in \Delta^0(z_1)\). The probability \( p_2(z_1) \) calculated according to formula (5.1) is
\[ p_2(z_1) = \sum_{(z_2, z_3) \in \Delta^0(z_1)} \alpha(p, z_1, z_2, z_3) \cdot p_{z_1, z_2, z_3}^0(z_1) \]
\[ = \sum_{(z_2, z_3) \in \Delta^0(z_1)} \alpha(p, z_1, z_2, z_3) \cdot \frac{\det[z_2, z_3]}{\sum_{j=1}^{3} \det[z_j, z_{j+1}]} \]
Substituting the values \( \alpha(p, z, z_2, z_3) \) given by (5.2) we get
\[ p_2(z_1) = \sum_{(z_2, z_3) \in \Delta^0(z_1)} \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}{\Phi(p)} p(z_1) p(z_2) p(z_3) \cdot \frac{\det[z_2, z_3]}{\sum_{j=1}^{3} \det[z_j, z_{j+1}]} \]
This proves Proposition 5.4.

\[ \Phi(p) = p(z_1). \]

\[ \sum_{(z_2, z_3) \in \Delta^0(z)} \frac{p(z_2)p(z_3) \cdot \det[z_2, z_3]}{\Phi(p)} = p(z_1). \]

Corollary 5.4.1: Any linear function \( f \) over \( \Theta^3(0, 0) \) has the following representation as a convex combination of its values at extreme points.

\[ f(p) = \sum_{(z_1, z_2, z_3) \in \Delta^0} \alpha(p, z_1, z_2, z_3) \cdot f(p_0^{z_1, z_2, z_3}) \]

with the coefficients \( \alpha(p, z_1, z_2, z_3) \), given by (5.2).

6. Constructing optimal strategies for Player 1

In this section we construct optimal strategies for Player 1 making use of the developed above decomposition for the initial distribution \( p \).

1. If the state chosen by chance move is \((0, 0)\), then Player 1 stops the game.

2. Let the state chosen by chance move be \( z \neq (0, 0) \), and let \( z = k \cdot w \), where \( k \in \mathbb{N} \) and \( w = (u, v) \) with \((u, v)\) being relatively prime pair of integers. If

\[ \sum_{k=1}^{\infty} k \cdot p(-k \cdot w) \geq \sum_{k=1}^{\infty} k \cdot p(k \cdot w), \]

then the state \( z \) belongs to the support of the distribution \( p_2^0 \) and does not belong to the support of the distribution \( p_3^0 \). Player 1 chooses a point \( z_2 = -l \cdot w \) by means of lottery with probabilities

\[ \frac{l \cdot p(-l \cdot w)}{\sum_{k=1}^{\infty} k \cdot p(-k \cdot w)}, \]

and further plays his optimal strategy \( \sigma^*(\cdot|z) \) for the state \( z \) in the two-point game \( G_\infty(p_0^{l \cdot w, -l \cdot w}) \).

3. Otherwise, if

\[ \sum_{k=1}^{\infty} k \cdot p(-k \cdot w) < \sum_{k=1}^{\infty} k \cdot p(k \cdot w). \]

then the state \( z \) belongs to the support of the both distributions \( p_2^0 \) and \( p_3^0 \) with probabilities

\[ \frac{\sum_{k=1}^{\infty} k \cdot p(-k \cdot w)}{\sum_{k=1}^{\infty} k \cdot p(k \cdot w)} \quad \text{and} \quad 1 - \frac{\sum_{k=1}^{\infty} k \cdot p(-k \cdot w)}{\sum_{k=1}^{\infty} k \cdot p(k \cdot w)} \]

correspondingly. Player 1 chooses a distributions \( p_2^0 \) or \( p_3^0 \) by means of lottery with these probabilities.
If the distribution $p^2$ is chosen, then further Player 1 behave as in the point 2.

If the distribution $p^3$ is chosen, then Player 1 chooses a pair $(z_2, z_3) \in \Delta^0(w)$ by means of lottery with probabilities

$$p(z_2)p(z_3) \cdot \det[z_2, z_3] \over \Phi(p),$$

and further plays his optimal strategy $\sigma^*(\cdot | z)$ for the state $z$ in the three-point game $G_{\infty}(p^{0, z_2, z_3}).$

As the optimal strategies $\sigma^*$ ensure Player 1 the gains equal to one half of the sum of component variances $D_p[u] + D_p[v]$ in the two and three-point games and as the sum of component variances is a linear function over $\Theta(0,0) \cap M^2$, where $M^2$ is the class of distributions with finite second moment, we obtain the following result.

**Theorem 6.1:** For any distribution $p \in \Theta(0,0) \cap M^2$ the compound strategy depicted above ensures Player 1 the gain $\frac{1}{2} \cdot (D_p[u] + D_p[v])$ in the game $G_{\infty}(p)$.

**References**


