PERVERSE SHEAVES ON INFINITE-DIMENSIONAL STACKS, AND AFFINE SPRINGER THEORY

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Abstract. The goal of this work is to construct a perverse t-structure on the ∞-category of ℓ-adic $\mathcal{L}G$-equivariant sheaves on the loop Lie algebra $\mathcal{L}\mathfrak{g}$ and to show that the affine Grothendieck-Springer sheaf $\mathcal{S}$ is perverse. Moreover, $\mathcal{S}$ is an intermediate extension of its restriction to the locus of "compact" elements with regular semi-simple reduction. Note that classical methods do not apply in our situation because $\mathcal{L}G$ and $\mathcal{L}\mathfrak{g}$ are infinite-dimensional ind-schemes.

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Introduction

0.1. Motivation and brief outline.

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0.1.1. The finite-dimensional case. Let k be an algebraically closed field, G a connected reductive group over k, $\mathfrak{g}$ the Lie algebra of G, B a Borel subgroup of G, $\mathfrak{b}$ the Lie algebra of B, and W the Weyl group of G. Let $\mathcal{B} := G/B$ be the flag variety and consider the variety

$$\tilde{\mathfrak{g}} := \{(gB, \gamma) \in \mathcal{B} \times \mathfrak{g} \,|(\mathrm{Ad}\, g^{-1})(\gamma) \in \mathfrak{b}\}.$$

The projection $\mathfrak{p}^{fin} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is known as the Grothendieck-Springer resolution, and its fibers $\mathcal{B}_\gamma$ are known as Springer fibers.

It was shown by Lusztig [Lus1, §3] that $\mathfrak{p}^{fin}$ is an $\mathrm{Ad}\, G$-equivariant small projective morphism, whose source is smooth and restriction to the regular semisimple locus is a Galois cover with Galois group W. Therefore the derived pushforward $\mathcal{S}^{fin} := \mathfrak{p}^{fin}_* \overline{\mathbb{Q}}_\ell[\dim(\tilde{\mathfrak{g}})]$ is an $\mathrm{Ad}\, G$-equivariant semisimple perverse sheaf on $\mathfrak{g}$. Moreover, $\mathcal{S}^{fin}$ equals to the intermediate extension of its restriction to the regular semisimple locus and it is equipped with an action of W. In particular, the action of W on $\mathcal{S}^{fin}$ induces an action of W on the cohomology of each $\mathcal{B}_\gamma$.

For each irreducible representation V of W, we denote by $\mathcal{S}^{fin}_V$ the isotypical component of $\mathcal{S}^{fin}$. Each $\mathcal{S}^{fin}_V$ is an $\mathrm{Ad}\, G$-equivariant irreducible perverse sheaf on $\mathfrak{g}$, and these sheaves are (Lie algebra analogs of) special cases of Lusztig's character sheaves [Lus3]. Character sheaves play a central role in the Lusztig's classification of irreducible characters of $G(\mathbb{F}_q)$ (see [Lus2, Lus4]).

A natural question is to develop an affine analogue of this theory. LusztigâĂŹs works [Lus6, Lus7] suggest that there exist affine analogs of character sheaves, and that these objects are closely related to characters of representations of p-adic groups.

0.1.2. The affine case. The Grothendieck-Springer fibration has a natural affine analog. Namely, let $\mathcal{L}^+(G)$ be the arc group of G, let $ev_G : \mathcal{L}^+(G) \to G$ be the evaluation map, and set $I := ev_G^{-1}(B)$. Let $\mathcal{L}\, G$ be the loop group of G, and let $\mathfrak{Fl} := \mathcal{L}\, G/I$ be the affine flag variety. Let $\mathfrak{C} \subset \mathcal{L}\, \mathfrak{g}$ be the locus of "compact elements" $\gamma \in \mathcal{L}\, \mathfrak{g}$, that is, those γ, whose "characteristic polynomial" has integral coefficients. More precisely, we define $\mathfrak{C} \subset \mathcal{L}\, \mathfrak{g}$ as the preimage $\mathfrak{C} := (\mathcal{L}\, \chi)^{-1}(\mathcal{L}^+(\mathfrak{c}))$, where $\mathfrak{c}$ is the Chevalley space of $\mathfrak{g}$, and $\mathcal{L}\, \chi : \mathcal{L}\, \mathfrak{g} \to \mathcal{L}\, \mathfrak{c}$ be the morphism, induced by the characteristic map $\chi : \mathfrak{g} \to \mathfrak{c}$.

Consider the ind-scheme

$$\widetilde{\mathfrak{C}} := \{(gI, \gamma) \in \mathfrak{Fl} \times \mathfrak{C} |\, \mathrm{Ad}(g)^{-1}(\gamma) \in \mathrm{Lie}(I)\},$$

which is the affine analog of $\widetilde{\mathfrak{g}}$. Then the projection $\mathfrak{p} : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ is an affine analog of the Grothendieck-Springer fibration, while fibers $\mathfrak{Fl}_\gamma$ of $\mathfrak{p}$ are the affine Springer fibers (see [KL]). Furthermore, Lusztig [Lus5] constructed an action of the extended affine Weyl group $\widetilde{W}$ of the cohomology on the $\mathfrak{Fl}_\gamma$'s, and a natural question is whether other aspects of classical Springer theory can be extended to this setting as well.

Note that it is impossible to study the fibration $\mathfrak{p}$ using classical algebro-geometric tools, because the source and the target are infinite-dimensional ind-schemes.

0.1.3. Letter of MacPherson. This project has begun with a letter from the second author to MacPherson in Summer 2009, in which he asks if, by considering the affine Grothendieck-Springer fibration, an appropriate counting of dimensions will tell us that this map is small. MacPherson formulated the notion of smallness which is applicable in our case, and provided the necessary computation which implies that $\mathfrak{p}$ is small (compare Proposition 7.3.2). Nevertheless, he conclude his letter by the following sentence:

"We don't have a theory of intersection homology that works in this context, so the general idea that the map is small doesn't help in constructing a Weyl group action, or reproducing the rest of Springer theory".

The main goal of this work is to establish such a theory. In addition, we generalize Lusztig's observation [Lus1, §3], and provide a supporting evidence of Lusztig's conjectures [Lus6, Lus7].

0.1.4. What is done in this paper. (a) By a *prestack (over k)*, we mean a contravariant functor from the category of affine schemes over k to groupoids. To every prestack $\mathcal{X}$, we associate a cocomplete stable ∞-category $\mathcal{D}(\mathcal{X})$ of ℓ-adic sheaves on $\mathcal{X}$, and for every morphism $f:\mathcal{X}\to\mathcal{Y}$ of prestacks, we associate a pullback functor $f^!:\mathcal{D}(\mathcal{Y})\to\mathcal{D}(\mathcal{X})$. In particular, for every prestack $\mathcal{X}$ we have a dualizing sheaf $\omega_{\mathcal{X}}\in\mathcal{D}(\mathcal{X})$, defined to be the !-pullback of $\overline{\mathbb{Q}}_l\in\mathcal{D}(\mathrm{pt})$.

(b) Let $\mathfrak{C}_\bullet\subset\mathfrak{C}$ be the locus of generically regular semisimple elements, and let $\mathfrak{p}_\bullet:\widetilde{\mathfrak{C}}_\bullet\to\mathfrak{C}_\bullet$ be the restriction of $\mathfrak{p}$ to $\mathfrak{C}_\bullet$. Then $\mathfrak{p}_\bullet$ is $\mathcal{L}G$-equivariant, so it induces a morphism $\overline{\mathfrak{p}}_\bullet:[\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]\to[\mathfrak{C}_\bullet/\mathcal{L}G]$ of quotient stacks, where we sheafify quotients with respect to étale topology.

(c) The projection $\overline{\mathfrak{p}}_\bullet$ is locally ind-fp-proper (see 0.4.5), therefore the pullback $\overline{\mathfrak{p}}_\bullet^!$ has a left adjoint $(\overline{\mathfrak{p}}_\bullet)_!$. We set $\mathcal{S}_\bullet:=(\overline{\mathfrak{p}}_\bullet)_!(\omega_{[\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]})\in\mathcal{D}([\mathfrak{C}_\bullet/\mathcal{L}G])$, and call it the *affine Grothendieck–Springer sheaf.*

(d) The main goal of this work is to define perverse t-structures on a certain class of infinite-dimensional prestacks (actually, even ∞-prestacks), which includes the quotient stacks $[\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]$ and $[\mathfrak{C}_\bullet/\mathcal{L}G]$, and to show that the affine Grothendieck-Springer sheaf $\mathcal{S}_\bullet$ is perverse. Moreover, $\mathcal{S}_\bullet$ is an intermediate extension of its restriction to a locus with regular semisimple reduction.

(e) In order to do this, we develop a dimension theory in the infinite-dimensional setting, introduce a class of (semi)-small morphisms, and show that the fibration $\overline{\mathfrak{p}}_\bullet:[\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]\to[\mathfrak{C}_\bullet/\mathcal{L}G]$ is small.

0.1.5. Remark. Contrary to the finite-dimensional case, it is crucial for our approach that we divide by the action $\mathcal{L}G$. For example, we don't know a framework in which the non-equivariant Grothendieck–Springer fibration $\mathfrak{p}_\bullet$ is small, and our perverse t-structure on $[\mathfrak{C}_\bullet/\mathcal{L}G]$ does not come from a t-structure on a non-equivariant category $\mathcal{D}(\mathfrak{C}_\bullet)$.

In the next four subsections we outline all necessary definitions, and provide more precise formulations of our constructions and results.

0.2. **(Topologically) placid ∞-stacks.** In this section we are going to introduce a class of objects, which admits canonical perverse t-structures.

0.2.1. Infinity-stacks. (a) Our basic geometric objects are ∞-*prestacks* (over k), defined as contravariant functors $\mathrm{Aff}_k^{op} \to \mathfrak{S}$ from the category Aff_k of affine schemes over k to the ∞-category of *spaces*. The collection of ∞-prestacks form an ∞-category PreSt_k, which contains the usual prestacks as a full subcategory, but has an advantage of being closed with respect to arbitrary homotopy colimits.

(b) Actually, we restrict ourselves to a subcategory $\mathrm{St}_k \subset \mathrm{PreSt}_k$ of ∞-*stacks*, that is, functors $\mathrm{Aff}_k^{op} \to \mathfrak{S}$ satisfying sheaf property with respect to étale topology. [1]

(c) We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of ∞-stacks is *surjective* or a *covering*, if it has sections locally for étale topology.

0.2.2. Placid ∞-stacks. Now we are going to introduce an important class of ∞-stacks, which is central for this work.

(a) We call an affine scheme X *globally placid*, if it has a presentation $X \simeq \lim_\alpha X_\alpha$ as a filtered limit of affine schemes of finite type over k with smooth transition maps.

(b) We call a morphism $f : X \to Y$ of affine schemes *strongly pro-smooth*, if X has a presentation $X \simeq \lim_\alpha X_\alpha$ over Y as a filtered limit of affine schemes such that all transition maps and all projections $X_\alpha \to Y$ are finitely presented and smooth.

(c) Mimicking Simpson's construction of geometric n-stacks, we construct the class of placid ∞-stacks and class of smooth morphisms between placid ∞-stacks. Namely, they are characterized as the smallest classes satisfied the following properties:

- The class of placid ∞-stacks contains globally placid affine schemes, and is closed under coproducts.
- The class of smooth morphisms contains strongly pro-smooth morphism between globally placid affines schemes, and is closed under compositions, coproducts and pullbacks.
- An ∞-stack $\mathcal{Y}$ is placid, if there exists a covering of ∞-stacks $f : \mathcal{X} \to \mathcal{Y}$ such that $\mathcal{X}$ and $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are placid, while both projections $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ are smooth.

[1]This restriction is not essential, because like in the classical setting, the inclusion $\mathrm{St}_k \subset \mathrm{PreSt}_k$ has a left adjoint, called the *sheafification*, while categories of ℓ-adic sheaves are not affected by sheafification (see 0.4.1(b)).

• A morphism $f : \mathcal{X} \to \mathcal{Y}$ between placid ∞-stacks is smooth, if for every smooth morphism $Y \to \mathcal{Y}$ from a globally placid affine scheme Y the fiber product $\mathcal{X} \times_{\mathcal{Y}} Y$ is placid, and the projection $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is smooth.

• A morphism $f : \mathcal{X} \to \mathcal{Y}$ between placid ∞-stacks is smooth, if there exists a smooth covering of placid ∞-stacks $\mathcal{Z} \to \mathcal{Y}$ such that the composition $\mathcal{Z} \to \mathcal{X} \to \mathcal{Y}$ is smooth.

(d) Similarly, replacing in (c) the class of strongly pro-smooth morphisms by the class of pro-étale morphisms, we construct classes of DM-placid ∞-stacks (where DM stands for "Deligne–Mumford") and pro-étale morphisms.

0.2.3. Perfect ∞-stacks and topological equivalences. In some places of this work we will want to "ignore" universal homeomorphisms. To make this procedure formal, we introduce the following definition.

(a) We call a morphism $f : \mathcal{X} \to \mathcal{Y}$ between ∞-stacks a *topological equivalence*, if it lies in the strong saturated closure (see 2.3.1(a)) of universal homeomorphisms between affine schemes.

Though a strong saturated closure is a very complicated notion in general, it turned out that topological equivalences can be described very explicitly:

(b) We call an affine scheme X *perfect*, if every universal homeomorphism $X' \to X$ from a reduced affine scheme X' is an isomorphism, and denote the category of perfect affine schemes of $\mathrm{Aff}_{\mathrm{perf},k}$. Notice that this notion coincides with the classical notion of perfect schemes when the characteristic of k is positive.

(c) We denote by $\mathrm{St}_{\mathrm{perf},k}$ the ∞-category of *perfect* ∞-stacks, defined as functors $\mathrm{Aff}^{op}_{\mathrm{perf},k} \to \mathfrak{S}$, satisfying sheaf condition with respect to étale topology. We have a restriction functor $\iota^* : \mathrm{St}_k \to \mathrm{St}_{\mathrm{perf},k}$ with a fully faithful left adjoint $\iota_! : \mathrm{St}_{\mathrm{perf},k} \to \mathrm{St}_k$.

(d) For every ∞-stack $\mathcal{X}$, we set $\mathcal{X}_{\mathrm{perf}} := \iota_!\iota^*(\mathcal{X})$ and call it *the perfection* of $\mathcal{X}$. Notice that this notion extends the classical perfection functor, when $\mathcal{X}$ is an (affine) scheme or an algebraic space.

(e) It it not difficult to see (see Lemma 2.3.6) that a morphism $f : \mathcal{X} \to \mathcal{Y}$ between ∞-stacks is a topological equivalence if and only if its restriction $\iota^*(f)$ is an equivalence of perfect ∞-stacks, or, equivalently the perfection $f_{\mathrm{perf}} : \mathcal{X}_{\mathrm{perf}} \to \mathcal{Y}_{\mathrm{perf}}$ is an equivalence. In other words, the ∞-category of perfect ∞-stacks can be described as the localization of the ∞-category of ∞-stacks by topological equivalences.

0.2.4. Topologically placid ∞-stacks. (a) For the purpose of the introduction, we call an ∞-stack $\mathcal{X}$ *topologically placid*, if there exists a placid ∞-stack $\mathcal{Y}$ and an isomorphism $\mathcal{X}_{\mathrm{perf}} \simeq \mathcal{Y}_{\mathrm{perf}}$. (Notice that this notion is more restrictive than in the main body of the paper).

(b) Next, we call a morphism $f : \mathcal{X} \to \mathcal{Y}$ of topologically placid ∞-stacks *topologically smooth*, if $f_{\mathrm{perf}} : \mathcal{X}_{\mathrm{perf}} \to \mathcal{Y}_{\mathrm{perf}}$ is isomorphic to a perfection of a smooth morphism between placid ∞-stacks. We also call f is a *topologically covering*, if the restriction $\iota^*(f) : \iota^*(\mathcal{X}) \to \iota^*(\mathcal{Y})$ is a covering in $\mathrm{St}_{\mathrm{perf},k}$.

(c) We call a topologically placid ∞-stack *topologically smooth*, if the it the projection $\mathcal{X} \to \mathrm{pt}$ is topologically smooth.

(d) Finally, we call a morphism $f : \mathcal{X} \to \mathcal{Y}$ of topologically placid ∞-stacks *topologically étale*, if for every topologically smooth morphism $Y \to \mathcal{Y}$ from a (topologically placid) affine scheme Y, the induced map $(\mathcal{X} \times_{\mathcal{Y}} Y)_{\mathrm{perf}} \to Y_{\mathrm{perf}}$ is isomorphic to a perfection of a pro-étale morphism between DM-placid ∞-stacks.

0.2.5. Extending classes of morphisms. (a) Let (P) be a class of morphisms of ∞-stacks $\mathcal{X} \to Y$, where Y is an affine scheme, which closed under pullbacks. Such a class gives rise to a class of morphisms $f : \mathcal{X} \to \mathcal{Y}$ of ∞-stacks, defined by the property that if for every morphism $Y \to \mathcal{Y}$ from an affine scheme Y, the pullback $f \times_{\mathcal{Y}} Y : \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ belongs to (P).

(b) In particular, we can talk about (fp)-representable morphisms of ∞-stacks, where "fp" stands for "finitely presented", fp-proper, that is, proper and finitely presented, (fp)-open/closed/locally closed embeddings, etc.

0.2.6. Constructible stratifications, stratified ∞-stacks, and perversity. It turned out that we need to consider a larger class of ∞-stacks, which we call *stratified.*

Let $\mathcal{X}$ be an ∞-stack, and let $\{\mathcal{X}\}_{\alpha\in\mathcal{I}}$ be a collection of fp-locally closed ∞-substacks (see 0.2.5).

(a) We say that $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ form a *finite constructible stratification* of $\mathcal{X}$, if $\mathcal{I}$ is finite, and there exists a full ordering $\alpha_1 < \ldots < \alpha_n$ of $\mathcal{I}$ and an increasing sequence of fp-open substacks $\emptyset = \mathcal{X}_0 \subsetneq \mathcal{X}_1 \subsetneq \ldots \subsetneq \mathcal{X}_n = \mathcal{X}$ such that $\mathcal{X}_{\alpha_i} \subset \mathcal{X}_i \smallsetminus \mathcal{X}_{i-1}$, and the embedding $\mathcal{X}_{\alpha_i} \hookrightarrow \mathcal{X}_i \smallsetminus \mathcal{X}_{i-1}$ is a topological equivalence for all $i = 1, \ldots, n$.

(b) More generally, we say that $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ form a *bounded* constructible stratification of $\mathcal{X}$, if $\mathcal{X}$ has a fp-open covering $\mathcal{X} = \cup_i \mathcal{U}_i$ such that $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I},\mathcal{X}_\alpha\subset\mathcal{U}_i}$ form a finite constructible stratification of $\mathcal{U}_i$.

(c) By a *stratified ∞-stack*, we mean an ∞-stack $\mathcal{X}$, equipped with a bounded constructible stratification $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ by topologically placid ∞-stacks. In this case, by a *perversity* of $\mathcal{X}$ we mean a function $p_\nu : \mathcal{I} \to \mathbb{Z}$, or what is the same a collection of integers $\{\nu_\alpha\}_{\alpha\in\mathcal{I}}$.

(d) Actually, having further applications in mind, in the main body of the paper we consider *unbounded* constructible stratifications as well.

0.3. **Dimension theory, and (semi)-small morphisms.** Our proof of perversity of the affine Grothendieck–Springer sheaf is based on the observation (essentially due to MacPherson) that the morphism $\overline{\mathfrak{p}}_\bullet$ is small. To define the notion of *small morphisms*, we introduce a notion of equidimensional morphisms between ∞-stacks.

0.3.1. Dimension function and (weakly) equidimensional morphisms.

(a) To every morphism $f : X \to Y$ of schemes of finite type over k we associate a dimension function $\underline{\dim}_f : X \to \mathbb{Z}$ defined by $\underline{\dim}_f(x) = \dim_x(X) - \dim_{f(x)}(Y)$.

(b) It is not difficult to see (see Corollary 3.1.7) that every Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\phi} & Y \end{array}$$

such that either f or ϕ are universally open, we have an equality $\underline{\dim}_\psi = g^*\underline{\dim}_\phi$.

(c) We call a morphism f *weakly equidimensional* (of relative dimension d), if $\underline{\dim}_f$ is locally constant, that is, constant on each connected component (constant function with value d). Moreover, we call f *equidimensional*, if in addition we have $\underline{\dim}_f(x) = \dim_x f^{-1}(f(x))$. Notice that every open weakly equidimensional morphism is automatically equidimensional (see Corollary 3.1.5).

(d) By the property (b), the classes of weakly equidimensional morphisms and equidimensional morphisms are stable under all pullbacks with respect to universally open morphisms, while the class of universally open equidimensional morphisms is stable under all pullbacks.

(e) We say that a locally closed subscheme $Y \subset X$ is of *pure codimension* d, if the inclusion map $Y \hookrightarrow X$ is weakly equidimensional of relative dimension $-d$.

0.3.2. (Weakly/universally open) equidimensional morphisms of ∞-stacks.

(a) Using observation 0.3.1(d) and the fact that smooth morphisms and universal homeomorphisms are universally open, we define the class of (weakly) equidimensional morphisms of relative dimension d, to the smallest class (P) of morphisms $f : \mathcal{X} \to \mathcal{Y}$ of topologically placid ∞-stacks, such that

- $\mathcal{P}$ contains the corresponding class 0.3.1(c) of morphisms of schemes of finite type over k;
- $\mathcal{P}$ stable under pullbacks with respect to topologically smooth morphisms and coproducts;
- a morphism $f : \mathcal{X} \to \mathcal{Y}$ belongs to (P), if there exists a topologically smooth covering $\mathcal{Y}' \to \mathcal{Y}$ such that the pullback $f \times_{\mathcal{Y}} \mathcal{Y}'$ belongs to (P);
- for every topologically étale covering $g : \mathcal{Z} \to \mathcal{X}$, the morphism $f : \mathcal{X} \to \mathcal{Y}$ belongs to (P) if and only if the composition $f \circ g : \mathcal{Z} \to \mathcal{Y}$ belongs to $\mathcal{P}$.

(b) Next, repeating the definition of 0.3.1(e), we can talk about fp-locally closed substacks of pure codimension d.

(c) Moreover, using the fact that universally open equidimensional morphisms is stable under all pullbacks, replacing topologically smooth morphisms in (a) by all morphisms, we define a class of universally open equidimensional morphisms of relative dimension d between (not necessarily topologically placid) ∞-stacks.

0.3.3. (Semi)-small morphisms. Finally, we define a class of (semi)-small morphisms of ∞-stacks, extending the classical notion. Following a suggestion of MacPherson, we do it using codimensions on the source rather than on the target. [2]

(a) Let $\mathcal{Y}$ be a stratified ∞-stack with constructible stratification $\{\mathcal{Y}_\alpha\}_\alpha$, let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of ∞-stacks, and $\mathcal{X}_\alpha := f^{-1}(\mathcal{Y}_\alpha)$ be the induced constructible stratification of $\mathcal{X}$. Assume that

- $\mathcal{X}$ is topologically placid;
- each $\mathcal{X}_\alpha \subset \mathcal{X}$ is of pure codimension b_α;
- each $f_\alpha : \mathcal{X}_\alpha \to \mathcal{Y}_\alpha$ is equidimensional of relative dimension δ_α.

(b) We say that f is *semi-small*, if for every $\alpha \in \mathcal{I}$ we have an inequality $\delta_\alpha \leq b_\alpha$.

(c) Moreover, $\mathcal{U} \subset \mathcal{Y}$ be an fp-open substack, which is a union of strata $\{\mathcal{Y}_\alpha\}_\alpha$. We say that f is $\mathcal{U}$*-small*, if for every $\alpha \in \mathcal{I}$ such that $\mathcal{X}_\alpha \in \mathcal{X} \smallsetminus \mathcal{U}$, we have a strict inequality $\delta_\alpha < b_\alpha$.

0.4. ℓ-adic sheaves on ∞-stacks, and perverse t-structures.

0.4.1. ℓ-adic sheaves on ∞-(pre)stacks. (a) To every ∞-prestack $\mathcal{X}$, we associate a (presentable) stable ∞-category $\mathcal{D}(\mathcal{X})$ of ℓ-adic sheaves on $\mathcal{X}$, and for every morphism $h : \mathcal{X} \to \mathcal{Y}$ of ∞-prestacks, we associate a pullback functor $f^! : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$ (compare [RS]). We carry out the construction in three steps:

- When X is an affine scheme of finite type over k, we denote by $\mathcal{D}_c(X)$ the ∞-derived category $\mathcal{D}^b_c(U, \overline{\mathbb{Q}}_l)$ of constructible ℓ-adic sheaves on X, and by $\mathcal{D}(X)$ the ind-category $\operatorname{Ind}\mathcal{D}_c(X)$.
- When X is an arbitrary affine scheme over k, we write X as a filtered limit $X \simeq \lim_\alpha X_\alpha$ of affine schemes of finite type and denote by $\mathcal{D}(X)$ the colimit $\operatorname{colim}_\alpha \mathcal{D}(X_\alpha)$, taken with respect to !-pullbacks. It is easy to see that the resulting ∞-category is independent of the presentation.
- Finally, for an arbitrary ∞-prestack $\mathcal{X}$, we denote by $\mathcal{D}(\mathcal{X})$ the limit category $\lim \mathcal{D}(X)$, taken over all morphisms $X \to \mathcal{X}$, where X is an affine scheme.

(b) Notice that the ∞-category $\mathcal{D}(\mathcal{X})$ is not affected by the étale sheafification, that is, if $\mathcal{X}^{sh}$ is the sheafification of $\mathcal{X}$, then the pullback $i^! : \mathcal{D}(\mathcal{X}^{sh}) \to \mathcal{D}(\mathcal{X})$, corresponding to the canonical morphism $i : \mathcal{X} \to \mathcal{X}^{sh}$, is an equivalence.

(c) Using the fact that universal homeomorphisms induce equivalences of étale sites, one shows that for every topological equivalence $f : \mathcal{X} \to \mathcal{Y}$ of ∞-stacks the induced map $f^! : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$ is an equivalence. In particular, for every ∞-stack $\mathcal{X}$, the pullback $\pi^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}_{\mathrm{perf}})$, corresponding to the projection $\pi : \mathcal{X}_{\mathrm{perf}} \to \mathcal{X}$ is an equivalence.

[2] see 6.4.2 for comparison with the classical notion.

0.4.2. Perverse t-structures on topologically placid ∞-stacks. For every topologically placid ∞-stack $\mathcal{X}$, we equip the ∞-category $\mathcal{D}(\mathcal{X})$ with a perverse t-structure. We carry out the construction in six steps:

(a) For every equidimensional affine scheme X of finite type over k, we equip $\mathcal{D}(\mathcal{X})$ with the perverse t-structure $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ obtained from the classical (middle dimensional) perverse t-structure by homological shift by $\dim X$ to the left. In other words, and object $K \in \mathcal{D}(\mathcal{X})$ is perverse in our t-structure if and only if $K[-\dim X]$ is perverse in the classical t-structure.

(b) Next, every affine of scheme X of finite type over k has a constructible stratification $\{X_i\}_i$ by locally closed equidimensional subschemes, where X_i is the set of all $x \in X$ such that $\dim_x(X) = i$. We denote by $\eta_i : X_i \hookrightarrow X$ the inclusion, and let ${}^p\mathcal{D}_c^{\leq 0}(X) \subset \mathcal{D}_c(X)$ (resp. ${}^p\mathcal{D}_c^{\geq 0}(X) \subset \mathcal{D}_c(X)$) be the set of all $K \in \mathcal{D}_c(\mathcal{X})$ such that $\eta_i^*(K) \in {}^p\mathcal{D}_c^{\leq 0}(X_i)$ (resp. $\eta_i^!(K) \in {}^p\mathcal{D}_c^{\geq 0}(X_i)$). Now the fact that $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ is indeed a t-structure follows from the gluing lemma of [BBD].

(c) For an affine scheme X of finite type over k, we equipp $\mathcal{D}(\mathcal{X})$ with the unique t-structure such that ${}^p\mathcal{D}^{\leq 0}(X) = \operatorname{Ind} {}^p\mathcal{D}_c^{\leq 0}(X)$ and similarly for ${}^p\mathcal{D}^{\geq 0}(X)$.

The main property of the t-structure we just constructed is that for every smooth morphism or a universal homeomorphism $f : X \to Y$ of affine schemes of finite type, the pullback $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$ is t-exact.

(d) We show that for every globally placid affine scheme X, there exists a unique t-structure on $\mathcal{D}(X)$ such that for every strongly pro-smooth morphism $f : X \to Y$ to an affine scheme Y of finite type over k, the pullback $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$ is t-exact.

(e) Then we show that for every placid ∞-stack $\mathcal{X}$, there exists a unique t-structure on $\mathcal{D}(\mathcal{X})$ such that for every smooth morphism $f : X \to \mathcal{X}$ from a globally placid affine scheme X over k, the pullback $f^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(X)$ is t-exact.

(f) Finally, we show that for every topologically placid ∞-stack $\mathcal{X}$, there exists a unique t-structure on $\mathcal{D}(\mathcal{X})$ such that for every isomorphism $\mathcal{X}_{\text{perf}} \simeq \mathcal{Y}_{\text{perf}}$, where $\mathcal{Y}$ is a placid ∞-stack, the natural equivalence $\mathcal{D}(\mathcal{X}) \simeq \mathcal{D}(\mathcal{X}_{\text{perf}}) \simeq \mathcal{D}(\mathcal{Y}_{\text{perf}}) \simeq \mathcal{D}(\mathcal{Y})$ is t-exact.

0.4.3. Infinity-stacks admitting gluing of sheaves. Now we are going to introduce a property of ∞-stacks, we will need to define perverse t-structure in 0.4.4.

(a) Let $\mathcal{X}$ be an ∞ stack, $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ be an fp-closed embedding, and $j : \mathcal{U} \hookrightarrow \mathcal{X}$ is the complementary fp-open embedding. Then the pullback $j^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{U})$ has a right adjoint $j_* : \mathcal{D}(\mathcal{U}) \to \mathcal{D}(\mathcal{X})$, while the pullback $i^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Z})$ has a left adjoint $i_! : \mathcal{D}(\mathcal{Z}) \to \mathcal{D}(\mathcal{X})$. Moreover, both adjoints are fully faithful, and satisfy usual properties. On the other hand, the left adjoints $j_!$ of $j^!$ and i^* of $i_!$ do not exist in general.

(b) We say that an ∞-stack $\mathcal{X}$ *admits gluing of sheaves*, if for every fp-open embedding $j:\mathcal{U}\hookrightarrow\mathcal{X}$, there exists a left adjoint $j_!$ of $j^!$, and that $j_!$ is fully faithful. Note that this assumption also implies which the existence of the left adjoint i^* of $i_!$ (see Lemma 6.1.3).

(c) The class of ∞-stacks, admitting gluing of sheaves includes two important classes of examples: topologically placid ∞-stacks, and quotient stacks $[X/H]$, where X is an ind-placid scheme (that is, X can be represented as a filtered colimit $X\simeq\operatorname{colim}_i X_i$, where each X_i is a placid scheme, and each transition maps are fp-closed embeddings), and H is an ind-placid group, (that is, group object in the category of ind-placid schemes).

0.4.4. Perverse t-structures on stratified ∞-stacks, admitting gluing of sheaves. Let $\mathcal{Y}=\{\mathcal{Y}_\alpha\}_{\alpha\in\mathcal{I}}$ be a stratified ∞-stack, admitting gluing of sheaves.

(a) For every embedding $\eta_\alpha:\mathcal{Y}_\alpha\hookrightarrow\mathcal{Y}$ we have two pullback functors $\eta_\alpha^*,\eta_\alpha^!:\mathcal{D}(\mathcal{Y})\to\mathcal{D}(\mathcal{Y}_\alpha)$.

(b) Using gluing lemma, for every perversity function p_ν on $\mathcal{Y}$, there exists a unique t-structure on $\mathcal{D}(\mathcal{Y})$ such that ${}^p\mathcal{D}^{\geq 0}(\mathcal{Y})$ is the set of all $K\in\mathcal{D}(\mathcal{Y})$ such that $\eta_\alpha^! K\in{}^p\mathcal{D}^{\geq -\nu_\alpha}(\mathcal{Y}_\alpha)$ for all α. Moreover, ${}^p\mathcal{D}^{\leq 0}(\mathcal{Y})$ is the set of all $K\in\mathcal{D}(\mathcal{Y})$ such that $\eta_\alpha^* K\in{}^p\mathcal{D}^{\leq -\nu_\alpha}(\mathcal{Y}_\alpha)$ for all α.

(c) Let $\mathcal{U}\subset\mathcal{Y}$ be an fp-open ∞-substack, equals to union of strata $\mathcal{Y}_\alpha$, and let $j:\mathcal{U}\hookrightarrow\mathcal{Y}$ be the inclusion map. Then it follows from the definition that the pullback $j^!:\mathcal{D}(\mathcal{X})\to\mathcal{D}(\mathcal{U})$ is t-exact. Moreover, by usual procedure we can define the intermediate extension functor $j_{!*}:\operatorname{Perv}(\mathcal{U})\to\operatorname{Perv}(\mathcal{X})$.

0.4.5. Locally ind-fp-proper morphisms. (a) Let Y be an affine scheme. We say that a morphism $f:\mathcal{X}\to Y$ of ∞-stacks is *ind-fp-proper*, if $\mathcal{X}$ has a presentation as a filtered colimit $\mathcal{X}\simeq\operatorname{colim}_\alpha\mathcal{X}_\alpha$, where each $\mathcal{X}_\alpha$ is an algebraic space, fp-proper over Y (see 0.2.5), and all transition maps are closed embeddings.

(b) More generally, we say that f is *locally ind-fp-proper*, if there exists an étale covering $Y'\to Y$ such that the pullback $f\times_Y Y'$ is ind-fp-proper.

(c) Both classes (a) and (b) are closed under all pullbacks, so construction 0.2.5 applies. So we can talk about (locally) ind-fp-proper morphisms between ∞-stacks.

0.4.6. First Main Theorem. For every semi-small morphism $f:\mathcal{X}\to\mathcal{Y}$, consider perversity $p_f:=\{\nu_\alpha\}_{\alpha\in\mathcal{I}}$ on $\mathcal{Y}$, defined by $\nu_\alpha:=b_\alpha+\delta_\alpha$ for all α.

Our first main result (Theorem 6.4.5) asserts that if $f:\mathcal{X}\to\mathcal{Y}$ is a locally ind-fp-proper semi-small morphism of ∞-stacks, where $\mathcal{X}$ is topologically smooth, while $\mathcal{Y}$ admits gluing of sheaves, then the pushforward $K:=f_!(\omega_{\mathcal{X}})$ is p_f-perverse. Moreover, if f is $\mathcal{U}$-small, and $j:\mathcal{U}\hookrightarrow\mathcal{Y}$ is an open embedding, then we have an isomorphism $K\simeq j_{!*}j^!(K)$.

0.5. **Affine Springer theory.**

0.5.1. The GKM stratification. Following Goresky-Kottwitz-MacPherson [GKM], we introduce a constructible stratification of $\mathfrak{C}_\bullet$, which plays a central role in what follows:

(a) Consider the $\mathcal{L}G$-invariant constructible stratification $\{\mathfrak{C}_d\}_{d\geq 0}$ of $\mathfrak{C}_\bullet$, where $\mathfrak{C}_d$ consists of points $\gamma\in\mathfrak{C}_\bullet(k)\subset\mathfrak{g}(k((t)))$ such that valuation of the discriminant $\mathfrak{D}(\gamma)$ equals d. By definition, this stratification is induced by the corresponding stratification $\{\mathfrak{c}_d\}_{d\geq 0}$ of the regular part $\mathcal{L}^+(\mathfrak{c})_\bullet:=\mathcal{L}^+(\mathfrak{c})_{\mathfrak{D}\neq 0}$ of $\mathcal{L}^+(\mathfrak{c})$.

(b) Using results of [GKM], every stratum $\mathfrak{c}_d$ decomposes as a disjoint union $\mathfrak{c}_d=\sqcup_{(w,\mathbf{r})}\mathfrak{c}_{w,\mathbf{r}}$ of connected components, parameterized by W-orbits of pairs $(w,\mathbf{r})$, where w is an element of W and $\mathbf{r}$ is a function $R\to\mathbb{Q}_{\geq 0}$ from the set of roots R of G.

(c) The decomposition of (b) induces a decomposition $\mathfrak{C}_d=\sqcup_{(w,\mathbf{r})}\mathfrak{C}_{w,\mathbf{r}}$ of $\mathfrak{C}_d$, thus induces constructible stratifications $\{\mathfrak{C}_{w,\mathbf{r}}\}_{w,\mathbf{r}}$ of $\mathfrak{C}_\bullet$, $\{\widetilde{\mathfrak{C}}_{w,\mathbf{r}}\}_{w,\mathbf{r}}$ of $\widetilde{\mathfrak{C}}_\bullet$, and $\{\mathrm{Lie}(I)_{w,\mathbf{r}}\}_{w,\mathbf{r}}$ of $\mathrm{Lie}(I)_\bullet$.

(d) Notice that the open stratum $\mathfrak{C}_0$ consists of the locus of points with regular semisimple reduction.

0.5.2. Geometry of the affine Springer fibration. We show that:

(a) The fibration $\mathfrak{p}:\widetilde{\mathfrak{C}}\to\mathfrak{C}$ is ind-fp-proper (see Lemma 8.1.4).

(b) Up to a topological equivalence, the restriction $\mathfrak{p}_0:\widetilde{\mathfrak{C}}_0\to\mathfrak{C}_0$ of $\mathfrak{p}$ is an étale Galois covering whose Galois group is the affine Weyl group $\widetilde{W}$ of G (see Corollary 8.2.6).

(c) For every GKM stratum $(w,\mathbf{r})$, the restriction $\mathfrak{p}_{w,\mathbf{r}}:\widetilde{\mathfrak{C}}_{w,\mathbf{r}}\to\mathfrak{C}_{w,\mathbf{r}}$ is *topologically representable*, that is, for every morphism $Y\to\mathfrak{C}_{w,\mathbf{r}}$ from an affine scheme Y, the perfectization $(\widetilde{\mathfrak{C}}_{w,\mathbf{r}}\times_{\mathfrak{C}_{w,\mathbf{r}}}Y)_{\mathrm{perf}}$ is an algebraic space. Furthermore, if Y is globally placid, then reduction $(\widetilde{\mathfrak{C}}_{w,\mathbf{r}}\times_{\mathfrak{C}_{w,\mathbf{r}}}Y)_{\mathrm{red}}$ is an algebraic space, locally finitely presented over Y (use Theorem 8.3.3).

(d) In addition, every $\mathfrak{p}_{w,\mathbf{r}}:\widetilde{\mathfrak{C}}_{w,\mathbf{r}}\to\mathfrak{C}_{w,\mathbf{r}}$ is universally open equidimensional of explicit relative dimension $\delta_{w,\mathbf{r}}$ (see Corollary 8.3.4).

0.5.3. The smallness of $\overline{\mathfrak{p}}_\bullet$. One of the main goals of this work is to show that affine Grothendieck–Springer fibration $\overline{\mathfrak{p}}_\bullet$ is small.

(a) By 0.5.2(a), the induced map $\overline{\mathfrak{p}}:[\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]\to[\mathfrak{C}_\bullet/\mathcal{L}G]$ is locally ind-fp-proper.

(b) Note that we have a natural isomorphism $[\widetilde{\mathfrak{C}}/\mathcal{L}G]\simeq[\mathrm{Lie}(I)/I]$. In particular, the stack $[\widetilde{\mathfrak{C}}/\mathcal{L}G]$ and hence also its open substack $[\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]$ is a smooth placid ∞-stack. Moreover, for every GKM stratum $(w,\mathbf{r})$, we have a canonical isomorphism $[\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\mathcal{L}G]\simeq[\mathrm{Lie}(I)_{w,\mathbf{r}}/I]$.

(c) The constructible stratification 0.5.1(c), gives rise to a constructible stratifications $\{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]\}_{w,\mathbf{r}}$ of $[\mathfrak{C}_\bullet/\mathcal{L}G]$. Moreover, every stratum $[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]$ is topologically placid (see Corollary 8.1.11), thus $[\mathfrak{C}_\bullet/\mathcal{L}G]$ is a stratified ∞-stack.

(d) Since $\mathfrak{C}$ is an ind-placid scheme, while $\mathcal{L}G$ is an ind-placid group, the quotient $[\mathfrak{C}_\bullet/\mathcal{L}G]$ satisfies gluing of sheaves (using 0.4.3(c)).

(e) Every stratum $\mathfrak{c}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{c})$ is of pure codimension $b_{w,\mathbf{r}}$ with explicit formula (see Proposition 7.3.2). Moreover, using observation (b) flatness of the map $\mathrm{Lie}(I) \to \mathcal{L}^+(\mathfrak{c})$ is flat (see Corollary 7.4.4), we conclude that that $[\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\mathcal{L}G] \subset [\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]$ is of pure codimension $b_{w,\mathbf{r}}$ as well.

(f) By 0.5.2(d), every $\overline{\mathfrak{p}}_{w,\mathbf{r}} : [\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\mathcal{L}G] \to [\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]$ is equidimensional of relative dimension $\delta_{w,\mathbf{r}}$.

(g) Using (c)-(f) we conclude that $\overline{\mathfrak{p}}_\bullet$ is $[\mathfrak{C}_0/\mathcal{L}G]$-small (see Corollary 7.3.5).

0.5.4. Perversity of $\mathcal{S}_\bullet$, and $\widetilde{W}$-action. (a) Since the fibration $\overline{\mathfrak{p}}_\bullet$ is locally ind-fp-proper and $[\mathfrak{C}_0/\mathcal{L}G]$-small (see 0.5.3), the result of 0.4.6(b) applies. Therefore the affine Grothendieck–Springer sheaf $\mathcal{S}_\bullet$ is perverse. Moreover, it is isomorphic to the intermediate extension of its restriction $\mathcal{S}_0$ to $[\mathfrak{C}_0/\mathcal{L}G]$.

(b) Using observation 0.5.2(b), the restriction $\mathcal{S}_0$ of $\mathcal{S}_\bullet$ to $[\mathfrak{C}_0/\mathcal{L}G]$ is equipped with a $\widetilde{W}$-action. Thus, by (a), the $\widetilde{W}$-action on $\mathcal{S}_0$ uniquely extends to an action on $\mathcal{S}_\bullet$. Furthermore, we have a natural algebra isomorphisms $\mathrm{End}(\mathcal{S}_\bullet) \simeq \mathrm{End}(\mathcal{S}_0) \simeq \overline{\mathbb{Q}}_l[\widetilde{W}]$.

0.5.5. The case of the affine Springer sheaf. Let $\overline{\mathfrak{p}}^u_\bullet : [\widetilde{\mathfrak{C}}^u_\bullet/\mathcal{L}G] \to [\mathfrak{C}^u_\bullet/\mathcal{L}G]$ be the restriction of $\overline{\mathfrak{p}}_\bullet$ to the topologically nilpotent locus. We denote by $\mathcal{S}^u_\bullet$ be the !-pullback of $\mathcal{S}$ and call it the *affine Springer sheaf*. We show that $\overline{\mathfrak{p}}^u_\bullet$ is semi-small, $\mathcal{S}^u_\bullet \simeq (\overline{\mathfrak{p}}^u_\bullet)_!(\omega_{[\widetilde{\mathfrak{C}}^u_\bullet/\mathcal{L}G]})$, and therefore the affine Springer sheaf $\mathcal{S}^u_\bullet$ is perverse.

0.6. **Possible extensions, generalizations and analogs.**

0.6.1. The derived coinvariants. For every representation V of $\widetilde{W}$, we can consider the derived V-isotypical component $\mathcal{S}_V \in \mathcal{D}([\mathfrak{C}_\bullet/\mathcal{L}G])$.

(a) We expect that every $\mathcal{S}_V$ is perverse. Moreover, we can show this result assuming purity of the homology of affine Springer fibers and a strengthening of a theorem of Yun [Yun2] about the compatibility of the $\widetilde{W}$-action on the affine Springer fibers and the action group of connected components of the centralizer. On the other hand, the $\mathcal{S}_V$'s are not intermediate extension of its restriction to $\mathfrak{C}_0$ in general.

(b) If V is finite-dimensional, then the corresponding $\mathcal{S}_V$ is "constructible", by which we mean in particular that all of its !-stalks are constructible.

0.6.2. Distributions. In this work we only construct t-structure on the category $\mathcal{D}([\mathfrak{C}_\bullet/\mathcal{L}G])$, while the affine Grothendieck-Springer sheaf $\mathcal{S}$ naturally lives on a larger category $\mathcal{D}([\mathfrak{C}/\mathcal{L}G])$. A natural problem would be to try to construct a t-structure on the whole of $\mathcal{D}([\mathfrak{C}/\mathcal{L}G])$ and to show that $\mathcal{S}$ is an intermediate extension of its restriction to $[\mathfrak{C}_\bullet/\mathcal{L}G]$. This would be a categorical analog of the well-known fact that many important invariant distributions on a p-adic group $G(F)$ are locally L^1, and therefore can be reconstructed from their restriction to $G(F)^{rs}$.

0.6.3. Mixed characteristic case. We expect that our results and techniques can be easily extended to the mixed characteristic case. In order to do this, one needs to use the mixed characteristic version of the Affine Grassmannian, introduced by Zhu [Zhu] and studied further by Bhatt–Scholze ([BS]). Actually, this is one of the reasons why we carried out all of our constructions in the setting of perfectly/topologically placid ∞-stacks.

0.7. **Plan of the paper.** This work consists of three main parts.

In the first part we introduce our main players, that is, (topologically) placid ∞-stacks and (weakly) equidimensional morphisms. Namely, in Section 1 we discuss a generalization of Simpson construction of n-geometric stacks. Then, in Section 2 we apply this construction to construct placid algebraic ∞-stacks and their perfect and topological analogs. We also introduce reduced and perfect ∞-stacks and study a notion of topological equivalence, which is central for this work. Finally, in section 3 we develop dimension theory, which is interesting for its own and is crucially used for the definition of the perverse t-structures.

In the second part of this work we study ∞-categories of ℓ-adic sheaves on ∞-stacks and introduce perverse t-structures. First, in Section 4 we introduce ∞-categories of ℓ-adic sheaves on arbitrary ∞-prestack and show various functorial properties and base change isomorphisms. Then, in Section 5 we introduce perverse t-structures on topologically placid ∞-stacks, and show their exactness properties. Finally, in Section 6, we study perverse t-structures on ∞-stacks, which "admit gluing of sheaves" and have constructible stratification by topologically placid ∞-stacks, and apply this in the case of (semi)-small morphisms.

In the last part of the work we extend parts of the classical Springer theory to the affine setting. Namely, in Section 7, we study the Goresky-Kottwitz-MacPherson stratification: first on the arc space of the Chevalley space, following very closely the results of [GKM], and then on $\operatorname{Lie} I$. Next, in Section 8, we study the geometry of the affine Grothendieck–Springer fibration, and apply the constructions and results from the previous parts in this case. In particular, we show that each GKM stratum $[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]$ is topologically placid, define the perverse t-structure on $[\mathfrak{C}_{\bullet}/\mathcal{L}G]$, study the structure of the fibration over each GKM stratum, show that the affine Grothendieck–Springer fibration is small, and deduce the perversity of $\mathcal{S}_{\bullet}$ from it. Finally, in section 9 we complete proofs of some of the results from Section 8.

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Part 1. Topologically placid ∞-stacks and dimension theory

1. Categorical preliminaries

In this section we will carry out certain categorical constructions, which will be needed for the construction of (topologically/perfectly) placid ∞-stacks later. Because of the technical nature of this section, we recommend to skip it during the first reading and come back to it when needed.

1.1. A version of Simpson's construction.

In this subsection we recall a general categorical construction, which is essentially due to Simpson [Si].

1.1.1. Set-up. (a) Let $\mathcal{C}$ be an ∞-category, admitting all fiber products. Assume that we are given

• a class Cov of morphisms in $\mathcal{C}$, called *coverings*, containing isomorphisms and closed under pullbacks and compositions;

• a class $\mathrm{Ob}_0(\mathcal{C})$ of objects in $\mathcal{C}$, closed under isomorphisms, and

• a class $\mathrm{Mor}^0_0(\mathcal{C})$ of morphisms in $\mathcal{C}$ between objects in $\mathrm{Ob}_0(\mathcal{C})$, which

(i) contains isomorphisms, and is closed under compositions;

(ii) closed under pullbacks with respect to morphisms between objects in $\mathrm{Ob}_0(\mathcal{C})$, that is, for every pair of morphisms $f : x \to y$ in $\mathrm{Mor}^0_0(\mathcal{C})$ and $y' \to y$ in $\mathrm{Mor}(\mathcal{C})$ with $y' \in \mathrm{Ob}_0(\mathcal{C})$, the fiber product $x \times_y y'$ in $\mathcal{C}$ exists, belongs to $\mathrm{Ob}_0(\mathcal{C})$, and the projection $x \times_y y' \to y'$ is in $\mathrm{Mor}^0_0(\mathcal{C})$.

1.1.2. Construction. Assume that we are in the situation of 1.1.1. By recursion, for every $n \geq 0$ we are going to construct a class of objects $\mathrm{Ob}_n(\mathcal{C}) \subset \mathrm{Ob}(\mathcal{C})$, a class of morphisms $\mathrm{Mor}^0_n(\mathcal{C})$ of the form $f : x \to y$, where $x \in \mathrm{Ob}_n(\mathcal{C})$ and $y \in \mathrm{Ob}_0(\mathcal{C})$, and a larger class $\mathrm{Mor}_n(\mathcal{C}) \subset \mathrm{Mor}(\mathcal{C})$.

Assume that classes $\mathrm{Ob}_n(\mathcal{C})$ and $\mathrm{Mor}^0_n(\mathcal{C})$ are constructed.

(a) Denote by $\mathrm{Mor}_n(\mathcal{C})$ the class of all morphisms $f : x \to y$ in $\mathrm{Mor}(\mathcal{C})$ such that for every morphism $y' \to y$ in $\mathrm{Mor}(\mathcal{C})$ with $y' \in \mathrm{Ob}_0(\mathcal{C})$, we have $x \times_y y' \in \mathrm{Ob}_n(\mathcal{C})$ and the projection $x \times_y y' \to y'$ is in $\mathrm{Mor}^0_n(\mathcal{C})$.

(b) Denote by $\mathrm{Ob}_{n+1}(\mathcal{C})$ the class of objects $x \in \mathrm{Ob}(\mathcal{C})$ for which there exists a covering $g : z \to x$ in $\mathrm{Mor}_n(\mathcal{C})$ with $z \in \mathrm{Ob}_0(\mathcal{C})$.

(c) Denote by $\mathrm{Mor}^0_{n+1}(\mathcal{C})$ the class of morphisms $f : x \to y$ with $x \in \mathrm{Ob}_{n+1}(\mathcal{C})$ and $y \in \mathrm{Ob}_0(\mathcal{C})$ for which there exists a covering $a : z \to x$ in $\mathrm{Mor}_n(\mathcal{C})$ with $z \in \mathrm{Ob}_0(\mathcal{C})$ such that $f \circ a : z \to y$ is in $\mathrm{Mor}^0_n(\mathcal{C})$.

1.1.3. Remark. By construction, $\mathrm{Mor}_n(\mathcal{C})$ is closed under all pullbacks.

The following technical but rather straightforward lemma summarizes basic properties of this construction.

Lemma 1.1.4. *For every $n \geq 0$, we have the following assertions:*

(a)$_n$ If $(f : x \to y) \in \mathrm{Mor}_n(\mathcal{C})$ and $y \in \mathrm{Ob}_0(\mathcal{C})$, then $f \in \mathrm{Mor}^0_n(\mathcal{C})$, thus $x \in \mathrm{Ob}_n(\mathcal{C})$.

(b)$_n$ If $(f : x \to y) \in \mathrm{Mor}_n(\mathcal{C})$ and $y \in \mathrm{Ob}_n(\mathcal{C})$, then $x \in \mathrm{Ob}_n(\mathcal{C})$.

(c)$_n$ The class $\mathrm{Mor}_n(\mathcal{C})$ is closed under compositions.

(d)$_n$ We have $\mathrm{Ob}_n(\mathcal{C}) \subset \mathrm{Ob}_{n+1}(\mathcal{C})$ and $\mathrm{Mor}_n(\mathcal{C}) \subset \mathrm{Mor}_{n+1}(\mathcal{C})$.

(e)$_n$ The class $\mathrm{Mor}^0_n(\mathcal{C})$ is closed under pullbacks with respect to morphisms between objects from $\mathrm{Ob}_0(\mathcal{C})$, and we have an inclusion $\mathrm{Mor}^0_n(\mathcal{C}) \subset \mathrm{Mor}_n(\mathcal{C})$.

Proof. (a)$_n$ Apply the definition of $\mathrm{Mor}_n(\mathcal{C})$ for the identity $\mathrm{id}_y : y \to y$.

The remaining assertions we will show by induction on n, that is, will assume that all assertions for $n-1$ are satisfied.

(b)$_n$. Note that (b)$_0$ follows from (a)$_0$, hence we can assume that $n > 0$. Since $y \in \mathrm{Ob}_n(\mathcal{C})$ there exists a covering $y' \to y$ in $\mathrm{Mor}_{n-1}(\mathcal{C})$ such that $y' \in \mathrm{Ob}_0(\mathcal{C})$. Then $x \times_y y' \to x$ is a covering, which belongs to $\mathrm{Mor}_{n-1}(\mathcal{C})$, and $x \times_y y' \to y'$ belongs to $\mathrm{Mor}^0_n(\mathcal{C})$, because $f \in \mathrm{Mor}_n(\mathcal{C})$. Thus $x \times_y y' \in \mathrm{Ob}_n(\mathcal{C})$. Therefore there exists a covering $x' \to x \times_y y'$ in $\mathrm{Mor}_{n-1}(\mathcal{C})$ such that $x' \in \mathrm{Ob}_0(\mathcal{C})$. Then $x' \to x \times_y y' \to x$ is a covering from $\mathrm{Mor}_{n-1}(\mathcal{C})$ by (c)$_{n-1}$. Thus $x \in \mathrm{Ob}_n(\mathcal{C})$, as claimed.

(c)$_n$. Let $f : x \to y$ and $g : y \to z$ be in $\mathrm{Mor}_n(\mathcal{C})$, and we want to show that $g \circ f \in \mathrm{Mor}_n(\mathcal{C})$. Taking pullback with respect to $z' \to z$ with $z' \in \mathrm{Ob}_0(\mathcal{C})$, we can assume that $z \in \mathrm{Ob}_0(\mathcal{C})$. Thus $y \in \mathrm{Ob}_n(\mathcal{C})$, hence $x \in \mathrm{Ob}_n(\mathcal{C})$ by (b)$_n$. We want to show that $g \circ f \in \mathrm{Mor}^0_n(\mathcal{C})$.

When $n = 0$, we have that $f, g \in \mathrm{Mor}^0_0(\mathcal{C})$, thus $g \circ f \in \mathrm{Mor}^0_0(\mathcal{C})$ by assumption 1.1.1(i).

When $n > 0$, we want to construct a covering $a : x' \to x$ in $\mathrm{Mor}_{n-1}(\mathcal{C})$ with $x' \in \mathrm{Ob}_0(\mathcal{C})$ such that $g \circ f \circ a : x' \to z$ is in $\mathrm{Mor}_{n-1}(\mathcal{C})$ as well. Since $g : y \to z$ is in $\mathrm{Mor}^0_n(\mathcal{C})$ there exists $y' \to y$ in $\mathrm{Mor}_{n-1}(\mathcal{C})$ such that $y' \to y \to z$ is in $\mathrm{Mor}^0_{n-1}(\mathcal{C})$.

In addition, $y' \times_y x \to x$ is in $\mathrm{Mor}_{n-1}(\mathcal{C})$ (since $\mathrm{Mor}_{n-1}(\mathcal{C})$ is closed under pullbacks), while $y' \times_y x \to y'$ is in $\mathrm{Mor}^0_n(\mathcal{C})$, thus $y' \times_y x \in \mathrm{Ob}_n(\mathcal{C})$.

Thus, there exists a covering $a : x' \to y' \times_y x$ in $\mathrm{Mor}_{n-1}(\mathcal{C})$ such that the composition $x' \to y' \times_y x \to y'$ is in $\mathrm{Mor}^0_{n-1}(\mathcal{C})$. Thus the compositions $x' \to y \to z$ and $x' \to y' \times_y x \to x$ is in $\mathrm{Mor}_{n-1}(\mathcal{C})$ by (c)$_{n-1}$.

(d)$_n$ Let $x \in \mathrm{Ob}_n(\mathcal{C})$. We want to show there exists a covering $a : x' \to x$ in $\mathrm{Mor}_n(\mathcal{C})$ such that $x' \in \mathrm{Ob}_0(\mathcal{C})$. If $n = 0$, we take f to be the identity map. If $n > 0$, then there exists a covering $f : x' \to x$ in $\mathrm{Mor}_{n-1}(\mathcal{C})$ such that $x' \in \mathrm{Ob}_0(\mathcal{C})$. So the assertion follows from (d)$_{n-1}$ for morphisms.

Let $x \to y \in \mathrm{Mor}_n(\mathcal{C})$, and we want to show that $x \to y \in \mathrm{Mor}_{n+1}(\mathcal{C})$. Taking pullback with respect to $y' \to y$ with $y' \in \mathrm{Ob}_0(\mathcal{C})$, we can assume that $y \in \mathrm{Ob}_0(\mathcal{C})$, thus $x \to y \in \mathrm{Mor}_n^0(\mathcal{C})$. We want to show that there exists $x' \to x$ in $\mathrm{Mor}_n(\mathcal{C})$ such that $x' \in \mathrm{Ob}_0(\mathcal{C})$ and $x' \to x \to y \in \mathrm{Mor}_n^0(\mathcal{C})$. If $n = 0$, the identity works. If $n > 0$ there exists $x' \to x$ in $\mathrm{Mor}_{n-1}(\mathcal{C})$ such that $x' \in \mathrm{Ob}_0(\mathcal{C})$ and $x' \to x \to y \in \mathrm{Mor}_{n-1}^0(\mathcal{C})$, so we conclude again from $(\mathrm{d})_{n-1}$.

$(\mathrm{e})_n$ By definition, the second assertion follows from the first, so we want to show that for every morphism $f : x \to y$ in $\mathrm{Mor}_n^0(\mathcal{C})$ and $a : y' \to y$ with $y' \in \mathrm{Ob}_0(\mathcal{C})$, the pullback $x \times_y y' \to y'$ is in $\mathrm{Mor}_n^0(\mathcal{C})$. If $n = 0$, the assertion is our assumption 1.1.1(ii). Let $n > 0$, and let $g : z \to x$ be a covering in $\mathrm{Mor}_{n-1}(\mathcal{C})$ such that the composition $z \to x \to y$ is in $\mathrm{Mor}_{n-1}^0(\mathcal{C})$. Then by $(\mathrm{c})_{n-1}$, the composition $z \times_y y' \to x \times_y y' \to y'$ is in $\mathrm{Mor}_{n-1}^0(\mathcal{C})$. In particular, $z \times_y y' \in \mathrm{Ob}_{n-1}(\mathcal{C})$. Moreover, $z \times_y y' \to x \times_y y'$ is a covering in $\mathrm{Mor}_{n-1}(\mathcal{C})$. If $n = 1$, we are done.

If $n > 1$, there exists a covering $z' \to z \times_y y'$ in $\mathrm{Mor}_{n-2}(\mathcal{C})$. Then (by $(\mathrm{c})_{n-1}$ and $(\mathrm{d})_{n-2}$) we conclude the composition $z' \to z \times_y y' \to x \times_y y'$ is a covering in $\mathrm{Mor}_{n-1}(\mathcal{C})$, while the composition $z' \to z \times_y y' \to x \times_y y' \to y'$ is in $\mathrm{Mor}_{n-1}(\mathcal{C})$. □

1.1.5. Notation. Objects from $\mathrm{Ob}_n(\mathcal{C})$ will be called *n-geometric* and morphisms from $\mathrm{Mor}_n(\mathcal{C})$ will be called *n-special*. Moreover, we call on object of (resp. a morphism) *geometric* (resp. *special*), if it is n-geometric (resp. n-special) for some n. Of course, these notions depend on clases $Cov, \mathrm{Ob}_0(\mathcal{C})$ and $\mathrm{Mor}_0^0(\mathcal{C})$ from 1.1.1.

Corollary 1.1.6. *If $x \to y$ is an $(n-1)$-special covering, and x is n-geometric, then y is n-geometric.*

Proof. Choose an $(n-1)$-special covering $z \to x$ with 0-geometric z. Then the composition $z \to x \to y$ is an $(n-1)$-special covering by Lemma 1.1.4(c), thus y is n-geometric by definition. □

Corollary 1.1.7. *For every special morphism $f : x \to y$ between 0-geometric objects, there exists a 0-special covering $a : z \to x$ with 0-geometric z such that the composition $f \circ a : z \to y$ is 0-special.*

Proof. Assume that f is n-special. By decreasing induction, we will show that for every m there exists an m-special covering $a : z \to x$ with 0-geometric z such that the composition $f \circ a : z \to y$ is m-special. When $m = n$, the identity map $a = \mathrm{id}_x$ satisfies the property.

It remains to show that if $m > 0$, then the assertion for m implies that for $m - 1$. By definition, there exists an $(m-1)$-special covering $z' \to z$ with 0-geometric z' such that the composition $z' \to z \to x$ is an $(m-1)$-special covering. Then $z' \to z \to x \to y$ is m-special by $(\mathrm{c})_m$ and $(\mathrm{d})_{m+1}$, so there exists an $(m-1)$-special covering $z'' \to z'$ such that the composition $z'' \to z' \to z \to x \to y$ is $(m-1)$-special. Since $z'' \to z' \to z \to x$ is an $(m-1)$-special covering by $(\mathrm{c})_{m-1}$, the induction step follows. □

1.1.8. Čech nerve. (a) Recall that to every morphism $f : x \to y$ one can associate its Čech complex $C(f) = \{x^{[m]}\}_{[m]\in\Delta_s^{op}}$, parameterized by the semi-simplicial category Δ_s, where each $x^{[m]}$ is defined to be the $(m+1)$-times fiber product $x\times_y \times \ldots \times_y x$ of x over y, and morphisms are projections $x^{[m']} \to x^{[m'']}$ corresponding to injective maps $[m''] \to [m']$.

(b) It follows from Lemma 1.1.4, that if x is n-geometric and f is n-special then all terms in the Čech complex $C(f)$ are n-geometric and all maps are n-special. In particular, we are going to apply this when y is $(n+1)$-geometric and x is 0-geometric.

(c) Assume now that $\mathcal{C}$ is an ∞-topos, that is, $\mathcal{C}$ is of the form $\mathrm{Shv}(\mathcal{A})$ (see 1.2.1 below), and $f : x \to y$ is surjective, that is, $f(a) : x(a) \to y(a)$ locally has a section. Then the canonical morphism $\mathrm{colim}_{[m]\in\Delta_s^{op}} x^{[m]} \to y$ an equivalence (use, for example, [Lu1, Prop. 7.2.1.14]). Therefore, by the observation (b), every $(n+1)$-geometric object y can be written as a colimit of n-geometric objects with respect to n-special morphisms. Similarly, every $(n+1)$-special morphism $y \to z$ with $z \in \mathrm{Ob}_0(\mathcal{C})$, can be written as colimit of n-special morphisms $x^{[m]} \to z$.

1.2. **The case of ∞-categories of sheaves.** In this subsection we will specify the construction of 1.1 to the case where $\mathcal{C}$ is an ∞-topos, that is, has the form $\mathcal{C} = \mathrm{Shv}(\mathcal{A})$ for some ∞-category $\mathcal{A}$ equipped with a Grothendieck topology.

1.2.1. Notation. Let $\mathfrak{S}$ be the ∞-category of spaces, which are often referred as ∞-groupoids. For every ∞-category $\mathcal{A}$, we denote by $\mathrm{PShv}(\mathcal{A})$ the ∞-category of functors $\mathcal{A}^{op} \to \mathfrak{S}$. Moreover, when $\mathcal{A}$ is equipped with a Grothendieck topology $\mathcal{T}$, we denote by $\mathrm{Shv}(\mathcal{A}) \subset \mathrm{PShv}(\mathcal{A})$ be the ∞-subcategory of sheaves in the $\mathcal{T}$-topology.

1.2.2. Assumptions. Let $\mathcal{A}$ be an ∞-category, is equipped with a Grothendieck topology $\mathcal{T}$, and let $\mathrm{Ob}_0(\mathcal{A}) \subset \mathrm{Ob}\,\mathcal{A}$ and $\mathrm{Mor}_0^0(\mathcal{A}) \subset \mathrm{Mor}(\mathcal{A})$ be classes of objects and morphisms, satisfying the following assumptions:

(a) The class $\mathrm{Ob}_0(\mathcal{A})$ is closed under isomorphisms, while $\mathrm{Mor}_0^0(\mathcal{A})$ contains isomorphisms, and is closed under compositions and pullbacks with respect to morphisms between objects in $\mathrm{Ob}_0(\mathcal{A})$.

(b) The topology $\mathcal{T}$ is *subcanonical*, that is, every representable presheaf is a sheaf.

(c) The class $\mathrm{Ob}_0(\mathcal{A}) \subset \mathrm{Ob}(\mathcal{C})$ is closed under *direct summands*, by which we mean that if $a \in \mathrm{Ob}_0(\mathcal{A})$ decomposes in $\mathcal{C}$ as a coproduct $a \simeq b \sqcup c$, then $b, c \in \mathrm{Ob}_0(\mathcal{A})$, and, moreover, inclusions $b \hookrightarrow a$ and $c \hookrightarrow a$ belong to $\mathrm{Mor}_0^0(\mathcal{A})$.

(d) Every $x \in \mathrm{Ob}_0(\mathcal{A})$ has basis of $\mathcal{T}$-coverings of the form $\{f_\alpha : x_\alpha \to x\}$ with $f_\alpha \in \mathrm{Mor}_0^0(\mathcal{A})$ for all α.

1.2.3. Remark. For our applications, $\mathcal{A}$ will be an ordinary category.

1.2.4. Construction. (a) To the data of 1.2.2 we associate the data of 1.1.1 as follows:

(i) Set $\mathcal{C} := \mathrm{Shv}(\mathcal{A})$, and let Cov be the class of all surjective morphisms in $\mathcal{C}$,

(ii) Let $\mathrm{Ob}_0(\mathcal{C})$ be the class of all objects of the form $\sqcup_\alpha a_\alpha$, with $a_\alpha \in \mathrm{Ob}_0(\mathcal{A})$ for all α.

(iii) Let $\mathrm{Mor}_0^0(\mathcal{C})$ be the class of all morphisms of the form $\sqcup_{\alpha,\beta} b_{\alpha,\beta} \to \sqcup_\alpha a_\alpha$, where each $b_{\alpha,\beta} \to a_\alpha$ is in $\mathrm{Mor}_0^0(\mathcal{A})$.

(b) We claim that the class of morphisms $\mathrm{Mor}_0^0(\mathcal{C})$ is closed under pullbacks with respect to morphisms between objects in $\mathrm{Ob}_0(\mathcal{C})$.

Indeed, we have to show that for every morphism $c = \sqcup_{\alpha,\beta} c_{\alpha,\beta} \to a = \sqcup_\alpha a_\alpha$ in $\mathrm{Mor}_0^0(\mathcal{C})$ and every morphism $b \to a$ in $\mathcal{C}$ with $b \in \mathrm{Ob}_0(\mathcal{C})$, the fiber product $b \times_a c \to b$ belongs to $\mathrm{Mor}_0^0(\mathcal{C})$.

By definition, b decomposes as $b = \sqcup_\beta b_\beta$ with $b_\beta \in \mathrm{Ob}_0(\mathcal{A})$. Since coproducts in $\mathcal{C}$ commute with pullbacks, we conclude that $b \times_a c$ decomposes as $b \times_a c \simeq \sqcup_\beta b_\alpha \times_a c$, so replacing b by b_β, we can assume that $b \in \mathrm{Ob}_0(\mathcal{A})$. Since coproducts in $\mathcal{C}$ commute with pullbacks, b decomposes in $\mathcal{C}$ as $b \simeq \sqcup_\alpha b_\alpha$, where $b_\alpha := b \times_a a_\alpha$ is in $\mathrm{Ob}_0(\mathcal{A})$ by the assumption 1.2.2(c). Thus $b \times_a c \to a$ decomposes as a coproduct of $\sqcup_\beta (b_\alpha \times_{a_\alpha} c_{\alpha,\beta}) \to b_\alpha$. Now the assertion follows from the fact that $\mathrm{Mor}_0^0(\mathcal{A})$ was closed under all pullbacks between objects in $\mathrm{Ob}_0(\mathcal{A})$.

(c) We claim that the class $\mathrm{Mor}_0^0(\mathcal{C})$ is closed under compositions.

Indeed, let $f : b \to a$ and $g : c \to b$ be morphisms in $\mathrm{Mor}_0^0(\mathcal{C})$ of the form $\sqcup_{\alpha,\beta} b_{\alpha,\beta} \to \sqcup_\alpha a_\alpha$ and $\sqcup_{\gamma,\delta} c_{\gamma,\delta} \to b_\gamma$, respectively. Then we have decompositions $b_\gamma \simeq \sqcup_{\alpha,\beta} (b_\gamma \times_b b_{\alpha,\beta})$ and $c_{\gamma,\delta} \simeq \sqcup_{\alpha,\beta} (c_{\gamma,\delta} \times_b b_{\alpha,\beta})$. By 1.2.2(c), the fiber products $b_\gamma \times_b b_{\alpha,\beta}$ and $c_{\gamma,\delta} \times_b b_{\alpha,\beta}$ are in $\mathrm{Ob}_0(\mathcal{A})$. Thus the composition $g \circ f : c \to a$ decomposes as $\sqcup_{\alpha,\beta,\gamma,\delta} (c_{\gamma,\delta} \times_b b_{\alpha,\beta}) \to \sqcup_\alpha a_\alpha$, where each composition

$$c_{\gamma,\delta} \times_b b_{\alpha,\beta} \to b_\gamma \times_b b_{\alpha,\beta} \to b_{\alpha,\beta} \to a_\alpha$$

is in $\mathrm{Mor}_0^0(\mathcal{A})$ by 1.2.2(a),(c).

(d) By (b) and (c), the assumptions of 1.1.1 are satisfied. Thus the construction 1.1.2 applies, and we can talk about n-geometric objects and n-special morphisms in $\mathcal{C}$.

Lemma 1.2.5. *Let $f : x \to y$ be a morphism in $\mathcal{C}$, and let $z \to y$ be covering in $\mathcal{C}$. Then f is n-special if and only if its pullback $x \times_y z \to z$ is n-special.*

Proof. Since n-special morphisms are stable under pullbacks, the "only if" assertion follows. Conversely, assume that $x \times_y z \to z$ is n-special. We want to show that $f : x \times_y t \to t$ is n-special for every morphism $t \to y$ with 0-geometric t. Since t has a form $t = \sqcup_\beta t_\beta$ with $t_\beta \in \mathrm{Ob}_0(\mathcal{A})$, we conclude that $x \times_y t \to t$ is the coproduct of $x \times_y t_\beta \to t_\beta$. Thus we can assume that $t \in \mathrm{Ob}_0(\mathcal{A})$.

Since $z \to y$ is covering, there exists a $\mathcal{T}$-covering $\{t_\alpha \to t\}_\alpha$ such that every composition $t_\alpha \to t \to y$ has a lifting to $t_\alpha \to z$. By our assumption 1.2.2(d), we can assume that every $t_\alpha \to t$ belongs $\mathrm{Mor}_0^0(\mathcal{A})$. Set $t' := \sqcup_\alpha t_\alpha \in \mathcal{C}$. Then the covering $t' \to t$ belong to $\mathrm{Mor}_0^0(\mathcal{C})$, and the composition $t' \to t \to y$ has a lifting to $t' \to z$.

Since $x \times_y z \to z$ is n-special, and since n-special morphisms are stable under pullbacks, the map $x \times_y t' \to t'$ is n-special. Thus there exists an $(n-1)$-special covering $t'' \to x \times_y t'$ such that the composition $t'' \to t'$ is $(n-1)$-special. Since $t' \to t$ is a 0-special covering, we get that the composition $t'' \to x \times_y t' \to x \times_y t$ is an $(n-1)$-special covering, and the composition $t'' \to t$ is $(n-1)$-special. Thus $x \times_y t \to t$ is n-special, and the proof is complete. □

As an application, we get a characterization of geometric objects and special morphisms in the spirit of 0.2.2(c).

Corollary 1.2.6. *The classes of geometric objects in $\mathcal{C}$ and special morphisms between geometric objects can be characterized as the smallest classes, containing* $\mathrm{Ob}_0(\mathcal{A})$ *and* $\mathrm{Mor}_0^0(\mathcal{A})$*, closed under coproducts and satisfying the following properties:*

(*i*) *The class of special morphisms is closed under compositions and pullbacks.*

(*ii*) *An object $y \in \mathcal{C}$ is geometric, if there exists a covering $f : x \to y$ such that x and $x \times_y x$ are geometric, while both projections $x \times_y x \to x$ are special.*

(*iii*) *A morphism $f : x \to y$ between geometric objects is special, if for every special morphism $z \to y$ with $z \in$* $\mathrm{Ob}_0(\mathcal{A})$ *the fiber product $z \times_y x$ is geometric, and the projection $z \times_y x \to z$ is special.*

(*iv*) *A morphism $f : x \to y$ between geometric objects is special, if there exists a special covering $z \to x$ such that the composition $z \to x \to y$ is special.*

Proof. First we claim that classes of geometric objects and special morphisms satisfy properties (i)-(iv). Indeed, (i) follows from Lemma 1.1.4(c),(e), (ii) follows from a combination of Lemma 1.2.5 and Corollary 1.1.6, while (iii) and (iv) follow essentially from definitions.

Conversely, by induction on n, we claim that any pair of classes $(\mathrm{Ob}', \mathrm{Mor}')$ satisfying (i)-(iv) contains classes of n-geometric objects and n-special morphisms between geometric objects for all n. This is clear for $n = 0$. Assume now $n > 0$.

By definition, for every n-geometric object y there exists an $(n-1)$-special covering $x \to y$ from 0-geometric x. Then the fiber product $x \times_y x$ is $(n-1)$-geometric, and both projections $x \times_y x \to x$ are $(n-1)$-special. Thus y belongs to Ob' by (ii) and induction.

Finally, let $f : x \to y$ be an n-special morphism between geometric objects, and let $z \to y$ be a special covering with 0-geometric z. Using (iii) and arguing as in Lemma 1.2.5, it suffices to show that the pullback $x \times_y z \to z$ belongs to Mor'. But this follows immediately from (iv) and induction. □

1.2.7. Restriction to a subcategory. (a) Let $\mathcal{A}' \subset \mathcal{A}$ be a full subcategory, compatible with $\mathcal{T}$, by which we mean that every $x \in \mathcal{A}'$ has a basic of covering of the form $\{x_\alpha \to x\}$ with $x_\alpha \in \mathcal{A}'$.

(b) Let $\iota : \mathcal{A}' \to \mathcal{A}$ be the inclusion. Then the restriction functor $\iota^* : \mathrm{PShv}(\mathcal{A}) \to \mathrm{PShv}(\mathcal{A}')$ induces the functor $\iota^* : \mathrm{Shv}(\mathcal{A}) \to \mathrm{Shv}(\mathcal{A}')$, whose left adjoint we denote by $\iota_! : \mathrm{Shv}(\mathcal{A}) \to \mathrm{Shv}(\mathcal{A}')$.

Lemma 1.2.8. *In the situation of 1.2.7, the functor $\iota_! : \mathrm{Shv}(\mathcal{A}') \to \mathrm{Shv}(\mathcal{A})$ is fully faithful, and its essential image consists of all $y \in \mathrm{Shv}(\mathcal{A})$ such that the counit $\iota_!\iota^* y \to y$ is an isomorphism.*

Proof. We have to show that the unit morphism $x \to \iota^*\iota_! x$ is an isomorphism for every $x \in \mathrm{Shv}(\mathcal{A}')$. Since ι^* and $\iota_!$ commute with (homotopy) colimits and every x is a colimit of representable objects $a \in \mathcal{A}'$, it suffices to show that each map $a \to \iota^*\iota_! a$ is an equivalence. By the Yoneda lemma and our assumption 1.2.2(b), $\iota_! a$ is the representable presheaf $\iota(a)$, so the assertion follows from the fact that $\iota : \mathcal{A}' \to \mathcal{A}$ is fully-faithful. The second assertion is standard. □

1.2.9. Assumptions. (a) In the situation of 1.2.7, assume that $\iota : \mathcal{A}' \to \mathcal{A}$ has a right adjoint ι^R. Assume furthermore that ι^R maps $\mathcal{T}$-coverings to $\mathcal{T}$-coverings.

(b) Let $\mathrm{Ob}_0(\mathcal{A}') \subset \mathrm{Ob}(\mathcal{A}')$ and $\mathrm{Mor}_0^0(\mathcal{A}') \subset \mathrm{Mor}(\mathcal{A}')$ be classes satisfying the assumptions of 1.2.2. Thus we can talk about n-geometric objects and n-special morphisms in $\mathcal{C}$ and $\mathcal{C}' := \mathrm{Shv}(\mathcal{A}')$.

Lemma 1.2.10. *In the situation of 1.2.9, assume that $\iota^R(\mathrm{Ob}_0(\mathcal{A})) \subset \mathrm{Ob}_0(\mathcal{A}')$ and $\iota^R(\mathrm{Mor}_0^0(\mathcal{A})) \subset \mathrm{Mor}_0^0(\mathcal{A}')$.*

(a) If x is n-geometric in $\mathcal{C}$, then $\iota^*(x)$ is n-geometric in $\mathcal{C}'$.

(b) If $f : x \to y$ is an n-special morphism between geometric objects in $\mathcal{C}$, then $\iota^*(f)$ is an n-special morphism in $\mathcal{C}'$.

Proof. We will show both assertions by induction on n.

(a) If $n = 0$, then x is a coproduct of objects in $\mathrm{Ob}_0(\mathcal{A})$. Since ι^* commutes with colimits, we can assume that $x \in \mathrm{Ob}_0(\mathcal{A})$. By definition, $\iota^*(x)$ is the representable sheaf $\iota^R(x)$. Thus, $\iota^*(x) = \iota^R(x) \in \mathrm{Ob}_0(\mathcal{A}')$ by assumption.

Assume now that $n > 0$, and choose an $(n-1)$-special covering $y \to x$ with 0-geometric y. Then $\iota^*(y) \to \iota^*(x)$ is a covering, by assumption, and it is $(n-1)$-special by the assertion (b) for $(n-1)$. As it was shown above that $\iota^*(y)$ is 0-geometric, we conclude that $\iota^*(x)$ is n-geometric by definition.

(b) Choose a covering $z \to y$ with 0-geometric z. Then $x \times_y z \to z$ is n-special, and $\iota^*(z) \to \iota^*(y)$ is a covering. So by Lemma 1.2.5 it suffices to show that the projection $\iota^*(x \times_y z) \simeq \iota^*(x) \times_{\iota^*(y)} \iota^*(z) \to \iota^*(z)$ is n-special. Thus, replacing $x \to y$ by $x \times_y z \to z$, we reduce ourself to the case, when y is 0-geometric, hence x is n-geometric.

Assume first that $n = 0$. In this case, $f : x \to y$ decomposes as a coproduct of $f_{\alpha,\beta} : x_{\alpha,\beta} \to y_\alpha$ from $\mathrm{Mor}_0^0(\mathcal{A})$. Therefore $\iota^*(f)$ decomposes as the coproduct of $\iota^*(f_{\alpha,\beta})$, and each of them belong to $\mathrm{Mor}_0^0(\mathcal{A}')$ by assumption. Therefore $\iota^*(f)$ is 0-special.

Assume now that $n > 0$, and choose an $(n-1)$-special covering $z \to x$ with 0-geometric z such that the composition $z \to x \to y$ is $(n-1)$-special. Then $\iota^*(z) \to \iota^*(x)$ and $\iota^*(z) \to \iota^*(x) \to \iota^*(y)$ are $(n-1)$-special by assumption. Since $\iota^*(z)$ is 0-geometric, we conclude that $\iota^*(f)$ is n-special, as claimed. □

1.3. **Passing to pro-categories.** In our application the ∞-category $\mathcal{A}$ from 1.2.2 will of the form $\mathcal{A} \simeq \mathrm{Pro}\,\mathcal{B}$ for some ∞-category $\mathcal{B}$. In this subsection, we will describe what kind of data on $\mathcal{B}$ gives rise to the data of 1.2.2.

1.3.1. Construction. (a) Let $\mathcal{B}$ be an ∞-category, and let $\mathcal{P}$ be a class of morphisms in $\mathcal{B}$ which contains isomorphisms, and closed under compositions and all pullbacks. In particular, for every morphism $x \to y$ in $\mathcal{P}$ and every morphism $z \to y$ in $\mathcal{B}$ the fiber product $x \times_y z$ exists in $\mathcal{B}$ and the projection $x \times_y z \to z$ is in $\mathcal{P}$.

(b) Let $\mathcal{A} := \mathrm{Pro}(\mathcal{B})$ be the pro-category of $\mathcal{B}$, and $\mathrm{Ob}_0(\mathcal{A})$ be the class of objects $x \in \mathrm{Ob}(\mathcal{A})$ which have presentations as filtered limits $x \simeq \lim_\alpha x_\alpha$, where all transition maps $x_\alpha \to x_\beta$ belong to $\mathcal{P}$.

(c) Notice that assumption (a) implies that if $f : x \to y$ is in $\mathcal{P} \subset \mathrm{Mor}(\mathcal{B}) \subset \mathrm{Mor}(\mathcal{A})$, then for every morphism $y' \to y$ in $\mathcal{A}$ the fiber product $x \times_y y'$ exists in $\mathcal{A}$. Explicitly, if $y' \simeq \lim_\alpha y'_\alpha$ is a presentation of y', then the projection $y' \to y$ factors through a morphism $y'_\alpha \to y$ for all sufficiently large α, and $\lim_\alpha(y'_\alpha \times_y x)$ is a presentation of $x \times_y y'$.

(d) We denote by $\widetilde{\mathcal{P}}$ the class of all morphisms $f' : x' \to y'$ in $\mathcal{A}$ of the form $f' \simeq y' \times_y f$ for some morphism $f : x \to y$ in $\mathcal{P}$ and a morphism $y' \to y$ such that $f' \simeq y' \times_y f$. Notice that the class $\widetilde{\mathcal{P}}$ also contains contains isomorphisms and is closed under compositions and pullbacks. Moreover, $\mathcal{P}$ is nothing but $\widetilde{\mathcal{P}}|_{\mathcal{B}}$, where the later is defined to be the class of all $f' : x' \to y'$ in $\widetilde{\mathcal{P}}$ such that $y' \in \mathcal{B}$.

(e) We denote by $\mathrm{Mor}_0(\mathcal{A})$ the class of all morphisms $f : x \to y$ in $\mathcal{A}$ such that x has a presentation as a filtered limit $x \simeq \lim_\alpha x_\alpha$ over y such that all projection maps $x_\alpha \to y$ and transition maps $x_\alpha \to x_\beta$ are in $\widetilde{\mathcal{P}}$. This class contains all isomorphisms and is closed under all pullbacks.

(f) Notice that $x \in \mathrm{Ob}(\mathcal{A})$ belongs to $\mathrm{Ob}_0(\mathcal{A})$ if and only if there exists a morphism $(x \to y) \in \mathrm{Mor}_0(\mathcal{A})$ with $y \in \mathrm{Ob}(\mathcal{B})$.

1.3.2. Remark. For our applications, $\mathcal{B}$ will be an ordinary category, in which case, $\mathcal{A}$ will be an ordinary category as well.

Lemma 1.3.3. *In the situation of 1.3.1,*

(a) The class $\mathrm{Mor}_0(\mathcal{A})$ *is closed under compositions.*
(b) For every $f : x \to y$ *in* $\mathrm{Mor}_0(\mathcal{A})$ *with* $y \in \mathrm{Ob}_0(\mathcal{A})$*, we have* $x \in \mathrm{Ob}_0(\mathcal{A})$*.*

Proof. Notice first that it follows from the observation 1.3.1(f) that assertion (b) follows from (a). Thus, it remains to show that for every $f : x \to y$ and $g : y \to z$ in $\mathrm{Mor}_0(\mathcal{A})$ we have $g \circ f \in \mathrm{Mor}_0(\mathcal{A})$.

Though it is not difficult to show this assertion directly by constructing a presentation of $x \to z$ from presentations of $x \to y$ and $y \to z$, we are going to deduce it from a standard fact that every pro-category has all filtered limits. Our argument is based on the following construction.

1.3.4. Construction. (a) Since the class $\widetilde{\mathcal{P}} \subset \mathrm{Mor}(\mathcal{A})$ contains all isomorphisms and is closed under compositions, we can view $\widetilde{\mathcal{P}}$ as a (non-full) subcategory of $\mathcal{A}$. Then the over-category $\widetilde{\mathcal{P}}/z$ is a subcategory of $\mathcal{A}/z$. Moreover, since $\mathcal{A}$ has all filtered limits, the inclusion $\widetilde{\mathcal{P}}/z \subset \mathcal{A}/z$ gives rise to the functor

$$\iota : \mathrm{Pro}(\widetilde{\mathcal{P}}/z) \to \mathrm{Pro}(\mathcal{A}/z) \xrightarrow{\lim} \mathcal{A}/z,$$

whose essential image is precisely the morphisms $(x \to z) \in \mathrm{Mor}_0(\mathcal{A})$.

(b) Since $\widetilde{\mathcal{P}}$ is closed under pullbacks, for every pair of morphisms $a \to b$ and $c \to b$ in $\widetilde{\mathcal{P}}$ there exists a fiber product $a \times_b c \in \mathcal{A}$ such that both projections $a \times_b c \to a$ and $a \times_b c \to c$ are in $\widetilde{\mathcal{P}}$. Mimicking the construction of 1.3.1(d), we denote by $\widetilde{\widetilde{\mathcal{P}}}$ the class of morphisms $\widetilde{f} : \widetilde{x} \to \widetilde{y}$ in $\mathrm{Mor}(\mathrm{Pro}(\widetilde{\mathcal{P}}/z))$ such that for every (or equivalently for some) presentation $\widetilde{y} \simeq \lim_{\alpha \in I} y_\alpha$ of $\widetilde{y}$, there exists $\beta \in I$ and a morphism $f_\beta : x_\beta \to y_\beta$ in $\widetilde{\mathcal{P}}/z$ such that $\widetilde{f} \simeq \lim_{\alpha > \beta}(y_\alpha \times_{y_\beta} f_\beta)$.

(c) By assumption, $y \in \mathcal{A}/z$ has a lift to a certain $\widetilde{y} \in \mathrm{Pro}(\widetilde{\mathcal{P}}/z)$.

(d) Since the class $\widetilde{\widetilde{\mathcal{P}}} \subset \mathrm{Mor}(\mathrm{Pro}(\widetilde{\mathcal{P}}/z))$ is closed under compositions, we can view $\widetilde{\widetilde{P}}$ as a subcategory of $\mathrm{Pro}(\widetilde{P}/z)$, and thus can consider category $\widetilde{\widetilde{\mathcal{P}}}/\widetilde{y} \subset \mathrm{Pro}(\widetilde{\mathcal{P}}/z)/\widetilde{y}$.

Claim 1.3.5. *Functor* ι *induces an equivalence of categories* $\overline{\iota} : \widetilde{\widetilde{\mathcal{P}}}/\widetilde{y} \xrightarrow{\sim} \widetilde{\mathcal{P}}/y$*.*

Let us finish the proof of the lemma assuming the claim. By the definition of $\mathrm{Mor}_0(\mathcal{A})$, there exists a presentation x as a filtered limit $x \simeq \lim_{\alpha \in I} x_\alpha$ over y such that all projections $x_\alpha \to y$ and all transition maps $x_\alpha \to x_\beta$ are in $\widetilde{\mathcal{P}}$. In other words, the assignment $x. : \alpha \mapsto x_\alpha$ is a functor $I \to \widetilde{\mathcal{P}}/y$. By Claim 1.3.5, functor $x.$ has a natural lift to a functor $\widetilde{x}. : I \to \widetilde{\widetilde{\mathcal{P}}}/\widetilde{y} : \alpha \mapsto \widetilde{x}_\alpha$. Since $\mathrm{Pro}(\widetilde{\mathcal{P}}/z)$ has all filtered limits, while ι preserves filtered limits, the limit $\widetilde{x} := \lim_\alpha \widetilde{x}_\alpha \in \mathrm{Pro}(\widetilde{\mathcal{P}}/z)$ exists and satisfies $\iota(\widetilde{x}) \simeq x$. Thus $x \to z$ is in $\mathrm{Mor}_0(\mathcal{A})$, as claimed. □

1.3.6. Remark. Our argument shows that if $\widetilde{y} \in \mathrm{Pro}(\widetilde{\mathcal{P}}/z)$ is a lift of $(y \to z) \in \mathrm{Mor}_0(\mathcal{A})$, then every morphism $f : x \to y$ in $\mathrm{Mor}_0(\mathcal{A})$ has a lift to a morphism $\widetilde{f} : \widetilde{x} \to \widetilde{y}$ in $\mathrm{Pro}(\widetilde{\mathcal{P}}/z)$.

It remains to show Claim 1.3.5.

Proof of Claim 1.3.5. Choose a presentation $\widetilde{y} \simeq \lim_\alpha y_\alpha$ of $\widetilde{y} \in \mathrm{Pro}(\widetilde{\mathcal{P}}/z)$. It clearly induces a presentation $y \simeq \lim_\alpha y_\alpha$ of $y \in \mathcal{A}/z$.

First, we will show that $\overline{\iota}$ is essentially surjective. Take any object $f : x \to y$ in $\widetilde{\mathcal{P}}/y$. By definition, there exists an index β and a morphism $f_\beta : x_\beta \to y_\beta$ from $\mathcal{P}$ such that $f \simeq y \times_{y_\alpha} f_\alpha$. Then the morphism $\widetilde{f} := \lim_{\alpha > \beta}(y_\alpha \times_{y_\beta} f_\beta)$ belongs to $\widetilde{\widetilde{\mathcal{P}}}/\widetilde{y}$ and satisfies $\overline{\iota}(\widetilde{f}) \simeq f$.

It remains to show that $\overline{\iota}$ is fully faithful. Let $f' : x' \to \widetilde{y}$ and $f'' : x'' \to \widetilde{y}$ be two objects in $\widetilde{\widetilde{\mathcal{P}}}/\widetilde{y}$ coming from morphisms $f'_\beta : x'_\beta \to y_\beta$ and $f''_\beta : x''_\beta \to y_\beta$ from $\widetilde{\mathcal{P}}/z$. Now the assertion follows from the fact that both $\mathrm{Hom}_{\widetilde{\widetilde{\mathcal{P}}}/\widetilde{y}}(f', f'')$ and $\mathrm{Hom}_{\widetilde{\mathcal{P}}/y}(\overline{\iota}(f'), \overline{\iota}(f''))$ are naturally isomorphic to

$$\mathrm{colim}_{\alpha > \beta} \mathrm{Hom}_{\widetilde{\mathcal{P}}/y_\alpha}(y_\alpha \times_{y_\beta} x'_\beta, y_\alpha \times_{y_\beta} x''_\beta).$$

□

1.3.7. Summary. (a) In the situation of 1.3.1, we denote by $\mathrm{Mor}_0^0(\mathcal{A})$ the class of all morphisms $f : x \to y$ in $\mathrm{Mor}_0(\mathcal{A})$ such that $y \in \mathrm{Ob}_0(\mathcal{A})$, (and hence $x \in \mathrm{Ob}_0(\mathcal{A})$ by Lemma 1.3.3). By construction, the pair $(\mathrm{Ob}_0(\mathcal{A}), \mathrm{Mor}_0^0(\mathcal{A}))$ satisfies all the assumptions of 1.2.2(a).

(b) Note that every Grothendieck topology $\mathcal{T}_\mathcal{B}$ on $\mathcal{B}$ induces a Grothendieck topology $\mathcal{T}$ on $\mathcal{A}$. Namely, for every presentation $x \simeq \lim_\alpha x_\alpha$, coverings of x are generated by coverings of the form $\{x \times_{x_\alpha} x_{\alpha,i}\}_i$, where $\{x_{\alpha,i} \to x_\alpha\}_i$ is a covering of x_α. In particular, if the Grothendieck topology $\mathcal{T}_\mathcal{B}$ is generated by morphisms belonging to $\mathcal{P}$, then $\mathcal{T}$ satisfies the assumption 1.2.2(d).

1.4. **Extension of classes of morphisms.** In this subsection we will outline a general procedure how to extend classes of morphisms in between objects in $\mathrm{Ob}_0(\mathcal{A})$ to correspodning classes of morphisms between geometric objects in $\mathcal{C}$. The constructions and results of this subsection will not be used before subsection 3.2.

1.4.1. $\mathcal{M}$-special morphisms. In the situation of 1.2.2, let $\mathcal{M} \supset \mathrm{Mor}_0^0(\mathcal{A})$ be a class of morphisms between objects in $\mathrm{Ob}_0(\mathcal{A})$, which is closed under compositions, and under pullbacks with respect to morphisms in $\mathrm{Mor}_0^0(\mathcal{A})$.

(a) We say that a morphism $f : x \to y$ in $\mathcal{C}$ between 0-geometric objects x and y is $\mathcal{M}_0^0$-*special*, if it decomposes as a coproduct $f = \sqcup_{\alpha,\beta} f_{\alpha,\beta} : \sqcup_{\alpha,\beta} x_{\alpha,\beta} \to \sqcup_\beta y_\beta$ where each $f_{\alpha,\beta} \in \mathcal{M}$.

(b) We say that a morphism $f : x \to y$ in $\mathcal{C}$ from a geometric x to a 0-geometric y is $\mathcal{M}_0$-*special*, if there exists a special covering $z \to x$ with 0-geometric z such that the composition $z \to x \to y$ is $\mathcal{M}_0^0$-special (see (a)).

(c) We say that a morphism $f : x \to y$ in $\mathcal{C}$ between geometric objects is $\mathcal{M}$-*special*, if for every special morphism $y' \to y$ with 0-geometric y', the pullback $x \times_y y' \to y'$ is $\mathcal{M}_0$-special (see (b)).

1.4.2. Remarks. (a) As in 1.2.4(b),(c), the assumptions on $\mathcal{M}$ imply that the class of $\mathcal{M}_0^0$-special morphisms from 1.4.1(a) contains $\mathrm{Mor}_0^0(\mathcal{C})$ and is closed under compositions and $\mathrm{Mor}_0^0(\mathcal{C})$-pullbacks.

(b) The class of $\mathcal{M}_0$-special morphisms from 1.4.1(b) is closed under pullbacks with respect to special morphisms $y' \to y$. Indeed, let $x \to y$ by $\mathcal{M}_0$-special, and let $z \to x$ be as in 1.4.1(b). By definition, there exists a 0-special covering $y'' \to y'$ such that the composition $y'' \to y' \to y$ is 0-special. Therefore, by (a) the composition $z \times_y y'' \to y''$ is $\mathcal{M}_0^0$-special. Hence the composition $z \times_y y'' \to y'' \to y'$ or, what is the same, $z \times_y y'' \to x \times_y y' \to y'$ is $\mathcal{M}_0^0$-special. Since $z \times_y y'' \to x \times_y y'$ is a special covering, we conclude that $x \times_y y' \to y'$ is $\mathcal{M}_0$-special, as claimed.

(c) By (b), every morphism from $\mathcal{M}_0$-special morphism is $\mathcal{M}$-special (see 1.4.1(c)).

(d) By definition, the class of $\mathcal{M}$-special morphisms is closed under pullbacks with respect to all special morphisms. Moreover, using (a) one sees that this class is closed under composition and contains special morphisms.

(e) Note that if $x \xrightarrow{f} y \xrightarrow{g} z$ are morphisms in $\mathcal{C}$ such that f is a special covering and $g \circ f$ is $\mathcal{M}$-special, then g is $\mathcal{M}$-special.

(f) Notice that a morphism $f := \sqcup_\alpha f_\alpha : x = \sqcup_\alpha x_\alpha \to y$ is $\mathcal{M}$-special if and only if every f_α is $\mathcal{M}$-special. Indeed, the "only if" assertion follows from the fact that the inclusion $x_\alpha \to \sqcup_\alpha x_\alpha$ is special.

As for the converse, passing to the pullback with respect to special morphism $y' \to y$ with 0-geometric y, we can assume that y is 0-geometric. Next, choose a special covering $g_\alpha : z_\alpha \to x_\alpha$ with 0-geometric z_α, such that $z_\alpha \to x_\alpha \to y$ is $\mathcal{M}_0^0$-special, and set $z := \sqcup_\alpha z_\alpha$. Then $g := \sqcup_\alpha g_\alpha : z \to x$ is a special covering, and the composition $z \to x \to y$ is $\mathcal{M}_0^0$-special, by definition. Hence f is $\mathcal{M}_0$-special, thus $\mathcal{M}$-special.

(g) By (f), a morphism $f : x \to y$ is $\mathcal{M}$-special if and only if the pullback $x \times_y y' \to y'$ is $\mathcal{M}$-special for every special morphism $y' \to y$ with $y' \in \mathrm{Ob}_0(\mathcal{A})$. Also by (e) and (f), a morphism $f : x \to y$ is $\mathcal{M}$-special if and only if the composition $x' \to x \to y'$ is $\mathcal{M}$-special for every special morphism $x' \to x$ with $x' \in \mathrm{Ob}_0(\mathcal{A})$.

Lemma 1.4.3. *Let $f : x \to y$ be a morphism in $\mathcal{C}$ such that pullback $x \times_y z \to z$ is $\mathcal{M}$-special for some special covering $z \to y$. Then f is $\mathcal{M}$-special.*

Proof. We want to show that $z \times_y t \to t$ is in $\mathcal{M}$ for every special morphism $t \to y$ with 0-geometric t. Set $t' := z \times_y t$. Then the projection $t' = z \times_y t \to z$ is a special morphism, thus the pullback $x \times_y t' \to t'$ is $\mathcal{M}$-special.

On the other hand the projection $t' = z \times_y t \to t$ is a special covering, therefore the composition $x \times_y t' \to t' \to t$, or (what is the same) $x \times_y t' \to x \times_y t \to t$ is $\mathcal{M}$-special (by 1.4.2(d)). Since $x \times_y t' \to x \times_y t$ is a special covering, we conclude that $x \times_y t \to t$ is in $\mathcal{M}$ by 1.4.2(e). □

1.4.4. $\mathcal{P}$-adapted classes. (a) Let $\mathcal{P}$ be as in 1.3.1(a), so the construction of 1.3.1 applies, and let $\mathcal{Q} \supseteq \mathcal{P}$ be a class of morphisms in $\mathcal{B}$, which is closed under compositions, and $\mathcal{P}$-pullbacks, that is, pullbacks with respect to morphisms from $\mathcal{P}$.

(b) We denote by $\mathcal{Q}_{\mathcal{A}}$ the class of morphism $f : x \to y$ in $\mathcal{A}$ with $x, y \in \mathrm{Ob}_0(\mathcal{A})$ such that for *every* two strongly pro-$\mathcal{P}$ presentations $y \simeq \lim_\alpha y_\alpha$ and $x \simeq \lim_\beta x_\beta$ the following condition is satisfied:

($\star$) for every α there exists β and a morphism $f_{\beta,\alpha} : x_\beta \to y_\alpha$ belonging $\mathcal{Q}$ such that $\mathrm{pr}_\alpha \circ f : x \to y \to y_\alpha$ factors as $f_{\beta,\alpha} \circ \mathrm{pr}_\beta : x \to x_\beta \to y_\alpha$.

(c) By definition, the class $\mathcal{Q}_{\mathcal{A}}$ is closed under compositions. Also, a morphism $f : x \to y$ is in $\mathcal{Q}_{\mathcal{A}}$ if and only if for every strongly pro-$\mathcal{P}$ presentation $y \simeq \lim_\alpha y_\alpha$, each composition $\mathrm{pr}_\alpha \circ f : x \to y_\alpha$ is in $\mathcal{Q}_{\mathcal{A}}$.

(d) We say that the class $\mathcal{Q}$ is $\mathcal{P}$*-adapted*, if for every $x \in \mathrm{Ob}_0(\mathcal{A})$, the identity map $\mathrm{id} : x \to x$ is in $\mathcal{Q}_{\mathcal{A}}$.

1.4.5. Remarks. (a) Notice that if $\mathcal{Q}$ is $\mathcal{P}$-adapted, $\mathcal{Q}' \supset \mathcal{Q}$ and $\mathcal{P}' \subset \mathcal{P}$, then $\mathcal{Q}'$ is $\mathcal{P}'$-adapted.

(b) If $\mathcal{Q}$ is $\mathcal{P}$-adapted, then in order a morphism $f : x \to y$ in $\mathcal{A}$ with $x, y \in \mathrm{Ob}_0(\mathcal{A})$ be in $\mathcal{Q}_{\mathcal{A}}$, it suffices to check that the condition ($\star$) of 1.4.4(b) is satisfied for *some* presentations $y \simeq \lim_\alpha y_\alpha$ and $x \simeq \lim_\beta x_\beta$.

(c) Since $\mathcal{Q}$ is closed under composition, we can view $\mathcal{Q}$ as a (non-full) subcategory of $\mathcal{B}$. Thus we have a natural functor $\iota : \mathrm{Pro}(\mathcal{Q}) \to \mathcal{A}$. Then $\mathcal{Q}$ is $\mathcal{Q}$-adapted if and only if ι induces an equivalence between $\mathrm{Pro}(\mathcal{Q})$ and a subcategory of $\mathcal{A}$.

Lemma 1.4.6. *Assume that $\mathcal{Q}$ is $\mathcal{P}$-adapted. Then the class $\mathcal{Q}_{\mathcal{A}}$ contains $\mathrm{Mor}_0^0(\mathcal{A})$ and is closed under pullbacks with respect to all morphisms in $\mathrm{Mor}_0^0(\mathcal{A})$.*

Proof. By remark 1.3.6, for every lift $\widetilde{y} \in \mathrm{Pro}(\mathcal{P})$ of $y \in \mathrm{Ob}_0(\mathcal{A})$, every morphism $(f : x \to y) \in \mathrm{Mor}_0^0(\mathcal{A})$ can be lifted to a a morphism $\widetilde{f} : \widetilde{x} \to \widetilde{y}$ in $\mathrm{Pro}(\mathcal{P})$. In other words, for some presentations of y and x the condition ($\star$) of 1.4.4(b) is satisfied with $f_{\beta,\alpha} \in \mathcal{P} \subset \mathcal{Q}$. Hence, by remark 1.4.5(c), this implies that f is in $\mathcal{Q}_{\mathcal{A}}$.

Next, let $g : z \to y$ be in $\mathrm{Mor}_0^0(\mathcal{A})$, while $f : x \to y$ is in $\mathcal{Q}_{\mathcal{A}}$. We want to show that the fiber product $x \times_y z$ in $\mathcal{A}$ exists, belongs to $\mathrm{Ob}_0(\mathcal{A})$, and the projection $x \times_y z \to z$ is in $\mathcal{Q}_{\mathcal{A}}$. Note that since $z \to y$ is in $\mathrm{Mor}_0(\mathcal{A})$, the fiber product $x \times_y z$

exists and $x \times_y z \to x$ is in $\mathrm{Mor}_0^0(\mathcal{A})$ (see 1.3.1(c)). It remains to show that the projection $x \times_y z \to z$ is in $\mathcal{Q}_{\mathcal{A}}$.

Choose a presentation $z \simeq \lim_\alpha z_\alpha$ over y such that each $z_\alpha \to y$ is in $\widetilde{\mathcal{P}}$. It suffices to show that each projection $x \times_y z \to z \to z_\alpha$ is in $\mathcal{Q}_{\mathcal{A}}$. Since this composition decomposes as $x \times_y z \to x \times_y z_\alpha \to z_\alpha$, and the first map is in $\mathrm{Mor}_0^0(\mathcal{A})$, it suffices to show that the map $x \times_y z_\alpha \to z_\alpha$ is in $\mathcal{Q}_{\mathcal{A}}$. Replacing $z \to y$ by $z_\alpha \to y$, we can assume that $z \to y$ in $\widetilde{\mathcal{P}}$.

Choose a presentation, $y \simeq \lim_\alpha y_\alpha$. Since $z \to y$ in $\widetilde{\mathcal{P}}$, it is a pullback of some morphism $z_\alpha \to y_\alpha$ in $\mathcal{P}$. In particular, z has a presentation $x \simeq \lim_{\beta>\alpha} x_\beta$ with $x_\beta \simeq x_\alpha \times_{y_\alpha} y_\beta$. By 1.4.4(c),(d), it suffices to show that $x \times_{y_\beta} z_\beta \simeq x \times_y z \to z \to z_\beta$ is in $\mathcal{Q}$. Thus, replacing $z \to y$ by $z_\alpha \to y_\alpha$, we can assume that $z \to y$ in $\mathcal{P}$.

Choose a presentation $x \simeq \lim_\beta x_\beta$. By definition, there exists β such that $x \to y$ factors through some $(x_\beta \to y) \in \mathcal{Q}$. Thus $x \times_y z \to z$ factors as $x \times_y z \to x_\beta \times_y z \to z$, which is in $\mathcal{Q}$, because $x \times_y z \to x_\beta \times_y z$ is in $\mathrm{Mor}_0^0(\mathcal{A})$, while $x_\beta \times_y z \to z$ is in $\mathcal{Q}$.

(b) Let $f : x \to y$ is in $\mathcal{Q}_{\mathcal{A}}$, and let $g : z \to y$ be as in the lemma. We want to show that the fiber product $x \times_y z$ in $\mathcal{A}$ exists, belongs to $\mathrm{Ob}_0(\mathcal{A})$, and the projection $x \times_y z \to z$ is in $\mathcal{Q}_{\mathcal{A}}$. Choose a strongly pro-$\mathcal{P}$ presentations $y \simeq \lim_\alpha y_\alpha$ and $x \simeq \lim_\beta x_\beta$ of y and z, respectively.

By our assumption on g, we have $g \simeq g_\alpha \times_{y_\alpha} y$ for α and some morphism $g_\alpha : z_\alpha \to y_\alpha$ in $\mathcal{B}$, thus z has a presentation $z \simeq \lim_{\alpha'>\alpha} y_{\alpha'} \times_{y_\alpha} z_\alpha$. By our assumption on f, the composition $x \to y \to y_\alpha$ factors as $x \to x_\beta \xrightarrow{f_{\beta,\alpha}} y_\alpha$ with $f_{\beta,\alpha} \in \mathcal{Q}$.

Then the fiber product $x \times_y z \simeq x \times_{y_\alpha} z_\alpha$ exists and has a strongly pro-$\mathcal{P}$ presentation $x \times_y z \simeq \lim_{\beta'>\beta} x_{\beta'} \times_{y_\alpha} z_\alpha$. Since $\mathcal{Q}$ is closed under all pullbacks, we thus conclude that the projection $x \times_y z \to z \to z_\alpha$ is in $\mathcal{Q}_{\mathcal{A}}$. Using 1.4.4(c), we thus conclude that the projection $x \times_y z \to z$ is in $\mathcal{Q}_{\mathcal{A}}$, as claimed. □

1.4.7. $\mathcal{Q}$-special morphisms. In the situation of 1.4.4, assume that $\mathcal{Q}$ is $\mathcal{P}$-adapted. By Lemma 1.4.6, the class $\mathcal{Q}_{\mathcal{A}}$ satisfies all the assumptions 1.4.1. In particular, $\mathcal{Q}$ gives rise to a class of $\mathcal{Q}_{\mathcal{A}}$-special morphisms between geometric objects of $\mathcal{C}$ which we will simply call $\mathcal{Q}$-special. By 1.4.4(d), the class of $\mathcal{Q}$-special morphisms is stable under pullbacks with respect to special morphisms.

1.4.8. Finitely presented morphisms. In the situation of 1.4.4, we say that

- a morphism $f : x \to y$ in $\mathcal{A}$ is *fp*, for the form $f \simeq f' \times_{y'} y$ for some morphism $f' : z' \to y'$ in $\mathcal{B}$.
- a morphism $f : x \to y$ in $\mathcal{A}$ is *fp*, if for every morphism $g : y_0 \to y$ with $y_0 \in \mathcal{A}$, we have $x \times_y y_0 \in \mathcal{A}$, and the projection $x \times_y y_0 \to y_0$ is fp.

Lemma 1.4.9. *Assume that the class $\mathcal{Q}$ from 1.4.4(a) is closed under all pullbacks in $\mathcal{B}$. Then*

(a) The class $\mathcal{Q}_{\mathcal{A}}$ is closed under pullbacks with respect all fp-morphisms in $\mathcal{A}$.

(b) The class of $\mathcal{Q}$-special morphisms is closed under pullbacks with respect all fp-morphisms in $\mathcal{C}$.

Proof. (a) Let $f : x \to y$ is in $\mathcal{Q}_{\mathcal{A}}$, and let $g : z \to y$ be fp. We want to show that the fiber product $x \times_y z$ in $\mathcal{A}$ exists, belongs to $\mathrm{Ob}_0(\mathcal{A})$, and the projection $x \times_y z \to z$ is in $\mathcal{Q}_{\mathcal{A}}$. Choose a strongly pro-$\mathcal{P}$ presentations $y \simeq \lim_\alpha y_\alpha$ and $x \simeq \lim_\beta x_\beta$ of y and z, respectively.

By our assumption on g, we have $g \simeq g_\alpha \times_{y_\alpha} y$ for α and some morphism $g_\alpha : z_\alpha \to y_\alpha$ in $\mathcal{B}$, thus z has a presentation $z \simeq \lim_{\alpha' > \alpha} y_{\alpha'} \times_{y_\alpha} z_\alpha$. By our assumption on f, the composition $x \to y \to y_\alpha$ factors as $x \to x_\beta \xrightarrow{f_{\beta,\alpha}} y_\alpha$ with $f_{\beta,\alpha} \in \mathcal{Q}$.

Then the fiber product $x \times_y z \simeq x \times_{y_\alpha} z_\alpha$ exists and has a strongly pro-$\mathcal{P}$ presentation $x \times_y z \simeq \lim_{\beta' > \beta} x_{\beta'} \times_{y_\alpha} z_\alpha$. Since $\mathcal{Q}$ is closed under all pullbacks, we thus conclude that the projection $x \times_y z \to z \to z_\alpha$ is in $\mathcal{Q}_{\mathcal{A}}$. Using 1.4.4(c), we thus conclude that the projection $x \times_y z \to z$ is in $\mathcal{Q}_{\mathcal{A}}$, as claimed.

(b) Let $f : x \to y$ is $\mathcal{Q}$-special, and let $g : z \to y$ be fp. We want to show that the projection $x \times_y z \to z$ is $\mathcal{Q}$-special. Using Lemma 1.4.3, we can take pullback with respect to a special morphism $y_0 \to y$ with $y_0 \in \mathcal{A}$, thus assuming that y (and hence also z) is in $\mathcal{A}$. Moreover, precomposing f with a special covering $x_0 \to x$ with 0-geometric x_0 and using 1.4.4(f), we can assume that $x \in \mathcal{A}$ and $f \in \mathcal{Q}_{\mathcal{A}}$. In this case, the assertion follows from (a). □

2. Placid ∞-stacks and their analogs

In this section we are going to introduce our basic geometric objects, namely placid ∞-stacks, and their perfect and topological analogs.

2.1. **Placid ∞-stacks.**

Let k be an algebraically closed field. In this subsection we will study the construction of Section 1 in the case when $\mathcal{B}$ is the category Aff_k^{ft} of affine schemes of finite type over k, equipped with étale topology, and $\mathcal{P}$ is the class of all smooth morphisms.

2.1.1. Globally placid affine schemes. (a) In the situation of 1.3.1, let $\mathcal{B}$ be the category Aff_k^{ft} of affine schemes of finite type over k, and $\mathcal{P} = \mathcal{P}_{sm}$ be the class of all smooth morphisms. Then $\mathcal{P}$ satisfies the assumptions of 1.3.1(a), thus the construction of 1.3.1 applies. In particular, we can form a category $\mathcal{A} := \mathrm{Pro}\,\mathcal{B}$ and can form a class of objects $\mathrm{Ob}_0(\mathcal{A}) \subset \mathrm{Ob}(\mathcal{A})$ and a class of morphisms $\mathrm{Mor}_0(\mathcal{A})$, which we are going now describe explicitly.

(b) Recall that the category $\mathcal{A}$ is canonically equivalent to the category Aff_k of affine schemes over k, while $\mathrm{Ob}_0(\mathcal{A})$ consists of all affine schemes X, which have presentations as filtered limits $X \simeq \lim_\alpha X_\alpha$, where every X_α is in Aff_k^{ft} and all transition maps $X_\alpha \to X_\beta$ are smooth. Such presentations will be called *placid.*

(c) Using [EGAIV, 8.9.1 and 17.7.8], one sees that $\widetilde{\mathcal{P}}$ is the class of all smooth finitely presented morphisms between affine schemes. Therefore a morphism $f : X \to Y$ in Aff_k belongs to $\mathrm{Mor}_0(\mathcal{A})$, if X has a presentation as a filtered limit $X \simeq \lim_\alpha X_\alpha$ over Y such that all projection maps $X_\alpha \to Y$ and transition maps $X_\alpha \to X_\beta$ are fp-smooth.

(d) We will call objects of $\mathrm{Ob}_0(\mathcal{A})$ *globally placid affine schemes*, and morphisms belonging $\mathrm{Mor}_0(\mathcal{A})$ *strongly pro-smooth*.

(e) As in 1.3.7(a), we define a subclass $\mathrm{Mor}_0^0(\mathcal{A}) \subset \mathrm{Mor}_0(\mathcal{A})$.

2.1.2. Remark. We use the term *globally placid* instead of simply *placid* both to emphasize the global nature of the definition and in order not to conflict with Definition 2.1.10 (compare remark 2.1.13(b)).

The following lemma will be needed to show that various constructions will be independent of a presentation.

Lemma 2.1.3. *Let $g : X \to Y$ be a flat map between globally placid affine schemes with presentations $X \simeq \lim_\alpha X_\alpha$ and $Y \simeq \lim_\beta Y_\beta$. Then for every β and every sufficiently large α the composition $X \xrightarrow{g} Y \xrightarrow{\mathrm{pr}_\beta} Y_\beta$ factors as $X \xrightarrow{\mathrm{pr}_\alpha} X_\alpha \xrightarrow{g_{\alpha,\beta}} Y_\beta$ with $g_{\alpha,\beta}$ flat. Furthermore, if g is strongly pro-smooth, then for every sufficiently large α, the morphism $g_{\alpha,\beta}$ is smooth.*

Proof. Since $X \simeq \lim_\alpha X_\alpha$ and Y_β is of finite type over k, there exists α such that $\mathrm{pr}_\beta \circ g : X \to Y_\beta$ factors as $X \xrightarrow{\mathrm{pr}_\alpha} X_\alpha \xrightarrow{g_{\alpha,\beta}} Y_\beta$. Thus $\mathrm{pr}_\beta = g_{\alpha',\beta} \circ \mathrm{pr}_{\alpha'}$ for every $\alpha' \geq \alpha$, and we would like to show that there exists $\alpha' \geq \alpha$ such that $g_{\alpha',\beta}$ is flat (resp. smooth).

Let $X'_\alpha \subset X_\alpha$ be the largest open subset such that $g'_{\alpha,\beta} := g_{\alpha,\beta}|_{X'_\alpha}$ is flat (resp. smooth). It suffices to show that the image of pr_α is contained in X'_α. Indeed, in this case, we would have a projection $\mathrm{pr}'_\alpha : X \to X'_\alpha$. Since $X \simeq \lim_{\alpha' \geq \alpha} X_{\alpha'}$, there exists $\alpha' \geq \alpha$ such that $\mathrm{pr}'_\alpha : X \to X'_\alpha$ factors as $X \xrightarrow{\mathrm{pr}_{\alpha'}} X_{\alpha'} \xrightarrow{\mathrm{pr}'_{\alpha',\alpha}} X'_\alpha$, and $\mathrm{pr}'_{\alpha',\alpha}$ is smooth. Therefore $g_{\alpha',\beta} = g'_{\alpha,\beta} \circ \mathrm{pr}'_{\alpha',\alpha}$ is flat (resp. smooth), as claimed.

Fix a point $x \in X$, and set $x_\alpha := \mathrm{pr}_\alpha(x) \in X_\alpha$. We want to show that $g_{\alpha,\beta}$ is flat (resp. smooth) at x_α. Set $y := g(x)$ and $y_\beta := \mathrm{pr}_\beta(y) \in Y_\beta$.

Notice that both $X \to X_\alpha$ and $Y \to Y_\beta$ are strongly pro-smooth, thus flat. Therefore the composition $X \to Y \to Y_\beta$ is flat, thus $\mathcal{O}_{X,x}$ is faithfully flat both as an $\mathcal{O}_{X_\alpha,x_\alpha}$-algebra and an $\mathcal{O}_{Y_\beta,y_\beta}$-algebra. Therefore $\mathcal{O}_{X_\alpha,x_\alpha}$ is a flat $\mathcal{O}_{Y_\beta,y_\beta}$-algebra, thus $g_{\alpha,\beta}$ is flat at x_α.

Assume now that g is strongly pro-smooth, thus $\mathrm{pr}_\beta \circ g : X \to Y_\beta$ is strongly pro-smooth. To show that $g_{\alpha,\beta}$ is smooth at x_α, it remains to show that x_α is a smooth point in the fiber $(X_\alpha)_{y_\beta}$ (see [St, Tag 01V9]). Furthermore, it suffices to show that any lift of x_α is a smooth point in the geometric fiber $(X_\alpha)_{\overline{y_\beta}}$. Taking base change with respect to the morphism $\overline{y_\beta} \to Y_\beta$, and replacing x and x_α by their lifts, we can assume that $Y_\beta = \operatorname{Spec} k$.

To show that x_α is a smooth point, it now remains to show that the sheaf of differentials $\Omega_{X_\alpha/k}$ is locally free at x_α, and the local ring $\mathcal{O}_{X_\alpha,x_\alpha}$ is reduced (see [St, Tag 04QP]).

Since X is strongly pro-smooth over $Y_\beta = \operatorname{Spec} k$, it is reduced. Therefore the local ring $\mathcal{O}_{X,x}$ is reduced. Since $\mathcal{O}_{X,x}$ is a faithfully flat $\mathcal{O}_{X_\alpha,x_\alpha}$-algebra, we therefore conclude that the canonical morphism $\mathcal{O}_{X_\alpha,x_\alpha} \to \mathcal{O}_{X,x}$ is injective, thus $\mathcal{O}_{X_\alpha,x_\alpha}$ is reduced.

Moreover, using faithful flatness of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{X_\alpha,x_\alpha}$ again, to show that $\Omega_{X_\alpha/k}$ is locally free at x_α, it suffices to show that the pullback $\operatorname{pr}_\alpha^* \Omega_{X_\alpha/k}$ is a flat $\mathcal{O}_X$-module. To prove this, we will show that there exists an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \operatorname{pr}_\alpha^* \Omega_{X_\alpha/k} \to \Omega_{X/k} \to \Omega_{X/X_\alpha} \to 0, \tag{2.1}$$

and both Ω_{X/X_α} and $\Omega_{X/k}$ are flat $\mathcal{O}_X$-modules.

Using the canonical identification $\Omega_{X/X_\alpha} \simeq \operatorname{colim}_{\alpha'>\alpha} \operatorname{pr}_{\alpha'}^* \Omega_{X_{\alpha'}/X_\alpha}$, the flatness of Ω_{X/X_α} follows the fact that each $\Omega_{X_{\alpha'}/X_\alpha}$ is a flat $\mathcal{O}_{X_{\alpha'}}$-module (because projection $\operatorname{pr}_{\alpha',\alpha} : X_{\alpha'} \to X_\alpha$ is smooth). The flatness assertion for $\Omega_{X/k}$ follows by the same argument, using the assumption that $X \to \operatorname{Spec} k$ is strongly pro-smooth.

Next, since every projection $\operatorname{pr}_{\alpha',\alpha} : X_{\alpha'} \to X_\alpha$ is smooth, we have an exact sequence

$$0 \to \operatorname{pr}_{\alpha',\alpha}^* \Omega_{X_\alpha/k} \to \Omega_{X_{\alpha'}/k} \to \Omega_{X_{\alpha'}/X_\alpha} \to 0$$

of $\mathcal{O}_{X_{\alpha'}}$-modules. Since projection $\operatorname{pr}_{\alpha'} : X \to X_{\alpha'}$ is flat, the pullback $\operatorname{pr}_{\alpha'}^*$ is exact, thus we have an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \operatorname{pr}_\alpha^* \Omega_{X_\alpha/k} \to \operatorname{pr}_{\alpha'}^* \Omega_{X_{\alpha'}/k} \to \operatorname{pr}_{\alpha'}^* \Omega_{X_{\alpha'}/X_\alpha} \to 0. \tag{2.2}$$

Since filtered colimits are exact, applying $\operatorname{colim}_{\alpha'}$ to (2.2), we get the exact sequence (2.1) we were looking for. ☐

Applying Lemma 2.1.3 to the identity map, we get the following consequence.

Corollary 2.1.4. *Let X be a globally placid affine scheme with two presentations $X \simeq \lim_\alpha X_\alpha$ and $X \simeq \lim_\alpha X'_\beta$. Then for every β and every sufficiently large α the projection $\operatorname{pr}_\beta : X \to X'_\beta$ factors as a composition $X \xrightarrow{\operatorname{pr}_\alpha} X_\alpha \xrightarrow{g_{\alpha,\beta}} X'_\beta$ with $g_{\alpha,\beta}$ smooth.*

Corollary 2.1.5. *Let $f : X \to Y$ is finitely presented smooth covering between globally placid affine schemes such that X is strongly pro-smooth. Then Y is strongly pro-smooth.*

Proof. Choose a placid presentation $Y \simeq \lim_\alpha Y_\alpha$ and strongly pro-smooth presentation $X \simeq \lim_\beta X_\beta$, and we want to show that some Y_α is smooth. Since f is finitely presented, there exists an index α and a smooth covering $f_\alpha : X_\alpha \to Y_\alpha$ such that $f \simeq f_\alpha \times_{Y_\alpha} Y$. Then $X \simeq \lim_{\alpha'>\alpha}(X_\alpha \times_{Y_\alpha} Y_{\alpha'})$ is another placid presentation of X, so it follows from Corollary 2.1.4 that for every β there exists α' such that the projection $X \to X_\beta$ factors through a smooth map $X_\alpha \times_{Y_\alpha} Y_{\alpha'} \to X_\beta$. Since X_β is smooth, we deduce that $X_\alpha \times_{Y_\alpha} Y_{\alpha'}$ is smooth. Since $X_\alpha \to Y_\alpha$ and hence also $X_\alpha \times_{Y_\alpha} Y_{\alpha'} \to Y_{\alpha'}$ is a smooth covering, we conclude that $Y_{\alpha'}$ is smooth. □

2.1.6. Infinity-stacks. (a) Let $\mathcal{A} = \mathrm{Aff}_k$ as before, equipped with the étale topology. In this case, the category of presheaves $\mathrm{PShv}(\mathcal{A})$ is usually called the category of ∞-prestacks over k and denoted by PreSt_k. By analogy, we call the category $\mathcal{C} = \mathrm{Shv}(\mathcal{A})$ the category of ∞-*stacks* over k and denote it by St_k.

(b) Let (P) be a class of morphisms $f : \mathcal{X} \to Y$ from an ∞-stack $\mathcal{X}$ to a affine scheme Y, closed under pullbacks. We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of ∞-stacks *belongs to* (P), if for every $Y \to \mathcal{Y}$, where Y is an affine scheme, the pullback $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ belongs to (P). In particular, we can talk about representable/schematic/affine (fp)-morphisms, where "fp" stands for "finitely presented".

2.1.7. Placid ∞-stacks. (a) Let $\mathrm{Ob}_0(\mathcal{A}) \subset \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Mor}_0^0(\mathcal{A}) \subset \mathrm{Mor}(\mathcal{A})$ be the classes of objects and morphisms, constructed in 2.1.1. We claim that all the assumptions of 1.2.2 are satisfied. Indeed, (a) and (d) follow from 1.3.7, (b) is standard, so it remains to show (c).

Notice that for every $F', F'' \in \mathrm{St}_k$, the coproduct $F = F' \sqcup F'' \in \mathrm{St}_k$ satisfies the property for every $X \in \mathrm{Aff}_k$, we have $F(X) = \sqcup_{X=X'\sqcup X''} F'(X') \times F''(X'')$. In particular, if X decomposes as $X \simeq F' \sqcup F''$, then the isomorphism $X \overset{\sim}{\to} F' \sqcup F''$ induces decomposition $X = X' \sqcup X''$ and isomorphisms $X' \overset{\sim}{\to} F', X'' \overset{\sim}{\to} F''$.

It remains to show that if X is globally placid, then X' and X'' also are. Indeed, let $X \simeq \lim_\alpha X_\alpha$ is a presentation of X, and let $f \in k[X]$ be the idempotent corresponding to X'. Then f comes from an idempotent $f_\beta \in k[X_\beta]$ for some index β, hence induces a decomposition $X_\beta = X'_\beta \sqcup X''_\beta$ of X_β. Hence X' is a globally placid affine scheme with presentation $X' \simeq \lim_{\alpha>\beta}(X_\alpha \times_{X_\beta} X'_\beta)$, and similarly for X''. Moreover, the embeddings $X' \hookrightarrow X, X'' \hookrightarrow X$ are finitely presented open embeddings, thus belong to $\mathrm{Mor}_0^0(\mathcal{A})$.

(b) By (a), the construction of 1.2.4 applies. The corresponding (n-)geometric objects of St_k will be called *(n-)placid*, and the corresponding (n-)special morphisms will be called *(n-)smooth*.

2.1.8. Remarks. (a) All ∞-stacks, considered in this work are actually usual 1-stacks. On the other hand, the introduction of ∞-stacks is necessary, because 1-stacks are not closed under homotopy colimits.

(b) In principle, one probably would like to sheafify for the fppf topology instead of étale. On the other hand, to work with étale topology is easier.

2.1.9. Examples. (a) Note that if $\mathcal{X}$ is an Artin stack of finite type over k, then it is placid. Moreover, every smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ between Artin stacks of finite type over k in the classical sense is also smooth in the sense of 2.1.7(b).

Indeed, since $\mathcal{X}$ has a (classically) smooth covering $X \to \mathcal{X}$ from an $X \in \mathrm{Aff}_k^{ft}$, and X is 0-placid, the first assertion follows from the second one. Note that a smooth morphism in Aff_k^{ft} is 0-smooth by definition, hence every affine smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ between Artin stacks of finite type over k is 0-smooth. Then, by a standard argument we see that every quasi-affine smooth f (between Artin stacks of finite type over k) is 1-smooth, every schematic smooth f is 2-smooth, every representable smooth f is 3-smooth, and every smooth f is 4-smooth.

(b) More generally, any locally finitely-presented morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, which is smooth in the classical sense is also smooth in the sense of 2.1.7(b). Indeed, as in (a), one reduces to the case of a smooth finitely-presented morphism $f : X \to Y$ of affine schemes. In this case, f is a pullback of a smooth morphism in Aff_k^{ft}. Hence it is 0-smooth, by definition.

Definition 2.1.10. (a) We call an affine scheme/scheme/algebraic space *(n-)placid*, if it is an (n-)placid as an ∞-stack.

(b) We call a scheme/algebraic space X *globally placid*, if it has a presentation as a filtered limits $X \simeq \lim_\alpha X_\alpha$ of schemes/algebraic spaces of finite type over k with smooth affine transition maps.

2.1.11. Remark. Note that the notation of Definition 2.1.10(b) is compatible with terminology of Definition 2.1.10(a). Namely, an affine scheme/scheme is globally placid if and only if it is globally placid as an algebraic space. Indeed, assume that an affine scheme/scheme X has a presentation $X \simeq \lim_\alpha X_\alpha$ as a filtered limit of algebraic spaces of finite type with affine transition maps. Then X_α is an affine scheme/scheme such for all sufficiently large α (see, for example, [Ry2, Prop 6.2 and Cor 6.3]).

Lemma 2.1.12. *(a) A globally placid algebraic space/scheme is placid.*

(b) If $f : X \to Y$ be an fp-morphism from an algebraic space X to a globally placid algebraic space Y. Then X is globally placid.

(c) Let $f : \mathcal{X} \to \mathcal{Y}$ be an fp-representable morphism of ∞-stacks such that $\mathcal{Y}$ is placid. Then $\mathcal{X}$ is placid.

Proof. (a) Let X be a globally placid algebraic space/scheme with placid presentation $X \simeq \lim_\alpha X_\alpha$, and let $X'_\alpha \to X_\alpha$ be an étale covering from an affine scheme X'_α. Then $X' := X \times_{X_\alpha} X'_\alpha$ has a presentation $X' \simeq \lim_{\beta > \alpha}(X_\beta \times_{X_\alpha} X'_\alpha)$, thus X' is a placid affine scheme, and $X' \to X$ is an fp-étale covering. Thus X is placid (use 2.1.9(b)), as claimed.

(b) Choose a placid presentation $Y \simeq \lim_\alpha Y_\alpha$. Since $X \to Y$ is finitely presented, it is a pullback of a morphism $X_\alpha \to Y_\alpha$ of algebraic spaces of finite type over k. Then $X \simeq \lim_{\beta>\alpha}(Y_\beta \times_{Y_\alpha} X_\alpha)$, thus X is a globally placid algebraic space.

(c) Choose a smooth covering $Y \to \mathcal{Y}$ from a 0-placid Y. Then the pullback $\mathcal{X} \times_\mathcal{Y} Y \to \mathcal{X}$ is a smooth covering, hence it suffices to show that $\mathcal{X} \times_\mathcal{Y} Y$ is placid. Thus we can assume that $\mathcal{Y}$ is 0-placid. In this case, we have a decomposition $\mathcal{Y} \simeq \sqcup_\alpha Y_\alpha$, where each Y_α is a globally placid affine scheme, which induces a decomposition $\mathcal{X} \simeq \sqcup_\alpha \mathcal{X} \times_\mathcal{Y} Y_\alpha$, therefore we can assume that $\mathcal{Y}$ is a globally placid affine scheme. In this case the assertion follows from a combination of (b) and (a). □

2.1.13. Remarks. (a) A disjoint union $X := X' \sqcup X''$ of two globally placid affine schemes is globally placid. Indeed, if $X' \simeq \lim_\alpha X'_\alpha$ and $X' \simeq \lim_\beta X'_\beta$ are placid presentations of X' and X'', then $X \simeq \lim_{\alpha,\beta}(X'_\alpha \sqcup X''_\beta)$ is a placid presentation of X.

(b) By definition, every globally placid affine scheme is 0-placid. Conversely, every 0-placid affine scheme is globally placid. Indeed, by definition, every 0-placid affine X is a disjoint union of $\sqcup_\alpha X_\alpha$ of globally placid affine schemes. Moreover, this disjoint union is finite, because X is quasi-compact. Hence X is globally placid by (a). On the other hand, we do not expect that every placid affine scheme is globally placid.

(c) Arguing as in 2.1.9 one can show that if a scheme/algebraic space X has a Zariski/étale covering by globally placid affine schemes, then X is placid. Again, we do not expect that the converse is true.

2.1.14. Example. Let H be a group-scheme acting on a 0-placid affine scheme X. Assume that H is 0-smooth, that is, the projection $H \to \mathrm{pt}$ is 0-smooth. Then the quotient stack $\mathcal{X} := [X/H]$ is 1-placid ∞-stack, and the projection $X \to \mathcal{X}$ is 0-smooth.

Indeed, the projection $\pi : X \to \mathcal{X}$ is a covering, so it remains to show that π is 0-smooth. By Lemma 1.2.5, it suffices to show that the projection $X \times_\mathcal{X} X \to X$ is 0-smooth. Since $X \times_\mathcal{X} X \simeq H \times X$, and $H \to \mathrm{pt}$ is 0-smooth, the assertion follows.

2.2. **Reduced ∞-stacks.**

2.2.1. The reduced ∞-substack. (a) Let $\mathrm{Aff}_{\mathrm{red},k} \subset \mathrm{Aff}_k$ be the category of reduced affine schemes over k. Then the inclusion $\iota : \mathrm{Aff}_{\mathrm{red},k} \hookrightarrow \mathrm{Aff}_k$ has a right adjoint $X \mapsto X_{\mathrm{red}}$.

(b) Recall that if $f : X \to Y$ is an étale morphism of affine schemes, and Y is reduced, then X is reduced as well (see [St, 03PC(8)]). Therefore the étale topology on Aff_k restricts to the étale topology on $\mathrm{Aff}_{\mathrm{red},k}$, thus the assumption 1.2.7 is satisfied. In particular, we can consider the ∞-category $\mathrm{St}_{\mathrm{red},k} := \mathrm{Shv}(\mathrm{Aff}_{\mathrm{red},k})$, we have the restriction map $\iota^* : \mathrm{St}_k \to \mathrm{St}_{\mathrm{red},k}$ with left adjoint $\iota_! : \mathrm{St}_{\mathrm{red},k} \to \mathrm{St}_k$.

(c) By definition, for every affine scheme $X \in \mathrm{St}_k$ we have $\iota^* X = X_{\mathrm{red}} \in \mathrm{St}_{\mathrm{red},k}$, thus $\iota_! \iota^* X = X_{\mathrm{red}} \in \mathrm{St}_k$.

(d) By analogy with (c), for every $\mathcal{X} \in \mathrm{St}_k$, we set $\mathcal{X}_{\mathrm{red}} := \iota_! \iota^* \mathcal{X}$ and call it the *reduced* ∞-stack of $\mathcal{X}$. By adjointness, we have a natural counit map $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$.

(e) We call an ∞-stack $\mathcal{X} \in \mathrm{St}_k$ *reduced*, if the counit map $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is an equivalence, and let $(\mathrm{St}_k)_{\mathrm{red}} \subset \mathrm{St}_k$ be the full subcategory of reduced ∞-stacks.

Lemma 2.2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ is an n-smooth morphism of placid ∞-stacks (see 2.1.7(b)). Then the induced morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{\mathrm{red}}$ is an isomorphism.*

Proof. First we show the assertion when $\mathcal{X}$ and $\mathcal{Y}$ are globally placid affine schemes, and f is strongly pro-smooth. When $\mathcal{X}, \mathcal{Y} \in \mathrm{Aff}_k^{ft}$ and f is smooth, the assertion is known. The general case follows from the fact that both functors $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{red}}$ and $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{\mathrm{red}}$ commute with limits. Namely, we first deduce the case when f is smooth and finitely presented, and then when f is strongly pro-smooth.

Assume now that $\mathcal{Y}$ is a globally placid affine scheme. We will show the assertion by induction on n.

If $n = 0$, then $\mathcal{X}$ decomposes as a coproduct $\mathcal{X} \simeq \sqcup_\alpha \mathcal{X}_\alpha$, where each $f_\alpha : \mathcal{X}_\alpha \to \mathcal{Y}$ is a strongly pro-smooth morphism between globally placid affine schemes. Since both functors $\cdot_{\mathrm{red}}$ and $\cdot \times_{\mathcal{Y}} \mathcal{Y}_{\mathrm{red}}$ commute with colimits, we reduce to the case $\mathcal{X}_\alpha \to \mathcal{Y}$, shown before.

Let now $n > 0$ and choose an $(n-1)$-smooth covering $\pi : \mathcal{Z} \to \mathcal{X}$ with 0-placid $\mathcal{Z}$ such that the composition $\mathcal{Z} \to \mathcal{X} \to \mathcal{Y}$ is $(n-1)$-smooth. By the observation 1.1.8, π gives rise to a presentation of $\mathcal{X}$ as a colimit of the Čech complex $\mathcal{X} \simeq \mathrm{colim}_{[m]} \mathcal{Z}^{[m]}$, where all $f^{[m]} : \mathcal{Z}^{[m]} \to \mathcal{X} \to \mathcal{Y}$ are $(n-1)$-smooth. By the induction hypothesis, the assertion holds for each $f^{[m]}$. Thus, using the commutativity of both sides with colimits, the assertion for f follows.

Assume now that $\mathcal{Y}$ is r-placid, and we argue by induction on r. When $r = 0$, then $\mathcal{Y}$ decomposes as a coproduct $\mathcal{Y} \simeq \sqcup_\alpha \mathcal{Y}_\alpha$, which induces the decomposition $\mathcal{X} \simeq \sqcup_\alpha \mathcal{X}_\alpha$, where $\mathcal{X}_\alpha = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_\alpha$, and the assertion for $\mathcal{X} \to \mathcal{Y}$ follows from that for $\mathcal{X}_\alpha \to \mathcal{Y}_\alpha$, shown above.

When $r > 0$, we apply 1.1.8 to write $\mathcal{Y}$ as a colimit of the Čech complex $\mathcal{Y} \simeq \mathrm{colim}_{[m]} \mathcal{Z}^{[m]}$, where each $\mathcal{Z}^{[m]}$ is $(r-1)$-placid. Then we reduce the assertion for f to that for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}^{[m]} \to \mathcal{Z}^{[m]}$, and we conclude by the induction hypothesis. □

Corollary 2.2.3. *(a) If $f : \mathcal{X} \to \mathcal{Y}$ is an n-smooth morphism (resp. covering) of placid ∞-stacks, then the induced morphism $f_{\mathrm{red}} : \mathcal{X}_{\mathrm{red}} \to \mathcal{Y}_{\mathrm{red}}$ is an n-smooth morphism (resp. covering) as well.*

(b)If $\mathcal{X}$ is a placid ∞-stack, then $\mathcal{X}_{\mathrm{red}}$ is a placid ∞-stack as well, and the morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is a finitely presented closed embedding.

Proof. (a) Since n-smooth morphisms/covering are closed under pullbacks, the assertion follows from Lemma 2.2.2.

(b) By Lemma 2.1.12, it suffices to show that $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is a fp-closed embedding. If $\mathcal{X} \in \mathrm{Aff}_k^{ft}$, the assertion is clear. Next, assume that $\mathcal{X}$ be a globally placid affine scheme. Then $\mathcal{X}$ admits a strongly pro-smooth morphism $\mathcal{X} \to X$ with $X \in \mathrm{Aff}_k^{ft}$. Then, $\mathcal{X}_{\mathrm{red}} \simeq \mathcal{X} \times_X X_{\mathrm{red}}$ by Lemma 2.2.2, thus the assertion for $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ follows from that for $X_{\mathrm{red}} \to X$.

In the general case, choose a smooth covering $X \to \mathcal{X}$ with 0-geometric X. Since X is a coproduct of globally placid affine schemes, we conclude that the map $\mathcal{X}_{\mathrm{red}} \times_{\mathcal{X}} X \simeq X_{\mathrm{red}} \to X$ is an fp-closed embedding (use Lemma 2.2.2).

We want to show that an arbitrary morphism $U \to \mathcal{X}$ from an affine scheme U, the induced morphism $\mathcal{X}_{\mathrm{red}} \times_{\mathcal{X}} U \to U$ is an fp-closed embedding. Since $X \to \mathcal{X}$ is a covering, there exists an étale covering $V \to U$ such that $V \to U \to \mathcal{X}$ factors through $X \to \mathcal{X}$. Thus by the proven above, $\mathcal{X}_{\mathrm{red}} \times_{\mathcal{X}} V \to V$ is an fp-closed embedding. Therefore by a faithfully flat descent, there exists an fp-closed embedding $U' \to U$ such that $U' \times_U V \simeq \mathcal{X}_{\mathrm{red}} \times_{\mathcal{X}} V$ over V.

It thus remains to show that $\mathcal{X}_{\mathrm{red}} \times_{\mathcal{X}} U \simeq U'$. But this follows from the fact that both sides are identified with a homotopy colimit $\mathrm{colim}_{[m]}(\mathcal{X}_{\mathrm{red}} \times_{\mathcal{X}} V^{[m]})$. □

2.2.4. Remarks. Arguing as in Corollary 2.2.3(b) it is not difficult to deduce from Lemma 2.2.2 that if $\mathcal{X}$ is a scheme (resp. algebraic space), then $\mathcal{X}_{\mathrm{red}}$ is the classical reduced scheme (resp. algebraic space) corresponding to $\mathcal{X}$.

2.3. **Perfect ∞-stacks, and topological equivalences.**

2.3.1. Topological equivalences. (a) Let S be a collection of morphisms in an ∞-category $\mathcal{C}$. Recall that the *saturated closure* of S is the smallest collection of morphisms $\overline{S} \supseteq S$, which is closed under homotopy colimits, pushouts and 2-out-of-3.

(b) We say that a morphism $f : \mathcal{Y} \to \mathcal{X}$ of ∞-stacks is a *topological equivalence*, if if it lies in the strong saturated closure of universal homeomorphisms between affine schemes.

Though in general, strong saturated closure is a very complicated notion, it turns out that topological equivalences can be described in very explicit terms.

2.3.2. Perfectly reduced schemes. Following [BGH], we call an affine scheme X over k is called *perfectly reduced* or simply *perfect*, if for every universal homeomorphism of affine schemes $X' \to X$ such that X' reduced is an isomorphism. In particular, every perfectly reduced affine scheme is reduced.

(b) Let $\mathcal{A}' := \mathrm{Aff}_{\mathrm{perf},k} \subset \mathrm{Aff}_k$ be the category of perfectly reduced affine schemes over k. Then the inclusion $\iota : \mathrm{Aff}_{\mathrm{perf}} \hookrightarrow \mathrm{Aff}_k$ has a right adjoint $X \mapsto X_{\mathrm{perf}}$ (see

[BGH, 14.3.2]). More precisely, X_{perf} is the inverse limit $\lim_{X'\to X} X'$ taken over all isomorphism classes of finitely presented universal homeomorphisms $X' \to X$.

2.3.3. Remark. Note that if the characteristic of k is zero (resp. positive), then perfectly reduced means absolutely weakly normal (see [Ry1, App. B]) (resp. perfect) (see [BS, Lem. 3.8]).

2.3.4. Étale morphisms and universal homeomeorphisms.

(a) Recall that for every universal homeomorphism $g : X \to Y$ in Aff_k the functor $Y' \mapsto X' := X \times_Y Y'$ induces an equivalence of categories between étale (affine) schemes over Y and étale (affine) schemes over X (see [St, 04DZ]).

(b) Recall that for every étale morphism $f : X \to Y$ of affine schemes, the canonical morphism $X_{\mathrm{perf}} \to X \times_Y Y_{\mathrm{perf}}$ is an isomorphism (see [Ry1, Prop B.6(ii)]), thus the induced morphism $f_{\mathrm{perf}} : X_{\mathrm{perf}} \to Y_{\mathrm{perf}}$ is étale.

(c) It follows from (b) that if $f : X \to Y$ is an étale morphism and Y is perfect, then X is perfect as well.

(d) If follows from (a) that every composition $X \overset{f}{\to} Y \overset{g}{\to} Z$, where g is universal homeomorphism and f is étale, decomposes as $X \overset{g'}{\to} Y' \overset{f'}{\to} Z$, where g' is universal homeomorphism and f' is étale.

2.3.5. Perfect ∞-stacks. Let $\mathcal{A} = \mathrm{Aff}_k$ be the category of affine schemes, equipped with an étale topology, and let $\mathcal{A}' := \mathrm{Aff}_{\mathrm{perf},k} \subset \mathcal{A}$ be the subcategory of perfect affine schemes.

(a) By 2.3.4(c), the étale topology on $\mathcal{A}$ restricts to the étale topology on $\mathcal{A}'$, thus as in 1.2.7 we can consider the ∞-category $\mathrm{St}_{\mathrm{perf},k} := \mathrm{Shv}(\mathrm{Aff}_{\mathrm{perf},k})$. In particular, we have the restriction map $\iota^* : \mathrm{St}_k \to \mathrm{St}_{\mathrm{perf},k}$ with left adjoint $\iota_! : \mathrm{St}_{\mathrm{perf},k} \to \mathrm{St}_k$.

(b) By definition, for every affine scheme $X \in \mathrm{St}_k$ we have $\iota^* X = X_{\mathrm{perf}} \in \mathrm{St}_{\mathrm{perf},k}$, thus $\iota_!\iota^* X = X_{\mathrm{perf}} \in \mathrm{St}_k$.

(c) By analogy with (c), for every $\mathcal{X} \in \mathrm{St}_k$, we set $\mathcal{X}_{\mathrm{perf}} := \iota_!\iota^*\mathcal{X}$ and call it the *perfectization* of $\mathcal{X}$. By adjointness, we have a natural counit map $\mathcal{X}_{\mathrm{perf}} \to \mathcal{X}$.

(d) We call an ∞-stack $\mathcal{X} \in \mathrm{St}_k$ *perfect*, if the counit map $\mathcal{X}_{\mathrm{perf}} \to \mathcal{X}$ is an equivalence, and let $(\mathrm{St}_k)_{\mathrm{perf}} \subset \mathrm{St}_k$ be the full subcategory of perfect ∞-stacks.

(e) If follows from Lemma 1.2.8 that functor $\iota_! : \mathrm{St}_{\mathrm{perf},k} \to \mathrm{St}_k$ induces an equivalence of ∞-categories $\mathrm{St}_{\mathrm{perf},k} \overset{\sim}{\to} (\mathrm{St}_k)_{\mathrm{perf}}$ with inverse $\iota^* : (\mathrm{St}_k)_{\mathrm{perf}} \overset{\sim}{\to} \mathrm{St}_{\mathrm{perf},k}$. Therefore we will not distinguish between these categories, and will refer to both of them as the ∞-category of perfect ∞-stacks.

Part (b) of the following result describes topological equivalences explicitly.

Lemma 2.3.6. *(a) For every $\mathcal{X} \in \mathrm{St}_k$, the counit maps $\mathcal{X}_{\mathrm{perf}} \to \mathcal{X}$ and $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ are topological equivalences.*

(b) A morphism $f : \mathcal{Y} \to \mathcal{X}$ in St_k is a topological equivalence, if and only if $f_{\mathrm{perf}} : \mathcal{Y}_{\mathrm{perf}} \to \mathcal{X}_{\mathrm{perf}}$ is an equivalence.

Proof. (a) Since topological equivalence are stable under homotopy colimits, we reduce (arguing as in Lemma 1.2.8) to the case when $\mathcal{X}$ is an affine scheme X. In this case the counit maps are simply $X_{\mathrm{perf}} \to X$ and $X_{\mathrm{red}} \to X$, respectively, which are universal homeomorphisms between affine schemes, thus topological equivalences.

(b) Assume that f_{perf} is an equivalence. Since topological equivalences are closed by 2-out-of-3, f is a topological equivalence by (a). It remains to show that if f is a topological equivalence, then f_{perf} is an equivalence. Note first that if $f : Y \to X$ is a universal homeomorphism of affine schemes, then $f_{\mathrm{perf}} : Y_{\mathrm{perf}} \to X_{\mathrm{perf}}$ is a universal homeomorphism between affine schemes such that Y_{perf} is reduced and X_{perf} is perfectly reduced. Therefore f_{perf} is isomorphism in this case. Thus, it suffices to show that the collection of morphisms f such that f_{perf} is an equivalence is closed by homotopy colimits, pushouts and 2-out-of-3. But this follows from the fact that the perfection functor $\iota_!\iota^* : \mathcal{X} \mapsto \mathcal{X}_{\mathrm{perf}}$ preserves homotopy colimits. □

Corollary 2.3.7. *(a) Topological equivalences are stable under pullbacks.*

(b) A morphism $f : \mathcal{Y} \to \mathcal{X}$ in St_k is a topological equivalence if and only if for every morphism $Z \to \mathcal{X}$ with Z affine the base change $\mathcal{Y}_Z := \mathcal{Y} \times_{\mathcal{X}} Z \to Z$ is a topological equivalence.

(c) Topological equivalences are stable under quotients, that is, if $f : \mathcal{Y} \to \mathcal{X}$ is an topological equivalence, equivariant with respect to an action of the ∞-group stack H, then the induced map $[f] : [\mathcal{Y}/H] \to [\mathcal{X}/H]$ is an topological equivalence.

(d) A morphism $f : \mathcal{Y} \to \mathcal{X}$ in St_k is a topological equivalence if and only if the morphism $f(Z) : \mathcal{Y}(Z) \to \mathcal{X}(Z)$ is an equivalence for every $Z \in \mathrm{Aff}_{\mathrm{perf},k}$.

Proof. (a) Since $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{\mathrm{perf}} \simeq \mathcal{X}_{\mathrm{perf}} \times_{\mathcal{Y}_{\mathrm{perf}}} \mathcal{Z}_{\mathrm{perf}}$, the assertion about pullbacks follows from Lemma 2.3.6(b).

(b) The "only if" assertion follows from (a). Since topological equivalences are stable under comotopy colimits, the "if" assertion follows from the fact that the pullbacks commute with homotopy colimits, and every ∞-stack is a colimit of affine schemes.

(c) Recall that $[\mathcal{Y}/H]$ is defined as the colimit $\mathrm{colim}_{[m]}(H^m \times \mathcal{Y})$. Since f is a topological equivalence, the induced map $\mathrm{Id}_{H^m} \times f$ is a topological equivalence by (a). As topological equivalences are closed under colimits, $[f]$ is a topological equivalence as well.

(d) By Lemma 2.3.6(b), we have to show that $f_{\mathrm{perf}} = \iota_!\iota^*(f)$ is an equivalence if and only if $\iota^*(f) : \iota^*(\mathcal{Y}) \to \iota^*(\mathcal{X})$ is an equivalence in $\mathrm{St}_{\mathrm{perf},k}$. Since $\iota_! : \mathrm{St}_{\mathrm{perf},k} \xrightarrow{\sim} (\mathrm{St}_k)_{\mathrm{perf}}$ is an equivalence of categories (see 2.3.5(e)), the assertion follows. □

2.3.8. Notation. Let (P) be a class of morphisms $f:\mathcal{X}\to Y$ from an ∞-stack $\mathcal{X}$ to a affine scheme Y, closed under pullbacks. Our main examples are classes of morphisms of algebraic spaces, and its subclasses of affine/schematic/fp-proper/finitely presented morphisms or (fp) open/closed/locally closed embeddings.

(a) We say that a morphism $\widetilde{f}:\widetilde{\mathcal{X}}\to Y$ is *topologically* (P), if there exists a morphism $f:\mathcal{X}\to Y$ from (P) and an isomorphism $\widetilde{\mathcal{X}}_{\mathrm{perf}}\simeq\mathcal{X}_{\mathrm{perf}}$ over Y.

In particular, we say that a morphism $\widetilde{f}:\widetilde{\mathcal{X}}\to Y$ from an ∞-stack $\mathcal{X}$ to a affine scheme Y, topologically *representable/schematic/affine*, if $\widetilde{\mathcal{X}}_{\mathrm{perf}}$ is an algebraic space/scheme/affine scheme. Furthermore, using 2.3.4(a), we see that a topologically representable $\widetilde{f}$ is *topologically locally fp*, if $\widetilde{\mathcal{X}}_{\mathrm{perf}}$ it has an étale covering by affine schemes, which are topologically fp over Y.

(b) Notice that for every morphism $Y'\to Y$ between affine schemes we have a natural isomorphism $(\mathcal{X}\times_Y Y')_{\mathrm{perf}}\simeq\mathcal{X}_{\mathrm{perf}}\times_{Y_{\mathrm{perf}}}Y'_{\mathrm{perf}}$ and similarly for $\cdot_{\mathrm{red}}$. Therefore classes of topologically (P) morphisms are closed under pullbacks, so construction 2.1.6(b) applies.

In particular, we can talk about topologically *representable/affine/schematic/fp-proper/fp/locally fp* morphisms and *(fp) open/closed/locally closed embeddings.*

2.3.9. Remark. (a) In the situation of 2.3.8(a) we will add the word *strongly* before *topological*, if there exists a stronger isomorphism $\widetilde{\mathcal{X}}_{\mathrm{red}}\simeq\mathcal{X}_{\mathrm{red}}$ over Y. In this case, we will add the word *strongly* in 2.3.8(b) as well.

(b) In the situation of 2.3.8(a), assume that Y is a globally placid affine scheme, and $X\to Y$ is a locally finitely presented morphism of algebraic spaces. Then X is a placid algebraic space (see Lemma 2.1.12), thus $X_{\mathrm{red}}\to X$ is finitely presented (Corollary 2.2.3), hence X_{red} is a locally fp algebraic space over Y. Therefore in this case, $f:\mathcal{X}\to Y$ is strongly topologically fp-proper/locally fp if and only if $\mathcal{X}_{\mathrm{red}}$ is an algebraic space, which is fp-proper/locally fp over Y.

2.4. **Topologically placid ∞-stacks.** In this subsection we will introduce a class of topologically placid ∞-stacks, which is more general than the one, considered in the introduction.

2.4.1. Uh-smooth morphisms. As in 2.1.1, let $\mathcal{B}$ be the category Aff_k^{ft}. We denote by $\mathcal{P}=\mathcal{P}_{uh-sm}$ be the smallest class of morphisms in $\mathcal{B}$ which is

(i) closed under compositions, contains smooth morphisms and universal homeomorphisms;

(ii) local in the étale topology and topology generated by universal homeomorphisms, that is, if $f:X\to Y$ and $\pi:X'\to X$ are morphisms in $\mathcal{B}$ such that π is either étale surjective or a universal homeomorphism and $f\circ\pi\in\mathcal{P}$, then $f\in\mathcal{P}$.

We call morphisms from $\mathcal{P}$ *uh-smooth*, where "uh" stands for "universal homeomorphisms".

2.4.2. Remark. For the purpose of this work one can replace $\mathcal{P} = \mathcal{P}_{uh-sm}$ by the smallest class satisfying 2.4.1(i) only.

For completeness, we now give two more explicit descriptions of the class $\mathcal{P}_{uh-sm}$.

Lemma 2.4.3. *For a morphism $f : X \to Y$ in Aff_k^{ft} the following are equivalent:*
(a) f belongs to $\mathcal{P}_{uh-sm}$;
(b) f can be completed to a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{g} & V & \xleftarrow{\pi} & V' \\ {\scriptstyle f}\big\downarrow & & & & \big\| \\ Y & \xleftarrow{f'} & Y' & \xleftarrow{\pi'} & V', \end{array}$$

where π and π' are universal homeomorphisms, g is étale surjective and f' is smooth.
(c) The induced map $f_{\mathrm{perf}} : X_{\mathrm{perf}} \to Y_{\mathrm{perf}}$ is a perfectly smooth *in the sense of* [Zhu]*, that is, every $x \in X_{\mathrm{perf}}$ has an étale neighborhood $p : U' \to X_{\mathrm{perf}}$ such that the composition $U' \xrightarrow{p} X_{\mathrm{perf}} \xrightarrow{f_{\mathrm{perf}}} Y_{\mathrm{perf}}$ factors as $U' \xrightarrow{g'} Y_{\mathrm{perf}} \times (\mathbb{A}^n)_{\mathrm{perf}} \xrightarrow{\mathrm{pr}} Y_{\mathrm{perf}}$, where g' is étale.*

Proof. (a) $\Longrightarrow$ (c) We have to show that the class of morphisms from (c) satisfy properties (i) and (ii) of 2.4.1. Clearly, (c) is closed under compositions. Since every smooth morphism Zariski locally decomposes as a composition $X \xrightarrow{g} Y \times \mathbb{A}^n \xrightarrow{\mathrm{pr}} Y$, where g is étale, we conclude that any smooth morphism belongs to (c). Since the functor $X \mapsto X_{\mathrm{perf}}$ maps universal homeomorphisms into isomorphisms, the class of (c) contains universal homeomorphisms and is local with respect to universal homeomorphisms. Finally étale local property follows from 2.3.4(b).

(c) $\Longrightarrow$ (b) Assume that f_{perf} is perfectly smooth. By 2.3.4(d), we conclude that the composition $U' \xrightarrow{p} X_{\mathrm{perf}} \to X$ decomposes as $U' \xrightarrow{\widetilde{\pi}} V \xrightarrow{g} X$, where g is étale and $\widetilde{\pi}$ is a universal homeomorphism, while the composition $U' \xrightarrow{\mathrm{pr} \circ g'} Y_{\mathrm{perf}} \to Y$, or what is the same,

$$U' \xrightarrow{g'} (Y \times \mathbb{A}^n)_{\mathrm{perf}} \to Y \times \mathbb{A}^n \xrightarrow{\mathrm{pr}} Y,$$

decomposes as $U' \xrightarrow{\widetilde{\pi}'} Y' \xrightarrow{f'} Y$, where f' is surjective and $\widetilde{\pi}'$ is an universal homeomorphism. Finally, by standard limit theorems ([EGAIV, 8.10.5] and [St, 0EUJ]), universal homeomorphisms $X' \xleftarrow{\widetilde{\pi}'} U' \xrightarrow{\widetilde{\pi}} V$ descend to universal homeomorphisms $X' \xleftarrow{\pi'} V' \xrightarrow{\pi} V$ with $V' \in \mathrm{Aff}_k^{ft}$.

(b) $\Longrightarrow$ (a) follows directly from the definition of $\mathcal{P}_{uh-sm}$. ☐

2.4.4. Topological version of globally placid affine schemes. In the situation of 1.3.1, let $\mathcal{B}$ be Aff_k^{ft} and let $\mathcal{P}$ be $\mathcal{P}_{uh-sm}$ (see 2.4.1).

(a) Note that the class $\mathcal{P}_{uh-sm}$ is closed under all pullbacks. Indeed, we have to show that the class of all morphisms $f : X \to Y$ in Aff_k^{ft} such that the pullback $f \times_Y Y' \in \mathcal{P}_{uh-sm}$ for all morphisms $Y' \to Y$ satisfies properties 2.4.1(i),(ii). But this follows from the fact that all classes involved (smooth, étale, universal homeomorphisms, etc) are closed under pullbacks. (Alternatively, the assertion can be shown by noticing that classes (b) and (c) of Lemma 2.4.3 are closed under pullbacks.)

(b) By (a), the construction of 1.3.1 applies (compare 2.1.1). In particular, we can form a class of objects $\mathrm{Ob}_0(\mathcal{A}) \subset \mathrm{Ob}(\mathcal{A})$ and a class of morphisms $\mathrm{Mor}_0^0(\mathcal{A}) \subset \mathrm{Mor}_0(\mathcal{A}) \subset \mathrm{Mor}(\mathcal{A})$.

(c) We will call objects of $\mathrm{Ob}_0(\mathcal{A})$ *globally uh-placid affine schemes*, and morphisms belonging $\mathrm{Mor}_0(\mathcal{A})$ *strongly pro-uh-smooth*.

2.4.5. Globally uh-placid algebraic spaces.

(a) We call a morphism $f : X \to Y$ in AlgSp_k^{ft} *uh-smooth*, if étale locally it is a uh-smooth morphism in Aff_k^{ft}.

(b) We call an algebraic space/scheme X *globally uh-placid*, if it has a presentation as a filtered limit $X \simeq \lim_\alpha X_\alpha$, where each $X_\alpha \in \mathrm{AlgSp}_k^{ft}$ and all transition maps are uh-smooth and affine.

(c) Alternatively, globally uh-placid algebraic spaces can be obtained by applying construction of 1.3.1 to the category $\mathcal{B} = \mathrm{AlgSp}_k^{ft}$ and class $\mathcal{P}$ of affine uh-smooth morphisms.

2.4.6. Perfectly placid ∞-stacks.

(a) As in 2.3.5, we set $\mathcal{A}' := \mathrm{Aff}_{\mathrm{perf},k} \subset \mathrm{Aff}_k = \mathcal{A}$ and have a natural identification $\iota_! : \mathrm{Shv}(\mathcal{A}') \simeq (\mathrm{St}_k)_{\mathrm{perf}}$.

(b) In the situation of 2.4.4, we set $\mathrm{Ob}_0(\mathcal{A}') := \mathrm{Ob}_0(\mathcal{A}) \cap \mathrm{Ob}(\mathcal{A}')$, and $\mathrm{Mor}_0^0(\mathcal{A}') := \mathrm{Mor}_0^0(\mathcal{A}) \cap \mathrm{Mor}(\mathcal{A}')$. We will call objects of $\mathrm{Ob}_0(\mathcal{A}')$ *globally perfectly placid affine schemes*, and morphisms belonging $\mathrm{Mor}_0(\mathcal{A}')$ *strongly perfectly pro-smooth*.

(c) Using Lemma 2.4.7(c) below and arguing as in 2.1.7(a), these data satisfies all the assumption of 1.2.2, therefore the construction of 1.2.4 applies. We will call the corresponding geometric objects of $\mathrm{St}_{\mathrm{perf},k}$ *perfectly placid*, and the corresponding special morphisms *perfectly smooth*.

(d) Recall (see 2.3.2) that the inclusion functor $\iota : \mathcal{A}' \to \mathcal{A}$ has the right adjoint $\iota^R : \mathcal{A} \to \mathcal{A}' \subset \mathcal{A} : X \mapsto X_{\mathrm{perf}}$.

Lemma 2.4.7. *(a) Every universal homeomorphism $f : X \to Y$ in Aff_k belongs to $\mathrm{Mor}_0(\mathcal{A})$.*

(b) Functor ι^R satisfies $\iota^R(\mathrm{Ob}_0(\mathcal{A})) \subset \mathrm{Ob}_0(\mathcal{A})$ and $\iota^R(\mathrm{Mor}_0(\mathcal{A})) \subset \mathrm{Mor}_0(\mathcal{A})$, thus $\iota^R(\mathrm{Mor}_0^0(\mathcal{A})) \subset \mathrm{Mor}_0^0(\mathcal{A})$.

(c) The class $\mathrm{Mor}_0^0(\mathcal{A}')$ is closed under all pullbacks between objects in $\mathrm{Ob}_0(\mathcal{A}')$.

Proof. (a) Assume that f is finitely presented. Then by the standard limit results, f is isomorphic to a pullback of an universal homeomorphism $f' : X' \to Y'$ in Aff_k^{ft}. Since $f' \in \mathcal{P} = \mathcal{P}_{uh-sm}$, we conclude that $f \in \widetilde{\mathcal{P}} \subset \mathrm{Mor}_0(\mathcal{A})$. In general, notice that X can be written as a filtered limit $X \simeq \lim_\alpha X_\alpha$ over Y such that each $X_\alpha \to Y$ is a finitely presented universal homeomorphism (see [St, Tag 0EUJ]). Then all transition maps $X_\beta \to X_\alpha$ are finitely-presented universal homeomorphisms as well, so $f \in \mathrm{Mor}_0(\mathcal{A})$ by definition.

(b) Note that for every $X \in \mathrm{Ob}_0(\mathcal{A})$, the projection $\pi : X_{\mathrm{perf}} \to X$ is a universal homeomorphism. Thus $\pi \in \mathrm{Mor}_0(\mathcal{A})$ by (a), hence $X_{\mathrm{perf}} \in \mathrm{Ob}_0(\mathcal{A})$ by Corollary 1.3.5. This shows the assertion for objects.

Similarly, for every morphism $f : X \to Y$ in $\mathrm{Mor}_0(\mathcal{A})$, the induced morphism $f_{\mathrm{perf}} : X_{\mathrm{perf}} \to Y_{\mathrm{perf}}$ decomposes as $X_{\mathrm{perf}} \to X \times_Y Y_{\mathrm{perf}} \to Y_{\mathrm{perf}}$. The first map $X_{\mathrm{perf}} \to X \times_Y Y_{\mathrm{perf}}$ is an universal homeomorphism, thus it belongs to $\mathrm{Mor}_0(\mathcal{A})$ by (a). The second map $X \times_Y Y_{\mathrm{perf}} \to Y_{\mathrm{perf}}$ is a base change of f, thus it belongs to $\mathrm{Mor}_0(\mathcal{A})$ as well. Since $\mathrm{Mor}_0(\mathcal{A})$ is closed under composition, the composition f_{perf} belongs to $\mathrm{Mor}_0(\mathcal{A})$, as claimed.

(c) Notice that the pullback of $(f : X \to Y) \in \mathrm{Mor}(\mathcal{A}')$ with respect to $(g : Z \to Y) \in \mathrm{Mor}(\mathcal{A}')$ is the composition $\widetilde{f} : (X \times_Y Z)_{\mathrm{perf}} \to X \times_Y Z \to Z$. Thus, arguing as in (b) one sees that it belongs to $\mathrm{Mor}_0(\mathcal{A})$ if $f \in \mathrm{Mor}_0(\mathcal{A})$. If in addition, $Z \in \mathrm{Ob}_0(\mathcal{A})$, then $\widetilde{f} \in \mathrm{Mor}_0^0(\mathcal{A}) \cap \mathrm{Mor}(\mathcal{A}')$, thus $\widetilde{f} \in \mathrm{Mor}_0^0(\mathcal{A}')$, as claimed. □

Corollary 2.4.8. *(a) For every placid $\mathcal{X} \in \mathrm{St}_k$, then its perfection $\mathcal{X}_{\mathrm{perf}} \in (\mathrm{St}_k)_{\mathrm{perf}} \subset \mathrm{St}_k$ is perfectly placid.*

(b) For every smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ of placid ∞-stacks, its perfection $f_{\mathrm{perf}} : \mathcal{X}_{\mathrm{perf}} \to \mathcal{Y}_{\mathrm{perf}}$ is perfectly smooth.

Proof. By Lemma 2.4.7(b) we get inclusions $\iota^R(\mathrm{Ob}_0(\mathcal{A})) \subset \mathrm{Ob}_0(\mathcal{A}')$ and $\iota^R(\mathrm{Mor}_0^0(\mathcal{A})) \subset \mathrm{Mor}_0^0(\mathcal{A}')$. Therefore both assertions follow from Lemma 1.2.10. □

Now we are ready to define topologically placid ∞-stacks.

Definition 2.4.9. (a) We call an ∞-stack $\mathcal{X}$ *topologically placid*, if its perfection $\mathcal{X}_{\mathrm{perf}} \in \mathrm{St}_k$ is perfectly placid.

(b) We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of ∞-stacks is *topologically smooth*, if its perfection $f_{\mathrm{perf}} : \mathcal{X}_{\mathrm{perf}} \to \mathcal{Y}_{\mathrm{perf}}$ is perfectly smooth.

(c) We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of ∞-stacks is a *topological covering*, if the restriction $\iota^* f : \iota^*\mathcal{X} \to \iota^*\mathcal{Y}$ is a covering in $\mathrm{St}_{\mathrm{perf},k}$.

2.4.10. Remarks. (a) Notice that if $f : \mathcal{X} \to \mathcal{Y}$ is a topological equivalence of ∞-stacks, then f_{perf} is an equivalence (by Lemma 2.3.6(b)). Thus $\mathcal{X}$ is topologically n-placid if and only if $\mathcal{Y}$ is.

(b) If $f : \mathcal{X} \to \mathcal{Y}$ is a covering, then $\iota^* f : \iota^*\mathcal{X} \to \iota^*\mathcal{Y}$ is a covering, thus f is a topological covering.

(c) Note that repeating arguments Lemma 2.1.12(c) word-by-word one shows that if $f : \mathcal{X} \to \mathcal{Y}$ be an fp-representable morphism of ∞-stacks such that $\mathcal{Y}$ is a topologically placid ∞-stack (resp. globally uh-placid algebraic space), then $\mathcal{X}$ is also so.

Corollary 2.4.11. *(a) Every placid ∞-stack $\mathcal{X}$ is topologically placid.*

(b) Every smooth morphism between placid ∞-stacks is topologically smooth.

Proof. Both assertions immediately follow from Corollary 2.4.8. □

Finally, we are going to modify slightly our constructions to define a class of topologically étale between topologically placid ∞-stacks, extending the corresponding notion from 0.2.4.

2.4.12. Topologically étale morphisms. (a) Replacing in 2.4.1 smooth morphisms by étale morphisms, we introduce the class of uh-étale morphisms in Aff_k^{ft}.

(b) We call a fp-morphism $X \to Y$ in Aff_k *uh-étale*, if if is a pullback of an uh-étale morphism in Aff_k^{ft}.

(c) We call a morphism $X \to Y$ in Aff_k *strongly pro-uh-étale*, if X has a presentation as a filtered limit $X \simeq \lim_\alpha X_\alpha$ over Y such that each projection $X_\alpha \to Y$ and each transition map $X_\alpha \to X_\beta$ is uh-étale.

(d) Consider a pair $(\mathrm{Ob}_0(\mathcal{A}), \mathrm{Mor}_0^0(\mathcal{A}))$, where $\mathrm{Ob}_0(\mathcal{A})$ is the class of globally perfectly placid affine schemes, and $\mathrm{Mor}_0^0(\mathcal{A})$ is the class of strongly pro-uh-étale morphisms between globally perfectly placid affine schemes. As in 2.4.6, this pair satisfies all the assumptions of 1.2.2, therefore the construction of 1.2.4 applies. We will call the corresponding geometric objects of $\mathrm{St}_{\mathrm{perf},k}$ *perfectly DM-placid*, where "DM" stands for "Deligne-Mumford", and special morphisms *perfectly étale*.

(e) More generally, we call a morphism $f : \mathcal{X} \to \mathcal{Y}$ of perfectly placid ∞-stacks *perfectly étale*, if for every perfectly smooth morphism $Y \to \mathcal{Y}$, where Y is a globally perfectly placid affine scheme, the pullback $f \times_{\mathcal{Y}} Y : \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is a perfectly étale morphism of perfectly DM-placid ∞-stacks.

(f) Finally, we call a morphism $f : \mathcal{X} \to \mathcal{Y}$ of topologically placid ∞-stacks *topologically étale*, if its perfection $f_{\mathrm{perf}} : \mathcal{X}_{\mathrm{perf}} \to \mathcal{Y}_{\mathrm{perf}}$ is perfectly étale.

3. Equidimensional morphisms

Our next goal is to introduce an important class of (weakly/universally open) equidimensional morphisms, first in the case of schemes of finite type, and then extend this notion to topologically placid ∞-stacks.

3.1. The case of schemes of finite type.

3.1.1. (Locally) equidimensional schemes and the canonical filtration. Let Y be a scheme of finite type over k.

(a) Recall that for every $y \in Y$, we can form the dimension Y at y, defined to be $\dim_y(Y) = \min_{U \ni y} \dim U$, where U runs over all open neighbourhoods of y. Alternatively, $\dim_y(Y)$ is the maximal of dimensions of irreducible components, containing y. We denote by $\underline{\dim}_Y : Y \to \mathbb{Z}$ the function $y \mapsto \dim_y(Y)$.

(b) Recall that Y is called *equidimensional*, if each irreducible component of Y is of the same dimension. Equivalently, this happens if and only if the dimension function $\underline{\dim}_Y$ is constant.

(c) For every $i \in \mathbb{Z}$, we set $Y_{\geq i} := \underline{\dim}_Y^{-1}(\{\geq i\})$, $Y_{\leq i} := \underline{\dim}_Y^{-1}(\{\leq i\})$ and $Y_i := \underline{\dim}_Y^{-1}(\{i\})$. By definition, each $Y_{\leq i} \subset Y$ is open, $Y_{\geq i} = Y \smallsetminus Y_{\leq i-1}$ is closed, and $Y_i = Y_{\geq i} \cap Y_{\leq i}$ is locally closed. Explicitly, each $Y_{\geq i}$ is the union of all irreducible components of Y of dimensions $\geq i$, and $Y_i = Y_{\geq i} \smallsetminus Y_{\geq i+1}$. In particular, Y_i is equidimensional of dimension i. Let $\eta_i : Y_i \hookrightarrow Y$ be the embedding.

(d) We say that Y is *locally equidimensional*, if the dimension function $\underline{\dim}_Y$ is locally constant. This happens if and only if each connected component of Y is equidimensional, or equivalently, if and only if each $Y_i \subset Y$ from (d) is a union of connected components.

3.1.2. Dimension function and (weakly) equidimensional morphisms.

(a) To every morphism $f : X \to Y$ be a morphism between schemes of finite type over k, we associate the dimension function $\underline{\dim}_f := \underline{\dim}_X - f^*\underline{\dim}_Y : X \to \mathbb{Z}$. Explicitly, for every $x \in X$ we have $\underline{\dim}_f(x) = \dim_x(X) - \dim_{f(x)}(Y)$.

(b) We call f *weakly equidimensional*, if the dimension function $\underline{\dim}_f$ is locally constant.

(c) We call f *equidimensional*, if f is weakly equidimensional, and we have an equality $\underline{\dim}_f(x) = \dim_x f^{-1}(f(x))$ for all $x \in X$.

(d) We say that a locally closed subscheme $X \subset Y$ is *pure of codimension d*, and write $\operatorname{codim}_X(Y) = d$, if the embedding $X \hookrightarrow Y$ is weakly equidimensional of constant dimension $-d$. For example, each stratum $Y_i \subset Y$ from 3.1.1(c) is of pure codimension 0, and $X \subset Y$ is of pure codimension $\dim Y - \dim X$, if both Y and X are equidimensional.

(e) For shortness, we will often call universally open equidimensional morphisms simply *uo-equidimensional*.

3.1.3. Remarks. (a) Our notion of an equidimensional morphism is slightly stronger than that of [EGAIV]. For example, an embedding of an irreducible component $i : X' \hookrightarrow X$ is always equidimensional in the sense of [EGAIV] but is not weakly equidimensional in our sense, if $\dim X' < \dim X$. On the other hand, both notions coincide, if f is dominant or open.

(b) Notice that f is automatically weakly equidimensional, if X and Y are locally equidimensional. Also every morphism $\iota : \mathrm{pt} \to X$ is weakly equidimensional.

(c) Explicitly, f is weakly equidimensional of dimension d if and only if for every $i \in \mathbb{N}$, we have $f(X_i) \subset Y_{i-d}$.

(d) Notice that a scheme X is locally equidimensional if and only if the structure morphism $X \to \mathrm{pt}$ is equidimensional.

Lemma 3.1.4. *For every morphism $f : X \to Y$ of schemes of finite type over k, we have an inequality $\underline{\dim}_f(x) \leq \dim_x f^{-1}(f(x))$. Moreover, this inequality is an equality, if f is an open map.*

Proof. The assertion is well-known (see, for example, [EGAIV, 14.2.1] or [St, 0B2L]). □

Lemma 3.1.4 immediately implies the following corollary.

Corollary 3.1.5. *If f is open and weakly equidimensional, then it is equidimensional.*

Lemma 3.1.6. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes of finite type over k.*

(a) If f is surjective, and $g \circ f$ is (universally) open, then g is (universally) open.

(b) We have an equality $\underline{\dim}_{g\circ f} = \underline{\dim}_f + f^*\underline{\dim}_g$.

(c) Assume that g is weakly equidimensional. Then f is weakly equidimensional if and only if $g \circ f$ is.

(d) Assume that f is open surjective. If f and $g \circ f$ are weakly equidimensional, then so is g.

(e) Assume that f and g are open, and f is surjective. If $g \circ f$ are weakly equidimensional, then so are f and g.

Proof. (a) and (b) are clear, and (c) follows from (b).

(d) By (b), the assumption implies that $f^*\underline{\dim}_g = \underline{\dim}_{g\circ f} - \underline{\dim}_f$ is locally constant. Since f is open and surjective, then $\underline{\dim}_g$ is locally constant as well.

(e) By Lemma 3.1.4, both functions $\underline{\dim}_f$ and $\underline{\dim}_g$ are upper semi-continuous, that is, the preimage of $\{\geq i\}$ is closed for all i. Then $f^*\underline{\dim}_g$ is upper semi-continuous as well. Since the sum $\underline{\dim}_{g\circ f} = \underline{\dim}_f + f^*\underline{\dim}_g$ is locally constant, we conclude that both function $\underline{\dim}_f$ and $f^*\underline{\dim}_g$ are lower semi-continuous as well. This implies that both $\underline{\dim}_f$ and $f^*\underline{\dim}_g$ are locally constant, and hence (as in (d)), function $\underline{\dim}_g$ is locally constant as well. □

Corollary 3.1.7. *Consider a Cartesian diagram of schemes of finite type over k*

(3.1)
$$\begin{array}{ccc} X' & \xrightarrow{\psi} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\phi} & Y \end{array}$$

such that either f and g are open or ϕ and ψ are open.

(a) Then we have an equality $\underline{\dim}_\psi = g^*\underline{\dim}_\phi$.

(b) In particular, if ϕ is weakly equidimensional, then ψ also is.

Proof. (a) For every $x' \in X'$, we set $x := g(x') \in X$, $y' := \psi(x') \in Y'$ and $y = \phi(x) = f(y') \in Y$. We want to show that $\underline{\dim}_\psi(x') = \underline{\dim}_\phi(x)$.

When ϕ and ψ are open, then by Lemma 3.1.4, we have to show the equality $\dim_{x'} \psi^{-1}(y') = \dim_x \phi^{-1}(y)$. Since our diagram is Cartesian, g induces an isomorphism $\psi^{-1}(y') \simeq \phi^{-1}(y) \times_y y'$, which implies the required equality.

Assume now f and g are open. Then, by the proven above, we have an equality $\underline{\dim}_g = \psi^*\underline{\dim}_f$. On the other hand, using Lemma 3.1.6(b) for $\phi \circ g \simeq f \circ \phi'$, we conclude that $\underline{\dim}_g + g^*\underline{\dim}_\phi = \underline{\dim}_\psi + \psi^*\underline{\dim}_f$, hence

$$\underline{\dim}_g - \psi^*\underline{\dim}_f = \underline{\dim}_\psi - g^*\underline{\dim}_\phi.$$

Sice the left hand vanishes by the proven above, the right hand side vanishes as well.

(b) The assertion follows easily from (a). Indeed, if $\underline{\dim}_\phi$ is locally constant, then $\underline{\dim}_\psi = \underline{\dim}_\phi \circ g$ is locally constant as well. □

Corollary 3.1.8. *The class of universally open equidimensional morphisms is closed under compositions and base change.*

Proof. While the first assertion follows from Lemma 3.1.4 and Lemma 3.1.6(c), the second one follows from Corollary 3.1.7. □

Corollary 3.1.9. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes of finite type over k.*

(a) If f and g are equidimensional, then so is $g \circ f$.

(b) Assume that f is open surjective. If f and $g \circ f$ are equidimensional, then so is g.

Proof. For every $x \in X, y := f(x) \in Y$ and $z := g(y) \in Z$, we let $f_z : f^{-1}(g^{-1}(z)) \to g^{-1}(z)$ be the restriction of f. Then

$$\dim_x f^{-1}(g^{-1}(z)) = \underline{\dim}_{f_z}(x) + \dim_y g^{-1}(z). \tag{3.2}$$

(a) Since $g \circ f$ is weakly equidimensional by Lemma 3.1.6(c), Lemma 3.1.4 implies that it remains to show that for $x \in X$ we have $\dim_x f^{-1}(g^{-1}(z)) \leq \underline{\dim}_{g\circ f}(x)$. Using Lemma 3.1.4 and the assumption that f and g are equidimensional, we conclude that $\underline{\dim}_{f_z}(x) \leq \dim_x f^{-1}(y) = \underline{\dim}_f(x)$ and $\dim_y g^{-1}(z) = \underline{\dim}_g(y)$. Therefore we conclude from (3.2) that

$$\dim_x f^{-1}(g^{-1}(z)) \leq \underline{\dim}_f(x) + \underline{\dim}_g(y) = \underline{\dim}_{g\circ f}(x),$$

as claimed.

(b) Since g is weakly equidimensional by Lemma 3.1.6(d), it remains to show that for every $y \in Y$ we have $\dim_y g^{-1}(z) = \underline{\dim}_g(y)$. Since f is open, its restriction

f_z is open as well. Therefore we conclude from Corollary 3.1.7(a) that for every $x \in f^{-1}(y)$, we have $\underline{\dim}_{f_z}(x) = \underline{\dim}_f(x)$. Since $g \circ f$ is equidimensional, we conclude that $\dim_x f^{-1}(g^{-1}(z)) = \underline{\dim}_{g\circ f}(x)$. Hence, by (3.2), we conclude that

$$\dim_y g^{-1}(z) = \underline{\dim}_{g\circ f}(x) - \underline{\dim}_f(x) = \underline{\dim}_g(y).$$

□

Lemma 3.1.10. *Every uh-smooth morphism between schemes of finite type over k is universally open equidimensional.*

Proof. We have to show that the class of universally open equidimensional morphisms satisfies properties (i),(ii) of 2.4.1. It is easy to see that it contains smooth morphisms and universal homeomorphisms, and is closed under compositions (by Corollary 3.1.8). Moreover, since it contains étale morphisms, property (ii) follows from Lemma 3.1.6(a),(d). □

3.1.11. Remark. Notice that every flat morphism between schemes of finite type is automatically universally open, but not necessarily weakly equidimensional. For example, condider the projection $X \to \mathrm{pt}$ from a non-locally equidimensional scheme.

3.2. **Extension to topologically placid ∞-stacks.** In this subsection we are going to define classes of equidimensional, weakly equidimensional and uo-equidimensional morphisms between topologically placid ∞-stacks. In order not to repeat the same arguments three times, we will introduce the following notation.

3.2.1. Notation. (a) Let $\mathcal{B} := \mathrm{Aff}_k^{ft}$, and let $\mathcal{P}_+$ be the class of all universally open and equidimensional morphisms. Then $\mathcal{P}_+$ is closed under compositions a pullbacks with respect to all morphisms in $\mathcal{B}$ (by Corollary 3.1.8). In particular, $\mathcal{P}_+$ satisfies all the assumptions of 1.3.1(a).

(b) Let $\mathcal{Q} \supset \mathcal{P}_+$ be a class of morphisms in $\mathcal{B}$ such that

• $\mathcal{Q}$ is closed under compositions and $\mathcal{P}_+$-pullbacks (compare 1.4.4).

• $\mathcal{Q}$ is $\mathcal{P}_+$*-local*, by which we mean that for every morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of schemes of finite type over k such that $f \in \mathcal{P}_+$ is surjective, and $g \circ f \in \mathcal{Q}$, we have $g \in \mathcal{Q}$.

3.2.2. Remark. Notice that if $\mathcal{Q}$ is $\mathcal{P}_+$ local, then for every composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of schemes of finite type over k such that $f \in \mathcal{P}_+$, and $g \circ f \in \mathcal{Q}$, we have $g|_{f(X)} \in \mathcal{Q}$. Indeed, apply the definition to the composition $X \xrightarrow{f} f(X) \xrightarrow{g|_{f(X)}} Z$.

Lemma 3.2.3. *Let $\mathcal{Q}$ be either $\mathcal{P}_+$ or one of the following classes of morphisms*

(i) (universally) open morphisms;

(ii) (weakly) equidimensional morphisms.

Then $\mathcal{Q}$ satisfies all the assumptions of 3.2.1(b).

Proof. Since every $\mathcal{Q}$ is closed under pullbacks with respect to universally open morphisms (by Corollary 3.1.7), we conclude that every $\mathcal{Q}$ is closed under $\mathcal{P}_+$-pullbacks. Next, the fact that every $\mathcal{Q}$ is closed under composition follows from Lemma 3.1.6(c) and Corollary 3.1.9(a), and the fact that $\mathcal{Q}$ is $\mathcal{P}_+$-local follows from Lemma 3.1.6(a),(d) and Corollary 3.1.9(b). □

The following lemma plays a central role in this work.

Lemma 3.2.4. *Let $\mathcal{Q}$ be either $\mathcal{P}_+$ or the class of (universally) open morphisms. Then $\mathcal{Q}$ is $\mathcal{Q}$-adapted (see 1.4.4).*

In other words, let $X \simeq \lim_\alpha X_\alpha$ and $X \simeq \lim_\beta X'_\beta$ be two presentations of an affine scheme X with all transition maps in $\mathcal{Q}$. Then for every β and every sufficiently large α the projection $\mathrm{pr}_\alpha : X \to X_\alpha$ factors as a composition $X \overset{\mathrm{pr}_\beta}{\longrightarrow} X'_\beta \overset{f_{\beta,\alpha}}{\longrightarrow} X_\alpha$ with $f_{\beta,\alpha} \in \mathcal{Q}$.

Proof. Since X_α is of finite type over k, there exists β such that $\mathrm{pr}_\alpha : X \simeq \lim_\beta X'_\beta \to X_\alpha$ factors through $f_{\beta,\alpha} : X'_\beta \to X_\alpha$. We claim that there exists $\delta > \beta$ such that the composition $X'_\delta \overset{\mathrm{pr}'_{\delta,\beta}}{\longrightarrow} X'_\beta \overset{f}{\to} X_\alpha$ belongs to $\mathcal{Q}$.

Note that the projection $\mathrm{pr}_\beta : X \to X'_\beta$ factors through $g : X_\gamma \to X'_\beta$. Moreover, increasing γ we can further assume that $\gamma > \alpha$ and the composition $X_\gamma \overset{g}{\to} X'_\beta \overset{f}{\to} X_\alpha$ is the transition map. In particular, $f \circ g \in \mathcal{Q}$.

Similarly, there exists $\delta > \beta$ such that $\mathrm{pr}_\gamma : X \to X_\gamma$ factors through $h : X'_\delta \to X_\gamma$ and such that $g \circ h : X'_\delta \to X'_\beta$ is the transition map. Thus $g \circ h \in \mathcal{Q}$ as well.

First we claim that if $\mathcal{Q}$ is the class of (universally) open morphisms, then the composition $f \circ g \circ h : X'_\delta \to X_\alpha$ belongs to $\mathcal{Q}$. Set $U \subset X'_\beta$ be the image of $g \circ h$. Since $g \circ h$ is open, we conclude that U is open. Since $f \circ g \circ h = (f|_U) \circ (g \circ h)$, it remains to show that $f|_U : U \to X_\alpha$ belongs to $\mathcal{Q}$. Set $V := g^{-1}(U) \subset X_\gamma$. It is an open subset, because U is. Note that the map $g|_V : V \to U$ is surjective, because $U = \mathrm{Im}(g \circ f) \subset \mathrm{Im}\, g$, and $(f|_U) \circ (g|_V) = (f \circ g)|_V$ belongs to $\mathcal{Q}$, because $f \circ g$ is. Therefore $f|_U$ belongs to $\mathcal{Q}$ by Lemma 3.1.6(a).

Now assume that $\mathcal{Q} = \mathcal{P}_+$. By the proven above, we can increase β, γ and δ if necessary, so that f and g are open. In this case, we claim that the composition $f \circ g \circ h$ is equidimensional. As before, it suffices to show that $f|_U$ is such. By our assumptions, $g|_V$ is open surjective, $f|_U$ is open and $(f|_U) \circ (g|_V) = (f \circ g)|_V$ is equidimensional. Therefore $f|_U$ is equidimensional by Lemma 3.1.6(e), and the proof is complete. □

3.2.5. Notation. (a) Let $\mathcal{P} = \mathcal{P}_{uh-sm}$ be the class of uh-smooth morphisms in $\mathcal{B}$, then $\mathcal{P} \subset \mathcal{P}_+$ (by Lemma 3.1.10).

(b) It follows from Lemma 3.2.4 and 1.4.5(a), that every class $\mathcal{Q}$ from 3.2.1 is $\mathcal{P}$-adapted, so the assumption of 1.4.7 are satisfied. Thus we can talk about $\mathcal{Q}$-special morphisms $f:\mathcal{X}\to\mathcal{Y}$ between perfectly placid ∞-stacks.

(c) We say that a morphism $f:\mathcal{X}\to\mathcal{Y}$ of topologically placid ∞-stacks is *topologically $\mathcal{Q}$-special*, if its perfection $f_{\mathrm{perf}}:\mathcal{X}_{\mathrm{perf}}\to\mathcal{Y}_{\mathrm{perf}}$ is $\mathcal{Q}$-special.

(d) By construction, the class of $\mathcal{Q}$-special morphisms from (b) is closed under pullbacks with respect to perfectly smooth morphisms. Therefore arguing as in Lemma 2.4.7(c), we conclude that the class of topologically $\mathcal{Q}$-special morphisms from (c) is closed under pullbacks with respect to topologically smooth morphisms

The following proposition provides a much simpler characterization of $\mathcal{Q}$-special morphisms in some cases.

Proposition 3.2.6. *Let $f:X\to Y$ be an fp-morphism between globally uh-placid algebraic spaces. Then the following are equivalent:*

(i) f is topologically $\mathcal{Q}$-special in the sense of 3.2.5(c).

(ii) There exists a uh-placid presentation $Y\simeq\lim_{\alpha}Y_{\alpha}$ an index α and a morphism $f_{\alpha}:X_{\alpha}\to Y_{\alpha}$ in $\mathcal{Q}$ such that $f\simeq f_{\alpha}\times_{Y_{\alpha}}Y$.

Before starting the proof of the proposition, we will give several equivalent reformulations of the condition (ii) of Proposition 3.2.6.

Lemma 3.2.7. *Let $f:X\to Y$ be an fp-morphism between globally uh-placid algebraic spaces. Then the following are equivalent:*

(a) There exists a uh-placid presentation $Y\simeq\lim_{\alpha}Y_{\alpha}$, an index α and a morphism $f_{\alpha}:X_{\alpha}\to Y_{\alpha}$ in $\mathcal{Q}$ such that $f\simeq f_{\alpha}\times_{Y_{\alpha}}Y$.

(b) For every uh-placid presentation $Y\simeq\lim_{\alpha}Y_{\alpha}$, an index α and a morphism $f_{\alpha}:X_{\alpha}\to Y_{\alpha}$ such that $f\simeq f_{\alpha}\times_{Y_{\alpha}}Y$, there exists $\beta>\alpha$ such that $f_{\beta}:=f_{\alpha}\times_{Y_{\alpha}}Y_{\beta}\in\mathcal{Q}$.

(c) For every presentation $Y\simeq\lim_{\alpha}Y_{\alpha}$, index α and a morphism f_{α} as in (b), there exists an open subset $U\subset X_{\alpha}$, containing $\mathrm{pr}_{\alpha}(X)$, such that $f_{\alpha}|_{U}$ belongs to $\mathcal{Q}$.

Proof. (a) $\Longrightarrow$ (b) By definition, there exists a uh-placid presentation $Y\simeq\lim_{\alpha'}Y'_{\alpha'}$ and a morphism $f_{\alpha'}:X'_{\alpha'}\to Y'_{\alpha'}$ in $\mathcal{Q}$ such that $f\simeq f'_{\alpha'}\times_{Y'_{\alpha'}}X'_{\alpha'}$. By Lemma 3.2.4, there exists $\beta>\alpha$ and a morphism $\mathrm{pr}_{\beta,\alpha'}:Y_{\beta}\to Y'_{\alpha'}$ in $\mathcal{P}_{+}$ such that $\mathrm{pr}'_{\alpha'}:Y\to Y'_{\alpha'}$ factors as $Y\to Y_{\beta}\to Y'_{\alpha'}$. Increasing β if necessary, we can guarantee that $f_{\beta}\simeq f'_{\alpha'}\times_{Y'_{\alpha'}}Y_{\beta}$. Since $f_{\alpha'}\in\mathcal{Q}$, and $\mathcal{Q}$ is closed under $\mathcal{P}_{+}$-pullbacks, we conclude that $f_{\beta}\in\mathcal{Q}$, as claimed.

(b) $\Longrightarrow$ (c) By (b), the composition $X_{\beta}\to Y_{\beta}\to Y_{\alpha}$ or (what is the same) $X_{\beta}\overset{\mathrm{pr}_{\beta,\alpha}}{\longrightarrow}X_{\alpha}\overset{f_{\alpha}}{\longrightarrow}Y_{\alpha}$ belongs to $\mathcal{Q}$. Therefore we conclude from remark 3.2.2 that $f_{\alpha}|_{\mathrm{pr}_{\beta,\alpha}(X_{\beta})}$ belongs to $\mathcal{Q}$. Since $\mathrm{pr}_{\beta,\alpha}(X_{\beta})\subset X_{\alpha}$ is an open subset containing $\mathrm{pr}_{\alpha}(X)$, we are done.

(c) $\Longrightarrow$ (a) Choose a presentation as in (c). Since $U\supset\mathrm{pr}_{\alpha}(X)$, the projection $\mathrm{pr}_{\alpha}:X\to X_{\alpha}$ defines a morphism $X\simeq\lim_{\beta>\alpha}X_{\beta}\to U$, which induces a morphism

$X_\beta \to U$ for some β. For such a β, the morphism $f_\beta : X_\beta \to Y_\beta$ is a pullback of $f_\alpha|_U$, thus belongs to $\mathcal{Q}$. $\square$

Now we are ready to show Proposition 3.2.6.

Proof of Proposition 3.2.6. (ii) $\Longrightarrow$ (i). Choose a presentation $Y \simeq \lim_\alpha Y_\alpha$ and a morphism $f_\alpha : X_\alpha \to Y_\alpha \in \mathcal{Q}$ be as in (ii). Then X has a presentation $X \simeq \lim_{\beta>\alpha} X_\beta$, with $X_\beta := X_\alpha \times_{Y_\alpha} Y_\beta$. Since $\mathcal{Q}$ is closed under $\mathcal{P}$-pullbacks, we conclude that $f_\beta := f_\alpha \times_{Y_\alpha} Y_\beta \in \mathcal{Q}$. Since $\mathcal{Q}$ is $\mathcal{P}$-adapted (by Lemma 3.2.4), we conclude that f is topologically $\mathcal{Q}$-special by remark 1.4.5(b).

(i) $\Longrightarrow$ (ii). Choose any uh-placid presentation $Y \simeq \lim_\alpha Y_\alpha$. Then there exists an index α and a morphism $f_\alpha : X_\alpha \to Y_\alpha$ such that $f \simeq f_\alpha \times_{Y_\alpha} Y$ (see [St, 01ZM]). By Lemma 3.2.7(c), it suffices to show that there exists an open neighborhood U of $\mathrm{pr}_\alpha(X) \subset X_\alpha$ such that $f_\alpha|_U$ is in $\mathcal{Q}$.

Since f is topologically $\mathcal{Q}$-special, there exists a topological smooth covering $g : Z \to X$ such that Z is a globally uh-placid affine scheme, and the composition $f \circ g : Z \to X \to Y$ is in $\mathcal{Q}_\mathcal{A}$ in the sense of 1.4.4(b). Hence there exists a presentation $Z \simeq \lim_\beta Z_\beta$ and a morphism $h_{\beta,\alpha} : Z_\beta \to Y_\alpha$ in $\mathcal{Q}$ such that $Z \to X \to Y \to Y_\alpha$ or (what is the same) $Z \to X \to X_\alpha \to Y_\alpha$ decomposes as $h_{\beta,\alpha} \circ \mathrm{pr}_\beta : Z \to Z_\beta \to Y_\alpha$. Increasing β, we can assume that $\mathrm{pr}_\alpha \circ g : Z \to X \to X_\alpha$ factor as $g_{\beta,\alpha} \circ \mathrm{pr}_\beta : Z \to Z_\beta \to X_\alpha$. Increasing β further we can assume that $h_{\beta,\alpha} = f_\alpha \circ g_{\beta,\alpha} : Z_\beta \to X_\alpha \to Y_\alpha$.

Since g and pr_α are topological smooth, we conclude from Lemma 3.1.10 and Lemma 3.2.4 that $g_{\beta,\alpha} \in \mathcal{P}_+$. Since $\mathcal{Q}$ is $\mathcal{P}_+$-local, we deduce that the restriction $f_\alpha|_{g_{\beta,\alpha}(Z_\beta)}$ is in $\mathcal{Q}$.

Thus it suffices to show that $\mathrm{pr}_\alpha(X) \subset g_{\beta,\alpha}(Z_\beta)$, which follows from inclusion

$$\mathrm{pr}_\alpha(X) = \mathrm{pr}_\alpha(g(Z)) = g_{\beta,\alpha}(\mathrm{pr}_\beta(Z)) \subset g_{\beta,\alpha}(Z_\beta),$$

where the first equality follows from the surjectivity of g. $\square$

3.2.8. Notation. (a) Using Lemma 3.2.3, we will say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of topologically placid ∞-stacks is *equidimensional/weakly equidimensional/uo-equidimensional/uo-special*, if it is topologically $\mathcal{Q}$-special (see 3.2.5(c)) when $\mathcal{Q}$ is the class of equidimensional/weakly equidimensional/universally open and equidimensional/universally open morphisms, respectively.

(b) Note that all classes in (a) are closed under pullbacks with respect to topologically smooth morphisms (see 3.2.5(d)), while the classes of uo-equidimensional/uo-special morphisms are closed under all fp-pullbacks (by Lemma 1.4.9 and Corollary 3.1.8).

(c) When f is an fp-morphism between globally uh-placid algebraic spaces, the classes of (a) have much more simple descriptions (by Proposition 3.2.6).

3.2.9. Remark. We use somewhat strange terminology *uo-special* rather than simply *universally open* (unlike for the remaining classes of morphisms) to make sure it does not conflict with the usual (topological) meaning.

3.3. **Dimension function.**

3.3.1. Notation.

(a) Let $\phi : X \to Y$ be a finitely presented morphism of globally uh-placid schemes. To this data, we associate a constructible dimension function $\underline{\dim}_f : X \to \mathbb{Z}$, defined as follows:

Choose a uh-placid presentation $Y \simeq \lim_\alpha Y_\alpha$. Since ϕ is finitely presented, it has a form $\phi \simeq \phi)_\alpha \times_{Y_\alpha} Y$ for some morphism $\phi_\beta : X_\beta \to Y_\beta$ of schemes of finite type (see [St, 01ZM]), and we set $\underline{\dim}_\phi := \pi_\beta^* \underline{\dim}_{\phi_\beta}$, where $\pi_\beta : X \to X_\beta$ is the projection. It remains to show that $\underline{\dim}_\phi$ is independent of all choices involved.

Since all transition maps $Y_\alpha \to Y_\beta$ is universally open (see 2.4.1), it follows from Corollary 3.1.7 that $\underline{\dim}_\phi$ will not change if we replace $\phi_\beta : X_\beta \to Y_\beta$ by $\phi_\beta \times_{Y_\beta} Y_\alpha : X_\beta \times_{Y_\beta} Y_\alpha \to Y_\alpha$ for some $\alpha > \beta$. From this we also deduce that $\underline{\dim}_\phi$ is independent of ϕ_β, because every two choices became isomorphic after a pullback to some Y_α. Finally, the independence of the presentation follows from Lemma 3.2.4 using Corollary 3.1.7 again.

(b) We call an fp-morphism $\phi : X \to Y$ of globally uh-placid schemes *weakly equidimensional of relative dimension* d, if the dimension function $\underline{\dim}_\phi$ is a constant function with value d.

(c) As in the case of schemes of finite type (see 3.1.2(d)), we say that a finitely presented locally closed subscheme $X \subset Y$ is of pure codimension d, if the embedding $\iota : X \hookrightarrow Y$ is weakly equidimensional of relative dimension $-d$.

Lemma 3.3.2. *An fp-morphism $\phi : X \to Y$ of globally uh-placid schemes is weakly equidimensional if and only if the function $\underline{\dim}_\phi$ is locally constant. Furthermore, ϕ is equidimensional if and only if the locally constant function $\underline{\dim}_\phi$ also satisfies $\underline{\dim}_\phi(x) = \dim_x \phi^{-1}(\phi(x))$ for all $x \in X$.*

Proof. Assume that $\phi : X \to Y$ is weakly equidimensional/equidimensional. Then ϕ of the form $\phi \simeq \phi_\alpha \times_{Y_\alpha} Y$, and ϕ_α is weakly equidimensional/equidimensional (see Proposition 3.2.6). Then $\underline{\dim}_{\phi_\alpha}$ is locally constant/and $\underline{\dim}_{\phi_\alpha}(x) = \dim_x \phi_\alpha^{-1}(\phi_\alpha(x))$ for every $x_\alpha \in X_\alpha$. Since $\underline{\dim}_\phi = \mathrm{pr}_\alpha^* \underline{\dim}_{\phi_\alpha}$, the first direction follows.

Conversely, it is easy to see or can be deduced from Lemma 3.2.3 that there exists the largest open subset $U \subset X_\alpha$ such that $\phi_\alpha|_U$ is weakly equidimensional/equidimensional. By Lemma 3.2.7(c), it suffices to show that U contains $\mathrm{pr}_\alpha(X)$.

By assumption $\phi \simeq \phi_\alpha \times_{Y_\alpha} Y$ satisfies that $\underline{\dim}_\phi = \mathrm{pr}_\alpha^* \underline{\dim}_{\phi_\alpha}$ is locally constant. Since basis of open subsets of Y are induced by open subsets of some X_β, and X is quasi-compact, there exists $\beta \geq \alpha$ such that ϕ_β is constant on each connected component of $\mathrm{pr}_\beta(X)$. Since $X_\beta \to X_\alpha$ is open, the same holds for $\beta = \alpha$. This shows the first assertion.

Next, the equality $\underline{\dim}_\phi(x) = \dim_x \phi^{-1}(\phi(x))$ for all $x \in X$ implies that

$$\underline{\dim}_{\phi_\alpha}(x) = \dim_x \phi_\alpha^{-1}(\phi_\alpha(x))$$

for all $x \in \mathrm{pr}_\alpha(X)$. Using Lemma 3.1.4 together with the upper semicontinuity of the right hand side, we conclude that the locus of points for which equality holds is open, which finishes the proof. □

We have the following extension of Corollary 3.1.7 to the uh-placid case.

Corollary 3.3.3. *Consider Cartesian diagram (3.1) such that Y and Y' are globally uh-placid schemes, ϕ is finitely presented, while f is uo-special (see 3.2.8(a)).*

(a) Then we have an equality $\underline{\dim}_\psi = g^*\underline{\dim}_\phi$.

(b) If in addition morphism ϕ is (weakly) equidimensional (of constant relative dimension d), then so is ψ.

Proof. Note that (b) is an immediate corollary of (a) and Lemma 3.3.2. Moreover, (a) is a formal consequence of a combination of Lemma 3.2.4 and Corollary 3.1.7. Indeed, choose uh-placid presentations $Y \simeq \lim_\alpha Y_\alpha$ and $Y' \simeq \lim_\beta Y'_\beta$. By the definition of $\underline{\dim}_\phi$, there exists an index α and a morphism $\phi_\alpha : X_\alpha \to Y_\alpha$ of schemes of finite type such that $\phi \simeq \phi_\alpha \times_{Y_\alpha} Y$ and $\underline{\dim}_\phi = \mathrm{pr}_\alpha^* \underline{\dim}_{\phi_\alpha}$.

Since f is uo-special, there exists a strongly pro-uh-smooth covering $\pi : Y'' \to Y'$ such that the composition $\widetilde{f} := f \circ \pi : Y' \to Y' \to Y$ is in $\mathcal{Q}_{\mathcal{A}}$, where $\mathcal{Q}_{\mathcal{A}}$ is the class corresponding to the class $\mathcal{Q}$ of universally open morphisms in the sense of 1.4.4(a). Since the assertion for f follows from the corresponding assertions for π and $\widetilde{f}$, we can assume that f is in $\mathcal{Q}_{\mathcal{A}}$.

Therefore there exists an index β and universally open morphism $f_{\beta,\alpha} : Y'_\beta \to Y_\alpha$ such that $\mathrm{pr}_\alpha \circ f : Y' \to Y \to Y_\alpha$ decomposes as $f_{\beta,\alpha} \circ \mathrm{pr}_\beta : Y' \to Y'_\beta \to Y_\alpha$.

Consider Cartesian diagram

$$\begin{array}{ccc} X'_\beta & \xrightarrow{\psi_\beta} & Y'_\beta \\ {\scriptstyle g_{\beta,\alpha}}\downarrow & & \downarrow{\scriptstyle f_{\beta,\alpha}} \\ X_\alpha & \xrightarrow{\phi_\alpha} & Y_\alpha \end{array}$$

Then we have an equality $g^*_{\beta,\alpha}\underline{\dim}_{\phi_\alpha} = \underline{\dim}_{\psi_\beta}$ by Corollary 3.1.7, which implies the equality

$$g^*\underline{\dim}_\phi = g^* \mathrm{pr}_\alpha^* \underline{\dim}_{\phi_\alpha} = \mathrm{pr}_\beta^* g^*_{\beta,\alpha}\underline{\dim}_{\phi_\alpha} = \mathrm{pr}_\beta^* \underline{\dim}_{\psi_\beta} = \underline{\dim}_\psi,$$

we were looking for. □

The following simple lemma motivates the definition 3.3.5 below.

Lemma 3.3.4. *Let $f : X \to Y$ be a topologically étale morphism (resp. covering) of globally uh-placid affine schemes. Then we have* $\dim X \leq \dim Y$ *(resp.* $\dim X = \dim Y$*).*

Proof. Using Corollary 1.1.7, there exists a strongly pro-uh-étale covering of globally uh-placid affine schemes $U \to X$ such that the composition $U \to X \to Y$ is strongly pro-uh-étale covering (resp. morphism). Thus we can assume that f is strongly pro-uh-étale. In this case, the assertion is standard. □

3.3.5. Equidimensional morphisms of relative dimension d.

(a) We call a morphism $f : \mathcal{X} \to Y$ from a topologically placid ∞-stack $\mathcal{X}$ to a globally uh-placid affine scheme Y *(weakly/uo) equidimensional morphisms of relative dimension* d, if there exists a topologically étale covering $\sqcup_\alpha U_\alpha \to \mathcal{X}$ from a disjoint union of affine schemes such that each composition $U_\alpha \to \mathcal{X} \to Y$ decomposes as $U_\alpha \overset{\pi_\alpha}{\to} Y_\alpha \overset{f'_\alpha}{\to} Y$, where π_α is topologically étale (see 2.4.12), and $f'_\alpha : Y'_\alpha \to Y$ is an fp-(weakly/uo) equidimensional affine morphism of relative dimension d (see 3.3.1(b)).

(b) We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of topologically placid ∞-stacks is *(weakly/uo) equidimensional of relative dimension* d, if for every topologically smooth morphism $Y \to \mathcal{Y}$ from a globally uh-placid affine scheme Y, the pullback $f \times_{\mathcal{Y}} Y : \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is (weakly/uo) equidimensional of relative dimension d in the sense of (a).

(c) It follows from Corollary 3.3.3(b), that the condition of (b) is automatically satisfied, if there exists a topologically smooth covering $Y \to \mathcal{Y}$ such that the pullback $f \times_{\mathcal{Y}} Y : \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is (weakly/uo) equidimensional of relative dimension d.

The following simple lemma will be useful later.

Lemma 3.3.6. *Let $f : Y \to X$ be an fp-morphism between strongly pro-smooth schemes such that Y is connected. Then f is a weakly equidimensional morphism of constant relative dimension.*

Proof. Choose a strongly pro-smooth presentation $X \simeq \lim_\alpha X_\alpha$. Since the morphism $f : Y \to X$ is finitely presented, it comes from an morphism $f_\alpha : Y_\alpha \to X_\alpha$. Then Y has a placid presentation $Y \simeq \lim_{\beta > \alpha} Y_\beta$ with $Y_\beta = Y_\alpha \times_{X_\alpha} X_\beta$. Since Y is strongly pro-smooth, it follows from Corollary 2.1.4 that Y_β is smooth, if β is sufficiently large. Moreover, since Y is connected, one can assume that Y_β is connected. Since Y_β and X_β are smooth, thus locally equidimensional, we conclude that the morphism $f_\alpha : Y_\beta \to X_\beta$ is of constant dimension (see 3.1.3(b)). Therefore its pullback $f : Y \to X$ is of constant relative dimension as well. □

Part 2. Sheaves on prestacks and perverse t-structures

4. Categories of sheaves on prestacks

4.1. Limits and colimits of ∞-categories.

4.1.1. Notation and convention. Let k be a field, let ℓ be a prime different from the characteristic of k.

(a) All categories are ∞-categories, all functors are ∞-functors between ∞-categories, and all limits and colimits are the homotopical one. In particular, if $\mathcal{C}$ is an ordinary category, we will view it as an ∞-category. We say that a morphism in $\mathcal{C}$ is an isomorphism, if it is an isomorphism in the homotopy category of $\mathcal{C}$.

(b) Let $\mathrm{Cat}_{\mathrm{st},\ell}$ be the ∞-category, whose objects are stable $\overline{\mathbb{Q}}_\ell$-linear small ∞-categories, and morphisms are exact functors, i.e. those that preserve finite colimits.

(c) Let $\mathrm{PrCat}_{\mathrm{st},\ell}$ be the ∞-category, whose objects are stable $\overline{\mathbb{Q}}_\ell$-linear presentable ∞-categories (see [Lu1, 5.5.0.1]), and morphisms are continuous functors, i.e., commuting with all small colimits.

(d) Recall that the ∞-categories $\mathrm{PrCat}_{\mathrm{st},\ell}$ and $\mathrm{Cat}_{\mathrm{st},\ell}$ have all limits and filtered colimits (see [Lu1, 4.2.4.8, 5.5.3.13, 5.5.3.18], [Lu2, 1.1.4.4, 1.1.4.6]) and there is a natural functor $\mathrm{Ind}:\mathrm{Cat}_{\mathrm{st},\ell}\to\mathrm{PrCat}_{\mathrm{st},\ell}:\mathcal{C}\mapsto\mathrm{Ind}(\mathcal{C})$, commuting with all small filtered colimits (compare [Lu1, 5.3.5.10] or [DG, 1.9.2]).

4.1.2. Adjoint theorem. (a) Let $\mathcal{I}$ be a small category and $\Psi:\mathcal{I}\to\mathrm{PrCat}_{\mathrm{st},\ell}$ a functor. In particular, for every $i\in\mathcal{I}$, we are given an ∞-category $\mathcal{C}_i$ and for every morphism $(i\overset{\alpha}{\to}j)\in\mathcal{I}$ we are given a functor $\psi_\alpha\in\mathrm{Funct}_{cont}(\mathcal{C}_i,\mathcal{C}_j)$.

(b) Suppose that for every morphism $\alpha:i\to j$ in $\mathcal{I}$, the functor ψ_α admits a continuous right adjoint ϕ_α. Since adjoints are compatible with compositions, the data $(\mathcal{C}_i,\phi_\alpha)$ extends to a functor $\Phi:\mathcal{I}^{op}\to\mathrm{PrCat}_{\mathrm{st},\ell}$ (see [Lu1, 5.5.3.4]).

The following result allows to rewrite a colimit as a limit and vice versa (see [Lu1, 5.5.3.3] or [DG, sect. 1.7-1.9]).

Theorem 4.1.3. *The colimit*

$$\mathcal{C}:=\mathrm{colim}\,\Psi=\mathrm{colim}_{i\in\mathcal{I}}\,\mathcal{C}_i\in\mathrm{PrCat}_{\mathrm{st},\ell}$$

exists and is canonically equivalent to the limit

$$\widehat{\mathcal{C}}:=\lim\Phi=\lim_{i\in\mathcal{I}^{op}}\mathcal{C}_i\in\mathrm{PrCat}_{\mathrm{st},\ell}\,.$$

Moreover, the equivalence $\mathcal{C}\overset{\sim}{\to}\widehat{\mathcal{C}}$ is uniquely characterized by the condition that for every $i\in\mathcal{I}$ the evaluation functor $ev_i:\widehat{\mathcal{C}}\to\mathcal{C}_i$ is the right adjoint to the tautological functor $\mathrm{ins}_i:\mathcal{C}_i\to\mathcal{C}$.

4.1.4. Filtered case. Assume $\mathcal{I}$ is filtered. Then one shows that for every $i, j \in \mathcal{I}$ the composition $ev_j \circ \text{ins}_i : \mathcal{C}_i \to \mathcal{C} \overset{\sim}{\to} \widehat{\mathcal{C}} \to \mathcal{C}_j$ can be written as a colimit

$$ev_j \circ \text{ins}_i \simeq \text{colim}_{\alpha:i\to k,\beta:j\to k}\, \phi_\beta \circ \psi_\alpha.$$

This gives another description of the equivalence $\mathcal{C} \overset{\sim}{\to} \widehat{\mathcal{C}}$ in this case.

Corollary 4.1.5. *For every object $c \in \mathcal{C}$, the assignment $i \mapsto \text{ins}_i \circ ev_i(c) \in \mathcal{C}$ gives rise to the functor $\mathcal{I} \to \mathcal{C}$, and the canonical map*

$$\text{colim}_{i\in\mathcal{I}}\, \text{ins}_i \circ ev_i(c) \to c \tag{4.1}$$

is an isomorphism.

Proof. Though the assertion is standard among specialists (compare [Ga, 0.8.3]), we sketch the argument for the convenience of the reader. Since $\mathcal{C} \simeq \lim_{i\in\mathcal{I}} \mathcal{C}_i$, for every $d \in \mathcal{C}$, we have a natural equivalence from the mapping space $\text{map}_{\mathcal{C}}(c, d)$ to

$$\lim_{i\in\mathcal{I}} \text{map}_{\mathcal{C}_i}(ev_i(c), ev_i(d)) \simeq \lim_{i\in\mathcal{I}} \text{map}_{\mathcal{C}}(\text{ins}_i \circ ev_i(c), d) \simeq \text{map}_{\mathcal{C}}(\text{colim}_{i\in\mathcal{I}}\, \text{ins}_i \circ ev_i(c), d),$$

the first of which follows from adjointness of ev_i and ins_i, and the second one by the definition of the colimit. The assertion now follows from Yoneda lemma. ☐

4.1.6. Compactly generated case. In the situation of 4.1.2(a), assume that each $\mathcal{C}_i$ is compactly generated, and denote by $\mathcal{C}_i^c \subset \mathcal{C}_i$ be the sub-category of compacts objects.

(a) Assume in addition that each ψ_α preserves compact objects. Then the functor Ψ defines a functor $\mathcal{I} \to \text{Cat}_{\text{st},\ell} : i \mapsto \mathcal{C}_i^c$, and we have a natural equivalence $\mathcal{C} \simeq \text{Ind}(\text{colim}_{i\in\mathcal{I}}\, \mathcal{C}_i^c)$ (compare 4.1.1(d)). In particular, $\mathcal{C}$ is compactly generated.

(b) Notice that assumption (a) is satisfied automatically in the situation of 4.1.2(b).

We finish this subsection by recalling a general result about existence of adjoints in a limit and colimit of categories.

4.1.7. Assumptions.

(a) Let Cat_ℓ be either $\text{Cat}_{\text{st},\ell}$ or $\text{PrCat}_{\text{st},\ell}$. Let $\mathcal{I}$ be a small category, and let $\mathcal{D}., \mathcal{C}.$ be two functors $\mathcal{I} \to \text{Cat}_\ell$. In particular, we are given categories $\mathcal{C}_i, \mathcal{D}_i \in \text{Cat}_\ell$ and functors $\mathcal{C}_\alpha : \mathcal{C}_i \to \mathcal{C}_j$ and $\mathcal{D}_\alpha : \mathcal{D}_i \to \mathcal{D}_j$ for every morphism $\alpha : i \to j$ in $\mathcal{I}$.

(b) Let $\Phi. : \mathcal{C}. \to \mathcal{D}.$ be a morphism in $\text{Funct}(\mathcal{I}, \text{Cat}_\ell)$. Then $\Phi.$ gives rise to

- a functor $\Phi_i : \mathcal{C}_i \to \mathcal{D}_i$ for every $i \in \mathcal{I}$ and
- an equivalence $\Phi_\alpha : \mathcal{D}_\alpha \circ \Phi_i \simeq \Phi_j \circ \mathcal{C}_\alpha$ for every morphism $\alpha : i \to j$ in $\mathcal{I}$.

(c) Assume that

- For every $i \in \mathcal{I}$ the morphism $\Phi_i : \mathcal{C}_i \to \mathcal{D}_i$ has a left adjoint Ψ_i.

- (Beck-Chevalley condition) For every morphism $\alpha : i \to j$ in $\mathcal{I}$ the base change morphism $BC_\alpha : \Psi_j \circ \mathcal{D}_\alpha \to \mathcal{C}_\alpha \circ \Psi_i$ obtained by adjointness from the counit map $\mathcal{D}_\alpha \to \mathcal{D}_\alpha \circ \Phi_i \circ \Psi_i \simeq \Phi_j \circ \mathcal{C}_\alpha \circ \Psi_i$ is an equivalence.

The following standard assertion will be central for what follows.

Proposition 4.1.8. *Assume that we are in the situation of 4.1.7.*

(a) The collection of Ψ_i and BC_α can be upgraded to morphisms of functors $\Psi. : \mathcal{D}. \to \mathcal{C}.$.

(b) The limit functor $\widehat{\Phi} = \lim_{i\in\mathcal{I}} \Phi_i : \lim_{i\in\mathcal{I}} \mathcal{C}_i \to \lim_{i\in\mathcal{I}} \mathcal{D}_i$ has a left adjoint $\widehat{\Psi}$, and the natural base change morphism

$$\Psi_i \circ ev_i^{\mathcal{D}} \to ev_i^{\mathcal{C}} \circ \widehat{\Psi} \tag{4.2}$$

is an equivalence for every $i \in \mathcal{I}$.

(c) Assume that $\mathcal{I}$ is filtered. Then the colimit functor $\Phi : \operatorname{colim}_{i\in\mathcal{I}} \mathcal{C}_i \to \operatorname{colim}_{i\in\mathcal{I}} \mathcal{D}_i$ has a left adjoint $\widehat{\Psi}$, and the natural base change morphism

$$\Psi \circ \operatorname{ins}_i^{\mathcal{D}} \to \operatorname{ins}_i^{\mathcal{C}} \circ \Psi_i \tag{4.3}$$

is an equivalence for every $i \in \mathcal{I}$.

4.1.9. Remarks. (a) One does not need the assumption that $\mathcal{I}$ is filtered in Proposition 4.1.8(c). However, in this case the notion of a colimit and the proof is much simpler and this is the only case, which is needed for this work.

(b) The notion of adjoint functors can be generalized to morphisms in an arbitrary $(\infty, 2)$-category ([GR]). One can show that in the situation of 4.1.7 morphism $\Phi.$ in the $(\infty, 2)$-category $\operatorname{Funct}(\mathcal{I}, \operatorname{Cat}_\ell)$ has a left adjoint $\Psi. : \mathcal{D}. \to \mathcal{C}.$ such that the base change morphism $\Psi_i \circ ev_i^{\mathcal{D}} \to ev_i^{\mathcal{C}} \circ \Psi.$ of functors $\mathcal{D}. \to \mathcal{C}_i$ is an equivalence for every $i \in \mathcal{I}$. Having this, to get Proposition 4.1.8 one has to observe that the functors $\lim : \operatorname{Funct}(\mathcal{I}, \operatorname{Cat}_\ell) \to \operatorname{Cat}_\ell$ and $\operatorname{colim} : \operatorname{Funct}(\mathcal{I}, \operatorname{Cat}_\ell) \to \operatorname{Cat}_\ell$ are functors of $(\infty, 2)$-categories.

4.2. Categories of ℓ-adic sheaves on qcqs schemes and algebraic spaces.

4.2.1. Sheaves on algebraic spaces of finite type. Let $\operatorname{AlgSp}_k^{ft}$ be the category of algebraic spaces of finite type over k.

(a) Recall that for every $X \in \operatorname{AlgSp}_k^{ft}$ we have a stable ∞-category $\mathcal{D}_c(X) := \mathcal{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ whose homotopy category $D_c(X)$ is $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ (compare [LZ1],[LZ2] or [GL]).

(b) Moreover, the correspondence $X \mapsto \mathcal{D}_c(X)$ naturally upgrades to a functor of ∞-categories $\mathcal{D}_c = \mathcal{D}_c^! : (\operatorname{AlgSp}_k^{ft})^{op} \to \operatorname{Cat}_{\mathrm{st},\ell}$, which associates to every map $f : X \to Y$ its !-pullback $f^! : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$. We also define functor $\mathcal{D} := \operatorname{Ind} \circ \mathcal{D}_c : (\operatorname{AlgSp}_k^{ft})^{op} \to \operatorname{PrCat}_{\mathrm{st},\ell}$.

(c) Note that for every morphism $f : X \to Y$ in AlgSp_k^{ft}, there exists a left adjoint $f_! : \mathcal{D}_c(X) \to \mathcal{D}_c(Y)$ of $f^!$. In addition, there is a left adjoint f^* of $f_!$, when f is proper, and a right adjoint f_* of $f^!$, when f is étale. Namely, we have to check the corresponding assertions for homotopy categories, which is standard.

4.2.2. Remark. Actually, functors f^* and f_* can be defined for all f (but not by adjunction), but we will not need this fact later.

Lemma 4.2.3. *Consider a Cartesian diagram of* AlgSp_k^{ft}

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle a}\big\downarrow & & {\scriptstyle b}\big\downarrow \\ X & \xrightarrow{f} & Y. \end{array} \tag{4.4}$$

(a) If b is étale, then the base change morphism $f^!b_* \to a_*\widetilde{f}^!$ is an isomorphism.

(b) If b is proper or f is uh-smooth, then the base change morphism $a_!\widetilde{f}^! \to f^!b_!$ is an isomorphism.

(c) If b is proper, and f is uh-smooth, then the base change morphism $a^*f^! \to \widetilde{f}^!b^*$, obtained from the isomorphism of (b), is an isomorphism.

Proof. Notice first that (a) and the first assertion of (b) follow from the proper base change. Next, assertions (b) and (c) for smooth f are standard, while when f is a universal homeomorphism they follow from the fact that f induces an equivalence of étale sites. It remains to show that if $g : Z \to X$ is an étale covering or a universal homeomorphism, then the assertion for $f \circ g$ implies that for f. As an illustration, let us show (b). Since $g^!$ is faithful, it suffices to show that the map $g^!a_!\widetilde{f}^! \to g^!f^!b_!$ is an isomorphism. Therefore it suffices to prove the assertion for $f \circ g$ and g (see the argument of Proposition 4.2.7(a) below). Since the assertion for g was shown above, we are done. □

4.2.4. Sheaves on qcqs algebraic spaces. Let AlgSp_k be the category of quasi-compact and quasi-separated algebraic spaces over k.

By applying the left Kan extension to the functors $\mathcal{D}_c$ and $\mathcal{D}$ from 4.2.1(b), we get functors $\mathcal{D}_c : \mathrm{AlgSp}_k^{op} \to \mathrm{Cat}_{\mathrm{st},\ell}$ and $\mathcal{D} : \mathrm{AlgSp}_k^{op} \to \mathrm{PrCat}_{\mathrm{st},\ell}$. In particular, for every morphism $f : X \to Y$ in AlgSp_k we get functors $f^! : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$ and $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$.

4.2.5. Remarks. (a) By the explicit description of the left Kan extension, for every $X \in \mathrm{AlgSp}_k$ we have a natural equivalence $\mathcal{D}_c(X) \simeq \mathrm{colim}_{X \to Y} \mathcal{D}_c(Y)$, taken over category $(X/\mathrm{AlgSp}_k^{ft})^{op}$, whose objects are morphisms $X \to Y$ with $Y \in \mathrm{AlgSp}_k^{ft}$, and similarly, for $\mathcal{D}(X)$.

(b) Since AlgSp_k^{ft} has finite limits, we conclude that the category $(X/\mathrm{AlgSp}_k^{ft})^{op}$ is filtered. By 4.1.6, we thus have a natural equivalence $\mathcal{D}(X) \simeq \mathrm{Ind}(\mathcal{D}_c(X))$, thus $\mathcal{D}(X)$ is compactly generated.

(c) Recall that every $X \in \mathrm{AlgSp}_k$ can be written as a filtered limit $X \simeq \lim_i X_i$, where each $X_i \in \mathrm{AlgSp}_k^{ft}$, and all transition maps are affine. Then we have a natural equivalence $\mathcal{D}_c(X) \simeq \mathrm{colim}_i^! \mathcal{D}_c(X_i)$, and similarly for $\mathcal{D}(X)$.

(d) Since passage to homotopy categories and to Ind commute with filtered colimits, we have an equivalence of homotopy categories $D_c(X) \simeq \mathrm{colim}_i^! D_c(X_i)$, and similarly for $\mathcal{D}$.

(e) Recall that if $f : X' \to X$ is a finitely presented morphism, then for every presentation $X \simeq \lim_i X_i$ as in (c), there exists an index i, a finitely presented map $f_i : X'_i \to X_i$ and an isomorphism $X' \tilde{\to} X \times_{X_i} X'_i$. Then X' can be written as a limit $X' \simeq \lim_{j \geq i} X'_j$ with $X'_j := X'_i \times_{X_i} X_j$, thus we have a canonical equivalence $\mathcal{D}_c(X') \simeq \mathrm{colim}_{j \geq i}^! \mathcal{D}_c(X'_j)$.

(f) By definition, for every morphism of qcqs algebraic spaces $f : X' \to X$ we have a !-pullback functor $f^! : \mathcal{D}_c(X) \to \mathcal{D}_c(X')$, but the other three functors $f^*, f_!, f_*$ are not defined in general. The following proposition asserts that more functors are defined by adjointness in some cases.

Proposition 4.2.6. *Let $f : X' \to X$ be a finitely presented morphism in* AlgSp_k.

(a) Assume that either f is proper or X is globally uh-placid. Then $f^! : \mathcal{D}_c(X) \to \mathcal{D}_c(X')$ has a left adjoint $f_!$.

(b) Assume that f is étale. Then $f^!$ has a right adjoint f_*.

(c) Assume that f is proper and X is globally uh-placid. Then the functor $f_!$ has a left adjoint f^*.

Proof. As in 4.2.5(e), we can choose presentations $X \simeq \lim_{i \in \mathcal{I}} X_i$ and $X' \simeq \lim_{i \in \mathcal{I}} X'_i$. Moreover, by the standard limit arguments (see [Ry2, Prop B3] and references within), we can assume that each projection $f_i : X'_i \to X_i$ is proper (resp. étale), if f is such, and the transition maps $\pi_{j,i} : X_j \to X_i$ are uh-smooth, if X is globally uh-placid. Since $\mathcal{D}_c(X) \simeq \mathrm{colim}_i^! \mathcal{D}_c(X_i)$ and $\mathcal{D}_c(X') \simeq \mathrm{colim}_i^! \mathcal{D}_c(X'_i)$, all assertions will be deduced from Proposition 4.1.8(c). Since adjoints $(h_i)_!$ are known to exist in for finite type algebraic spaces (see Lemma 4.2.3), we will only have to show that the Beck-Chevalley condition in 4.1.7(c) is satisfied. Consider the Cartesian diagram

$$\begin{array}{ccc} X'_j & \xrightarrow{f_j} & X_j \\ {\scriptstyle \mathrm{pr}'_{j,i}} \downarrow & & {\scriptstyle \mathrm{pr}_{j,i}} \downarrow \\ X'_i & \xrightarrow{f_i} & X_i \end{array} \qquad (4.5)$$

(a) We want to apply Proposition 4.1.8(c) to the morphism $\Phi_\cdot = f_\cdot^!$ of functors $\mathcal{I}^{op} \to \mathrm{Cat}_{\mathrm{st},\ell} : i \mapsto \mathcal{D}_c(X_i), (i \to j) \mapsto \mathrm{pr}_{j,i}^!$. We have to show that the base change morphism $(f_j)_! \,\mathrm{pr}'^!_{j,i} \to \mathrm{pr}'^!_{j,i}(f_i)_!$ is an isomorphism, when f_i is proper or $\mathrm{pr}_{j,i}$ is uh-smooth. This follows from Lemma 4.2.3(b).

(b) Now we want to apply Proposition 4.1.8(c) to the morphism $\Phi_\cdot = f_\cdot^!$ of functors $\mathcal{I}^{op} \to \mathrm{Cat}_{\mathrm{st},\ell} : i \mapsto \mathcal{D}_c(X_i)^{op}, (i \to j) \mapsto \mathrm{pr}_{j,i}^!$. The assumptions of 4.1.7(c) are satisfied since the base change $\mathrm{pr}_{j,i}^!(f_i)_* \to (f_j)_* \,\mathrm{pr}'^!_{j,i}$ is an isomorphism, when f_i is étale (see Lemma 4.2.3(a)).

(c) We want to apply Proposition 4.1.8(c) to the morphism $\Phi_\cdot = (f_\cdot)_!$ of functors $\mathcal{I}^{op} \to \mathrm{Cat}_{\mathrm{st},\ell} : i \mapsto \mathcal{D}_c(X_i), (i \to j) \mapsto \mathrm{pr}_{j,i}^!$. This follows from the fact that the base change map $f_j^* \,\mathrm{pr}_{j,i}^! \to \mathrm{pr}'^!_{j,i}\, f_i^*$ is an isomorphism, when $\mathrm{pr}_{j,i}$ is uh-smooth (by Lemma 4.2.3(c)). ☐

The adjoint maps from Proposition 4.2.6 satisfy the following base change formulas.

Proposition 4.2.7. *Consider Cartesian diagram of qcqs algebraic spaces (4.4) such that b is finitely presented.*

(a) If b is étale, then the base change morphism $f^! b_* \to a_* \widetilde{f}^!$ is an isomorphism.

(b) If b is proper, then the base change morphism $a_! \widetilde{f}^! \to f^! b_!$ is an isomorphism.

(c) If Y is globally uh-placid and f is strongly pro-uh-smooth, then the base change morphism $a_! \widetilde{f}^! \to f^! b_!$ is an isomorphism.

(d) If b is proper, Y is globally uh-placid and f is strongly pro-uh-smooth, then the base change morphism $b^* \widetilde{f}^! \to f^! a^*$, induced from the isomorphism of (b), is an isomorphism.

Proof. (a) We want to show that the map $f^! b_*(K) \to a_* \widetilde{f}^!(K)$ is an isomorphism for every $K \in \mathcal{D}_c(\widetilde{Y})$. Assume first that Y and $\widetilde{Y}$ are of finite type. Then we can assume that $Y = X_{i_0}$ for some presentation $X \simeq \lim X_i$ of X. Then $\widetilde{Y} = \widetilde{X}_{i_0}$, where $\widetilde{X}_i := X_i \times_Y \widetilde{Y}$, so our assertion follows from Proposition 4.1.8(c), because our base change is simply the map (4.3).

In the general case, choose presentations $Y \simeq \lim_i Y_i$ and $\widetilde{Y} \simeq \lim_i \widetilde{Y}_i$ is 4.2.4(e) and choose i_0 such that K is a pullback of some $K_0 \in \mathcal{D}_c(\widetilde{Y}_{i_0})$. Then the assertion for K follows from the assertion for K_0 applied to the right and the exterior square of the Cartesian diagram

$$\begin{array}{ccccc}
\widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} & \xrightarrow{\widetilde{p}_{i_0}} & \widetilde{Y}_{i_0} \\
\Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle b_{i_0}} \\
X & \xrightarrow{f} & Y & \xrightarrow{p_{i_0}} & Y_{i_0}.
\end{array}$$

Namely, we have to show that the morphism $f^!b_*\widetilde{p}^!_{i_0}(K_0) \to a_*\widetilde{f}^!\widetilde{p}^!_{i_0}(K_0)$ is an isomorphism. But for this suffices to show that in the composition

$$f^!p^!_{i_0}(b_{i_0})_*(K_0) \to f^!b_*\widetilde{p}^!_{i_0}(K_0) \to a_*\widetilde{f}^!\widetilde{p}^!_{i_0}(K_0)$$

the first map and the composition are isomorphisms. In other words, we have to show the assertion for $p_{i_0} : Y \to X_{i_0}$ and $p_{i_0} \circ f : X \to Y_{i_0}$ instead of f.

(b)-(d) The proofs of (b)-(d) are essentially identical to that of (a), except that in the case when Y a globally uh-placid we only consider presentations $Y \simeq \lim_i Y_i$, where all transition maps $Y_j \to Y_i$ are uh-smooth.

□

4.2.8. Sheaf property. The functors $\mathcal{D}_c : \mathrm{AlgSp}_k^{op} \to \mathrm{Cat}_{\mathrm{st},\ell}$ and $\mathcal{D} : \mathrm{AlgSp}_k^{op} \to \mathrm{PrCat}_{\mathrm{st},\ell}$ are "sheaves" in the étale topology (and even for h-topology see, for example, [RS] or [Va]). In other words, for every fp-étale covering $\pi : X \to Y$ in AlgSp_k, the induced map $\mathcal{D}_c(Y) \to \lim_{[m]} \mathcal{D}_c(X^{[m]})$ is an equivalence, and similarly for $\mathcal{D}$.

For convenience of the reader, we will sketch the argument. When π has a section, the assertion is standard. In the general case, we show first that the pullback $\underline{\pi}^! : \mathcal{D}_c(Y) \to \lim_{[m]} \mathcal{D}_c(X^{[m]})$ has a right adjoint $\underline{\pi}_*$. For this we for every m, let π_m is the projection $X^{[m]} \to Y$. Then π_m is fp-étale, so $\pi_m^!$ has a right adjoint $(\pi_m)_*$. Therefore $\underline{\pi}^!$ has a right adjoint $\underline{\pi}_*$, which sends $\underline{K} := \{K_m\}_m \in \lim_{[m]} \mathcal{D}_c(X^{[m]})$ to $\lim_m(\pi_m)_*(K_m)$.

Next, we claim that the unit $K \to \underline{\pi}_*\underline{\pi}^!K$ is an isomorphism, that is, the map $K \to \lim_{[m]}(\pi_m)_*\pi_m^!(K)$ is an isomorphism. Since $\pi^!$ is faithful, it suffices to check the isomorphism after we apply $\pi^!$. Since $\pi^!$ commutes with limits (because it has a left adjoint $\pi_!$), and with $(\pi_m)_*$ (by Proposition 4.2.7(a)), we reduce to the corresponding assertion for the projection $X \times_Y X \to X$. Since it has a section (the diagonal $X \to X \times_Y X$), we are done.

Finally, we claim that the counit maps $\underline{\pi}^!\underline{\pi}_*(\underline{K}) \to \underline{K}$ is an isomorphism. It suffices to show that the map $\pi^!(\lim_m(\pi_m)_*(K_m)) \to K_0$ is an isomorphism. As above, the assertion follows from the commutativity of $\pi^!$ commutes with limits and $(\pi_m)_*$.

4.3. Sheaves on ∞-(pre)stacks.

4.3.1. Construction. (a) Applying the right Kan extension to the functors $\mathcal{D}_c$ and $\mathcal{D}$ from 4.2.4, we get functors

$$\mathcal{D}_c : \mathrm{PShv}(\mathrm{AlgSp}_k)^{op} \to \mathrm{Cat}_{\mathrm{st},\ell} \text{ and } \mathcal{D} : \mathrm{PShv}(\mathrm{AlgSp}_k)^{op} \to \mathrm{PrCat}_{\mathrm{st},\ell}\,.$$

Moreover, using sheaf property 4.2.8, these functors factor through the category $\mathrm{Shv}(\mathrm{AlgSp}_k)$ of sheaves in the étale topology.

(b) Notice that the inclusion $\iota : \mathrm{Aff}_k \hookrightarrow \mathrm{AlgSp}_k$ gives rise to the commutative diagram of categories

$$\begin{array}{ccccccc}
\mathrm{Aff}_k & \longrightarrow & \mathrm{St}_k & \longrightarrow & \mathrm{PreSt}_k & \longrightarrow & \mathrm{St}_k \\
{\scriptstyle\iota}\big\downarrow & & {\scriptstyle\iota^*}\big\uparrow & & & & {\scriptstyle\iota^*}\big\uparrow \\
\mathrm{AlgSp}_k & \longrightarrow & \mathrm{Shv}(\mathrm{AlgSp}_k) & \longrightarrow & \mathrm{PShv}(\mathrm{AlgSp}_k) & \longrightarrow & \mathrm{Shv}(\mathrm{AlgSp}_k),
\end{array}$$

Moreover, it is a standard fact that the restriction functor $\iota^* : \mathrm{Shv}(\mathrm{AlgSp}_k) \to \mathrm{Shv}(\mathrm{Aff}_k) = \mathrm{St}_k$ is an equivalence of categories. Therefore the functors $\mathcal{D}_.$ from (a) can be viewed as functors from St_k. Precomposing this with the projection $\mathrm{PreSt}_k \to \mathrm{St}_k$, we can view them as functors $\mathcal{D}_c : \mathrm{PreSt}_k^{op} \to \mathrm{Cat}_{\mathrm{st},\ell}$ and $\mathcal{D} : \mathrm{PreSt}_k^{op} \to \mathrm{PrCat}_{\mathrm{st},\ell}$.

4.3.2. Properties. (a) By the Yoneda lemma, for every $\mathcal{X} \in \mathrm{PShv}(\mathrm{AlgSp}_k)$ the natural morphism $\mathrm{colim}_{X\to\mathcal{X}} X \to \mathcal{X}$, where the colimit is taken over all morphism $X \to \mathcal{X}$, where $X \in \mathrm{AlgSp}_k$, is an equivalence. Therefore the natural functors $\mathcal{D}_.(\mathcal{X}) \to \lim_{X\to\mathcal{X}} \mathcal{D}_.(X)$ is an equivalence. In particular, functors $\mathcal{D}_c : \mathrm{PShv}(\mathrm{AlgSp}_k)^{op} \to \mathrm{Cat}_{\mathrm{st},\ell}$ and $\mathcal{D} : \mathrm{PShv}(\mathrm{AlgSp}_k)^{op} \to \mathrm{PrCat}_{\mathrm{st},\ell}$ preserve limits.

(b) We claim that the induced functors $\mathcal{D}_c : \mathrm{St}_k^{op} \simeq \mathrm{Shv}(\mathrm{AlgSp}_k)^{op} \to \mathrm{Cat}_{\mathrm{st},\ell}$ and $\mathcal{D} : \mathrm{St}_k^{op} \simeq \mathrm{Shv}(\mathrm{AlgSp}_k)^{op} \to \mathrm{PrCat}_{\mathrm{st},\ell}$ preserve limits. Indeed, we want to show that if $\mathcal{X} \simeq \mathrm{colim}_{\mathrm{Shv}(\mathrm{AlgSp}_k)} \mathcal{X}_\alpha$, then the natural map $\mathcal{D}_.(\mathcal{X}) \to \lim \mathcal{D}_.(\mathcal{X}_\alpha)$ is an equivalence. Set $\mathcal{X}' := \mathrm{colim}_{\mathrm{PShv}(\mathrm{AlgSp}_k)} \mathcal{X}_\alpha$. Then $\mathcal{X}$ is the sheafification of $\mathcal{X}'$, thus the natural map $\mathcal{D}_.(\mathcal{X}) \to \mathcal{D}_.(\mathcal{X}')$ is an equivalence (by 4.2.8). Since the natural map $\mathcal{D}_.(\mathcal{X}') \to \lim \mathcal{D}_.(\mathcal{X}_\alpha)$ is an equivalence (by (a)), the assertion follows.

(c) Since the map $\mathrm{PreSt}_k^{op} \to \mathrm{St}_k^{op}$ is limit preserving, we obtain from (b) that the functors $\mathcal{D}_c : \mathrm{PreSt}_k^{op} \to \mathrm{Cat}_{\mathrm{st},\ell}$ and $\mathcal{D} : \mathrm{PreSt}_k^{op} \to \mathrm{PrCat}_{\mathrm{st},\ell}$ are limits preserving. Therefore arguing as in (a) one can show that they are equivalent to the right Kan extension of their restriction to Aff_k^{op}.

(d) Let $f : \mathcal{X} \to \mathcal{Y}$ be a surjective morphism of ∞-stacks (that is, it has sections locally for étale topology). Then $\mathcal{Y}$ is the colimit of the Čech-complex with terms $\mathcal{X}^{[m]}$ (see 1.1.8(c)). Hence we conlude by (b) $\mathcal{D}_.(\mathcal{Y})$ is the limit of the corresponding co-bar complex with terms $\mathcal{D}_c(\mathcal{X}^{[m]})$. In particular, the pullback $f^! : \mathcal{D}_.(\mathcal{Y}) \to \mathcal{D}_.(\mathcal{X})$ is faithful.

4.3.3. Remark. Notice that the inclusion $\mathcal{D}_c(\mathcal{X}) \hookrightarrow \mathcal{D}(\mathcal{X})$ induces a functor $\mathrm{Ind}(\mathcal{D}_c(\mathcal{X})) \to \mathcal{D}(\mathcal{X})$, which is an equivalence, when $\mathcal{X} \in \mathrm{AlgSp}_k$, but not in general.

4.3.4. Ind-algebraic spaces. (a) We call an ∞-stack X an *ind-algebraic space/ind-scheme*, if X can be written as a filtered colimit $X \simeq \mathrm{colim}_\alpha X_\alpha$ of qcqs algebraic spaces/schemes, where all of the transition maps are fp-closed embeddings. By definition, we have a canonical equivalence $\mathcal{D}(X) \simeq \lim^!_\alpha \mathcal{D}(X_\alpha)$.

(b) Recall that for every fp-closed embedding $i : X_\alpha \to X_\beta$ in AlgSp_k the functor $i^!$ has a left adjoint $i_!$ (see Proposition 4.2.6(a)). Then it follows from Theorem 4.1.3 that we have a natural equivalence $\mathcal{D}(X) \simeq \mathrm{colim}^!_\alpha \mathcal{D}(X_\alpha)$

(c) It follows from (b) and 4.1.6 that $\mathcal{D}(X)$ is compactly generated, and we have a natural equivalence $\mathcal{D}(X) \simeq \mathrm{Ind}(\mathrm{colim}^!_\alpha \mathcal{D}_c(X_\alpha))$.

4.3.5. Remark. Note that in the situation of 4.3.4 we have a fully faithful morphism $\mathrm{colim}^!_\alpha \mathcal{D}_c(X_\alpha) \hookrightarrow \mathcal{D}_c(X)$, which is not an equivalence. In particular, we have natural functors $\mathrm{Ind}(\mathrm{colim}^!_\alpha \mathcal{D}_c(X_\alpha)) \hookrightarrow \mathrm{Ind}(\mathcal{D}_c(X)) \to \mathcal{D}(X)$, the first of which is fully faithful, the second one is essentially surjective, and the composition is an equivalence.

We finish this subsection by assertion that topological equivalences do not change categories of sheaves.

Proposition 4.3.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ a topological equivalence between ∞-stacks. Then the induced maps $f^! : \mathcal{D}_c(\mathcal{Y}) \to \mathcal{D}_c(\mathcal{X})$ and $f^! : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$ are equivalences.*

Proof. Since f is a topological equivalence, the induced map $f_{\mathrm{perf}} : \mathcal{X}_{\mathrm{perf}} \to \mathcal{Y}_{\mathrm{perf}}$ is an equivalence (by Lemma 2.3.6). Thus the proposition is an immediate corollary Lemma 4.3.7 below. □

Lemma 4.3.7. *For every ∞-stack $\mathcal{X}$ the canonical functors $\pi^! : \mathcal{D}_c(\mathcal{X}) \to \mathcal{D}_c(\mathcal{X}_{\mathrm{perf}})$ and $\pi^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}_{\mathrm{perf}})$, induced by the projection $\pi : \mathcal{X}_{\mathrm{perf}} \to \mathcal{X}$, are equivalences.*

Proof. We will write $\mathcal{D}_.$ to treat both $\mathcal{D}$ and $\mathcal{D}_c$. Since $\mathcal{X}$ as a colimit of affine schemes $\mathcal{X} \simeq \mathrm{colim}\, U$, we have $\mathcal{D}_.(\mathcal{X}) \simeq \lim \mathcal{D}_.(U)$. Since $\iota_! \iota^* : \mathcal{X} \mapsto \mathcal{X}_{\mathrm{perf}}$ preserves colimits, we get equivalence $\mathcal{X}_{\mathrm{perf}} \simeq \mathrm{colim}\, U_{\mathrm{perf}}$, hence $\mathcal{D}_.(\mathcal{X}_{\mathrm{perf}}) \simeq \lim \mathcal{D}_.(U_{\mathrm{perf}})$. Therefore it suffices to show the induced map $\mathcal{D}_.(U) \to \mathcal{D}_.(U_{\mathrm{perf}})$ is an equivalence. Since both U and U_{perf} are affine schemes, in this case we have $\mathcal{D} \simeq \mathrm{Ind}\, \mathcal{D}_c$, so the assertion for $\mathcal{D}$ follows from that for $\mathcal{D}_c$.

Since $\pi : U_{\mathrm{perf}} \to U$ is a universal homeomorphism, U_{perf} has a presentation as filtered limit $U_{\mathrm{perf}} \simeq \lim_{U'} U'$, where each $U' \to U$ is an fp-universal homeomorphism (see [St, Tag 0EUJ]). Then $\mathcal{D}_c(U_{\mathrm{perf}})$ is filtered colimit $\mathcal{D}_c(U_{\mathrm{perf}}) \simeq \mathrm{colim}_{U'} \mathcal{D}_c(U')$, so it suffices to show that each $\mathcal{D}_c(U) \to \mathcal{D}_c(U')$ is an equivalence.

Note that every fp-universal homeomorphism $U' \to U$ comes from a universal homeomorphism between finite type schemes $U'_0 \to U_0$ by [EGAIV, 8.10.5]. Writing U as a limit $U \simeq \lim U_i$ over U_0, we get that $U' \simeq \lim_i U'_i$ with $U'_i = U_i \times_{U_0} U'_0$. Thus it suffices to show that each functor $\pi^! : \mathcal{D}_c(U_i) \to \mathcal{D}_c(U'_i)$ is an equivalence. Since $U'_i \to U_i$ is a universal homeomorphism between finite type affine schemes, the assertion follows from the fact that π induces an equivalence between étale sites on U'_i and U_i. □

Since $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is a topological equivalence (see 2.3.1(a)), we get the following corollary.

Corollary 4.3.8. *For every ∞-stack $\mathcal{X}$ the canonical functors $\pi^! : \mathcal{D}_c(\mathcal{X}) \to \mathcal{D}_c(\mathcal{X}_{\mathrm{red}})$ and $\pi^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}_{\mathrm{red}})$, induced by the projection $\pi : \mathcal{X}_{\mathrm{red}} \to \mathcal{X}$, are equivalences.*

4.4. **Base changes.**

Definition 4.4.1. (a) We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ from ind-algebraic space $\mathcal{X}$ to an affine scheme Y *ind-fp-proper*, if $\mathcal{X}$ has a presentation a filtered colimit $\mathcal{X} \simeq \operatorname{colim}_\alpha X_\alpha$ such that each X_α is fp-proper over Y.

(b) We say that a morphism $f : \mathcal{X} \to Y$ from an ∞-stack $\mathcal{X}$ to an affine scheme Y *locally ind-fp-proper*, if there exists an étale covering $Y' \to Y$ such that the base change $f \times_Y Y' : \mathcal{X} \times_Y Y' \to Y'$ is ind-fp-proper.

(c) Notice that classes of morphisms in (a) and (b) are stable under all pullbacks, therefore construction of 2.1.6(b) applies. In particular, we can talk about locally ind-fp-proper morphisms of ∞-stacks.

4.4.2. Example. Let $f : \mathcal{X} \to \mathcal{Y}$ be a locally ind-fp-proper morphism between ∞-stacks, which is equivariant with respect to an action of an ∞-groups stack H. Then the induced morphism $\overline{f} : [\mathcal{X}/H] \to [\mathcal{Y}/H]$ is locally ind-fp-proper.

Proof. Indeed, let $Y \to [\mathcal{Y}/H]$ be any morphism with Y affine. By definition, there exists an étale covering $Y' \to Y$ such that the composition $Y' \to Y \to [\mathcal{Y}/H]$ lifts to a morphism $Y' \to \mathcal{Y}$. Thus it suffices to show that the pullback $\overline{f} \times_{[\mathcal{Y}/H]} \mathcal{Y}$ is locally ind-fp-proper. Since as in the classical case, we have an isomorphism $\overline{f} \times_{[\mathcal{Y}/H]} \mathcal{Y} \simeq f$ (use 9.2.2), the assertion follows. ☐

Proposition 4.4.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a locally ind-fp-proper morphism of ∞-stacks. Then the pullback $f^!$ has a left adjoint $f_!$, satisfying base change. More precisely, for every Cartesian diagram of prestacks*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \xrightarrow{g} & \mathcal{X} \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{\mathcal{Y}} & \xrightarrow{h} & \mathcal{Y} \end{array}$$

the base change map

$$\widetilde{f}_! \widetilde{g}^! \to h^! f_! \tag{4.6}$$

is an isomorphism.

Proof. Our argument is almost identical to the one outlined in [Ga, Prop. 1.5.2].

Step 1. It is enough to show the assertion when $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are affine schemes.

Proof. Write $\mathcal{Y}$ as a colimit $\mathcal{Y} \simeq \operatorname{colim} U$ of affine schemes. It induces a presentation $\mathcal{X} \simeq \mathcal{X} \times_{\mathcal{Y}} U$, and every $f_U : \mathcal{X} \times_{\mathcal{Y}} U \to U$ is locally ind-fp-proper. If every $f_U^!$ has a left adjoint $(f_U)_!$, satisfying base change, then Proposition 4.1.8(b) implies that the left adjoint of $f^!$ exists and satisfies the base change for morphisms $U \to \mathcal{Y}$ with U affine.

To see the base change in general, notice that since $\mathcal{D}(\widetilde{\mathcal{Y}}) \simeq \lim \mathcal{D}(U)$ taken over all morphisms $U \to \widetilde{\mathcal{Y}}$ with affine U. Therefore it suffices to show that for every morphism $\alpha : U \to \widetilde{\mathcal{Y}}$ the base change morphism $\alpha^! \tilde{f}_! \tilde{g}^! \to \alpha^! h^! f_!$. Arguing as in Proposition 4.2.7(a), it thus suffices to show the base change for the morphisms $\alpha : U \to \widetilde{\mathcal{Y}}$ and $h \circ \alpha : U \to \mathcal{Y}$, shown above. □

Step 2. The assertion holds, if f is fp-proper.

Proof. Arguing as in Step 1, one reduces the assertion to the case when $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are affine. In this case, the existence of $f_!$ was shown in Proposition 4.2.6(a), and the base change property was shown in Proposition 4.2.7(b). □

Step 3. The assertion holds, if $\mathcal{Y} \simeq \operatorname{colim}_\alpha Y_\alpha$ is an ind-algebraic space, and f is the inclusion $f = i_\alpha : Y_\alpha \to \mathcal{Y}$.

Proof. Since i_α is fp-proper, the assertion follows from Step 2. □

Step 4. The assertion holds when $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are affine and f is ind-fp-proper.

Proof. Choose a presentation $\mathcal{X} \simeq \operatorname{colim} X_\alpha$ of $\mathcal{X}$ over $\mathcal{Y}$, let $i_\alpha : X_\alpha \hookrightarrow X$ be the embedding, and set $f_\alpha := f \circ i_\alpha : X_\alpha \to \mathcal{Y}$. By Step 3, the adjoint $(i_\alpha)_!$ exists and satisfies base change.

By the adjoint function theorem [Lu1], to show the existence of $f_!$ it suffices to show that $f^!$ preserves all small limits. Since $\mathcal{D}(\mathcal{X}) \simeq \lim_\alpha \mathcal{D}(X_\alpha)$ and $i_\alpha^!$ preserves all limits by Step 3, it suffices to show that the composition $f_\alpha^! = i_\alpha^! \circ f^! : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(X_\alpha)$ preserves all small limits. Since f_α is fp-proper, the pullback $f_\alpha^!$ has a left adjoint $(f_\alpha)_!$ by Proposition 4.2.6(a). Therefore $f_\alpha^!$ preserves all small limits, and the existence of $f_!$ follows.

Recall (see Corollary 4.1.5) that for every $K \in \mathcal{D}(X)$ we have a canonical isomorphism $\operatorname{colim}_\alpha (i_\alpha)_! i_\alpha^! K \to K$. Since all functors in (4.6) preserves small colimits, it suffices to check that the induced map $\tilde{f}_! \tilde{g}^! (i_\alpha)_! \to h^! f_! (i_\alpha)_!$ is an isomorphism. As in the proof of Proposition 4.2.7(a) it suffices to show that $(i_\alpha)_!$ and $(f_\alpha)_!$ satisfy base change. Since f_α are fp-proper, the assertion follows from Step 2 and 3. □

Step 5. Completion of the proof.

By Step 1, we can assume that $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are affine. Choose an étale covering $\pi : U \to \mathcal{Y}$ such that the base change $f \times_{\mathcal{Y}} U : \mathcal{X} \times_{\mathcal{Y}} U \to U$ is ind-fp-proper. Then $\mathcal{D}(\mathcal{Y})$ is a limit $\lim \mathcal{D}(U^{[m]})$, where $U^{[m]}$ is the Čech-complex, corresponding to the covering $U \to \mathcal{Y}$ (see 4.3.2(d)), and also $\mathcal{D}(\mathcal{X}) \simeq \lim \mathcal{D}(\mathcal{X}^{[m]})$, where $\mathcal{X}^{[m]} := \mathcal{X} \times_{\mathcal{Y}} U^{[m]}$. Since every induced morphism $\mathcal{X}^{[m]} \to U^{[m]}$ is ind-fp-proper, we conclude from Step 4 and Proposition 4.1.8(b) (as in Step 1) that $f^!$ has a left adjoint, which satisfies base change for the morphism π.

To show base change in general, we set $\widetilde{U} := \widetilde{\mathcal{Y}} \times_{\mathcal{Y}} U$. Arguing as in Step 1, it suffices to show the base change with respect to the morphisms $\widetilde{\pi} : \widetilde{U} \to \widetilde{\mathcal{Y}}$ and $\widetilde{U} \to \widetilde{\mathcal{Y}} \to \mathcal{Y}$, that is, $\widetilde{U} \to U \to \mathcal{Y}$ Since pullbacks of f to U (and hence also to $\widetilde{U}$) are ind-fp-proper, the base change for π and $\widetilde{\pi}$ was shown in the previous paragraph, while the base change for the morphism $\widetilde{U} \to U$ was shown in Step 4. □

4.4.4. Remark. Actually, as in [Ga] one can consider a more general notion of *pseudo-proper* morphisms, in which we do not require in Definition 4.4.1(a) that the colimit $\operatorname{colim}_{\alpha} X_{\alpha}$ is filtered and no restriction on the transition maps. The assertion Proposition 4.4.3 also hold for pseudo-proper morphisms as well. Namely, all steps in the argument except Step 3 work word-by-word. Though an analog of Step 3 is not difficult as well, one seems to need a slightly more general categorical framework of $(\infty, 2)$-categories to give an honest proof of it.

Proposition 4.4.5. *(a) Let $\mathcal{X}$ by a topological n-placid ∞-stack, and let $h : \mathcal{X}' \to \mathcal{X}$ be an fp-representable morphism.*

Then there exists a left adjoint $h_! : \mathcal{D}(\mathcal{X}') \to \mathcal{D}(\mathcal{X})$ of $h^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}')$. Moreover, if in addition h is proper, then there exists a left adjoint $h^* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}')$ of $h_!$.

(b) Let $f : \mathcal{Y} \to \mathcal{X}$ be a topologically n-smooth morphism, let $\widetilde{h} : \mathcal{Y}' \to \mathcal{Y}$ and $f' : \mathcal{Y}' \to \mathcal{X}'$ be the pullbacks of h and f, respectively.

Then the base change morphism $\widetilde{h}_! f'^! \to f^! h_!$ is an isomorphism. Moreover, if h is fp-proper, then the induced base change morphism $\widetilde{h}^* f^! \to f'^! h^*$ is an isomorphism as well.

Proof. The proof goes by induction on n.

(a)$_0$. If $n = 0$, then $\mathcal{X}$ decomposes as a coproduct $\mathcal{X} \simeq \sqcup_{\alpha} X_{\alpha}$ of globally uh-placid schemes X_{α}, and $\mathcal{X}'$ decomposes as a $\mathcal{X}' \simeq \sqcup_{\alpha} X'_{\alpha}$. Thus we are reduced to the case when $\mathcal{X} = X_{\alpha}$ is a globally uh-placid affine scheme. In this case the assertion was shown in Proposition 4.2.6(a),(c).

(b)$_0$. Arguing as in (a)$_0$, we reduce to the case of when $f : Y \to X$ is strongly pro-uh-smooth morphism between globally uh-placid algebraic spaces. In this case, it is enough to show the $\mathcal{D}_c$ version instead of $\mathcal{D}$, and the assertion was shown in Proposition 4.2.7(c),(d).

Form now on, we will assume that assertions (a)$_n$ and (b)$_n$ are satisfied.

(a)$_{n+1}$ Choose a topologically n-smooth covering $p : X \to \mathcal{X}$ with topologically 0-placid X. Then, as in 1.1.8, the covering gives rise to the presentation $\mathcal{X} \simeq$

$\mathrm{colim}_{[m]} X^{[m]}$, where each $X^{[m]}$ an n-topologically placid ∞-stack, and all transition maps are n-topologically smooth. Moreover, h induces an fp-representable morphism $X'^{[m]} \to X^{[m]}$ for all $[m]$, which is proper if h is such. Since assumptions (a)$_n$ and (b)$_n$ hold by the induction hypothesis, the assumptions of Proposition 4.1.8(b) are satisfied. Therefore there exists an adjoint $h_!$ of $h^!$ (and also h^* of $h_!$, if h is proper), which satisfies the base change with respect to p.

(b)$_{n+1}$ Assume first that there exists a topologically n-smooth covering $p : Y \to \mathcal{Y}$ such that the composition $f \circ p : Y \to \mathcal{X}$ is topologically n-smooth, and let $p' : Y' \to \mathcal{Y}'$ and $h_Y : Y' \to Y$ be its base changes. Notice, that this assumption is satisfied automatically, if f is topologically n-smooth.

Since $p^!$ is faithful, to show that $\widetilde{h}_! f'^! \to f^! h_!$ is an isomorphism, it suffices to show that the pullback $p^! \widetilde{h}_! f'^! \to p^! f^! h_!$ is an isomorphism. Since we have seen during the proof of (a)$_{n+1}$ that the base change $(h_Y)_! p'^! \to p^! \widetilde{h}_!$ is an isomorphism, it suffices to show that the base change $(h_Y)_!(f' \circ p')^! \to (f \circ p)^! h_!$ is an isomorphism. Since $f \circ p : Y \to \mathcal{X}$ is topologically n-smooth by assumption, it can be completed to a topologically n-smooth covering. So the first assertion follows from (a), while the proof of the second assertion is similar.

In the general case, choose a topologically n-smooth covering $p : X \to \mathcal{X}$ with topologically 0-placid X, and let $p_Y : \mathcal{Y} \times_{\mathcal{X}} X \to \mathcal{Y}$ and $f_X : \mathcal{Y} \times_{\mathcal{X}} X \to X$ be pullbacks. Then $p_Y^!$ is faithful, so it suffices to show the base change with respect to $p_Y^!$ and $p_Y^! \circ f^!$. Since p, hence also p_Y is topologically n-smooth, the assertion for $p_Y^!$ follows from the particular case, shown above. Next $p_Y^! \circ f^! \simeq f_X^! \circ p^!$, and the assertion for $p^!$ was shown in (a). Hence it remains to show the assertion for $f_X^!$. In other words, we can assume that $\mathcal{X}$ is topologically 0-placid.

Since f is topologically $(n+1)$-smooth, there exists an n-topologically smooth covering $p : Y \to \mathcal{Y}$ such that the composition $f \circ p : Y \to \mathcal{X}$ is n-topologically smooth. Thus the assertion follows from the proven above. □

4.5. (Fp) locally closed pushforwards.

4.5.1. Complementary ∞-substacks, and support.

(a) Let $\mathcal{X}$ be an ∞-stack, and let $\mathcal{Y} \subset \mathcal{X}$ be an ∞-substack, that is, $\mathcal{Y}$ is an ∞-stack, and that $\mathcal{Y}(U) \subset \mathcal{X}(U)$ is a *subspace*, that is, a union of connected components for every $U \in \mathrm{Aff}_k$.

(b) For every $U \in \mathrm{Aff}_k$, consider the subspace $(\mathcal{X} \smallsetminus \mathcal{Y})(U) \subset \mathcal{X}(U)$ consisting of all morphisms $a : U \to \mathcal{X}$ such that $U \times_{\mathcal{X}} \mathcal{Y} = \emptyset$. We claim that $\mathcal{X} \smallsetminus \mathcal{Y} \subset \mathcal{X}$ is an ∞-substack.

Indeed, to see that it is an ∞-prestack, notice that every morphism $V \to U$ in Aff_k induces a morphism $V \times_{\mathcal{X}} \mathcal{Y} \to U \times_{\mathcal{X}} \mathcal{Y}$. Therefore we have $V \times_{\mathcal{X}} \mathcal{Y} = \emptyset$, if

$U \times_{\mathcal{X}} \mathcal{Y} = \emptyset$. To see that it is an ∞-stack, notice that if $V \to U$ is an étale covering, then $V \times_{\mathcal{X}} \mathcal{Y} \to U \times_{\mathcal{X}} \mathcal{Y}$ is surjective. Therefore we have $U \times_{\mathcal{X}} \mathcal{Y} = \emptyset$, if $V \times_{\mathcal{X}} \mathcal{Y} = \emptyset$.

(c) By definition, for every morphism $f : \mathcal{X}' \to \mathcal{X}$ of ∞-stacks, we have a natural identification $(\mathcal{X} \smallsetminus \mathcal{Y}) \times_{\mathcal{X}} \mathcal{X}' \simeq \mathcal{X}' \smallsetminus (\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}')$.

(d) Notice that we always have an inclusion $\mathcal{Y} \subset \mathcal{X} \smallsetminus (\mathcal{X} \smallsetminus \mathcal{Y})$, but we don't have an equality in general.

4.5.2. The case of open and closed embeddings.

(a) Notice that if $\mathcal{X}$ is a scheme X and $\mathcal{U}$ is an open subscheme U, then the reduced complement $(\mathcal{X} \smallsetminus \mathcal{U})_{\mathrm{red}}$ (4.5.1(b)) is the reduced closed subsheme $(X \smallsetminus U)_{\mathrm{red}} \subset X$. Therefore it follows from 4.5.1(c) that if $\mathcal{U} \subset \mathcal{X}$ is a (fp)-open ∞-substack, then the complement $\mathcal{X} \smallsetminus \mathcal{U} \subset \mathcal{X}$ is a topologically (fp)-closed substack.

(b) Conversely, if $\mathcal{Z} \subset \mathcal{X}$ is a topologically (fp)-closed ∞-substack, then the complement $\mathcal{X} \smallsetminus \mathcal{Z} \subset \mathcal{X}$ is a complementary (fp)-open ∞-substack. Indeed, using 4.5.1(c), one reduces to the case when $\mathcal{X} = X$ is an scheme. In this case, $\mathcal{Z}_{\mathrm{red}} \subset X$ is a closed subscheme, and $\mathcal{X} \smallsetminus \mathcal{Z} = \mathcal{X} \smallsetminus \mathcal{Z}_{\mathrm{red}}$ is an (fp)-open subscheme.

(c) It follows from 4.5.1(c) and the scheme case that we always have an equality $\mathcal{U} = \mathcal{X} \smallsetminus (\mathcal{X} \smallsetminus \mathcal{U})$ when $\mathcal{U}$ is open, and $\mathcal{Z}_{\mathrm{red}} = (\mathcal{X} \smallsetminus (\mathcal{X} \smallsetminus \mathcal{U}))_{\mathrm{red}}$ when $\mathcal{Z} \subset \mathcal{X}$ is topologically closed.

Lemma 4.5.3. *Let be $j : \mathcal{U} \hookrightarrow \mathcal{X}$ an fp-open embedding with a complementary topologically fp-closed embedding $i : \mathcal{Z} \hookrightarrow \mathcal{X}$. Then*

(a) There exists a right adjoint j_* of $j^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{U})$, which preserves $\mathcal{D}_c$ and satisfies base change.

(b) There exists a left adjoint $i_!$ of $i^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Z})$, which preserves $\mathcal{D}_c$ and satisfies base change.

(c) Functors $i_!$ and j_* are fully faithful, and $j^! i_! \simeq 0$.

(d) For every $K \in \mathcal{D}(\mathcal{X})$, the unit and counit maps extend to a fibered sequence

$$i_! i^! K \to K \to j_* j^! K.$$

Proof. (a) A presentation $\mathcal{X} \simeq \operatorname{colim} X$ of $\mathcal{X}$ as a colimit of affine schemes, induces a presentation $\mathcal{U} \simeq \operatorname{colim}_X X_{\mathcal{U}}$, where $X_{\mathcal{U}} := X \times_{\mathcal{X}} \mathcal{U}$ is an fp-open subscheme of X. In particular, $j^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{U})$ is a limit $\lim_X j_X^! : \lim_X \mathcal{D}(X) \to \lim_X \mathcal{D}(X_{\mathcal{U}})$ and similarly for $\mathcal{D}_c$. Since the pullback $j_X^! : \mathcal{D}_c(X) \to \mathcal{D}_c(X_{\mathcal{U}})$ has a right adjoint (see Proposition 4.2.6(b)), which satisfies base change (see Proposition 4.2.7(a)) the existence of j_* follows from Proposition 4.1.8(b), applies to $\mathcal{D}^{op}_{\cdot}$ as in the proof of Proposition 4.2.6(b). To show the assertion about the base change, we argue as in Proposition 4.4.3.

(b) The argument is similar, except we use Corollary 4.3.8 and Proposition 4.2.6(a) and Proposition 4.2.7(b) instead. Notice that all assertions except the one about $\mathcal{D}_c$ can be easily deduced from Corollary 4.3.8 and Proposition 4.4.3.

(c) We have to show that the morphisms $\mathrm{Id} \to i^! i_!$ and $\mathrm{Id} \to j_* j^!$ are isomorphisms, and $j^! i_! \simeq 0$. Since all functors are defined as limits of the corresponding functors in the case of qcqs schemes, we immediately reduce to the case of qcqs schemes. In this case, $\mathcal{D} \simeq \operatorname{Ind} \mathcal{D}_c$, so we reduce to the case of $\mathcal{D}_c$. Next, using Corollary 4.3.8, we can assume that $i : \mathcal{Z} \to \mathcal{X}$ is fp-closed. In this case, all functors are colimits of the corresponding functors between schemes of finite type, hence we reduce to this case. In this case, the assertions are standard.

(d) Let K' be the fiber of the unit map $K \to j_* j^! K$. We have to show that the counit map $i_! i^! K \to K$ factors canonically as a composition $i_! i^! K \xrightarrow{\sim} K' \to K$.

Since $j^! i_! \simeq 0$, the composition $i_! i^! K \to K \to j_* j^! K$ is naturally equivalent to zero. Therefore the counit map $i_! i^! K \to K$ factors canonically as $i_! i^! K \to K' \to K$. It remains to show that $i_! i^! K \to K'$ is an equivalence.

Using equivalence $\mathcal{D}(\mathcal{X}) \simeq \lim_X \mathcal{D}(X)$ and observing that $i_! i^! K \to K'$ is a limit of the corresponding morphisms in $\mathcal{D}(X)$, we reduce the assertion to the case of schemes. Next, using Corollary 4.3.8, we can assume that i is finitely presented. Next, we reduce the assertion to $\mathcal{D}_c$ and observe that $i_! i^! K \to K'$ comes from a corresponding morphism for schemes of finite type. In this case the assertion is well-known. □

4.5.4. Sheaves with support. (a) Let $\mathcal{X}$ be an ∞-stack, let $\mathcal{Y} \subset \mathcal{X}$ be an ∞-substack, and let $\iota : \mathcal{X} \smallsetminus \mathcal{Y} \to \mathcal{X}$ be the inclusion. Let $\mathcal{D}_{\mathcal{Y}}(\mathcal{X}) \subset \mathcal{D}(\mathcal{X})$ be the full ∞-subcategory consisting of $K \in \mathcal{D}(\mathcal{X})$ such that $\iota^! K \simeq 0$, and say that objects $K \in \mathcal{D}_{\mathcal{Y}}(\mathcal{X})$ are supported on $\mathcal{Y}$.

(b) Notice that for every morphism $f : \mathcal{X}' \to \mathcal{X}$ we have an inclusion $f^!(\mathcal{D}_{\mathcal{Y}}(\mathcal{X})) \subset \mathcal{D}_{\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'}(\mathcal{X}')$. Indeed, this follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{X}' \smallsetminus \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{X} \smallsetminus \mathcal{Y} & \longrightarrow & \mathcal{X}. \end{array}$$

(c) Notice that canonical isomorphism $\mathcal{D}(\mathcal{X}) \simeq \lim_{X \to \mathcal{X}} \mathcal{D}(X)$ induces an isomorphism $\mathcal{D}_{\mathcal{Y}}(\mathcal{X}) \simeq \lim_{X \to \mathcal{X}} \mathcal{D}_{X \times_{\mathcal{X}} \mathcal{Y}}(X)$.

Proof. We have to show that if $K \in \mathcal{D}(\mathcal{X})$ corresponds to a compatible system $\{K_X \in \mathcal{D}(X)\}_{X \to \mathcal{X}}$, then $K \in \mathcal{D}_{\mathcal{Y}}(\mathcal{X})$ if and only if $K_X \in \mathcal{D}_{X \times_{\mathcal{X}} \mathcal{Y}}(X)$ for every $X \to \mathcal{X}$. The "only if" assertion follows from (b). Conversely, assume that $K_X \in \mathcal{D}_{X \times_{\mathcal{X}} \mathcal{Y}}(X)$ for every $X \to \mathcal{X}$, and we want to show that $\iota^! K \simeq 0$, that is, for every $a : X \to \mathcal{X} \smallsetminus \mathcal{Y} \subset \mathcal{X}$ we have $K_X := a^! K \simeq 0$. By assumption, $X \times_{\mathcal{X}} \mathcal{Y} = \emptyset$, then $K_X \in \mathcal{D}_{\emptyset}(X) = \{0\}$. □

Lemma 4.5.5. *Let $\eta : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a topologically fp-locally closed embedding. Then $\eta^!$ induces an equivalence of categories $\eta^! : \mathcal{D}_{\mathcal{Y}}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$.*

Proof. The equivalence $\mathcal{X} \simeq \operatorname{colim}_{X \to \mathcal{X}} X$ induces an equivalence $\mathcal{Y} \simeq \operatorname{colim}_{X \to \mathcal{X}}(\mathcal{Y} \times_{\mathcal{X}} X)$, and hence equivalences $\mathcal{D}_{\mathcal{Y}}(\mathcal{X}) \simeq \lim_{X \to \mathcal{X}} \mathcal{D}_{X \times_{\mathcal{X}} \mathcal{Y}}(X)$ (see 4.5.4(c)) and $\mathcal{D}(\mathcal{Y}) \simeq \lim_{X \to \mathcal{X}} \mathcal{D}(X \times_{\mathcal{X}} \mathcal{Y})$. Thus it suffices to show that η induces an equivalence $\eta^!_X : \mathcal{D}_{X \times_{\mathcal{X}} \mathcal{Y}}(X) \to \mathcal{D}(X \times_{\mathcal{X}} \mathcal{Y})$. In other words, we reduce to the case then $\mathcal{X}$ is an affine scheme X.

Then using Corollary 4.3.8 we can assume that η is an fp-locally closed embedding, that is, η decomposes as $Y \xrightarrow{j} Z \xrightarrow{i} X$, where i (resp. j) is an fp-closed (resp. open) embedding. Next we observe that it is enough to show the assertion separately for $\eta = i$ and $\eta = j$.

We claim that both assertions easily follow from Lemma 4.5.3. Assume first that $\eta = i$. Since the left adjoint $i_!$ is fully faithful, the unit map $\mathrm{Id} \to i^! i_!$ is an isomorphism. So it suffices to show that $i_!$ induces an equivalence $\mathcal{D}(Z) \xrightarrow{\sim} \mathcal{D}_Z(X)$. Since $j^! i_! \simeq 0$, the image of $i_!$ lies inside $\mathcal{D}_Z(X)$. Conversely, if $K \in \mathcal{D}_Z(X)$ we have $j^! K \simeq 0$, then the map $i_! i^! K \to K$ is an isomorphism (by Lemma 4.5.3(d)), thus K lies in the essential image of $i_!$. Since $i_!$ is fully faithful, we are done.

In the case $\eta = j : U \hookrightarrow X$, the argument is similar. Namely, since the right adjoint j_* is fully faithful, the counit map $j^! j_* \to \mathrm{Id}$ is an isomorphism, so it suffices to show that j_* induces an equivalence $\mathcal{D}(U) \xrightarrow{\sim} \mathcal{D}_U(X)$. We complete as before. □

4.5.6. Functor η_*. In the situation of Lemma 4.5.5, we denote by $\eta_* : \mathcal{D}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{Y}}(\mathcal{X}) \subset \mathcal{D}(\mathcal{X})$ the inverse of $\eta^! : \mathcal{D}_{\mathcal{Y}}(\mathcal{X}) \xrightarrow{\sim} \mathcal{D}(\mathcal{Y})$.

4.5.7. Examples. Arguing as in Lemma 4.5.5, one can show that if η is an fp-open (resp. topologically fp-closed) embedding $j : \mathcal{U} \hookrightarrow \mathcal{X}$ (resp. $i : \mathcal{Z} \hookrightarrow \mathcal{X}$), then η_* coincides with j_* (resp. $i_!$).

Indeed, using the fact that $i^! j_* \simeq 0$ (resp. $j^! i_! \simeq 0$), one sees that j_* (resp. $i_!$) induces a functor $j_* : \mathcal{D}(\mathcal{U}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{U}}(\mathcal{X})$ (resp. $i_! : \mathcal{D}(\mathcal{Z}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{Z}}(\mathcal{X})$). Next, the fact that j_* (resp. $i_!$) is fully faithful implies that the unit $\mathrm{Id} \to j_* j^!$ (resp. counit $i_! i^! \to \mathrm{Id}$) is an isomorphism, hence by Lemma 4.5.5 the functor j_* (resp. $i^!$) is the inverse of the equivalence $j^! : \mathcal{D}_{\mathcal{U}}(\mathcal{X}) \xrightarrow{\sim} \mathcal{D}(\mathcal{U})$ (resp. $i^! : \mathcal{D}_{\mathcal{Z}}(\mathcal{X}) \xrightarrow{\sim} \mathcal{D}(\mathcal{Z})$).

Lemma 4.5.8. *Let $\eta : \mathcal{X} \to \mathcal{Y}$ be a topologically fp-locally closed embedding. Then for every Cartesian diagram of ∞-stacks*

$$\begin{array}{ccc} \widetilde{\mathcal{Y}} & \xrightarrow{\widetilde{\eta}} & \widetilde{\mathcal{X}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{\eta} & \mathcal{X}. \end{array}$$

Then we have a canonical isomorphism

$$f^! \eta_* \simeq \widetilde{\eta}_* g^!.$$

Proof. Notice that for every $K \in \mathcal{D}(\mathcal{Y})$, we have $\widetilde{\eta}_* g^!(K) \in \mathcal{D}_{\widetilde{\mathcal{Y}}}(\widetilde{\mathcal{X}})$, $\eta_*(K) \in \mathcal{D}_{\mathcal{Y}}(\mathcal{X})$, thus $f^!\eta_*(K) \in \mathcal{D}_{\widetilde{\mathcal{Y}}}(\widetilde{\mathcal{X}})$ (by 4.5.4(b)). Therefore by Lemma 4.5.5, it suffices to construct an isomorphism $\widetilde{\eta}^! f^! \eta_* \simeq \widetilde{\eta}^! \widetilde{\eta}_* g^!$. Since $\eta^! \eta_* \simeq \mathrm{Id}$ and $\widetilde{\eta}^! \widetilde{\eta}_* \simeq \mathrm{Id}$, the composition

$$\widetilde{\eta}^! f^! \eta_* \simeq g^! \eta^! \eta_* \simeq g^! \simeq \widetilde{\eta}^! \widetilde{\eta}_* g^!$$

does the job. □

Corollary 4.5.9. *Let $\eta : \mathcal{Y} \xrightarrow{\eta'} \mathcal{Z} \xrightarrow{\eta''} \mathcal{X}$ be a composition of topologically fp-locally closed embeddings. Then the functor η_* coincides with the composition $\eta''_* \circ \eta'_*$.*

Proof. Since $\eta'_* : \mathcal{D}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{Y}}(\mathcal{Z})$ is the inverse of $\eta'^! : \mathcal{D}_{\mathcal{Y}}(\mathcal{Z}) \xrightarrow{\sim} \mathcal{D}(\mathcal{Y})$, it suffices to check that η''_* induces an equivalence $\mathcal{D}_{\mathcal{Y}}(\mathcal{Z}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{Y}}(\mathcal{X})$, inverse to $\eta''^!$. Since $\eta''_*(\mathcal{D}_{\mathcal{Y}}(\mathcal{Z})) \subset \mathcal{D}_{\mathcal{Y}}(\mathcal{X})$ (by Lemma 4.5.8 for η''), the assertion follows from the fact that $\eta''_* : \mathcal{D}(\mathcal{Z}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{Z}}(\mathcal{X})$ is an equivalence, inverse to $\eta''^!$. □

4.5.10. Decomposable case. (a) We call a topologically fp-locally closed embedding $\eta : \mathcal{X} \to \mathcal{Y}$ *decomposable*, if it decomposes as a composition $\mathcal{Y} \xrightarrow{i} \mathcal{U} \xrightarrow{j} \mathcal{X}$, where i (resp. j) is a topologically fp-closed (resp. fp-open) embedding.

(b) Notice that the class of decomposable topologically fp-locally closed embeddings is closed under compositions. For this we have to show that a composition $\eta : \mathcal{Y} \xrightarrow{j} \mathcal{Z} \xrightarrow{i} \mathcal{X}$, where i (resp. j) is a topologically fp-closed (resp. fp-open) embedding is decomposable. Since j is a fp-open embedding, the ∞-substack $\mathcal{T} := \mathcal{Z} \smallsetminus \mathcal{Y} \subset \mathcal{Z}$ is topologically fp-closed, and j induce an isomorphism $\mathcal{Y} \simeq \mathcal{Z} \smallsetminus \mathcal{T}$. Therefore η decomposes as $\eta : \mathcal{Y} \simeq \mathcal{Z} \smallsetminus \mathcal{Y} \xrightarrow{i} \mathcal{X} \smallsetminus \mathcal{T} \hookrightarrow \mathcal{X}$ of a topologically fp-closed embedding and an fp-open embedding. Thus η is decomposable.

(c) Conversely, for every decomposable fp-locally closed embedding $\eta : \mathcal{Y} \xrightarrow{i} \mathcal{U} \xrightarrow{j} \mathcal{X}$, set $\mathcal{V} := \mathcal{U} \smallsetminus \mathcal{Y}$. Then η decomposes as $\eta : \mathcal{Y} \to \mathcal{U} \smallsetminus \mathcal{V} \xrightarrow{j} \mathcal{X} \smallsetminus \mathcal{U} \hookrightarrow \mathcal{X}$ of a topological equivalence, a fp-open embedding and a topologically fp-closed embedding.

(d) Using Corollary 4.5.9 and 4.5.7, we get that every decomposable $\eta = j \circ i$ as in (a), the functor η_* coincides with the composition $j_* \circ i_!$.

(e) Let $\eta = j \circ i$ be a decomposable topologically fp-locally closed embedding of topologically placid ∞-stacks. Then the pushforward $\eta_* = j_* \circ i_!$ (by (d)) has a left adjoint $\eta^* = i^* \circ j^!$ (use Proposition 4.4.5(a) for i^*). Moreover, η^* satisfies base change with respect to topologically smooth morphisms (by Proposition 4.4.5(b)).

4.5.11. Remarks. (a) Since every fp-locally closed embedding of schemes η has a decomposition as in 4.5.7(b), we can define η_* by the formula $i_! \circ j_*$. Moreover, it is not difficult to see that this composition is independent of the decomposition, thus η_* is well defined.

(b) Moreover, since $i_!$ and j_* commute with all !-pullbacks, one show that factors η_* from (a) commute with !-pullbacks, thus give rise to functors η_* in general.

(c) Though the definition of η_* using (a) and (b) is the standard way of doing it, we feel that our way is more intrinsic, because it does not use any choices.

(d) Repeating arguments Proposition 4.4.5 one can show that in the situation of 4.5.10(e) the left adjoint η^* exists and the base change holds without the decomposable assumption.

(e) By an argument, similar to (a) and (b) one can define h_* for every (topologically) fp-representable morphism between prestacks, which generalizes functors $i_* := i_!$ and j_*, defined in Lemma 4.5.3 and is compatible with compositions and satisfies base change. We do not need this fact for this work.

4.6. **Endomorphisms of $\omega_{\mathcal{X}}$.**

4.6.1. (Classical) presheaves on fSets**.**

(a) Let fSets be the category of finite sets, and Pro(fSets) the category of pro-finite sets. By definition, we have a natural embedding

$$\mathrm{Pro}(\mathrm{fSets}) \hookrightarrow \mathrm{PSh}_{\lim}(\mathrm{fSets})^{op} : X \mapsto \mathrm{Hom}_{\mathrm{Pro}(\mathrm{fSets})}(X, -),$$

where $\mathrm{PSh}_{\lim}(\mathrm{fSets}) := \mathrm{Funct}_{\lim}(\mathrm{fSets}, \mathrm{Sets})$ is the category of limit preserving functors.

(b) Recall that the restriction functor $\iota^* : \mathrm{PSh}(\mathrm{Sets})^{op} \to \mathrm{PSh}(\mathrm{fSets})^{op}$ has a left Kan extension $\iota_! : \mathrm{PSh}(\mathrm{fSets})^{op} \to \mathrm{PSh}(\mathrm{Sets})^{op}$, which is fully faithful, and induces a functor $\mathrm{PSh}_{\lim}(\mathrm{fSets})^{op} \to \mathrm{PSh}_{\lim}(\mathrm{Sets})^{op}$.

(c) For every $F \in \mathrm{PSh}_{\lim}(\mathrm{fSets})$ and $A \in \mathrm{Sets}$, we set $A^F := (\iota_! F)(A) \in \mathrm{Sets}$. This is compatible with the standard notation for representable functors. Since F and hence also $\iota_! F$ preserves limits, we conclude that for every algebra A, the corresponding set A^F is naturally an A-algebra.

4.6.2. Functor $\underline{\pi}_0$. (a) Recall that to every $X \in \mathrm{Aff}_k$, one can associate a profinite set $\pi_0(X)$. In other words, π_0 is a functor $\mathrm{Aff}_k \to \mathrm{Pro}(\mathrm{fSets}) \subset \mathrm{PSh}_{\lim}(\mathrm{fSets})^{op}$.

(b) Let $\underline{\pi}_0 : \mathrm{PreSt}_k \to \mathrm{PSh}(\mathrm{fSets})^{op}$ be the left Kan extension of π_0. Explicitly, for every $\mathcal{X} \in \mathrm{PreSt}_k$, we have $\underline{\pi}_0(\mathcal{X}) \simeq \lim_{X \to \mathcal{X}, X \in \mathrm{Aff}_k} \pi_0(X) \in \mathrm{PSh}(\mathrm{fSets})$. In particular, we have $\underline{\pi}_0(\mathcal{X}) \in \mathrm{PSh}_{\lim}(\mathrm{fSets})$.

(c) We say that $\mathcal{X}$ is *connected*, if $\underline{\pi}_0(\mathcal{X}) = \mathrm{pt} \in \mathrm{fSets} \subset \mathrm{PSh}(\mathrm{fSets})^{op}$.

4.6.3. Remarks. (a) By definition, the functor $\underline{\pi}_0 : \mathrm{PreSt}_k \to \mathrm{PSh}(\mathrm{fSets})^{op}$ preserves colimits. One can show that its restriction $\underline{\pi}_0 : \mathrm{St}_k \to \mathrm{PSh}(\mathrm{fSets})^{op}$ preserves colimits as well.

(b) Using (a) one can show (arguing as in Corollary 4.6.5 below) that if $\mathcal{X} \to \mathcal{Y}$ is surjective map, and $\mathcal{X}$ is connected, then $\mathcal{Y}$ is connected.

Lemma 4.6.4. *For every $\mathcal{X} \in \mathrm{St}_k$, the endomorphism algebra $\mathrm{End}_{\mathcal{D}(\mathcal{X})}(\omega_{\mathcal{X}})$ is a discrete $\overline{\mathbb{Q}}_l$-algebra, canonically isomorphic to $\overline{\mathbb{Q}}_l^{\pi_0(\mathcal{X})}$. Moreover, for every $\overline{\mathbb{Q}}_l$-vector space V, we have a canonical isomorphism of $\mathrm{End}_{\mathcal{D}(\mathcal{X})}(\omega_{\mathcal{X}})$-modules*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{X})}(\omega_{\mathcal{X}}, V \otimes_{\overline{\mathbb{Q}}_l} \omega_{\mathcal{X}}) \simeq \mathrm{End}_{\mathcal{D}(\mathcal{X})}(\omega_{\mathcal{X}}) \otimes_{\overline{\mathbb{Q}}_l} V. \tag{4.7}$$

Proof. Note first that if $X \in \mathrm{Aff}_k^{ft}$, then we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}(X)}(\omega_X, \omega_X) \simeq \mathrm{Hom}_{\mathcal{D}(X)}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}_l) \simeq \overline{\mathbb{Q}}_l^{\pi_0(X)},$$

where the first isomorphism follows from the Verdier duality, and the second one from the fact that the constant sheaf $\overline{\mathbb{Q}}_l$ has no negative self-exts.

Next, let $X \in \mathrm{Aff}_k$, and choose a presentation $X \simeq \lim_\alpha X_\alpha$ as a filtered limit, where $X_\alpha \in \mathrm{Aff}_k^{ft}$ for all α. Then $\mathrm{End}(\omega_X) \simeq \mathrm{colim}_\alpha \mathrm{End}(\omega_{X_\alpha})$ by [Ro]. Thus it is a discrete $\overline{\mathbb{Q}}_l$-algebra, being a filtered colimit of discrete spaces, which by the proven above is isomorphic to $\mathrm{colim}_\alpha \overline{\mathbb{Q}}_l^{\pi_0(X_\alpha)} = \overline{\mathbb{Q}}_l^{\pi_0(X)}$.

Then, for an arbitrary $\mathcal{X} \in \mathrm{St}_k$, the identification $\mathcal{X} \simeq \mathrm{colim}_{X \to \mathcal{X}} X$, gives an identification $\mathcal{D}(\mathcal{X}) \simeq \lim_{X \to \mathcal{X}} \mathcal{D}(X)$, under which $\omega_{\mathcal{X}}$ corresponds to the compatible system of the ω_X's. Thus $\mathrm{End}(\omega_{\mathcal{X}}) \simeq \lim_{X \to \mathcal{X}} \mathrm{End}(\omega_X)$, hence it is a discrete algebra isomorphic to $\lim_{X \to \mathcal{X}} \overline{\mathbb{Q}}_l^{\pi_0(X)} \simeq \overline{\mathbb{Q}}_l^{\pi_0(\mathcal{X})}$.

For an arbitrary $\overline{\mathbb{Q}}_l$-vector space V, the isomorphism (4.7) for $X \in \mathrm{Aff}_k$ follows from the fact that $\omega_X \in \mathcal{D}_c(X)$ is compact in $\mathcal{D}(X)$. Finally, isomorphism (4.7) for an arbitrary $\mathcal{X}$ follows from that for $X \in \mathrm{Aff}_k$ using the fact that tensor product with a fixed vector space commute with all limits. □

Corollary 4.6.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ is surjective map in St_k such that $\mathrm{End}(\omega_{\mathcal{X}}) \simeq \overline{\mathbb{Q}}_l$. Then $\mathrm{End}(\omega_{\mathcal{Y}}) \simeq \overline{\mathbb{Q}}_l$.*

Proof. Since f is surjective, the natural morphism $\mathrm{colim}_{[m]}(\mathcal{X}^{[m]}) \to \mathcal{Y}$ is an isomorphism. Thus, as in the proof of Lemma 4.6.4, the induced map of discrete $\overline{\mathbb{Q}}_l$-algebras $\mathrm{End}(\omega_{\mathcal{Y}}) \to \lim_{[m]} \mathrm{End}(\omega_{\mathcal{X}^{[m]}})$ is an isomorphism. Since $\mathrm{Hom}_{\Delta_s}([0], [m]) \neq \emptyset$ for every m, we conclude that the pullback $f^! : \mathrm{End}(\omega_{\mathcal{Y}}) \to \mathrm{End}(\omega_{\mathcal{X}})$ is injective. Since $\mathrm{End}(\omega_{\mathcal{X}^{[0]}}) = \mathrm{End}(\omega_{\mathcal{X}}) \simeq \overline{\mathbb{Q}}_l$, by assumption, we thus conclude that $\mathrm{End}(\omega_{\mathcal{Y}}) \simeq \overline{\mathbb{Q}}_l$, as claimed. □

4.6.6. Quotient by a discrete group.

(a) Let Γ be a discrete group acting on ∞-stack $\mathcal{X}$, let $\mathcal{Y} := [\mathcal{X}/\Gamma]$ the quotient ∞-stack, and let $f : \mathcal{X} \to \mathcal{Y}$ be the projection.

(b) Notice that since the trivial Γ-torsor $\Gamma \times \mathcal{X} \to \mathcal{X}$ is clearly ind-fp-proper, we conclude from 4.4.2 that f is locally ind-fp-proper.

(c) By (b) and Proposition 4.4.3, the pullback $f^! : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$ admits a left adjoint $f_! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$, satisfying base change.

Lemma 4.6.7. *In the situation of 4.6.6, we have a natural isomorphism of $\overline{\mathbb{Q}}_l$-algebras*

$$\mathrm{End}(f_!(\omega_{\mathcal{X}})) \simeq \overline{\mathbb{Q}}_\ell[\Gamma] \otimes_{\overline{\mathbb{Q}}_l} \overline{\mathbb{Q}}_l^{\pi_0(\mathcal{X})}.$$

Proof. The group action of Γ on $\mathcal{X}$ over $\mathcal{Y}$ induces a group homomorphism $\Gamma \to \mathrm{Aut}(f_!(\omega_{\mathcal{X}}))$, commuting with the action of $\mathrm{End}(\omega_{\mathcal{X}})$. Hence it induces a homomorphism of $\overline{\mathbb{Q}}_l$-algebras

$$\overline{\mathbb{Q}}_l[\Gamma] \otimes_{\overline{\mathbb{Q}}_l} \mathrm{End}(\omega_{\mathcal{X}}) \to \mathrm{End}(f_!(\omega_{\mathcal{X}})). \tag{4.8}$$

Since $\mathrm{End}(\omega_{\mathcal{X}}) \simeq \overline{\mathbb{Q}}_l^{\pi_0(\mathcal{X})}$ (by Lemma 4.6.4), it now suffices to show that (4.8) is an isomorphism of $\overline{\mathbb{Q}}_l$-vector spaces. Since f is a Γ-torsor (by 9.2.2(c)), we have a Cartesian diagram

$$\begin{array}{ccc} \Gamma \times \mathcal{X} & \xrightarrow{a} & \mathcal{X} \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

Since $f_!$ commutes with base change, we get a natural isomorphism

$$f^! f_!(\omega_{\mathcal{X}}) \simeq \mathrm{pr}_! \, a^!(\omega_{\mathcal{X}}) \simeq \mathrm{pr}_!(\omega_{\Gamma\times\mathcal{X}}) \simeq \overline{\mathbb{Q}}_l[\Gamma] \otimes_{\overline{\mathbb{Q}}_l} \omega_{\mathcal{X}}.$$

Therefore by adjunction we have an isomorphism

$$\mathrm{End}(f_!(\omega_{\mathcal{X}})) \simeq \mathrm{Hom}(\omega_{\mathcal{X}}, f^! f_!(\omega_{\mathcal{X}})) \simeq \mathrm{Hom}(\omega_{\mathcal{X}}, \overline{\mathbb{Q}}_l[\Gamma] \otimes_{\overline{\mathbb{Q}}_l} \omega_{\mathcal{X}}) \simeq \overline{\mathbb{Q}}_l[\Gamma] \otimes_{\overline{\mathbb{Q}}_l} \mathrm{End}(\omega_{\mathcal{X}}),$$

where the last isomorphism follows from (4.7). Unwinding the definitions, one sees that this isomorphism coincides with the canonical homomorphism (4.8), we started from. □

Corollary 4.6.8. *$f : \mathcal{X} \to \mathcal{Y}$ be a morphism of ∞-stacks, and let Γ be a discrete group acting on $\mathcal{X}$ over $\mathcal{Y}$ such that the induced map $[f] : [\mathcal{X}/\Gamma] \to \mathcal{Y}$ is a topological equivalence. Then $f^! : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$ has a left adjoint $f_! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$, and we have a natural isomorphism of $\overline{\mathbb{Q}}_l$-algebras*

$$\mathrm{End}(f_!(\omega_{\mathcal{X}})) \simeq \overline{\mathbb{Q}}_\ell[\Gamma] \otimes_{\overline{\mathbb{Q}}_l} \overline{\mathbb{Q}}_l^{\pi_0(\mathcal{X})}.$$

Proof. Set $\mathcal{Y}' := [\mathcal{X}/\Gamma]$, and let $f' : \mathcal{X} \to \mathcal{Y}'$ be the projection. Since $[f] : \mathcal{Y}' \to \mathcal{Y}$ is a topological equivalence, the pullback $[f]^! : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{Y}')$ is an equivalence (by Proposition 4.3.6), hence has a left adjoint $[f]_!$. Therefore $f^! \simeq f'^! \circ [f]^!$ has a left adjoint $f_! := [f]_! \circ f'_!$. Now the assertion follows from Lemma 4.6.7 and the observation that $[f]_!$ is an equivalence. □

5. Perverse t-structures on topologically placid ∞-stacks

5.1. **Generalities.**

5.1.1. Recollections. Let $\mathcal{D}$ be a stable ∞-category.

(a) Recall (see Lurie [Lu2, 1.2.1]) that a *t-structure* on $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full subcategories of $\mathcal{D}$ satisfying certain properties. In particular, the embedding $\mathcal{D}^{\geq 0} \to \mathcal{D}$ (resp. $\mathcal{D}^{\leq 0} \to \mathcal{D}$) has a left (resp. right) adjoint

$$\tau_{\geq 0} : \mathcal{D} \to \mathcal{D}^{\geq 0} \text{ (resp. } \tau_{\leq 0} : \mathcal{D} \to \mathcal{D}^{\leq 0}).$$

Similarly, we define truncation functors $\tau_{\geq 1}$ and $\tau_{\leq -1}$. Notice that

$$x \in \mathcal{D}^{\leq 0} (\text{resp. } x \in \mathcal{D}^{\geq 0}) \text{ if and only if } \tau_{\geq 1}(x) = 0 \text{ (resp. } \tau_{\leq -1}(x) = 0). \tag{5.1}$$

(b) Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be an exact functor between stable ∞-categories equipped with t-structures. Recall that F is *right (resp. left) t-exact*, if F satisfies $F(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$ (resp. $F(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$), and it is called *t-exact*, if it is both left and right t-exact.

(c) Every t-exact F commutes with truncation functors. Indeed, for each object $x \in \mathcal{D}_1$, functor F maps the fiber sequence $\tau_{\leq 0}(x) \to x \to \tau_{\geq 1}(x)$ to the fiber sequence $F(\tau_{\leq 0}(x)) \to F(x) \to F(\tau_{\geq 1}(x))$. Since $F(\tau_{\leq 0}(x)) \in \mathcal{D}_2^{\leq 0}$ and $F(\tau_{\geq 1}(x)) \in \mathcal{D}_2^{\geq 1}$ by assumption, we conclude that $F(\tau_{\leq 0}(x)) \simeq \tau_{\leq 0}(F(x))$ and $F(\tau_{>0}(x)) \simeq \tau_{>0}(F(x))$, as claimed.

(d) Recall that F is called *faithful*, if $F(x) \not\simeq 0$ when $x \not\simeq 0$.

Lemma 5.1.2. *(a) For every t-structure on $\mathcal{D}$ has a unique extension to a t-structure on* $\operatorname{Ind}\mathcal{D}$ *such that* $\operatorname{Ind}(\mathcal{D})^{\geq 0}$ *is closed under filtered colimits. Explicitly,* $\operatorname{Ind}(\mathcal{D})^{\leq 0} = \operatorname{Ind}(\mathcal{D}^{\leq 0})$ *and* $\operatorname{Ind}(\mathcal{D})^{\geq 0} = \operatorname{Ind}(\mathcal{D}^{\geq 0})$.

(b) Let $\mathcal{D}$ be a stable ∞-category with a t-structure. Then $\mathcal{D}^{\leq 0}$ is closed under all colimits that exist in $\mathcal{D}$ and $\mathcal{D}^{\geq 0}$ is closed under all limits that exist in $\mathcal{D}$.

(c) Assume that $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a t-exact and faithful functor between stable ∞-categories. Then for every object $x \in \mathcal{D}_2$ we have

$$x \in \mathcal{D}_1^{\leq 0} \text{ if and only if } F(x) \in \mathcal{D}_2^{\leq 0}$$

and similarly for $\mathcal{D}_i^{\geq 0}$.

(d) Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ and $G : \mathcal{D}_2 \to \mathcal{D}_3$ be functors between stable ∞-categories, equipped with t-structures such that G is t-exact and faithful. Then F is t-exact if and only if $G \circ F$ is.

(e) The t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is uniquely determined by $\mathcal{D}^{\geq 0}$. Namely, an object $x \in \mathcal{D}$ belongs to $\mathcal{D}^{\leq 0}$ if and only if $\operatorname{Hom}_{\mathcal{D}}(x, y) \simeq 0$ *for every $y \in \mathcal{D}^{\geq 1}$.*

Proof. (a),(b) follow from [GR, 4.1.2.4] and [Lu2, Cor.1.2.1.6], respectively.

(c) By (5.1), we have $x \in \mathcal{D}_1^{\leq 0}$ if and only if $\tau_{\geq 1}(x) \simeq 0$, while $F(x) \in D_2^{\leq 0}$ if and only if $\tau_{\geq 1}(F(x)) \simeq 0$. Since $\tau_{\geq 1}(F(x)) \simeq F(\tau_{\geq 1}(x))$ (because F is t-exact), we have to show that $\tau_{\geq 1}(x) \simeq 0$ if and only if $F(\tau_{\geq 1}(x)) \simeq 0$. Since F is faithful, we are done.

(d) The "if" assertion follows from (c), while the converse is clear.

(e) is standard. □

Lemma 5.1.3. *Let $\mathcal{I}$ be a category and $\Psi : \mathcal{I} \to \mathrm{Cat}_{\mathrm{st},\ell}$ a functor. Assume that for every object $a \in \mathcal{I}$ the category $\mathcal{D}_a$ is equipped with a t-structure, and for every morphism $\alpha : a \to b$ in $\mathcal{I}$ the induced functor $\psi_\alpha : \mathcal{D}_a \to \mathcal{D}_b$ is t-exact. Then*

(a) Assume that $\mathcal{I}$ is filtered. Then there exists a unique t-structure on $\mathcal{D} := \mathrm{colim}_{a\in\mathcal{I}} \mathcal{D}_a$ such that every functor $\mathrm{ins}_a : \mathcal{D}_a \to \mathcal{D}$ is t-exact. Explicitly, $\mathcal{D}^{\leq 0} := \mathrm{colim}_{a\in\mathcal{I}} \mathcal{D}_a^{\leq 0}$ and similarly for $\mathcal{D}^{\geq 0}$.

(b) There exists a unique t-structure on $\mathcal{D} := \lim_{a\in\mathcal{I}^{op}} \mathcal{D}_a$ such that every functor $ev_a : \mathcal{D} \to \mathcal{D}_a$ is t-exact. Explicitly, $\mathcal{D}^{\leq 0} = \lim_{a\in\mathcal{I}} \mathcal{D}_a^{\leq 0}$ and similarly for $\mathcal{D}^{\geq 0}$.

Proof. (a) Let us prove that subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, defined as $\mathcal{D}^{\leq 0} := \mathrm{colim}_{a\in\mathcal{I}} \mathcal{D}_a^{\leq 0}$ and $\mathcal{D}^{\geq 0} := \mathrm{colim}_{a\in\mathcal{I}} \mathcal{D}_a^{\geq 0}$, equip $\mathcal{D}$ with a t-structure. Recall that every $x \in \mathcal{D}$ is of the form $x = \mathrm{ins}_a(x_a)$ for some $x_a \in \mathcal{D}_a$. By assumption, there exists a fibered sequence $\tau_{\leq 0}x_a \to x_a \to \tau_{\geq 1}x_a$ in $\mathcal{D}_a$ with $\tau_{\leq 0}x_a \in \mathcal{D}_a^{\leq 0}$ and $\tau_{\geq 1}x_a \in \mathcal{D}_a^{\geq 1}$. Applying ins_a, we get the corresponding fiber sequence for x.

It remains to show that for $x \in \mathcal{D}^{\leq 0}$ and $y \in \mathcal{D}^{\geq 1}$, we have $\mathrm{Hom}(x, y) \simeq 0$. Since $\mathcal{I}$ is filtered, x and y come from $x_a \in \mathcal{D}_a^{\leq 0}$ and $y_a \in \mathcal{D}_a^{\geq 1}$. As the colimit is filtered, it follows from [Ro, 0.4] that

$$\mathrm{Hom}_{\mathcal{D}}(x, y) \simeq \mathrm{colim}_{\alpha\in a\setminus\mathcal{I}} \mathrm{Hom}_{\mathcal{D}_b}(\psi_\alpha(x_a), \psi_\alpha(y_a)),$$

As ψ_α are t-exact, and $a\setminus\mathcal{I}$ filtered, thus weakly contractible [Sr, Cor. 3.9], we conclude that

$$\mathrm{Hom}_{\mathcal{D}}(x, y) \simeq \mathrm{colim}_{\alpha\in a\setminus\mathcal{I}} \mathrm{pt} \simeq \mathrm{pt},$$

as wished.

(b) We want to show that subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, defined as $\mathcal{D}^{\leq 0} := \lim_{a\in\mathcal{I}} \mathcal{D}_a^{\leq 0}$ and $\mathcal{D}^{\geq 0} := \lim_{a\in\mathcal{I}} \mathcal{D}_a^{\geq 0}$ equip $\mathcal{D}$ with a t-structure. First we claim that for every $x \in \mathcal{D}^{\leq 0}$ and $y \in \mathcal{D}^{\geq 1}$ we have $\mathrm{Hom}(x, y) \simeq 0$. Indeed, using for example [DG, 1.6.2] and [Lu1, 3.3.3.2] one has

$$\mathrm{Hom}(x, y) \simeq \lim_{a\in\mathcal{I}} \mathrm{Hom}(ev_a x, ev_a y). \tag{5.2}$$

Now, $ev_a x \in \mathcal{D}_a^{\leq 0}$ and $ev_a y \in \mathcal{D}_a^{\geq 1}$, all spaces on the right hand side are contractible. So the assertion follows from the standard fact that a (homotopy) limit of contractible spaces is contractible.

Next we claim that the inclusion functor $\mathcal{D}^{\geq 0} \hookrightarrow \mathcal{D}$ has a left adjoint $\tau_{\geq 0}$. Namely, since every $\mathcal{D}_a$ is equipped with a t-structure, the inclusion $\mathcal{D}_a^{\geq 0} \hookrightarrow \mathcal{D}_a$ has a left adjoint, and since every ψ_α is t-exact, these left adjoints satisfy the Beck-Chevalley condition (use 5.1.1(c)). Therefore the existence of $\tau_{\geq 0} : \mathcal{D} \to \mathcal{D}^{\geq 0}$ follows from Proposition 4.1.8(b). Now for every $x \in \mathcal{D}$, let $\tau_{\geq 1}x$ be the cofiber of the counit map $\tau_{\leq 0}x \to x$. It suffices to show that $\tau_{\geq 1}x \in \mathcal{D}^{\geq 1}$. But this follows from the fact that cofiber in the limit category is a compatible system of cofibers, that the cofiber of each $\tau_{\leq 0}(ev_a(x)) \to ev_a(x)$ lies in $\mathcal{D}_a^{\geq 1}$, and $\mathcal{D}^{\geq 1} = \lim_{a\in\mathcal{I}} \mathcal{D}_a^{\geq 1}$. □

The following assertion is not needed for the perversity of the affine Springer sheaf.

Proposition 5.1.4. *Let $\Psi : \mathcal{I} \to \mathrm{PrCat}_{\mathrm{st},\ell}$ be a functor $a \mapsto \mathcal{D}_a$. Assume that $\mathcal{I}$ is filtered, for every object $a \in \mathcal{I}$ the category $\mathcal{D}_a$ is equipped with a t-structure such that $\mathcal{D}_a^{\geq 0}$ is closed under filtered colimits, and for every morphism $\alpha : a \to b$ in $\mathcal{I}$ the induced functor $\psi_\alpha : \mathcal{D}_a \to \mathcal{D}_b$ is t-exact, and has a continuous right adjoint ϕ_α.*

Then there exists a unique t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D} := \mathrm{colim}_{a\in\mathcal{I}} \mathcal{D}_a$ such that $\mathcal{D}^{\geq 0}$ is closed under filtered colimits and every functor $\mathrm{ins}_a : \mathcal{D}_a \to \mathcal{D}$ is t-exact.

5.1.5. Remarks. (a) For every $a \in \mathcal{I}$, denote by $ev_a : \mathcal{D} \to \mathcal{D}_a$ the right adjoint of ins_a (which is automatically continuous by Theorem 4.1.3). It follows from the proof below that

$$\mathcal{D}^{\geq 0} = \{x \in \mathcal{D} \,|\, ev_a(x) \in \mathcal{D}_a^{\geq 0} \text{ for all } a \in \mathcal{I}\}. \tag{5.3}$$

Furthermore, this is the only t-structure on $\mathcal{D}$ satisfying this property (see Lemma 5.1.2(e)).

(b) For applications we currently have in mind, all categories $\mathcal{D}_a$ are compactly generated. In this case, Proposition 5.1.4 can be deduced from a combination of Lemma 5.1.2(a) and Lemma 5.1.3(a).

Namely, let $\mathcal{D}_a^c \subset \mathcal{D}_a$ be the subcategory of compact objects. Then $\mathcal{D}_a \simeq \mathrm{Ind}\, \mathcal{D}_a^c$, while the assumption that the right adjoints ϕ_α are continuous implies that Ψ induces a functor $\mathcal{I} \to \mathrm{Cat}_{\mathrm{st},\ell} : a \mapsto \mathcal{D}_a^c$. Hence we have a natural equivalence $\mathcal{D} \simeq \mathrm{Ind}\, \mathcal{D}^c$ with $\mathcal{D}^c := \mathrm{colim}_{a\in\mathcal{I}} \mathcal{D}_a^c$.

Next, the assumption that each $\mathcal{D}_a^{\geq 0}$ is closed under filtered colimits implies that the t-structure on $\mathcal{D}_a$ induces a t-structure on $\mathcal{D}_a^c$. Hence Lemma 5.1.3(a) provides us with a t-structure on $\mathcal{D}^c$, while Lemma 5.1.2(a) provides us with a t-structure on $\mathcal{D}$ such that $\mathcal{D}^{\geq 0}$ is closed under filtered colimits.

Proof. Let $\mathcal{D}^{\leq 0} \subset \mathcal{D}$ be the smallest full subcategory, containing $\mathrm{ins}_a(x_a)$ with $x_a \in \mathcal{D}_a^{\leq 0}$ and closed under all colimits, and let $\mathcal{D}^{\geq 0} \subset \mathcal{D}$ be the full subcategory, defined by (5.3). We claim that pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defines a t-structure on $\mathcal{D}$.

First of all, we have to check that for every $x \in \mathcal{D}^{\leq 0}$ and $y \in \mathcal{D}^{\geq 1}$ we have $\mathrm{Hom}(x, y) \simeq 0$. By the definition of $\mathcal{D}^{\leq 0}$, we can assume that $x = \mathrm{ins}_a(x_a)$ with $x_a \in \mathcal{D}_a^{\leq 0}$. In this case, we have

$$\mathrm{Hom}(x, y) = \mathrm{Hom}(\mathrm{ins}_a(x_a), y) \simeq \mathrm{Hom}(x_a, ev_a(y)) \simeq 0,$$

because $x_a \in \mathcal{D}_a^{\leq 0}$ (by assumption), and $ev_a(y) \in \mathcal{D}_a^{\geq 1}$ (by (5.3)).

Next, we are going to show that for every $x \in \mathcal{D}$ there exists a fibre sequence $x_{\leq 0} \to x \to x_{\geq 1}$ with $x_{\leq 0} \in \mathcal{D}^{\leq 0}$ and $x_{\geq 1} \in \mathcal{D}^{\geq 1}$. By Corollary 4.1.5, for every $x \in \mathcal{D}$, we have a natural functor $\mathcal{I} \to \mathcal{D} : a \mapsto \mathrm{ins}_a(x_a)$ with $x_a := ev_a(x) \in \mathcal{D}_a$, and that the natural map $\mathrm{colim}_a \, \mathrm{ins}_a(x_a) \to x$ is an isomorphism.

Recall that the perverse t-structure on $\mathcal{D}_a$ we get a fibred sequence

$$S_a : \tau_{\leq 0}(x_a) \to x_a \to \tau_{\geq 1}(x_a)$$

with $\tau_{\leq 0}(x_a) \in \mathcal{D}_a^{\leq 0}$ and $\tau_{\geq 1}(x_a) \in \mathcal{D}_a^{\geq 1}$.

We claim that the functor $a \mapsto \mathrm{ins}_a(x_a)$ extends to the functor $a \mapsto \mathrm{ins}_a(S_a)$. It suffices to show that a collection of morphisms $x_a \to \tau_{\geq 1}(x_a)$ gives rise to a morphism $\mathrm{ins}_a(x_a) \to \mathrm{ins}_a(\tau_{\geq 1}(x_a))$ of functors $\mathcal{I} \to \mathcal{D}$.

The main point is to show that the assignment $a \mapsto \mathrm{ins}_a(\tau_{\geq 1}(x_a))$ is functorial in $a \in \mathcal{I}$. In other words, we want to show that every morphism $\alpha : a \to b$ in $\mathcal{I}$ induces a canonical morphism $\mathrm{ins}_a(\tau_{\geq 1}(x_a)) \to \mathrm{ins}_b(\tau_{\geq 1}(x_b))$.

Since $\mathrm{ins}_a \simeq \mathrm{ins}_b \circ \psi_\alpha$, it suffices to construct a morphism $\psi_\alpha(\tau_{\geq 1}(x_a)) \to \tau_{\geq 1}(x_b)$, or, by adjointness, a morphism $\iota_\alpha : \tau_{\geq 1}(x_a) \to \phi_\alpha(\tau_{\geq 1}(x_b))$. Since ψ_α is t-exact, we conclude that ϕ_α is left t-exact. Thus $\phi_\alpha(\tau_{\geq 1}(x_b)) \in \mathcal{D}_a^{\geq 1}$, so the natural morphism

$$\mathrm{Hom}(\tau_{\geq 1}(x_a), \phi_\alpha(\tau_{\geq 1}(x_b))) \to \mathrm{Hom}(x_a, \phi_\alpha(\tau_{\geq 1}(x_b))) \simeq \mathrm{Hom}(\psi_\alpha(x_a), \tau_{\geq 1}(x_b)),$$

induced by the morphism $\mathrm{pr}_{\leq 1} : x_a \to \tau_{\geq 1}(x_a)$, is an isomorphism, and we define $\iota_\alpha : \tau_{\geq 1}(x_a) \to \phi_\alpha(\tau_{\geq 1}(x_b))$ to be the morphism corresponding to the composition

$$\psi_\alpha(x_a) \simeq \psi_\alpha \circ \phi_\alpha(x_b) \overset{counit}{\longrightarrow} x_b \to \tau_{\geq 1}(x_b).$$

Taking the colimit $\mathrm{colim}_a \, \mathrm{ins}_a(S_a)$, we get a fibred sequence

$$x_{\leq 0} := \mathrm{colim}_a \, \mathrm{ins}_a(\tau_{\leq 0}(x_a)) \to x \to x_{\geq 1} := \mathrm{colim}_a \, \mathrm{ins}_a(\tau_{\geq 1}(x_a)).$$

Since $\tau_{\leq 0}(x_a) \in \mathcal{D}_a^{\leq 0}$, the definition of $\mathcal{D}^{\leq 0}$ implies that $x_{\leq 0} \in \mathcal{D}^{\leq 0}$.

Next we show that $x_{\geq 1} \in \mathcal{D}^{\geq 1}$, that is, $ev_b(x_{\geq 1}) \in \mathcal{D}_b^{\geq 1}$ for all b. Since ev_b commutes with all (filtered) colimits, and $\mathcal{D}_b^{\geq 0}$ is closed under filtered colimits, we conclude that $\mathcal{D}^{\geq 1}$ is closed under all filtered colimits. Thus it suffices to show that for every $y_a \in \mathcal{D}_a^{\geq 1}$, we have $\mathrm{ins}_a(y_a) \in \mathcal{D}^{\geq 1}$, that is, we have $ev_b \circ \mathrm{ins}_a(y_a) \in \mathcal{D}_b^{\geq 1}$ for all $b \in \mathcal{I}$.

Since $ev_b \circ \mathrm{ins}_a$ is a filtered colimit $\mathrm{colim}_{\alpha:a\to c, \beta:b\to c} \, \phi_\beta \circ \psi_\alpha$ (see 4.1.4), and $\mathcal{D}_b^{\geq 1}$ is closed under all filtered colimits, it suffices to show that $\phi_\beta \circ \psi_\alpha(y_a) \in \mathcal{D}_b^{\geq 1}$. But this follows from the fact $y_a \in \mathcal{D}_a^{\geq 1}$, while both ϕ_β and ψ_α are left t-exact.

This completes the proof that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure. Moreover, in the course of the proof we shown that ins_a is left t-exact and that $\mathcal{D}^{\geq 0}$ is closed under filtered colimits. Furthermore, ins_a is right t-exact by the definition of $\mathcal{D}^{\leq 0}$.

Assume now that $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$ is another t-structure on $\mathcal{D}$ such that $\mathcal{D}'^{\geq 0}$ is closed under filtered colimits and every functor $\mathrm{ins}_a : \mathcal{D}_a \to \mathcal{D}$ is t-exact. We are going to show that in this case we have inclusions $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}'^{\leq 0}$ and $\mathcal{D}^{\geq 0} \subseteq \mathcal{D}'^{\geq 0}$, therefore both inclusions have to be equalities (say, by Lemma 5.1.2(c)).

First of all, for every $x_a \in \mathcal{D}_a^{\geq 0}$ we have $\mathrm{ins}_a(x_a) \in \mathcal{D}'^{\leq 0}$, because ins_a is t-exact. Since $\mathcal{D}'^{\leq 0}$ is closed under all colimits, the first inclusion follows from the definition of $\mathcal{D}^{\leq 0}$. Next, for every $x \in \mathcal{D}^{\geq 0}$ we have $ev_a(x) \in \mathcal{D}_a^{\geq 0}$ by (5.3), thus $\mathrm{ins}_a(ev_a(x)) \in \mathcal{D}'^{\leq 0}$, because ins_a is t-exact. Hence $x \simeq \mathrm{colim}_a \, \mathrm{ins}_a(ev_a(x)) \in \mathcal{D}'^{\leq 0}$, because $\mathcal{D}'^{\leq 0}$ is closed under filtered colimits. □

5.2. **Extension of t-structures.** Let Cat_{st} be the ∞-category of (small) stable ∞-categories.

Lemma 5.2.1. *In the situation of 1.2.4, assume that we are given a limit-preserving functor $\mathcal{D} : \mathcal{C}^{op} \to \mathrm{Cat}_{st}$, and that $\mathcal{D}(x)$ is equipped with a t-structure for every $x \in \mathrm{Ob}_0(\mathcal{A})$ such that $f^! := \mathcal{D}(f)$ is t-exact for every $f \in \mathrm{Mor}_0^0(\mathcal{A})$. Then*

(a)$_0$. For every 0-geometric x, there exists a unique t-structure on $\mathcal{D}(x)$ such that for every decomposition $x \simeq \sqcup_\alpha x_\alpha$ with $x_\alpha \in \mathrm{Ob}_0(\mathcal{A})$, each pullback $i_\alpha^! : \mathcal{D}(x) \to \mathcal{D}(x_\alpha)$, corresponding to the embedding $i_\alpha : x_\alpha \to x$, is t-exact.

(a)$_{n,n>0}$. For every n-geometric $x \in \mathrm{Ob}(\mathcal{C})$ there exists a unique t-structure on $\mathcal{D}(x)$ such that for every $(n-1)$-special morphism $f : y \to x$ in $\mathrm{Mor}(\mathcal{C})$ with $y \in \mathrm{Ob}_0(\mathcal{A})$ the functor $f^!$ is t-exact.

(b)$_{n,n\geq 0}$. For every n-special morphism $f : y \to x$ in $\mathrm{Mor}(\mathcal{C})$ with n-geometric x, the functor $f^!$ is t-exact.

Proof. (a)$_0$ Recall that every 0-geometric x decomposes as $x \simeq \sqcup_\alpha x_\alpha$ with $x_\alpha \in \mathrm{Ob}_0(\mathcal{A})$. Therefore $\mathcal{D}(x)$ decomposes as a product $\prod_\alpha \mathcal{D}(x_\alpha)$, so there exists a unique t-structure on $\mathcal{D}(x)$ such that every pullback $i_\alpha^! : \mathcal{D}(x) \to \mathcal{D}(x_\alpha)$ is t-exact.

We claim that this t-structure is independent of the decomposition. Indeed, let $x \simeq \sqcup_\beta y_\beta$ be another decomposition with $y_\beta \in \mathrm{Ob}_0(\mathcal{A})$, and let $j_\beta : y_\beta \to x$ be the inclusion. We want to show that the pullback $j_\beta^! : \mathcal{D}(x) \to \mathcal{D}(y_\beta)$ is t-exact.

For every α, β, we set $y_{\alpha,\beta} := y_\beta \times_x x_\alpha$. Then we have a decomposition $y_\beta \simeq \sqcup_\alpha y_{\alpha,\beta}$, which implies that each $y_{\alpha,\beta} \in \mathrm{Ob}_0(\mathcal{A})$ and each embedding $i_{\alpha,\beta} : y_{\alpha,\beta} \hookrightarrow y_\beta$ belongs to $\mathrm{Mor}_0^0(\mathcal{A})$ (see 1.2.2(c)). By symmetry, each $j_{\alpha,\beta} : y_{\alpha,\beta} \hookrightarrow x_\alpha$ belongs to $\mathrm{Mor}_0^0(\mathcal{A})$ as well.

Note that since $\mathcal{D}(y_\beta) \simeq \prod_\alpha \mathcal{D}(y_{\alpha,\beta})$ and each pullback $i_{\alpha,\beta}^! : \mathcal{D}(y_\beta) \to \mathcal{D}(y_{\alpha,\beta})$ is exact (because $i_{\alpha,\beta} \in \mathrm{Mor}_0^0(\mathcal{A})$), in order to show that $j_\beta^!$ is t-exact, it suffices to show that the composition $\mathcal{D}(x) \xrightarrow{j_\beta^!} \mathcal{D}(y_\beta) \xrightarrow{i_{\alpha,\beta}^!} \mathcal{D}(y_{\alpha,\beta})$ is t-exact.

Finally, $i_{\alpha,\beta}^! \circ j_\beta^!$ decomposes as $\mathcal{D}(x) \xrightarrow{i_\alpha^!} \mathcal{D}(x_\alpha) \xrightarrow{j_{\alpha,\beta}^!} \mathcal{D}(y_{\alpha,\beta})$, the pullback $i_\alpha^!$ is t-exact, by the construction of t-structure on $\mathcal{D}(x)$, and $j_{\alpha,\beta}^!$ is t-exact, because $j_{\alpha,\beta} \in \mathrm{Mor}_0^0(\mathcal{A})$.

(b)$_0$ Every 0-special f decomposes as a disjoint union of $f_{\alpha,\beta} : y_{\alpha,\beta} \to x_\alpha$ from $\mathrm{Mor}_0^0(\mathcal{A})$. Since every $f_{\alpha,\beta}^!$ is t-exact by assumption, the t-exactness of $f^!$ follows.

It thus remains to show assertions (a)$_{n+1}$ and (b)$_{n+1}$ for all $n \geq 0$. By induction, we can assume that for every n-geometric x, the category $\mathcal{D}(x)$ is equipped with an t-structure, and for every n-special morphism f between n-geometric objects, the pullback $f^!$ is t-exact. Indeed, for $n = 0$ this follows from assertions (a)$_0$ and (b)$_0$, shown above.

(a)$_{n+1}$. Choose an n-special covering $f : y \to x$ with $y \in \mathrm{Ob}_0(\mathcal{C})$. Then all terms in the Čech resolution $\{y^{[m]}\}_{[m]}$ are n-geometric (see 1.1.8) and all morphisms are n-special. Moreover, since $\mathcal{D}$ is limit preserving, the canonical map $\mathcal{D}(x) \to \lim_{[m]} \mathcal{D}(y^{[m]})$ is an equivalence. By the induction hypothesis, each $\mathcal{D}(y^{[m]})$ are equipped with a t-structure, and each transition maps are t-exact. Therefore it follows from Lemma 5.1.3(b) that there exists a unique t-structure on $\mathcal{D}(x)$ such that all $(f^{[m]})^!$ are t-exact.

In particular, the pullback $f^!$ is t-exact and faithful, thus the t-structure on $\mathcal{D}(x)$ is uniquely characterized by the property that $f^!$ is t-exact (by Lemma 5.1.2(c)). Furthermore, it follows from (a)$_0$ that if $y \simeq \sqcup_\alpha y_\alpha$ with $y_\alpha \in \mathrm{Ob}_0(\mathcal{A})$, and $f_\alpha := f \circ i_\alpha : y_\alpha \to x$, this t-structure is characterized by the property that each $f_\alpha^!$ is t-exact. Since every f_α is n-special, the uniqueness property follows.

By the above observation, it suffices to show that for every n-special morphism $f' : y' \to x$ with 0-geometric y', the pullback $f'^! : \mathcal{D}(x) \to \mathcal{D}(y')$ is t-exact. Set $y'' := y \times_x y'$. Then $\mathrm{pr}' : y'' \to y'$ is an n-special covering between n-geometric objects, hence the pullback $\mathrm{pr}'^!$ is t-exact and faithful by induction. Thus, by Lemma 5.1.2(d), it suffices to show that the composition $\mathcal{D}(x) \to \mathcal{D}(y') \to \mathcal{D}(y'')$ is t-exact. But this composition decomposes as $\mathcal{D}(x) \overset{f^!}{\to} \mathcal{D}(y) \overset{\mathrm{pr}^!}{\to} \mathcal{D}(y'')$, which is t-exact, because $f^!$ is t-exact by construction, and $\mathrm{pr}^!$ is exact because $\mathrm{pr} : y'' \to y$ is an n-special morphism between n-geometric objects.

(b)$_{n+1}$ Since x is $(n+1)$-geometric, and f is $(n+1)$-special, we conclude that y is $(n+1)$-geometric (by Lemma 1.1.4(b)). Choose an n-special covering $g : z \to y$ with 0-geometric z. Thus, by the characterization of the t-structure on $\mathcal{D}(y)$, the pullback $g^!$ is t-exact and faithful. Thus, by Lemma 5.1.2(d), it remains to show that the pullback $(f \circ g)^!$ is t-exact. If $f \circ g : z \to x$ is n-special, the assertion follows from the characterizing property of the t-structure on $\mathcal{D}(x)$. In particular, this finishes the proof in the case when f is n-special.

In the general case, let $h : t \to x$ be an n-special covering with 0-geometric t. Then $t \times_x y \to t$ is $(n+1)$-special, while $t \times_x y \to y$ is an n-special covering. Hence the pullback $\mathcal{D}(y) \to \mathcal{D}(t \times_x y)$ is t-exact and faithful by the proven above, thus it suffices to show the t-exactness of the composition $\mathcal{D}(x) \to \mathcal{D}(y) \to \mathcal{D}(t \times_x y)$, or, what is the same, of $\mathcal{D}(x) \xrightarrow{h^!} \mathcal{D}(t) \xrightarrow{\widetilde{f}^!} \mathcal{D}(t \times_x y)$. Since $h^!$ is t-exact by $(\mathrm{a})_{n+1}$, while $\widetilde{f}^!$ is t-exact by the particular case, shown above, the assertion follows. □

Corollary 5.2.2. *In the situation Lemma 5.2.1,*

(a) For every geometric $x \in \mathcal{C}$, there exists a unique t-structure on $\mathcal{D}(x)$ such that for every special $f : y \to x$ with $y \in \mathrm{Ob}_0(\mathcal{A})$ the functor $f^!$ is t-exact.

(b) For every special morphism $f : y \to x$ with geometric x, the functor $f^!$ is t-exact.

Proof. First of all the uniqueness assertion in (a) follows immediately from Lemma 5.2.1(a), so it suffices to construct t-structures, which satisfy (b).

For every geometric x, choose n such that x is n-geometric, and equip $\mathcal{D}(x)$ with the t-structure from Lemma $5.2.1(\mathrm{a})_n$. We claim that this t-structure is independent of n. Notice that x is m-geometric for all $m > n$ (by Lemma 1.1.4(d)), so it suffices to show that the t-structure on $\mathcal{D}(x)$ from Lemma $5.2.1(\mathrm{a})_m$ satisfies the property of the t-structure on $\mathcal{D}(x)$ from Lemma $5.2.1(\mathrm{a})_n$. If $n > 0$, this follows from the fact that every $(m-1)$-special morphism is $(n-1)$-special (by Lemma 1.1.4(d)), while for $n = 0$, this follows from the fact that every embedding $x_{\alpha'} \hookrightarrow \sqcup_\alpha x_\alpha$ is 0-special (by 1.2.2(c)).

Finally, for every special morphism $f : y \to x$ with geometric x choose n such that x are n-geometric and f is n-special. Then $f^!$ is t-exact by Lemma 5.2.1(b). □

Lemma 5.2.3. *In the situation of 1.3.1, assume that we are given a functor $\mathcal{D} : \mathcal{A}^{op} \to \mathrm{Cat}_{st}$ preserving filtered colimits and that $\mathcal{D}(x)$ is equipped with t-structure for every $x \in \mathrm{Ob}(\mathcal{B})$ such that*

(i) functor $f^!$ is t-exact for every $f \in \mathcal{P} \subset \mathrm{Mor}(\mathcal{B})$.

(ii) for every $x \in \mathrm{Ob}_0(\mathcal{A})$ with two presentations $x \simeq \lim_\alpha x_\alpha$ and $x \simeq \lim_\beta x'_\beta$ as in 1.3.1(b) and every β there exists α such that the projection $\mathrm{pr}'_\beta : x \to x'_\beta$ factors through a morphism $f : x_\alpha \to x'_\beta$, whose pullback $f^!$ is left t-exact.

Then

(a) For every $x \in \mathrm{Ob}_0(\mathcal{A})$ there exists a unique t-structure on $\mathcal{D}(x)$ such that for every morphism $(f : x \to y) \in \mathrm{Mor}_0^0(\mathcal{A})$ with $y \in \mathrm{Ob}(\mathcal{B})$, the functor $f^! : \mathcal{D}(y) \to \mathcal{D}(x)$ is t-exact.

(b) Moreover, for every morphism $f \in \mathrm{Mor}_0^0(\mathcal{A})$, the pullback $f^!$ is t-exact.

Proof. (a) Fix a presentation $x \simeq \lim_\alpha x_\alpha$ as in 1.3.1(b). Since $\mathcal{D}$ commutes with filtered colimits, the natural map $\mathrm{colim}_\alpha \mathcal{D}(x_\alpha) \to \mathcal{D}(x)$ is an equivalence. Hence, by Lemma 5.1.3(a) there exists a unique t-structure $(\mathcal{D}^{\leq 0}(x), \mathcal{D}^{\geq 0}(x))$ on $\mathcal{D}(x)$ such that the pullback $\mathrm{pr}_\alpha^! : \mathcal{D}(x_\alpha) \to \mathcal{D}(x)$ is t-exact for every α. Explicitly, $\mathcal{D}^{\geq 0}(x)$ is the essential image of $\mathrm{colim}_\alpha \mathcal{D}^{\geq 0}(x_\alpha) \to \mathcal{D}(x)$ and similarly for $\mathcal{D}^{\leq 0}(x)$.

We claim that this t-structure is independent of the presentation.

Let $x \simeq \lim_\beta x'_\beta$ be another presentation. Then, by (ii), every projection $\mathrm{pr}'_\beta : x \to x'_\beta$ decomposes as $x \xrightarrow{\mathrm{pr}_\alpha} x_\alpha \xrightarrow{f} x'_\beta$, with left t-exact $f^!$. Therefore $\mathrm{pr}'^!_\beta = \mathrm{pr}^!_\alpha \circ f^!$ is left t-exact as well, that is, we have an inclusion

$$\mathrm{pr}'^!_\beta(\mathcal{D}^{\geq 0}(x'_\beta)) \subset \mathcal{D}^{\geq 0}(x) \text{ for all } \beta. \tag{5.4}$$

Now let $(\mathcal{D}'^{\leq 0}(x), \mathcal{D}'^{\geq 0}(x))$ be the t-structure on $\mathcal{D}(x)$, corresponding to the presentation $x \simeq \lim_\beta x'_\beta$, and we want to show that it coincides with $(\mathcal{D}^{\leq 0}(x), \mathcal{D}^{\geq 0}(x))$. By Lemma 5.1.2(e), it suffices to show that $\mathcal{D}'^{\geq 0}(x) = \mathcal{D}^{\geq 0}(x)$. Since the inclusion $\mathcal{D}'^{\geq 0}(x) \subset \mathcal{D}^{\geq 0}(x)$ follows from (5.4), and the opposite inclusion follows by symmetry, we get the assertion.

Now let us show that the t-structure we constructed is the unique t structure such that $f^!$ is t-exact for every $(f : x \to y) \in \mathrm{Mor}^0_0(\mathcal{A})$ with $y \in \mathrm{Ob}(\mathcal{B})$. First of all, for such an f, there is a presentation $x \simeq \lim_\alpha x_\alpha$ such that $f = \mathrm{pr}_{\alpha_0}$ for some α_0. In particular, $f^!$ is t-exact. Conversely, since $\mathrm{pr}_\alpha : x \to x_\alpha$ belongs to $\mathrm{Mor}^0_0(\mathcal{A})$ for every α, and our t-structure is the unique t-structure for which all $\mathrm{pr}^!_\alpha$ are t-exact, and the proof is complete.

(b) We want we show that for every $A \in \mathcal{D}^{\leq 0}(y)$ we have $f^! A \in \mathcal{D}^{\leq 0}(x)$ and similarly for $\mathcal{D}^{\geq 0}$. Choose a presentation $y \simeq \lim_\alpha y_\alpha$. Then, by construction, $\mathcal{D}^{\leq 0}(y) \simeq \mathrm{colim}_\alpha \mathcal{D}^{\leq 0}(y_\alpha)$, thus $A \simeq \mathrm{pr}^!_\alpha(A_\alpha)$ for some $A_\alpha \in \mathcal{D}^{\leq 0}(y_\alpha)$. Thus it suffices to show that $f^! \circ \mathrm{pr}^!_\alpha = (\mathrm{pr}_\alpha \circ f)^!$ is t-exact.

But $\mathrm{pr}_\alpha \in \mathrm{Mor}^0_0(\mathcal{A})$ by the definition of $\mathrm{Mor}^0_0(\mathcal{A})$, and $f \in \mathrm{Mor}^0_0(\mathcal{A})$, by assumption. Therefore $\mathrm{pr}_\alpha \circ f \in \mathrm{Mor}^0_0(\mathcal{A})$, and $(\mathrm{pr}_\alpha \circ f)^!$ is t-exact by (a). □

5.3. *t*-structures on schemes of finite type over *k*.

5.3.1. Classical (middle-dimensional) perverse *t*-structures.

(a) For a scheme Y of finite type over k we denote by $({}^{p_{cl}}\mathcal{D}^{\leq 0}_c(Y), {}^{p_{cl}}\mathcal{D}^{\geq 0}_c(Y))$ the classical, that is, middle dimensional perverse t-structure on $\mathcal{D}_c(Y)$.

(b) Let $f : X \to Y$ be a morphism of schemes of finite type over k such that all non-empty fibers of f are of dimension $\leq d$. Then functors $f^*[d]$ and $f_![d]$ are right t-exact, that is, preserve $\mathcal{D}^{\leq 0}_c$, while $f^![-d]$ and $f_*[-d]$ are left t-exact (see [BBD, 4.2.4]).

5.3.2. Glueing of *t*-structures. Let Y be a scheme of finite type over k, and assume that we are given a stratification $Y = \cup_\alpha Y_\alpha$ of Y by locally closed subschemes, and let $\eta_\alpha : Y_\alpha \to Y$ be the embedding. Suppose that we are given a t-structure $(\mathcal{D}^{\leq 0}_c(X_\alpha), \mathcal{D}^{\geq 0}_c(Y_\alpha))$ on each Y_α. Then by the gluing lemma [BBD, Thm. 1.4.10] and induction on the number of strata, there exists the unique t-structure $(\mathcal{D}^{\leq 0}_c(Y), \mathcal{D}^{\geq 0}_c(Y))$ on $\mathcal{D}_c(Y)$ such that all functors η^*_α are right t-exact, and all functors $\eta^!_\alpha$ are left t-exact.

Explicitly, for $K \in \mathcal{D}_c(Y)$, we have $K \in \mathcal{D}_c^{\leq 0}(Y)$ (resp. $K \in \mathcal{D}_c^{\geq 0}(Y)$) if and only if $\eta_\alpha^* K \in \mathcal{D}_c^{\leq 0}(Y_\alpha)$ (resp. $\eta_\alpha^! K \in \mathcal{D}_c^{\geq 0}(Y_\alpha)$) for all α.

5.3.3. !-adapted perverse t-structure (see Remark 5.3.6 below for the explanation of the term). Let Y be a scheme of finite type over k.

(a) Assume that Y is equidimensional of dimension d. We define ${}^p\mathcal{D}_c^{\leq 0}(Y)$ (resp. ${}^p\mathcal{D}_c^{\geq 0}(Y)$) be the set of all $K \in \mathcal{D}_c(Y)$ such that $K[-d]$ belongs to ${}^{p_{cl}}\mathcal{D}_c^{\leq 0}(Y)$ (resp. ${}^{p_{cl}}\mathcal{D}_c^{\geq 0}(Y)$). In other words, $({}^p\mathcal{D}_c^{\leq 0}(Y), {}^p\mathcal{D}_c^{\geq 0}(Y))$ is $({}^{p_{cl}}\mathcal{D}_c^{\leq -d}(Y), {}^{p_{cl}}\mathcal{D}_c^{\geq -d}(Y))$, that is, the classical perverse t-structure, shifted by $\dim Y$ to the left.

(b) Let now Y be arbitrary, and let Y_i be the canonical equidimensional stratification from 3.1.1(c). We define ${}^p\mathcal{D}_c^{\leq 0}(Y)$ (resp. ${}^p\mathcal{D}_c^{\geq 0}(Y)$) to be the set of all $K \in \mathcal{D}_c(Y)$ such that $\eta_i^* K \in {}^p\mathcal{D}_c^{\leq 0}(Y_i)$ (resp. $\eta_i^! K \in {}^p\mathcal{D}_c^{\geq 0}(Y_i)$) for all i. Then $({}^p\mathcal{D}_c^{\leq 0}(Y), {}^p\mathcal{D}_c^{\geq 0}(Y))$ is t-structure by the gluing lemma (see 5.3.2).

5.3.4. Renormalized $*$-pullback. Let $X \in \mathrm{Aff}_k^{ft}$, and $K \in \mathcal{D}_c(X)$.

(a) For every $d \in \mathbb{Z}$ we set $K\langle d\rangle := K[2d](d) \in \mathcal{D}(X)$. More generally, to every locally constant function $\underline{d} : X \to \mathbb{Z}$, we associate an object $K\langle \underline{d}\rangle \in \mathcal{D}(X)$ such that for every connected component $X^0 \subset X$, we have $K\langle \underline{d}\rangle|_{X^0} := K|_{X^0}\langle \underline{d}(X_0)\rangle$.

(b) For every weakly equidimensional morphism $f : X \to Y$ in Aff_k^{ft}, we define functor $f^{*,ren} : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$ by $f^{*,ren}(K) := f^*(K)\langle \underline{\dim}_f\rangle$.

Lemma 5.3.5. *(a) Let $f : X \to Y$ be an equidimensional morphism in Aff_k^{ft}. Then $f^{*,ren}$ is right t-exact, while $f^!$ is left t-exact.*

(b) If $f : X \hookrightarrow Y$ is a weakly equidimensional locally closed embedding of dimension $-d$ (see 3.1.2(d)), then the pullback $f^*[-d] : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$ (resp. $f^![-d] : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$) are right (rest. left) t-exact.

(c) If $f : X \to Y$ is uh-smooth, then the pullback $f^!$ is t-exact. In particular, $f^!$ is t-exact, if f is smooth or a universal homeomorphism.

Proof. (a) Replacing X by its connected component, we can assume that f is equidimensional, that is, there exists $d \in \mathbb{N}$ such that $\underline{\dim}_f(x) = d$ for all $x \in X$. Then $f^{*,ren} = f^*\langle d\rangle$, all non-empty fibers of f are equidimensional of dimension d, and morphism f induces a morphism $f_i : X_i \to Y_{i-d}$ for all i. We want to show that for every $K \in {}^p\mathcal{D}_c^{\leq 0}(Y)$ we have $f^{*,ren}(K) \in {}^p\mathcal{D}_c^{\leq 0}(X)$.

Assume first that Y is equidimensional, and hence X is equidimensional as well. Then our assumption $K \in {}^p\mathcal{D}_c^{\leq 0}(Y) = {}^{p_{cl}}\mathcal{D}_c^{\leq -\dim Y}(Y)$ implies (by 5.3.1(b)) that $f^*(K) \in {}^{p_{cl}}\mathcal{D}_c^{\leq d-\dim Y}(X) = {}^p\mathcal{D}_c^{\leq d+\dim X-\dim Y}(X)$. Since $\dim X - \dim Y = d$, this implies that $f^*(K) \in {}^p\mathcal{D}_c^{\leq 2d}(X)$, thus $f^{*,ren}(K) \in {}^p\mathcal{D}_c^{\leq 0}(X)$, as claimed. In particular, the assertion holds for each morphism $f_i : X_i \to Y_{i-d}$.

In the general case, our assumption $K \in {}^p\mathcal{D}_c^{\leq 0}(Y)$ implies that $\eta_{i-d}^* K \in {}^p\mathcal{D}_c^{\leq 0}(Y_{i-d})$ for all i. Therefore, by the assertion for f_i, we conclude that

$$\eta_i^*(f^{*,ren}(K)) \simeq \eta_i^*(f^*(K))\langle d\rangle \simeq f_i^*(\eta_{i-d}^* K)\langle d\rangle \simeq f_i^{*,ren}(\eta_{i-d}^* K) \in {}^p\mathcal{D}_c^{\leq 0}(X_i)$$

for all i, thus $f^{*,ren}(K) \in {}^p\mathcal{D}_c^{\leq 0}(X)$. The proof of the assertion for $f^!$ is similar.

(b) The argument is similar to (a) but simpler. Namely, as in (a), one reduces to the case when Y is equidimensional. In this case, the assertion follows from the fact that f^* (resp. $f^!$) is right (resp. left) t-exact for the classical perverse t-structure.

(c) If f is smooth or a universal homeomorphism, then f is equidimensional, and we have a canonical isomorphism $f^{*,ren} \xrightarrow{\sim} f^!$. Thus the assertion follows from (a).

To show the assertion for uh-smooth morphisms, we have to check that morphisms f for which $f^!$ is t-exact satisfy properties (i),(ii) of 2.4.1. Now (i) was shown above, while (ii) follows from Lemma 5.1.2(d) together with the observation that if π is étale surjective or a universal homeomorphism, then $\pi^!$ is faithful and t-exact (by above). □

5.3.6. Remarks. (a) The reason why we consider this t-structure rather than the standard one is to guarantee that for (uh)-smooth morphisms the !-pullback is t-exact. This will enable us to define perverse t-structures on (topologically) placid ∞-stacks later.

(b) By a standard argument, one can show that if f is a uh-smooth morphism, then we have a canonical isomorphism $f^{*,ren} \xrightarrow{\sim} f^!$. This would give a slightly more conceptual explanation why $f^!$ is t-exact in this case. Moreover, the same applies to a more general class of so-called *cohomologically smooth* morphisms.

We finish this section by a partial generalization of Lemma 5.3.5(a).

Lemma 5.3.7. *If all non-empty fibers of $f : X \to Y$ are of dimension $\leq d$, then the functor $f^*[2d]$ is right t-exact.*

Proof. Assume first that $f : X \hookrightarrow Y$ is a locally closed embedding. In this case, we have to show that f^* is right t-exact, that is, $f^*(K) \in {}^p\mathcal{D}_c^{\leq 0}(X)$ for all $K \in {}^p\mathcal{D}_c^{\leq 0}(Y)$.

Observe that there exists a constructible stratification X_α of X such that both embeddings $\eta_\alpha : X_\alpha \hookrightarrow X$ and $f \circ \eta_\alpha : X_\alpha \hookrightarrow Y$ are weakly equidimensional of constant dimensions. Indeed, let X_i and Y_j be the the canonical equidimensional stratifications from 3.1.1(c), and take $\{X_\alpha\}_\alpha$ be the union of the canonical stratifications of $X_i \cap Y_j$. Since $f^*(K)$ is an extension of $(\eta_\alpha)_! \eta_\alpha^* f^*(K)$, it suffices to show that $(\eta_\alpha)_! \eta_\alpha^* f^*(K) \in {}^p\mathcal{D}_c^{\leq 0}(X)$ for all α.

We let $-a_\alpha$ and $-b_\alpha$ be the dimensions $\underline{\dim}_{\eta_\alpha}$ and $\underline{\dim}_{f\circ\eta_\alpha}$, respectively. By Lemma 5.3.5(b) and adjunction, we conclude that $(f \circ \eta_\alpha)^*$ and $(\eta_\alpha)_!$ send ${}^p\mathcal{D}_c^{\leq 0}$ to ${}^p\mathcal{D}_c^{\leq -b_\alpha}$ and ${}^p\mathcal{D}_c^{\leq a_\alpha}$, respectively. Since $X_\alpha \subset X \subset Y$, we conclude that $a_\alpha \leq b_\alpha$, thus $(\eta_\alpha)_! \eta_\alpha^* f^*(K) \in {}^p\mathcal{D}_c^{\leq 0}(X)$, as claimed.

Assume now that X and Y are equidimensional. Then the argument of Lemma 5.3.5(a) implies that for every $K \in {}^p\mathcal{D}_c^{\leq 0}(Y)$ we have $f^*(K) \in {}^p\mathcal{D}_c^{\leq d+\dim X-\dim Y}(X)$. Since for every $x \in X$ we have $\dim X - \dim Y = \underline{\dim}_f(x) \leq \dim_x f^{-1}(f(x)) \leq d$, we thus have $f^*(K) \in {}^p\mathcal{D}_c^{\leq 2d}(X)$, as claimed.

The general case now follows from the two cases shown above. Indeed, let $\eta_i : X_i \hookrightarrow X$ be as in 3.1.1(c). Since $f^*(K) \in {}^p\mathcal{D}_c^{\leq 2d}(X)$ if and only if $\eta_i^* f^*(K) \in {}^p\mathcal{D}_c^{\leq 2d}(X_i)$ for every i, replacing f by $f \circ \eta_i$, we may assume that X is equidimensional. Then, by a similar argument, we can assume that X is irreducible. Then the closure $\overline{f(X)} \subset Y$ is irreducible, hence equidimensional, and f decomposes as $X \xrightarrow{g} \overline{f(X)} \xrightarrow{\iota} Y$.

Finally, since ι^* and $g^*[2d]$ are right t-exact, by the particular cases, shown above, their composition $f^*[2d]$ is right t-exact as well. □

5.4. **Perverse t-structures on globally uh-placid affine schemes.** Our goal now it to apply the results from the previous two subsections to define perverse t-structures globally placid affine schemes and its uh/perfect analogs.

Proposition 5.4.1. *For every $Y \in \mathrm{Aff}_k^{ft}$, we equip the category $\mathcal{D}_c(Y)$ is equipped with the perfect t-structure, defined in 5.3.3. Then*

(a) For every globally uh-placid affine scheme X, there exists a unique t-structure on $\mathcal{D}_c(X)$ such that for every strongly pro-uh-smooth morphism $f : X \to Y$ with $Y \in \mathrm{Aff}_k^{ft}$, the pullback $f^! : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$ is t-exact.

(b) Moreover, the t-structures from (a) satisfy the property that for every strongly pro-uh-smooth morphism $f : X \to Y$ between globally uh-placid affine schemes, the pullback $f^! : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$ is t-exact.

Proof. Assume that we are in the situation of 2.4.4, that is, $\mathcal{B} := \mathrm{Aff}_k^{ft}$, $\mathcal{A} = \mathrm{Aff}_k$, and $\mathcal{P} = \mathcal{P}_{uh-sm}$ is the class of uh-smooth morphisms. We would like to apply Lemma 5.2.3 to the data consisting of the functor $\mathcal{D}_c : (\mathrm{Aff}_k)^{op} \to \mathrm{Cat}_{st,\ell}$ from 4.2.1 and perverse t-structures on $\mathcal{D}_c(Y)$ constructed in 5.3.3. It remains to check that all the assumptions Lemma 5.2.3 are satisfied. Now, $\mathcal{D}_c$ commutes with filtered colimits, because it defined as a left Kan extension, and (i) follows from Lemma 5.3.5(c). Since every uh-smooth morphisms are universally open and equidimensional (by Lemma 3.1.10), the assertion follows from a combination of Lemma 3.2.4 and Lemma 5.3.5(a). Now Lemma 5.2.3 applies, and the assertion follows. □

We will apply the above construction in the case of globally placid (and perfectly placid) affine schemes.

5.4.2. Two particular cases. Since a globally placid (and perfectly placid) affine schemes X are globally uh-placid, Proposition 5.4.1(a) provides a t-structure on $\mathcal{D}_c(X)$ in both these cases. Moreover, since every strongly pro-(perfectly) smooth morphism $f : X \to Y$ between globally (perfectly) placid affine schemes is strongly pro-uh-smooth, the pullback $f^!$ is t-exact by Proposition 5.4.1(b).

Lemma 5.4.3. *Let X is a globally placid affine scheme.*

(a) The perverse t-structure on $\mathcal{D}_c(X)$ from 5.4.2 can be characterized as the unique t-structure on $\mathcal{D}_c(X)$ such that for every strongly pro-smooth morphism $f: X \to Y$ with $Y \in \mathrm{Aff}_k^{ft}$, the pullback $f^! : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$ is t-exact.

(b) The perfection X_{perf} is a globally perfectly placid affine scheme, and the pullback $\pi^! : \mathcal{D}_c(X) \to \mathcal{D}_c(X_{\mathrm{perf}})$, corresponding to the projection $\pi : X_{\mathrm{perf}} \to X$ is t-exact.

Proof. (a) Choose a placid presentation $X \simeq \lim_\alpha X_\alpha$. Since every projection $\mathrm{pr}_\alpha : X \to X_\alpha$ is strongly pro-smooth, so it follows from Proposition 5.4.1 that every pullback $\mathrm{pr}_\alpha^! : \mathcal{D}_c(X_\alpha) \to \mathcal{D}_c(X)$ is t-exact. On the other hand, it follows from Lemma 5.1.3(a), that there exists a unique t-structure on $\mathcal{D}_c(X) \simeq \mathrm{colim}_\alpha \mathcal{D}_c(X_\alpha)$, satisfying this property.

(b) Since $\pi : X_{\mathrm{perf}} \to X$ is strongly pro-uh-smooth (by Lemma 2.4.7(a)), the assertion follows from Proposition 5.4.1(b). □

5.4.4. Remark. It follows from Lemma 5.4.3(a), that t-structures on globally placid affine schemes can be also constructed directly by applying Lemma 5.2.3 in the situation of 2.1.1.

5.4.5. Perverse t-structures on $\mathcal{D}(X)$.

(a) Recall that for every affine scheme X, the ∞-category $\mathcal{D}(X)$ is the ind-category $\mathrm{Ind}\,\mathcal{D}_c(X)$ (see 4.2.1). Therefore every t-structure $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ on $\mathcal{D}_c(X)$ gives rise to a unique t-structure $({}^p\mathcal{D}^{\leq 0}(X), {}^p\mathcal{D}^{\geq 0}(X))$ on $\mathcal{D}(X)$ such that ${}^p\mathcal{D}^{\geq 0}(X) = \mathrm{Ind}({}^p\mathcal{D}_c^{\geq 0}(X))$ and similarly for ${}^p\mathcal{D}^{\leq 0}(X)$ (see Lemma 5.1.2(a)). In particular, the subcategory ${}^p\mathcal{D}^{\leq 0}(X) \subset \mathcal{D}(X)$ is closed under filtered colimits, and for every morphism $f : X \to Y$ such that the pullback $f^! : \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$ is t-exact, the corresponding functor $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$ is t-exact as well.

(b) By (a), for every globally (uh)-placid affine scheme X, the perverse t-structure $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ on $\mathcal{D}_c(X)$ defined in Proposition 5.4.1 (or 5.4.4) gives rise to the perverse t-structure $({}^p\mathcal{D}^{\leq 0}(X), {}^p\mathcal{D}^{\geq 0}(X))$ on $\mathcal{D}(X)$.

5.5. **Perverse t-structures on perfect and topologically placid ∞-stacks.** We will write $\mathcal{D}_.(\mathcal{X})$ to refer both to $\mathcal{D}_c(\mathcal{X})$ and $\mathcal{D}(\mathcal{X})$.

Proposition 5.5.1. *For every globally perfectly placid affine scheme X, we equip $\mathcal{D}_.(X)$ with t-structure, constructed in 5.4.2 and 5.4.5.*

(a) For every perfectly placid ∞-stack $\mathcal{X}$, there exists a unique t-structure on $\mathcal{D}_.(\mathcal{X})$ such that for every perfectly smooth morphism $f : X \to \mathcal{X}$ from a globally perfectly placid affine scheme X, the pullback $f^!$ is t-exact.

(b) Moreover, the t-structures from (a) satisfy the property that for every perfectly smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ between perfectly placid ∞-stacks, the pullback $f^!$ is t-exact.

Proof. In the notation of 2.4.6, take $\mathcal{A} = \mathrm{Aff}_{\mathrm{perf},k}$ equipped with étale topology, while $\mathrm{Ob}_0(\mathcal{A})$ and $\mathrm{Mor}_0^0(\mathcal{A})$ are the classes of globally perfectly placid affine schemes and strongly pro-perfectly smooth morphisms. Then the construction of 1.2.4 applies, and $\mathcal{C}$ is the ∞-category $\mathrm{St}_{\mathrm{perf},k}$ of perfect ∞-stacks over k, which identify with $(\mathrm{St}_k)_{\mathrm{perf}} \subset \mathrm{St}_k$ as in 2.3.5(e).

We would like to apply Lemma 5.2.1 to the restriction $\mathcal{D}.|_{(\mathrm{St}_k)_{\mathrm{perf}}}$ of the functor $\mathcal{D}. : (\mathrm{St}_k)^{op} \to \mathrm{Cat}_{st}$ from 4.3.1. By construction, it commutes with limits. Moreover, by Proposition 5.4.1(b) that the assumption of Lemma 5.2.1 is satisfied, and the assertion follows. □

5.5.2. Perverse t-structures for topologically placid ∞-stacks.

(a) By definition, if $\mathcal{X}$ is a topologically placid ∞-stack, then its perfection $\mathcal{X}_{\mathrm{perf}}$ is a perfectly placid ∞-stack, thus $\mathcal{D}.(\mathcal{X}_{\mathrm{perf}})$ is equipped with a t-structure by Proposition 5.5.1(a).

(b) Recall that the pullback $\pi^! : \mathcal{D}.(\mathcal{X}) \to \mathcal{D}.(\mathcal{X}_{\mathrm{perf}})$ is an equivalence of categories (see Lemma 4.3.7). Thus there exists a unique t-structure on $\mathcal{D}.(\mathcal{X})$ such that the pullback $\pi^!$ is t-exact.

(c) Let $f : \mathcal{X} \to \mathcal{Y}$ be a topologically smooth morphism between topologically placid ∞-stacks. Then $f_{\mathrm{perf}} : \mathcal{X}_{\mathrm{perf}} \to \mathcal{Y}_{\mathrm{perf}}$ is a perfectly smooth morphism between perfectly placid ∞-stacks, hence $f^!_{\mathrm{perf}}$ is t-exact by Proposition 5.5.1(b). Thus, by the definition of t-structures in (b), the pullback $f^! : \mathcal{D}.(\mathcal{Y}) \to \mathcal{D}.(\mathcal{X})$ is t-exact.

(d) Notice that if $\mathcal{X}$ is perfectly placid, then $\mathcal{X}$ is perfect, hence the projection $\pi : \mathcal{X}_{\mathrm{perf}} \to \mathcal{X}$ is the equivalence between perfectly placid ∞-stacks. Thus, by Proposition 5.5.1(b), the t-structure on $\mathcal{D}.(\mathcal{X})$, given in (b), coincides with the t-structure from $\mathcal{D}.(\mathcal{X})$ from Proposition 5.5.1(a).

(e) For every topologically placid ∞-stack $\mathcal{X}$, the subcategory ${}^p\mathcal{D}^{\geq 0}(\mathcal{X}) \subset \mathcal{D}(\mathcal{X})$ is closed under filtered colimits. Indeed, choose a topologically smooth covering $f : X \to \mathcal{X}$, where X is a disjoint union of globally perfectly placid affine schemes. Since $f^!$ is t-exact, faithful and commutes with colimits, we reduce the problem to the case when $\mathcal{X}$ is a globally perfectly placid affine scheme (compare Lemma 5.1.2(c). In this case, the assertion follows by construction (see 5.4.5).

5.5.3. Perverse t-structures on placid ∞-stacks. Since every placid ∞-stack $\mathcal{X}$ is topologically placid, the construction of 5.5.2(b) provides $\mathcal{D}.(\mathcal{X})$ with a t-structure. Moreover, since every smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ between placid affine schemes is topologically smooth, the pullback $f^!$ is t-exact by 5.5.2(c).

Lemma 5.5.4. *For every globally placid affine scheme X, we equip $\mathcal{D}.(X)$ with t-structure, constructed in 5.4.2 and 5.4.5, and let $\mathcal{X}$ be a placid ∞-stack.*

Then the perverse t-structure on $\mathcal{D}.(\mathcal{X})$ from 5.5.3 can be characterized as the unique t-structure on $\mathcal{D}.(\mathcal{X})$ such that for every smooth morphism $f : X \to \mathcal{X}$ from a globally placid affine scheme X, the pullback $f^!$ is t-exact.

Proof. Choose a smooth covering $f : X \to \mathcal{X}$, where X is a disjoint union $X \simeq \sqcup_\alpha X_\alpha$ of globally placid affine schemes, and let $f_\alpha : X_\alpha \to \mathcal{X}$ be the restriction of f. Then, by Lemma 5.1.2(c) (or Lemma 5.2.1), there exists at most one t-structure on $\mathcal{X}$ such that all pullbacks $f_\alpha^!$ are t-exact.

It now remains to show that for every smooth morphism $f : X \to \mathcal{X}$ from a globally placid affine scheme X, the pullback $f^! : \mathcal{D}.(\mathcal{X}) \to \mathcal{D}.(X)$ is t-exact. Note that since $\pi^! : \mathcal{D}.(X) \to \mathcal{D}.(X_{\text{perf}})$ is an equivalence and t-exact (by Lemma 5.4.3(b)), it suffices to show that the composition $\pi^! \circ f^! : \mathcal{D}.(\mathcal{X}) \to \mathcal{D}.(X_{\text{perf}})$ is t-exact. Since this composition can be rewritten as a composition

$$\mathcal{D}.(\mathcal{X}) \xrightarrow{\pi^!} \mathcal{D}.(\mathcal{X}_{\text{perf}}) \xrightarrow{f^!_{\text{perf}}} \mathcal{D}.(X_{\text{perf}}),$$

the first map of which is t-exact by the definition of t-structure in 5.5.2(b). Finally, our assumption on f implies that $f_{\text{perf}} : X_{\text{perf}} \to \mathcal{X}_{\text{perf}}$ is a perfectly smooth morphism from a global perfectly placid affine scheme (by Corollary 2.4.8), thus $f^!_{\text{perf}}$ is t-exact by Proposition 5.5.1. □

5.5.5. Remarks. It follows from Lemma 5.5.4, that t-structures on placid ∞-stacks can be also constructed directly by applying Lemma 5.2.1 in the situation of 2.1.7.

5.6. ***t*-exactness properties.**

Lemma 5.6.1. *(a) Let $\mathcal{X}$ be a topologically placid ∞-stack. Then $\omega_{\mathcal{X}} \in {}^p\mathcal{D}^{\geq 0}(\mathcal{X})$.*

(b) Let $f : \mathcal{X} \to \mathcal{Y}$ be a equidimensional morphism (see 3.2.8(a)) of topologically placid ∞-stacks. Then the functor $f^!$ is left t-exact.

(c) Let $f : \mathcal{X} \to \mathcal{Y}$ be an fp-proper morphism of topologically placid ∞-stacks, equidimensional of relative dimension d. Then the functor $f^*\langle d\rangle$ is right t-exact.

(d) Let $f : \mathcal{X} \to \mathcal{Y}$ be a decomposable (see 4.5.10 and remark 5.6.2) fp-locally closed embedding of topologically placid ∞-stacks of relative dimension $-d$. Then the pullback $f^*[-d]$ (resp. $f^![-d]$) is right (resp. left) t-exact.

(e) In the situation of (d), assume that $\mathcal{Y}$ is topologically smooth. Then $f^*(\omega_{\mathcal{Y}}) \in {}^p\mathcal{D}^{\leq -2d}(\mathcal{X})$.

5.6.2. Remark. The only reason we assume that f is decomposable in (d) is because we showed the existence of f^* only in this case (see 4.5.10(e)). In particular, using remark 4.5.11(d) one can show that assertion (d) holds without this assumption as well.

Proof. For shortness, we will omit the word "topological" and will show all assertion in the placid case only.

(a) Assume first that $X \in \text{Aff}_k^{ft}$, and let $\pi : X \to \text{pt}$ be the projection. If X is locally equidimensional, then $\omega_X = \pi^!(\omega_{\text{pt}}) \in {}^p\mathcal{D}^{\geq 0}(X)$ by Lemma 5.3.5(a). In the general case, let X_i be the equidimensional stratification from 3.1.1(c). Since the assertion holds for each X_i, we have $\eta_i^!(\omega_X) \simeq \omega_{X_i} \in {}^p\mathcal{D}^{\geq 0}(X_i)$, thus $\omega_X \in {}^p\mathcal{D}^{\geq 0}(X)$.

Next, let $X \in \mathrm{Aff}_k$ be a globally placid affine scheme with a placid presentation $X \simeq \lim_\alpha X_\alpha$. Then $\omega_X \simeq \mathrm{pr}_\alpha^!(\omega_{X_\alpha}) \in {}^p\mathcal{D}^{\geq 0}(X)$, because $\omega_{X_\alpha} \in {}^p\mathcal{D}^{\geq 0}(X_\alpha)$, and $\mathrm{pr}_\alpha^!$ is t-exact.

Finally, for an arbitrary $\mathcal{X}$, choose a smooth covering $f = \sqcup_\alpha f_\alpha : \sqcup_\alpha X_\alpha \to \mathcal{X}$, where each X_α is a globally placid affine scheme. Then, by the proven above, $f_\alpha^!(\omega_\mathcal{X}) \simeq \omega_{X_\alpha} \in {}^p\mathcal{D}^{\geq 0}(X_\alpha)$ for all α, therefore $\omega_\mathcal{X} \in {}^p\mathcal{D}^{\geq 0}(\mathcal{X})$, as claimed.

(b) Choose a smooth covering $Y \to \mathcal{Y}$, where $Y \simeq \sqcup_\alpha Y_\alpha$, and each Y_α is a globally placid (affine) scheme. Since it suffices to show a result after a base change to each Y_α, we can assume that $\mathcal{Y}$ is a globally placid affine scheme Y. Next, choose a smooth covering $X \to \mathcal{X}$, where $X \simeq \sqcup_\alpha X_\alpha$, and each X_α is a globally placid affine scheme. Since it suffices to show the assertion for each $X_\alpha \to X \to \mathcal{X} \to Y$, we can assume that $\mathcal{X}$ is a globally placid affine scheme X.

In this case, it suffices to show the assertion for $\mathcal{D}_c$. Choose placid presentations $X \simeq \lim_\alpha X_\alpha$ and $Y \simeq \lim_\beta Y_\beta$. Then $\mathcal{D}_c(\mathcal{Y}) \simeq \mathrm{colim}_\beta \mathcal{D}_c(Y_\beta)$, so it suffices to show the left t-exactness of each $f^! \circ \mathrm{pr}_\beta^! \simeq (\mathrm{pr}_\beta \circ f)^!$.

Replacing X by a smooth strongly pro-smooth covering if necessary, we can assume that $\mathrm{pr}_\beta \circ f$ decomposes as $X \overset{\mathrm{pr}_\alpha}{\longrightarrow} X_\alpha \overset{f_{\alpha,\beta}}{\longrightarrow} Y_\beta$, where $f_{\alpha,\beta}$ is equidimensional. Then $\mathrm{pr}_\alpha^!$ is t-exact, because pr_α is strongly pro-smooth, and while $f_{\alpha,\beta}^!$ is left t-exact by Lemma 5.3.5(a). Hence $(\mathrm{pr}_\beta \circ f)^! \simeq \mathrm{pr}_\alpha^! \circ f_{\alpha,\beta}^!$ is left t-exact, as claimed.

(c)-(e) By Proposition 4.4.5(b), the pullback f^* satisfies the base change with respect to smooth !-pullbacks. Thus (as in (b)), we can assume that $\mathcal{Y}$ is a globally placid affine scheme Y. Then $X := \mathcal{X}$ is an algebraic space, fp over Y.

(c) As in (b), we choose a placid presentation $Y \simeq \lim_\alpha Y_\alpha$, and it suffices to show the right t-exactness of $f^*\langle d\rangle \circ \mathrm{pr}_\alpha^! : \mathcal{D}_c(Y_\alpha) \to \mathcal{D}_c(X)$ for all sufficiently large α. Since f is fp-proper, we can assume that f is a pullback of a proper equidimensional morphism $f_\alpha : X_\alpha \to Y_\alpha$. Since pr_α is strongly pro-smooth we have an isomorphism $f^*\langle d\rangle \circ \mathrm{pr}_\alpha^! \simeq \mathrm{pr}_\alpha^! \circ f_\alpha^*\langle d\rangle$. Since $\mathrm{pr}_\alpha^!$ is t-exact, because $\mathrm{pr}_\alpha : X \to X_\alpha$ is strongly pro-smooth, it remains to show that $f_\alpha^*\langle d\rangle$ is right t-exact. Since $f_\alpha^*\langle d\rangle \simeq f_\alpha^{*,ren}$, it is right t-exact by Lemma 5.3.5(a).

(d) Arguing as in (c), we reduce the assertion to the corresponding assertion for Sch_k^{ft}. In this case, the assertion follows from Lemma 5.3.5(b).

(e) Assume first that $Y \in \mathrm{AlgSp}_k^{ft}$, and let $\mathrm{pr}_Y : Y \to \mathrm{pt}$ and $\mathrm{pr}_X : X \to \mathrm{pt}$ be the projections. Since Y is smooth, pr_Y is equidimensional, and we have a canonical isomorphism $\omega_Y \simeq \mathrm{pr}_Y^!(\omega_{\mathrm{pt}}) \simeq \mathrm{pr}_Y^{*,ren}(\omega_{\mathrm{pt}})$. Therefore we have

$$f^*(\omega_Y)\langle d\rangle \simeq f^{*,ren}(\omega_Y) \simeq f^{*,ren}(\mathrm{pr}_Y^{*,ren}(\omega_{\mathrm{pt}})) \simeq \mathrm{pr}_X^{*,ren}(\omega_{\mathrm{pt}}) \in {}^p\mathcal{D}^{\leq 0}(X)$$

by Lemma 5.3.5(a), thus $f^*(\omega_Y) \in {}^p\mathcal{D}^{\leq -2d}(X)$.

In the general case, we choose a strongly pro-smooth presentation $Y \simeq \lim_\alpha Y_\alpha$. Then, in the notation of the proof of (c), we have

$$f^*(\omega_Y) \simeq f^*(\mathrm{pr}_\alpha^!(\omega_{Y_\alpha})) \simeq \mathrm{pr}_\alpha^!(f_\alpha^*(\omega_{Y_\alpha})).$$

Hence it belongs to ${}^p\mathcal{D}^{\leq -2d}(X)$, because $f_\alpha^*(\omega_{Y_\alpha}) \in {}^p\mathcal{D}^{\leq -2d}(X_\alpha)$, by the previous case, shown above, and $\mathrm{pr}_\alpha^!$ is t-exact. □

5.6.3. Generalizations.

(a) Extending the construction of 5.3.4, one can define renormalized pullback $f^{*,ren}$ for every weakly equidimensional morphism $f : \mathcal{X} \to \mathcal{Y}$ of topologically placid ∞-stacks. Namely, it is characterized by the condition that it is compatible with composition and satisfies $f^{*,ren} \simeq f^!$ when f is topologically smooth.

(b) Lemma 5.6.1(c) has a generalization asserting that for every equidimensional morphism $f : \mathcal{X} \to \mathcal{Y}$ of topologically placid ∞-stacks, the renormalized $*$-pullback $f^{*,ren}$ is right t-exact. Namely, as soon as functors $f^{*,ren}$ are constructed, this can be shown by repeating the arguments of Lemma 5.6.1(b),(c) almost word-by-word.

(c) Lemma 5.6.1(e) has a generalization asserting that if $f : \mathcal{X} \to \mathcal{Y}$ is a pro-weakly equidimensional morphism of topologically placid ∞-stacks and $\mathcal{Y}$ is topologically smooth, then $f^{*,ren}(\omega_\mathcal{Y}) \in {}^p\mathcal{D}^{\leq 0}(\mathcal{X})$.

(d) Moreover, assertion (c) actually easily follows from (b). Indeed, since $\pi_\mathcal{Y} : \mathcal{Y} \to \mathrm{pt}$ is topologically smooth, we have $\omega_\mathcal{Y} \simeq \pi_\mathcal{Y}^!(\omega_{\mathrm{pt}}) \simeq \pi_\mathcal{Y}^{*,ren}(\omega_{\mathrm{pt}})$, thus

$$f^{*,ren}(\omega_\mathcal{Y}) \simeq f^{*,ren}(\pi_\mathcal{Y}^{*,ren}(\omega_{\mathrm{pt}})) \simeq \pi_\mathcal{X}^{*,ren}(\omega_{\mathrm{pt}}) \in {}^p\mathcal{D}^{\leq 0}(\mathcal{X})$$

by (b), because $\pi_\mathcal{X} : \mathcal{X} \to \mathrm{pt}$ is weakly equidimensional by assumption, thus equidimensional.

The following lemma will play a central role later (see Theorem 6.4.5).

Lemma 5.6.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism between topologically placid ∞-stacks, which is locally ind-fp-proper, and equidimensional of relative dimension d. Then the functor $f_![-2d]$ is left t-exact.*

Proof. Replacing f by its the pullback with respect to a topologically smooth morphism $Y \to \mathcal{Y}$, we can assume that $\mathcal{Y}$ is a globally uh-placid affine scheme Y, and $f : \mathcal{X} \to Y$ is ind-fp-proper. Choose a presentation $\mathcal{X} \simeq \mathrm{colim}_\alpha X_\alpha$, where each $f_\alpha : X_\alpha \to Y$ is fp-proper and all transition maps are fp-closed embeddings.

Denote by $i_\alpha : X_\alpha \to \mathcal{X}$ the inclusion. By Corollary 4.1.5, for every $K \in \mathcal{D}(\mathcal{X})$, we have a natural isomorphism $K \simeq \mathrm{colim}_\alpha (i_\alpha)_! i_\alpha^! K$, which induces an isomorphism $f_!(K) \simeq \mathrm{colim}_\alpha (f_\alpha)_! i_\alpha^!(K)$. Since filtered colimits are t-exact, it suffices to show that each composition $(f_\alpha)_! i_\alpha^![-2d]$ is left t-exact.

Next, since f_α is an fp-proper morphism between globally placid algebraic spaces, $(f_\alpha)_!$ has a left adjoint f_α^* (by Proposition 4.2.6(c)). Therefore passing to left adjoints, it suffices to show that each composition $(i_\alpha)_! f_\alpha^*[2d]$ is right t-exact.

For every $\beta > \alpha$, consider the inclusion $i_{\alpha,\beta} : X_\alpha \to X_\beta$. It suffices to show that each composition $(i_{\alpha,\beta})_! f_\alpha^*[2d]$ is right t-exact. Consider the open embedding $j_{\alpha,\beta} : X_\beta \smallsetminus X_\alpha \hookrightarrow X_\beta$. Since $(i_{\alpha,\beta})_! f_\alpha^* \simeq (i_{\alpha,\beta})_! i_{\alpha,\beta}^* f_\beta^*$, we have a fibered sequence

$$f_\beta^*[2d] \to (i_{\alpha,\beta})_! f_\alpha^*[2d] \to (j_{\alpha,\beta})_! j_{\alpha,\beta}^! f_\beta^*[2d+1].$$

Therefore it suffices to show that functors on the left and the right are right t-exact. Since $j_{\alpha,\beta}^!$ if t-exact, and hence $(j_{\alpha,\beta})_!$ is right t-exact (by adjointness), it suffices to show that each functor $f_\beta^*[2d]$ is right t-exact. But this follows from a combination of Lemma 5.3.7 and Claim 5.6.5. □

Claim 5.6.5. *All non-empty fibers of each f_β are of dimension $\leq d$.*

Proof. By definition of equidimensional morphisms of relative dimension d (see 3.3.5), there exists a topologically étale covering $a : U \to \mathcal{X}$ such that $U = \sqcup_i U_i$ is a disjoint union of affine schemes, and each composition $U_i \to \mathcal{X} \to Y$ decomposes as $U_i \overset{\pi_i}{\to} Y_i' \overset{f_i'}{\to} Y$, where π_i is topologically étale, and $f_i' : Y_i' \to Y$ is fp-affine equidimensional of relative dimension d.

Fix $y \in Y$. Then every non-empty fiber $Y'_{i,y} = f_i'^{-1}(y)$ is equidimensional of dimension d, while $\pi_i : U_i \to Y_i'$ induces a topologically étale morphism of fibers $U_{i,y} \to Y'_{i,y}$. In particular, it follows from Lemma 3.3.4 that every non-empty $U_{i,y}$ is of dimension d, thus every non-empty fiber $U_y = \sqcup_i U_{i,y}$ is of dimension d.

Next, since $X_\beta \subset \mathcal{X}$ is an fp-closed subscheme, the pullback $U_\beta := U \times_{\mathcal{X}} X_\beta$ is an fp-closed subscheme of U, hence while the map $a_y : U_{\beta,y} \to X_{\beta,y}$ of fibers, induced by a, is a topologically étale covering.

Assume that $X_{\beta,y} := f_\beta^{-1}(y)$ is non-empty. Then $U_{\beta,y}$ and hence also U_y are non-empty, and it follows from Lemma 3.3.4 that $\dim X_{\beta,y} = \dim U_{\beta,y} \leq \dim U_y = d$. □

6. Stratified ∞-stacks, semi-small maps, and perversity

In this section we will define a larger class of ∞-stacks, which admit perverse t-structures. We will also introduce (semi)-small maps and extend classical (finite dimensional) results to this setting.

6.1. ∞-stacks admitting gluing of sheaves.

Definition 6.1.1. We say that an ∞-stack $\mathcal{X}$ *admits gluing of sheaves*, if for every fp-open embedding $j : \mathcal{U} \hookrightarrow \mathcal{X}$ there exists a fully faithful left adjoint $j_! : \mathcal{D}(\mathcal{U}) \to \mathcal{D}(\mathcal{X})$ of $j^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{U})$.

6.1.2. Remark. We will see later that ∞-stacks satisfying gluing of sheaves in the sense of Definition 6.1.1, satisfies the gluing of sheaves in the sense of [BBD].

Lemma 6.1.3. *Let $\mathcal{X}$ be an ∞-stack, admitting gluing of sheaves, let $j : \mathcal{U} \hookrightarrow \mathcal{X}$ be an fp-open embedding with a complementary topologically fp-closed embedding $i : \mathcal{Z} \to \mathcal{X}$. Then*

(a) There exists a left adjoint i^* of $i_! : \mathcal{D}(\mathcal{Z}) \to \mathcal{D}(\mathcal{X})$.

(b) We have $i^* \circ j_! \simeq 0$.

(c) For every $K \in \mathcal{D}(\mathcal{X})$ the unit and counit maps extend to a fibered sequence

$$j_! j^! K \to K \to i_! i^* K.$$

Proof. All assertions are rather straightforward applications of Lemma 4.5.3.

(a) We have seen during the proof of Lemma 4.5.5 that $i_!$ induces an equivalence $\mathcal{D}(\mathcal{Z}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{Z}}(\mathcal{X}) \subset \mathcal{D}(\mathcal{X})$. Thus to define a functor i^*, it suffices to define a functor $i_! i^* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}_{\mathcal{Z}}(\mathcal{X})$. Consider functor $i_! i^* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X})$, which sends K to the cofiber $\operatorname{Cof}(j_! j^! K \to K)$. Then $j^! \operatorname{Cof}(j_! j^! K \to K) \simeq \operatorname{Cof}(j^! j_! j^! K \to j^! K) \simeq 0$, where the last isomorphism follows from the assumption that $j_!$ is fully faithful thus $j^! j_! \simeq \mathrm{Id}$. Hence the image of functor $i_! i^*$ lies in $\mathcal{D}_{\mathcal{Z}}(\mathcal{X})$, as claimed.

To show that i^* is the left adjoint of $i_!$, we have to construct a functorial isomorphism $\operatorname{Hom}(i^* K, L) \simeq \operatorname{Hom}(K, i_! L)$. Since $i_!$ is fully faithful, we get isomorphisms

$$\operatorname{Hom}(i^* K, L) \simeq \operatorname{Hom}(i_! i^* K, i_! L) \simeq \operatorname{Hom}(\operatorname{Cof}(j_! j^! K \to K), i_! L) \simeq$$

$$\simeq \operatorname{Fib}(\operatorname{Hom}(K, i_! L) \to \operatorname{Hom}(j_! j^! K, i_! L)) \simeq \operatorname{Hom}(K, i_! L),$$

where the last isomorphism holds, since $\operatorname{Hom}(j_! j^! K, i_! L) \simeq \operatorname{Hom}(j^! K, j^! i_! L) \simeq 0$ (use Lemma 4.5.3(c)).

(b) Since $i^* \circ j_!$ is the left adjoint of $j^! \circ i_! \simeq 0$, we are done.

(c) follows from our construction of i^*. ☐

Corollary 6.1.4. *(a) Let $\mathcal{X}$ be an ∞-stack admitting gluing of sheaves, and let $\eta : \mathcal{Y} \to \mathcal{X}$ be a decomposable topologically fp-locally closed embedding (see 4.5.10).*

Then the pullback $\eta^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$ has a fully faithful left adjoint $\eta_! : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$, while the pushforward functor $\eta_* : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$ from 4.5.6 has a left adjoint $\eta^* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$.

(b) Moreover, suppose that we have a Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{Y}} & \xrightarrow{\widetilde{\eta}} & \widetilde{\mathcal{X}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{\eta} & \mathcal{X}, \end{array}$$

where $\widetilde{\mathcal{X}}$ is an ∞-stack admitting gluing of sheaves as well, and functors $f^!$ and $g^!$ admit left adjoints $f_!$ and $g_!$, respectively. Then we have a canonical isomorphism

$$\eta^* f_! \simeq g_! \widetilde{\eta}^*.$$

Proof. (a) By definition, η decomposes as $\mathcal{Y} \xrightarrow{i} \mathcal{U} \xrightarrow{j} \mathcal{X}$, where j (resp. i) is an fp-open (resp. topologically fp-closed) embedding.

Recall that $j^!$ has a fully faithful left adjoint $j_!$ (because $\mathcal{X}$ admits gluing of sheaves), j_* has a left adjoint $j^!$ (by the definition of j_*), $i^!$ has a fully faithful left adjoint $i_!$ (by Lemma 4.5.3(b),(c)), while $i_!$ has a left adjoint i^* (by Lemma 6.1.3).

Therefore the composition $\eta^! = i^! \circ j^!$ has a left adjoint $\eta_! := j_! \circ i_!$, while the composition $\eta_* = j_* \circ i_!$ (see 4.5.10(d)) has a left adjoint $\eta^* := i^* \circ j^!$.

(b) Since $\eta^* f_!$ and $g_! \widetilde{\eta}^*$ are left adjoints of functors $f^! \eta_*$ and $\widetilde{\eta}_* g^!$, respectively, the assertion follows from Lemma 4.5.8. □

Lemma 6.1.5. *(a) Assume that $\mathcal{X}$ admits gluing of sheaves, and let $\eta : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a decomposable topologically fp-locally closed embedding (see 4.5.6). Then $\mathcal{Y}$ admits gluing of sheaves as well.*

(b) Assume that $\mathcal{X}$ has a presentation as a filtered colimit $\mathcal{X} = \operatorname{colim}_\alpha \mathcal{X}_\alpha$ such that each $\mathcal{X}_\alpha$ satisfies gluing of sheaves and each transition map is an fp-open embedding. Then $\mathcal{X}$ satisfies gluing of sheaves as well.

Proof. (a) Let $j : \mathcal{U} \hookrightarrow \mathcal{Y}$ be an fp-open embedding. Then $\nu := \eta \circ j : \mathcal{U} \to \mathcal{X}$ is a decomposable topologically fp-locally closed embedding as well (see 4.5.10(b). Therefore the pullback $\nu^!$ has a fully faithful left adjoint $\nu_! : \mathcal{D}(\mathcal{U}) \to \mathcal{D}(\mathcal{X})$, while η_* has a left adjoint $\eta^* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$ (by Corollary 6.1.4(b)). We claim that the composition $j_! : \eta^* \nu_! : \mathcal{D}(\mathcal{U}) \to \mathcal{D}(\mathcal{Y})$ is a fully faithful left adjoint of $j^!$.

By construction, $j_!$ is a left adjoint of $\nu^! \eta_* \simeq j^!(\eta^! \eta_*) \simeq j^!$, where $\eta^! \eta_* \simeq \mathrm{Id}$ by the definition of η_* (see 4.5.6). Finally, since $\eta_!$ and $\nu_! = \eta_! \circ j_!$ are fully faithful, we conclude that $j_!$ is fully faithful, as claimed.

(b) Let $j : \mathcal{Y} \hookrightarrow \mathcal{X}$ be an fp-open embedding. Then the presentation $\mathcal{X} = \operatorname{colim}_\alpha \mathcal{X}_\alpha$ induces the presentation $\mathcal{Y} = \operatorname{colim}_\alpha \mathcal{Y}_\alpha$, and the induced maps $j_\alpha : \mathcal{Y}_\alpha \hookrightarrow \mathcal{X}_\alpha$ are fp-open embeddings. Therefore $\mathcal{D}(\mathcal{X}) \simeq \lim_\alpha \mathcal{D}(\mathcal{X}_\alpha)$, $\mathcal{D}(\mathcal{Y}) \simeq \lim_\alpha \mathcal{D}(\mathcal{Y}_\alpha)$, and the pullback $j_\alpha^! : \mathcal{D}(\mathcal{X}_\alpha) \to \mathcal{D}(\mathcal{Y}_\alpha)$ has a left adjoint $(j_\alpha)_! : \mathcal{D}(\mathcal{Y}_\alpha) \hookrightarrow \mathcal{D}(\mathcal{X}_\alpha)$ by our assumption on $\mathcal{X}_\alpha$. Thus in order to apply Proposition 4.1.8(b) and to conclude the proof, one has to show that the Beck–Chevalley condition is satisfied, that is, the base change map $(j_\beta)_! \pi^!_{\beta,\alpha} \to \pi^!_{\beta,\alpha}(j_\alpha)_!$, where $\pi_{\beta,\alpha}$ denote the transition maps $\mathcal{X}_\beta \to \mathcal{X}_\alpha$ and $\mathcal{X}_\beta \to \mathcal{X}_\alpha$, is an isomorphism. Passing to right adjoints, the assertion follows from Lemma 4.5.3(a). □

Now we are going to provide two classes of ∞-stacks, admitting gluing of sheaves.

Lemma 6.1.6. *Every topological placid ∞-stack $\mathcal{X}$ admits gluing of sheaves.*

Proof. By Proposition 4.4.5(a), for every fp-open embedding $j : \mathcal{U} \hookrightarrow \mathcal{X}$ there exists a left adjoint $j_!$ of $j^! : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}')$. It remains to show that $j_!$ is fully faithful, that is, the unit map $\mathrm{Id} \to j^! j_!$ is an isomorphism. Choose a topologically smooth covering $p : X \to \mathcal{X}$ with topologically 0-placid X. It suffices to show that the map $p^! \to p^! j^! j_!$ is an isomorphism. Using Proposition 4.4.5(b), we have to show that $p^! \to j^! j_! p^!$ is an isomorphism, thus reducing to the case $\mathcal{X} = X$. Passing to a connected component, we can thus assume that $\mathcal{X}$ is globally uh-placid affine scheme X. Choosing a presentation $X \simeq \lim_\alpha X_\alpha$, we reduce to a case when $X \in \mathrm{Aff}_k^{ft}$. In this case, the assertion is well-known. □

6.1.7. Ind-placid schemes. We call an ind-scheme X *ind-placid*, if it has a presentation of the form $X \simeq \mathrm{colim}_\alpha X_\alpha$ such that every algebraic space X_α is globally placid.

Proposition 6.1.8. *Let X be an ind-placid algebraic space, and let H be an ind-placid group, that is a group object in ind-placid algebraic spaces, acting on X. Then the quotient $\mathcal{X} = [X/H]$ admits gluing of sheaves.*

Proof. Our strategy will be similar to Lemma 4.5.3, though some extra care will be needed. We set $U := \mathcal{U} \times_{\mathcal{X}} X$. Then $U \hookrightarrow X$ is an fp-open embedding, and we have a natural equivalence $\mathcal{U} \simeq [U/H]$.

First we will show the assertions when $H = 1$, thus $\mathcal{X} = X$ is an ind-placid algebraic space. Choose a presentation $\mathcal{X} = \mathrm{colim}_\alpha X_\alpha$, where each X_α is a placid algebraic space, and all transition maps are fp-closed embeddings. This presentation induce a presentation $\mathcal{U} = \mathrm{colim}_\alpha U_\alpha$ of $\mathcal{U}$. Then $\mathcal{D}(\mathcal{X}) \simeq \mathrm{colim}_\alpha \mathcal{D}(X_\alpha)$ (see 4.3.4), and similarly for $\mathcal{U}$.

Since left adjoint $j_!$ exist for placid algebraic spaces (see Proposition 4.2.6(a)), in order to apply Proposition 4.1.8(c), we have to check that the Beck-Chevalley condition is satisfied. Explicitly, we have to show that for the Cartesian diagram

$$\begin{array}{ccc} U_\alpha & \xrightarrow{j_\alpha} & X_\alpha \\ {\scriptstyle\iota}\downarrow & & {\scriptstyle\iota}\downarrow \\ U_\beta & \xrightarrow{j_\beta} & X_\beta \end{array}$$

the base change morphism $(j_\alpha)_! \iota_! \to \iota_! (j_\beta)_!$ is a isomorphism. But this follows from the fact that all functors involved are left adjoints of !-pullbacks, and the diagram is commutative. Next, the fully-faithfulness of $j_!$ follows the corresponding assertion in the case when X is a algebraic space of finite type, in which case it is standard.

In the general case, using the Čech complex, corresponding to the projection $X \to \mathcal{X}$, we get equivalences $\mathcal{D}(\mathcal{X}) \simeq \lim_{[m]} \mathcal{D}(H^m \times T)$ (see 4.3.3(c)), and similarly for $\mathcal{D}(\mathcal{U})$.

By the case of ind-placid algebraic spaces, shown above, there exists a left adjoint $j_!$ of $j^! : \mathcal{D}(H^m \times X) \to \mathcal{D}(H^m \times U)$. Thus, in order to apply Proposition 4.1.8(b) and to finish the proof, we have to show that $j_!$ satisfy base change with respect to pullbacks $\eta^! : \mathcal{D}(H^n \times X) \to \mathcal{D}(H^m \times X)$.

Notice that every morphism $H^m \times X \to H^n \times X$ decomposes as a composition of the action morphisms $H \times X \to X : (h, x) \mapsto h(x)$, multiplications morphisms $H \times H \to H$ and projections. Since the action morphism $H \times X \to X$ decomposes as a composition of the isomorphism $H \times X \xrightarrow{\sim} H \times X : (h, x) \mapsto (h, h(x))$ and the projection, it suffices to show that $j_!$ satisfy base change with respect to pullbacks, corresponding to projections. Thus the assertion follows from Lemma 6.1.9 below. $\square$

Lemma 6.1.9. *Consider the Cartesian diagram*

$$\begin{array}{ccc} U \times Y & \xrightarrow{j} & X \times Y \\ {\scriptstyle \mathrm{pr}_Y}\downarrow & & {\scriptstyle \mathrm{pr}_Y}\downarrow \\ U & \xrightarrow{j} & X \end{array}$$

where Y is an placid ind-algebraic space. Then the base change morphism $j_! \operatorname{pr}_Y^! \to \operatorname{pr}_Y^! j_!$ is an isomorphism.

Proof. Assume first that X and Y are placid algebraic spaces. In this case, the assertion for $\mathcal{D} = \operatorname{Ind} \mathcal{D}_c$ follows from that for $\mathcal{D}_c$. When X and Y are algebraic spaces of finite type, the assertion for $\mathcal{D}_c$ is well known. Namely, passing to Verdier duals, we have to show that the natural morphism $j_* K \boxtimes \overline{\mathbb{Q}}_\ell \to j_*(K \boxtimes \overline{\mathbb{Q}}_\ell)$ is a isomorphism, known also as Kunneth formulas (see [SGA5, Exp III, Prop 1.7.4]). In the general case, notice that every object of $\mathcal{D}_c(X)$ comes from some algebraic space of finite type, so the assertion follows from the one for algebraic spaces of finite type.

Next we assume that X is placid algebraic space, but Y is a placid ind-algebraic space. Choose presentation $Y = \operatorname{colim}_\alpha Y_\alpha$, and let $\eta_\alpha : Y_\alpha \to Y$ be an inclusion. Then we have a natural equivalence $\operatorname{colim}_\alpha \eta_{\alpha,!} \operatorname{pr}_{Y_\alpha}^! \simeq \operatorname{colim}_\alpha \eta_{\alpha,!} \eta_\alpha^! \operatorname{pr}_Y^! \simeq \operatorname{pr}_Y^!$. Since $j_!$ commutes with colimits and $\eta_{\alpha,!}$, the assertion for X and Y follows from the corresponding assertion for X and Y_α, shown before. Finally, the extension to the case when X is a placid ind-algebraic space is similar. $\square$

6.2. **Stratified ∞-stacks.**

6.2.1. Notation. (a) Let $\mathcal{X}$ be an ∞-stack, and let $\{\mathcal{X}_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of non-empty disjoint topologically fp-locally closed ∞-substacks of $\mathcal{X}$, that is, $\mathcal{X}_\alpha \cap \mathcal{X}_\beta = \emptyset$ for every $\alpha \neq \beta$ in $\mathcal{I}$.

(b) For every ∞-substack $\mathcal{X}' \subset \mathcal{X}$, we set $\mathcal{I}_{\mathcal{X}'} := \{\alpha \in \mathcal{I} \,|\, \mathcal{X}_\alpha \subseteq \mathcal{X}'\}$.

(c) We say that $\mathcal{X}'$ is $\{\mathcal{X}_\alpha\}_\alpha$*-adapted*, if for every $\alpha \in \mathcal{I} \smallsetminus \mathcal{I}_{\mathcal{X}'}$, we have $\mathcal{X}_\alpha \cap \mathcal{X}' = \emptyset$. In other words, $\mathcal{X}'$ is $\{\mathcal{X}_\alpha\}_\alpha$-adapted if and only if for every $\alpha \in \mathcal{I}$ we have either $\mathcal{X}_\alpha \subseteq \mathcal{X}'$ or $\mathcal{X}_\alpha \cap \mathcal{X}' = \emptyset$, or equivalently, $\mathcal{X}_\alpha \subseteq \mathcal{X} \smallsetminus \mathcal{X}'$.

(d) By definition, the class of $\{\mathcal{X}_\alpha\}_\alpha$-adapted ∞-substacks is closed under arbitrary intersections and complements.

6.2.2. Constructible stratification. In the situation of 6.2.1(a),

(a) we say that $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ form a *finite constructible stratification* of $\mathcal{X}$, if $\mathcal{I}$ is finite, and there exists an full ordering $\alpha_1 < \ldots < \alpha_n$ of $\mathcal{I}$ and an increasing sequence of fp-open substacks $\emptyset = \mathcal{X}_0 \subsetneq \mathcal{X}_1 \subsetneq \ldots \subsetneq \mathcal{X}_n = \mathcal{X}$ such that $\mathcal{X}_{\alpha_i} \subseteq \mathcal{X}_i \smallsetminus \mathcal{X}_{i-1}$, and the embedding $\mathcal{X}_{\alpha_i} \hookrightarrow \mathcal{X}_i \smallsetminus \mathcal{X}_{i-1}$ is a topological equivalence for all $i = 1, \ldots, n$.

(b) we say that $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ form a *bounded constructible stratification* of $\mathcal{X}$, if $\mathcal{X}$ can be represented as a filtered colimit $\mathcal{X} = \operatorname{colim}_{U\in\mathcal{J}}\mathcal{X}_U$ such that each $\mathcal{X}_U$ is an fp-open $\{\mathcal{X}_\alpha\}_\alpha$-adapted substack of $\mathcal{X}$, and $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}_{\mathcal{X}_U}}$ form a finite constructible stratification of $\mathcal{X}_U$.

(c) we say that $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ form a *constructible stratification* of $\mathcal{X}$, if $\mathcal{X}$ can be represented as a filtered colimit $\mathcal{X} \simeq \operatorname{colim}_{\lambda\in\Lambda}\mathcal{X}_\lambda$ such that each $\mathcal{X}_\lambda$ is a topologically fp-closed $\{\mathcal{X}_\alpha\}_\alpha$-adapted substack of $\mathcal{X}$, and $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}_{\mathcal{X}_\lambda}}$ form a bounded constructible stratification of $\mathcal{X}_\lambda$.

6.2.3. Remarks. In the situation of 6.2.1,

(a) a collection $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ form a finite constructible stratification of $\mathcal{X}$ if and only if there exists $\beta \in \mathcal{I}$ such that $\mathcal{X}_\beta \subset \mathcal{X}$ is topologically fp-closed, and $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}\smallsetminus\beta}$ form a finite constructible stratification of $\mathcal{X} \smallsetminus \mathcal{X}_\beta$.

Indeed, if such a β exists, then the embedding $\mathcal{X} \hookrightarrow \mathcal{X} \smallsetminus (\mathcal{X} \smallsetminus \mathcal{X}_\beta)$ is a topological equivalence (see 4.5.2(c)). Conversely, in the situation of 6.2.2(a), we have $\mathcal{X}_{n-1} = \mathcal{X} \smallsetminus \mathcal{X}_{\alpha_n}$, so $\beta = \alpha_n$ satisfies the required property.

(b) assume that $\mathcal{Z} \subset \mathcal{X}$ is a topologically fp-closed $\{\mathcal{X}_\alpha\}_\alpha$-adapted substack such that $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}_\mathcal{Z}}$ (resp. $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}_{\mathcal{X}\smallsetminus\mathcal{Z}}}$) form a finite constructible stratification of $\mathcal{Z}$ (resp. $\mathcal{X} \smallsetminus \mathcal{Z}$). Then $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ form a finite constructible stratification of $\mathcal{X}$.

Indeed, this easily follows from (a) by induction on the cardinality of $\mathcal{I}_\mathcal{Z}$.

The following lemma summarizes simple properties of the notions we introduced.

Lemma 6.2.4. *Assume that $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ form a (finite/bounded) constructible stratification of $\mathcal{X}$.*

(a) For a morphism $f : \mathcal{Y} \to \mathcal{X}$ of ∞-stacks, the collection $\{f^{-1}(\mathcal{X}_\alpha)\}_{\alpha\in\mathcal{I},f^{-1}(\mathcal{X}_\alpha)\neq\emptyset}$ form a (finite/bounded) constructible stratification of $\mathcal{Y}$.

(b) If $\mathcal{X}'$ is $\{\mathcal{X}_\alpha\}_\alpha$-adapted, then $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}_{\mathcal{X}'}}$ form a (finite/bounded) constructible stratification of $\mathcal{X}'$.

(c) If $\{\mathcal{X}_{\alpha,\beta}\}_{\beta\in\mathcal{J}_\alpha}$ form a finite constructible stratification of $\mathcal{X}_\alpha$ for all $\alpha \in \mathcal{I}$, then $\{\mathcal{X}_{\alpha,\beta}\}_{\alpha\in\mathcal{I},\beta\in\mathcal{J}_\alpha}$ form a (finite/bounded) constructible stratification of $\mathcal{X}$.

(d) For every $\alpha \in \mathcal{I}$, the embedding $\eta_\alpha : \mathcal{X}_\alpha \hookrightarrow \mathcal{X}$ is decomposable (see 4.5.10).

Proof. (a) In the case of a finite stratification, note that the sequence of fp-open substacks $\mathcal{X}_i \subset \mathcal{X}$ from 6.2.2(a) induces a sequence of fp-open substacks $f^{-1}(\mathcal{X}_i) \subset \mathcal{Y}$.

So the assertion follows because $f^{-1}(\mathcal{X}_i \smallsetminus \mathcal{X}_{i-1}) \simeq f^{-1}(\mathcal{X}_i) \smallsetminus f^{-1}(\mathcal{X}_{i-1})$ (see 4.5.1(c)), and topological equivalences are stable under pullbacks (by Corollary 2.3.7(a)).

For the general case, notice that the presentation $\mathcal{X} = \operatorname{colim}_{U\in\mathcal{J}} \mathcal{X}_U$ from 6.2.2(b), induces a similar presentation $\mathcal{Y} = \operatorname{colim}_{U\in\mathcal{J}} f^{-1}(\mathcal{X}_U)$, while the presentation $\mathcal{X} = \operatorname{colim}_{\lambda\in\Lambda} \mathcal{X}_\lambda$ from 6.2.2(c), induces a similar presentation $\mathcal{Y} = \operatorname{colim}_{\lambda\in\Lambda} f^{-1}(\mathcal{X}_\lambda)$.

(b) follows immediately from (a).

(c) Choosing a presentation $\mathcal{X} \simeq \operatorname{colim}_{\lambda\in\Lambda} \mathcal{X}_\lambda$ from 6.2.2(c) and a presentation $\mathcal{X} = \operatorname{colim}_{U\in\mathcal{J}} \mathcal{X}_U$ from 6.2.2(b), we reduce to the case when $\mathcal{I}$ is finite. Then by 6.2.3(a), there exists $\alpha' \in \mathcal{I}$ such that $\mathcal{X}_{\alpha'} \subset \mathcal{X}$ is topologically fp-closed and $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}\smallsetminus\alpha'}$ form a finite constructible stratification of $\mathcal{X} \smallsetminus \mathcal{X}_{\alpha'}$. By assumption and induction on $|\mathcal{I}|$, we conclude that $\{\mathcal{X}_{\alpha,\beta}\}_{\alpha\in\mathcal{I}\smallsetminus\alpha',\beta\in\mathcal{J}_\alpha}$ (resp. $\{\mathcal{X}_{\alpha',\beta}\}_{\beta\in\mathcal{J}_{\alpha'}}$) form a finite constructible stratification of $\mathcal{X} \smallsetminus \mathcal{X}_{\alpha'}$ (resp. $\mathcal{X}_{\alpha'}$). Now the assertion follows from 6.2.3(b).

(d) Notice that a presentation $\mathcal{X} \simeq \operatorname{colim}_\lambda \mathcal{X}_\lambda$ from 6.2.2(c), induces a presentation $\mathcal{X}_\alpha \simeq \operatorname{colim}_\lambda(\mathcal{X}_\lambda \cap \mathcal{X}_\alpha)$. Therefore there exists $\lambda \in \Lambda$ such that $\mathcal{X}_\alpha \cap \mathcal{X}_\lambda \neq \emptyset$. Since $\mathcal{X}_\lambda \subset \mathcal{X}$ is $\{\mathcal{X}_\alpha\}_\alpha$-adapted, this implies that $\mathcal{X}_\alpha \subset \mathcal{X}_\lambda$, thus η_α factors as $\mathcal{X}_\alpha \hookrightarrow \mathcal{X}_\lambda \hookrightarrow \mathcal{X}$. Since $\mathcal{X}_\lambda \subset \mathcal{X}$ is a topologically fp-closed embedding, we can thus replace $\mathcal{X}$ by $\mathcal{X}_\lambda$, thus assuming that the stratification is bounded. Next, arguing similarly, one shows that in the situation of 6.2.2(b), there exists $U \in \mathcal{J}$ such that $\mathcal{X}_\alpha \subset \mathcal{X}_U$, and reduce to the case of finite stratification.

In this case, in the notation of 6.2.2(a), there exists i such that $\alpha = \alpha_i$. Then η_α decomposes as a composition $\mathcal{X}_{\alpha_i} \hookrightarrow \mathcal{X}_i \hookrightarrow \mathcal{X}$ of a topologically fp-closed embedding, and an fp-open embedding. □

6.2.5. Stratified ∞-stacks and perversity function.

(a) We call an ∞-stack $\mathcal{X}$ an *$\mathcal{I}$-stratified* (or simply *stratified*), if it is equipped with a constructible stratification $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ such that each $\mathcal{X}_\alpha$ is topologically placid.

(b) By a *perversity* on an $\mathcal{I}$-stratified ∞-stack $\mathcal{X}$, we mean a function $p_\nu : \mathcal{I} \to \mathbb{Z} : \alpha \mapsto \nu_\alpha$, or, what is the same, a collection $p_\nu = \{\nu_\alpha\}_{\alpha\in\mathcal{I}}$ of integers.

6.2.6. Remark. Note that if $(\mathcal{X}, \{\mathcal{X}_\alpha\}_\alpha)$ is a stratified ∞-stack, which admits gluing of sheaves, then every $\mathcal{D}(\mathcal{X}_\alpha)$ is equipped with a (!-adapted) perverse t-structure ${}^p\mathcal{D}(\mathcal{X}_\alpha)$ (see 5.5.2). It also follows from Lemma 6.2.4(d) and Corollary 6.1.4 that we have two pullback functors $\eta_\alpha^!, \eta_\alpha^* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}_\alpha)$.

Proposition 6.2.7. *Let $(\mathcal{X}, \{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}})$ be a stratified ∞-stack, admitting gluing of sheaves, and equipped with a perversity $p_\nu = \{\nu_\alpha\}$.*

(a) Assume that the stratification is bounded. Then there exists a unique t-structure $({}^{p_\nu}\mathcal{D}^{\leq 0}(\mathcal{X}), {}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}))$ on $\mathcal{D}(\mathcal{X})$ such that

$$(6.1) \qquad {}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}) = \{K \in \mathcal{D}(\mathcal{X}) \mid \eta_\alpha^! K \in {}^p\mathcal{D}^{\geq -\nu_\alpha}(\mathcal{X}_\alpha) \text{ for all } \alpha \in \mathcal{I}\},$$

$$(6.2) \qquad {}^{p_\nu}\mathcal{D}^{\leq 0}(\mathcal{X}) = \{K \in \mathcal{D}(\mathcal{X}) \mid \eta_\alpha^* K \in {}^p\mathcal{D}^{\leq -\nu_\alpha}(\mathcal{X}_\alpha) \text{ for all } \alpha \in \mathcal{I}\}.$$

Moreover, the subcategory ${}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X})\subset\mathcal{D}(\mathcal{X})$ is closed under filtered colimits.

(b) In the general case, there exists a unique t-structure $({}^{p_\nu}\mathcal{D}^{\leq 0}(\mathcal{X}),{}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}))$ on $\mathcal{D}(\mathcal{X})$ satisfying (6.1).

Proof. (a) Assume first that $\mathcal{I}$ is finite. In this case, the assertion follows from the gluing theorem [BBD, Thm.1.4.10] by induction on $|\mathcal{I}|$:

Since $|\mathcal{I}|=1$ the assertion is clear, we may assume that $|\mathcal{I}|>1$. By 6.2.3(a), there exists $\beta\in\mathcal{I}$ such that $Z:=\mathcal{X}_\beta\subset\mathcal{X}$ is topologically fp-closed, and $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}\smallsetminus\beta}$ form a constructible stratification of $\mathcal{U}:=\mathcal{X}\smallsetminus\mathcal{Z}$. Then $(\mathcal{U},\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}\smallsetminus\beta})$ is a stratified ∞-stack, admitting gluing of sheaves, with a perversity function $p'_\nu=\{\nu_\alpha\}_{\alpha\in\mathcal{I}'}$ (see Lemma 6.3.1 below). Therefore, by the induction hypothesis, there exists a unique t-structure $({}^{p'_\nu}\mathcal{D}^{\leq 0}(\mathcal{U}),{}^{p'_\nu}\mathcal{D}^{\geq 0}(\mathcal{U}))$ on $\mathcal{U}$ satisfying (6.1) and (6.2) for $\alpha\in\mathcal{I}\smallsetminus\beta$.

Now let $i:\mathcal{Z}\hookrightarrow\mathcal{X}$ and $j:\mathcal{U}\hookrightarrow\mathcal{X}$ be the corresponding topologically fp-closed and fp-open embeddings. Since $\mathcal{X}$ admits gluing of sheaves, we conclude from Lemma 4.5.3 and Lemma 6.1.3 that all the assumptions of [BBD, 1.4.3] are satisfied. Therefore by [BBD, Thm.1.4.10] there exists a unique t-structure $({}^{p_\nu}\mathcal{D}^{\leq 0}(\mathcal{X}),{}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}))$ on $\mathcal{D}(\mathcal{X})$ such that $K\in\mathcal{D}(\mathcal{X})$ belongs to ${}^{p_\nu}\mathcal{D}^{\leq 0}(\mathcal{X})$ (resp. ${}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X})$) if and only if we have $j^*K\in{}^{p'_\nu}\mathcal{D}^{\leq 0}(\mathcal{U})$ and $i^*K\in{}^{p}\mathcal{D}^{\leq -\nu_\alpha}(\mathcal{Z})$ (resp. $j^!K\in{}^{p'_\nu}\mathcal{D}^{\geq 0}(\mathcal{U})$ and $i^!K\in{}^{p}\mathcal{D}^{\geq -\nu_\alpha}(\mathcal{Z})$). This finishes the argument when $\mathcal{I}$ is finite.

In the general case, $\mathcal{X}$ can be written as a filtered colimit $\mathcal{X}\simeq\operatorname{colim}_U\mathcal{X}_U$, where each $\mathcal{X}_U\subset\mathcal{X}$ is an fp-open substack having a finite constructible stratification $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}_U}$. Since each $\mathcal{X}_U$ admits a gluing of sheaves (see Lemma 6.1.5(a)), we deduce from the finite case shown above that each $\mathcal{D}(\mathcal{X}_U)$ is equipped with a unique t-structure, satisfying (6.1) and (6.2) (for $\alpha\in\mathcal{I}_{\mathcal{X}_U}$). Furthermore, equalities (6.1) and (6.2) imply that for every $\mathcal{X}_U\subset\mathcal{X}_{U'}$ the restriction functor $\mathcal{D}(\mathcal{X}_{U'})\to\mathcal{D}(\mathcal{X}_U)$ is t-exact. Therefore it follows from Lemma 5.1.3(b) that there exists a unique t-structure on $\mathcal{D}(\mathcal{X})$, satisfying (6.1) and (6.2) for all $\alpha\in\mathcal{I}$.

Finally, since every functor $\eta_\alpha^!$ commutes with colimits, the last assertion follows from (6.1) and the corresponding assertion for topologically placid ∞-stacks (see 5.5.2(e)).

(b) By assumption, $\mathcal{X}$ can be written as a filtered colimit $\mathcal{X}\simeq\operatorname{colim}_{\lambda\in\Lambda}\mathcal{X}_\lambda$, where each $\mathcal{X}_\lambda$ has a bounded stratification by $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}_{\mathcal{X}_\lambda}}$, and all transition maps $i_{\lambda,\mu}:\mathcal{X}_\lambda\to\mathcal{X}_\mu$ are topologically fp-closed embeddings.

Then, by (a), each $\mathcal{D}(\mathcal{X}_\lambda)$ has a perverse t-structure $({}^{p_\nu}\mathcal{D}^{\leq 0}(\mathcal{X}_\lambda),{}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}_\lambda))$ satisfying (6.1) and such that the subcategory ${}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}_\alpha)\subset\mathcal{D}(\mathcal{X}_\alpha)$ is closed under filtered colimits. Moreover, each pushforward $(i_{\lambda,\mu})_!:\mathcal{D}(\mathcal{X}_\lambda)\to\mathcal{D}(\mathcal{X}_\mu)$ is t-exact (see Lemma 6.3.6(a) below), and has a continuous right adjoint $i^!_{\lambda,\mu}$.

Therefore all the assumptions of Theorem 4.1.3 and Proposition 5.1.4 are satisfied, hence the limit=colimit category $\mathcal{D}(\mathcal{X})=\lim_\lambda\mathcal{D}(\mathcal{X}_\lambda)$ is equipped with a canonical t-structure $({}^{p_\nu}\mathcal{D}^{\leq 0}(\mathcal{X}),{}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}))$. Let $i_\lambda:\mathcal{X}_\lambda\hookrightarrow\mathcal{X}$ be the inclusion. Then the formula (5.3) says in our case that

$$(6.3)\qquad {}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}) = \{K \in \mathcal{D}^{\leq 0}(\mathcal{X}) \,|\, i^!_\lambda K \in {}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}_\lambda) \text{ for all } \lambda \in \Lambda\}.$$

Combining (6.3) and equality (6.1) for each ${}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X}\lambda \in \Lambda)$, we conclude that equality (6.1) holds for ${}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{X})$. Finally, the uniqueness assertion follows from (6.1) and Lemma 5.1.2(e). □

6.2.8. The "canonical" perversity by codimension.

(a) Let $\mathcal{X}$ be an topologically placid ∞-stacks, and let $\{\mathcal{X}_\alpha\}_{\alpha\in\mathcal{I}}$ be a bounded constructible stratification. Then every $\mathcal{X}_\alpha$ is topologically placid (see 2.4.10(c)), therefore $\mathcal{X}$ is an $\mathcal{I}$-stratified ∞-stack.

(b) Assume now that each $\mathcal{X}_\alpha \subset \mathcal{X}$ is of pure codimension ν_α. We denote by p_{can} the *canonical* perversity $p_{\text{can}} := \{\nu_\alpha\}_\alpha$ on $\mathcal{X}$.

The following lemma explains why we call this perversity *canonical*.

Lemma 6.2.9. *In the situation of 6.2.8, the canonical t-structure ${}^{p_{\text{can}}}\mathcal{D}(\mathcal{X})$ on $\mathcal{D}(\mathcal{X})$, defined by the perversity p_{can}, coincides with the $!$-adapted perverse t-structure ${}^{p}\mathcal{D}(\mathcal{X})$.*

Proof. Using 6.2.6 and Lemma 5.6.1(d), our assumption on $\eta_\alpha : \mathcal{X}_\alpha \to \mathcal{X}$ imply that for every $K \in {}^{p}\mathcal{D}^{\leq 0}(\mathcal{X})$ (resp. $K \in {}^{p}\mathcal{D}^{\geq 0}(\mathcal{X})$), we have $\eta^*_\alpha K \in {}^{p}\mathcal{D}^{\leq -\nu_\alpha}(\mathcal{X}_\alpha)$ (resp. $\eta^!_\alpha K \in {}^{p_!}\mathcal{D}^{\geq -\nu_\alpha}(\mathcal{X}_\alpha)$). Therefore by formulas (6.1) and (6.2), we have inclusions ${}^{p}\mathcal{D}^{\leq 0}(\mathcal{X}) \subset {}^{p_{\text{can}}}\mathcal{D}^{\leq 0}(\mathcal{X})$ and ${}^{p}\mathcal{D}^{\geq 0}(\mathcal{X}) \subset {}^{p_{\text{can}}}\mathcal{D}^{\geq 0}(\mathcal{X})$. But then both inclusions have to be equalities (see Lemma 5.1.2(c)), and the assertion follows. □

6.3. **Functorial properties.** Below we show that many of the properties of the classical perverse t-structure extend to our setting almost word-by-word.

Lemma 6.3.1. *Let $(\mathcal{X}, \{\mathcal{X}_\alpha\}_\alpha)$ be an $\mathcal{I}$-stratified ∞-stack, admitting gluing of sheaves, and let $j : \mathcal{U} \to \mathcal{X}$ an fp-open immersion. Then $(\mathcal{U}, \{j^{-1}(\mathcal{X}_\alpha)\}_\alpha)$ is an $\mathcal{I}$-stratified ∞-stack, admitting gluing of sheaves as well.*

Moreover, if $p_\nu = \{\nu_\alpha\}_\alpha$ is a perversity on $\mathcal{X}$, and $p'_\nu = \{\nu_\alpha\}_\alpha$ is the corresponding perversity on $\mathcal{U}$, then the functor $j^!$ is t-exact, $j_!$ is right t-exact and j_* is left t-exact.

Proof. Since $\mathcal{U}$ admits gluing of sheaves by Lemma 6.1.5, the first two assertions follow from the fact every fp-open ∞-substack of a topologically placid ∞-stack is topologically placid (see 2.4.10(c)).

When $|\mathcal{I}| = 1$, the ∞-stack $\mathcal{X}$ is topologically placid, and j is topologically smooth. In this case, the t-exactness of $j^!$ is clear (see 5.5.2(c)), while the t-exactness assertions for $j_!$ and j_* follow by adjunction.

In the general case, it suffices to show that $j^!$ and j_* are left t-exact. Using (6.1) together with the fact that functor j_* satisfies base change (see Lemma 4.5.3(a)), we reduce to the case of $|\mathcal{I}| = 1$, shown above. □

6.3.2. Bounded case. When the stratification is bounded, then the argument can be slightly simplified. Namely, t-exactness of $j^!$ follows from (6.1) and (6.2) and the $|\mathcal{I}| = 1$ case, while the t-exactness properties of $j_!$ and j_* follow by adjunction.

6.3.3. The intermediate extension. Let $\mathcal{X}$ be a stratified ∞-stack, admitting gluing of sheaves and equipped with perversity p, let $j : \mathcal{U} \to \mathcal{X}$ an fp-open immersion, and let p' be the induced perversity on $\mathcal{U}$.

(a) For every $K \in \mathrm{Perv}^{p'}(U)$, we define

$$j_{!*}K := \mathrm{Im}({}^pH^0(j_!K) \to {}^pH^0(j_*K)),$$

induced by the canonical map $\theta : j_!K \to j_*K$, and call it *the intermediate extension* of K. In particular, we have a canonical surjection $\theta_1 : {}^pH^0(j_!K) \to j_{!*}K$ and injection $\theta_2 : j_{!*}K \to {}^pH^0(j_*K)$.

(b) We say that $K \in \mathcal{D}(\mathcal{X})$ is *supported on* $\mathcal{X} \smallsetminus \mathcal{U}$, if $K \in \mathcal{D}_{\mathcal{X}\smallsetminus\mathcal{U}}(\mathcal{X})$, that is, $j^!K \simeq 0$.

Corollary 6.3.4. *In the situation of 6.3.3, let* $K \in \mathrm{Perv}^{p'}(\mathcal{U})$*. Then*

(a) The kernel of $\theta_1 : {}^pH^0(j_!K) \to j_{!*}K$ *and cokernel of* $\theta_2 : j_{!*}K \to {}^pH^0(j_*K)$ *are supported on* $\mathcal{X} \smallsetminus \mathcal{U}$*.*

(b) The perverse sheaf ${}^pH^0(j_!K)$ *(resp.* ${}^pH^0(j_*B)$*) has no non-zero quotients (resp. subobjects) supported on* $\mathcal{X} \smallsetminus \mathcal{U}$*.*

(c) The intermediate extension $j_{!*}(K) \in \mathrm{Perv}^p(\mathcal{X})$ *is the unique perverse sheaf* $\widetilde{K} \in \mathrm{Perv}^p(\mathcal{X})$ *such that* $j^!(\widetilde{K}) \simeq K$ *and* $\widetilde{K}$ *has no non-zero subobjects and quotients, supported on* $\mathcal{X} \smallsetminus \mathcal{U}$*.*

Proof. All assertions formally follow from Lemma 6.3.1 and adjunctions.

(a) Follows from the fact that $j^!$ is t-exact and $j^!(\theta)$ is an isomorphism.

(b) Assume that $L \in \mathrm{Perv}^p(\mathcal{X})$ is supported on $\mathcal{X} \smallsetminus \mathcal{U}$, that is, $j^!L \simeq 0$. As $j_!(K) \in {}^p\mathcal{D}^{\leq 0}(\mathcal{X})$ and $j_*(K) \in {}^p\mathcal{D}^{\geq 0}(\mathcal{X})$ (by Lemma 6.3.1), we have isomorphisms

$$\mathrm{Hom}({}^pH^0(j_!K), L) \simeq \mathrm{Hom}(j_!K, L) \simeq \mathrm{Hom}(K, j^!L) \simeq 0$$

and

$$\mathrm{Hom}(L, {}^pH^0(j_*K)) \simeq \mathrm{Hom}(L, j_*K) \simeq \mathrm{Hom}(j^!L, K) \simeq 0.$$

(c) Since $j^!$ is t-exact, we have $j^!j_{!*}K \simeq K$. Next if L is a subobject (resp. quotient) of $j_{!*}K$, supported on $\mathcal{X} \smallsetminus \mathcal{U}$, then L is a subobject (resp. quotient) of ${}^pH^0(j_*K))$ (resp. ${}^pH^0(j_!K))$). So $L \simeq 0$ by (b) (resp. (a)).

Conversely, let $\widetilde{K} \in \mathrm{Perv}^p(\mathcal{X})$ such that $j^!\widetilde{K} \simeq K$ and $\widetilde{K}$ has no non-zero subobjects and quotients in supported on $\mathcal{X}\smallsetminus\mathcal{U}$. By adjunction, the isomorphism $j^!\widetilde{K} \simeq K$ gives rise to morphisms $j_!K \to \widetilde{K} \to j_*K$, hence to morphisms

$${}^pH^0(j_!K) \xrightarrow{a} \widetilde{K} \xrightarrow{b} {}^pH^0(j_*K).$$

We want to show that a is surjective, while b is injective. Since $j^!$ is t-exact, we deduce that $\operatorname{Coker} a$ and $\operatorname{Ker} b$ are supported on $\mathcal{X} \smallsetminus \mathcal{U}$. Hence both of them are zero by the assumption on $\widetilde{K}$. $\square$

Corollary 6.3.5. *In the situation of 6.3.3, let $A, B \in \operatorname{Perv}^{p'}(\mathcal{U})$. Then the pullback $j^! : \operatorname{Hom}(j_{!*}A, j_{!*}B) \to \operatorname{Hom}(A, B)$ is an isomorphism.*

Proof. As $j_!A \in {}^p\mathcal{D}^{\leq 0}(\mathcal{X})$ and $j_*B \in {}^p\mathcal{D}^{\geq 0}(\mathcal{X})$, we obtain natural isomorphisms

$$\operatorname{Hom}(j_!A, j_*B) \xrightarrow{\sim} \operatorname{Hom}({}^pH^0(j_!A), {}^pH^0(j_*B)) \xleftarrow{\sim} \operatorname{Hom}(j_{!*}A, j_{!*}B),$$

where the isomorphism on the right follows from Corollary 6.3.4(a),(b). Since the map $A \to j^!j_!A$ is an isomorphism, $j^!$ induces an isomorphism

$$\operatorname{Hom}(j_!A, j_*B) \simeq \operatorname{Hom}(j^!j_!A, B) \simeq \operatorname{Hom}(A, B),$$

thus the assertion follows. $\square$

Finally, when $j : \mathcal{U} \hookrightarrow \mathcal{X}$ is an fp-open embedding of a $\{\mathcal{X}_\alpha\}_\alpha$-adapted ∞-substack, we have the following result.

Lemma 6.3.6. *Let $(\mathcal{X}, \{\mathcal{X}_\alpha\}_\alpha)$ be an $\mathcal{I}$-stratified ∞-stack, equipped with perversity p_ν. Let $j : \mathcal{U} \hookrightarrow \mathcal{X}$ is an fp-open inclusion of an $\{\mathcal{X}_\alpha\}_\alpha$-adapted ∞-substack, and let $i : \mathcal{Z} := \mathcal{X} \smallsetminus \mathcal{U} \hookrightarrow \mathcal{X}$ be the complementary topologically fp-closed embedding. Equip $\mathcal{U}$ and $\mathcal{Z}$ with the induced perversities, and let $K \in \operatorname{Perv}^{p_\nu}(\mathcal{U})$.*

(a) The functor $i_!$ is t-exact, functor $i^!$ is left t-exact, while i^* is right t-exact.

(b) The intermediate extension $j_{!*}K \in \operatorname{Perv}^p(\mathcal{X})$ is the unique perverse extension $\widetilde{K}$ of K such that $i^*\widetilde{K} \in {}^{p_\nu}\mathcal{D}^{\leq -1}(\mathcal{Z})$ and $i^!\widetilde{K} \in {}^{p_\nu}\mathcal{D}^{\geq 1}(\mathcal{Z})$.

(c) Assume that the stratification is bounded. Then $j_{!*}K \in \operatorname{Perv}^{p_\nu}(\mathcal{X})$ is the unique perverse extension $\widetilde{K}$ of K such that for all $\alpha \in \mathcal{I} \smallsetminus \mathcal{I}_{\mathcal{U}}$, we have

$$\eta_\alpha^*\widetilde{K} \in {}^p\mathcal{D}^{\leq -\nu_\alpha - 1}(\mathcal{X}_\alpha) \text{ and } \eta_\alpha^!\widetilde{K} \in {}^p\mathcal{D}^{\geq -\nu_\alpha + 1}(\mathcal{X}_\alpha).$$

Proof. (a) By adjunction, it suffices to show that $i^!$ and $i_!$ are left t-exact. Both assertions immediately follow from formula (6.1) and identity $i^!i_! \simeq \mathrm{Id}$.

(b) By Corollary 6.3.4(c), it suffices to show that a perverse sheaf $\widetilde{K} \in \operatorname{Perv}^{p_\nu}(\mathcal{X})$ has no non-zero subobjects (resp. quotients) supported on $\mathcal{Z}$ if and only if $i^!\widetilde{K} \in {}^{p_\nu}\mathcal{D}^{\geq 1}(\mathcal{Z})$ (resp. $i^*\widetilde{K} \in {}^{p_\nu}\mathcal{D}^{\leq -1}(\mathcal{Z})$).

Notice that if $L \in \operatorname{Perv}^{p_\nu}(\mathcal{X})$ is supported on $\mathcal{Z}$, then $L \simeq i_!M$ for some $M \in \operatorname{Perv}^{p_\nu}(\mathcal{Z})$. Indeed, since $j^!L \simeq 0$, we have $L \simeq i_!M$ with $M := i^!L$ (by Lemma 4.5.3(d)). Since $i_!$ is fully faithful, we conclude that $M = i^!L \simeq i^*L$. Finally, since L is perverse, we conclude from (a) that M is perverse.

Now all assertions are easy. By (a), we have $i^!\widetilde{K} \in {}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{Z})$. Then for every $M \in \operatorname{Perv}^{p_\nu}(\mathcal{Z})$, we have an equivalence

$$\operatorname{Hom}(i_!M, \widetilde{K}) \simeq \operatorname{Hom}(M, i^!\widetilde{K}) \simeq \operatorname{Hom}(M, {}^{p_\nu}H^0(i^!\widetilde{K})).$$

It follows that if $i^!\widetilde{K}\in {}^{p_\nu}\mathcal{D}^{\geq 1}(\mathcal{Z})$, then $\operatorname{Hom}(i_!M,\widetilde{K})\simeq 0$. Thus $\widetilde{K}\in\operatorname{Perv}^{p_\nu}(\mathcal{X})$ has no non-zero subobjects supported on $\mathcal{Z}$. Conversely, if $i^!\widetilde{K}\in {}^{p_\nu}\mathcal{D}^{\geq 0}(\mathcal{Z})\smallsetminus {}^{p_\nu}\mathcal{D}^{\geq 1}(\mathcal{Z})$, then ${}^{p_\nu}H^0(i^!\widetilde{K}))\not\simeq 0$, thus there exists a nonzero morphism $a: i_!M\to\widetilde{K}$ (corresponding to the identity map of ${}^{p_\nu}H^0(i^!\widetilde{K})$). Then the image of a is the quotient of $i_!M$, hence is supported on $\mathcal{Z}$.

The proof of the second assertion is similar.

(c) Note that when the stratification is bounded we have $i^*\widetilde{K}\in {}^{p_\nu}\mathcal{D}^{\leq -1}(\mathcal{Z})$ if and only if $\eta_\alpha^*\widetilde{K}\in {}^{p}\mathcal{D}^{\leq -\nu_\alpha-1}(\mathcal{X}_\alpha)$ for every $\alpha\in\mathcal{I}_\mathcal{Z}=\mathcal{I}\smallsetminus\mathcal{I}_\mathcal{U}$ (by (6.2)) and similarly for $i^!\widetilde{K}\in {}^{p_\nu}\mathcal{D}^{\geq 1}(\mathcal{Z})$. Now the assertion follows from (b). □

6.4. **Semi-small morphisms.**

6.4.1. Notation.

(a) Let $\mathcal{Y}$ a stratified ∞-stack with bounded constructible stratification $\{\mathcal{Y}_\alpha\}_\alpha$, let $f:\mathcal{X}\to\mathcal{Y}$ be a morphism of ∞-stacks, and $\mathcal{X}_\alpha:=f^{-1}(\mathcal{Y}_\alpha)$ be the induced constructible stratification of $\mathcal{X}$. Then we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_\alpha & \xrightarrow{\widetilde{\eta}_\alpha} & \mathcal{X} \\ {\scriptstyle f_\alpha}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \mathcal{Y}_\alpha & \xrightarrow{\eta_\alpha} & \mathcal{Y}. \end{array} \tag{6.4}$$

(b) Assume that $\mathcal{X}$ is topologically placid, that each $\mathcal{X}_\alpha\subset\mathcal{X}$ is of pure codimension b_α, and each $f_\alpha:\mathcal{X}_\alpha\to\mathcal{Y}_\alpha$ is equidimensional of relative dimension δ_α (see 3.3.5).

(c) We say that f is *semi-small*, if for every $\alpha\in\mathcal{I}$ we have an inequality $\delta_\alpha\leq b_\alpha$.

(d) Let $\mathcal{U}\subset\mathcal{Y}$ be a $\{\mathcal{Y}_\alpha\}_\alpha$-adapted fp-open substack. We say that a semi-small map is $\mathcal{U}$-*small*, if for every $\alpha\in\mathcal{I}\smallsetminus\mathcal{I}_\mathcal{U}$, we have a strict inequality $\delta_\alpha<b_\alpha$.

6.4.2. Remarks. Assume that $f:X\to Y$ is a dominant morphism of irreducible schemes of finite type over k all of whose fibers are equidimensional.

(a) One can show that there exists a constructible stratification Y_α of Y such that each Y_α is irreducible and each $f_\alpha:=f|_{Y_\alpha}:X_\alpha\to Y_\alpha$ is equidimensional of relative dimension δ_α. Then X_α is equidimensional, thus is of pure codimension b_α in X. In other words, f satisfies the assumptions of 6.4.1.

(b) Recall that classically a morphism f is called *semi-small*, we have inequalities

$$\operatorname{codim}_Y(Y_\alpha):=\dim Y-\dim Y_\alpha\geq 2\delta_\alpha \text{ for all } \alpha,$$

and we claim that they are is equivalent to our inequalities $\delta_\alpha\leq b_\alpha$. It suffices to show that

$$\operatorname{codim}_Y(Y_\alpha)=b_\alpha+\delta_\alpha \text{ for all } \alpha. \tag{6.5}$$

In the case of open stratum Y_{α_0}, both inequalities imply that $\delta_\alpha = b_\alpha = 0$, thus f is generically finite. Hence $\dim X = \dim Y$, so identity (6.5) follows from equalities $\dim X_\alpha = \dim Y_\alpha + \delta_\alpha$ and $\dim X = \dim X_\alpha + b_\alpha$.

(c) Equality (6.5) also implies that a morphism f is *small* in the classical sense if and only if it is Y_{α_0}-small in our case.

(d) Note that though our assumptions 6.4.1(b) are never satisfied when not all fibers of $f : X \to Y$ are equidimensional, it is possible to modify them in order to include a more general case as well.

6.4.3. Perversity, induced by f.

In the situation of 6.4.1(a),(b), we consider perversity $p_f := \{\nu_\alpha\}_{\alpha\in\mathcal{I}}$, defined by $\nu_\alpha := b_\alpha + \delta_\alpha$ for all α. Then f is semi-small, if and only if we have

$$2\delta_\alpha \leq \nu_\alpha \leq 2b_\alpha \text{ for every } \alpha \in \mathcal{I}. \tag{6.6}$$

Moreover, f is $\mathcal{U}$-small, if and only if we have

$$2\delta_\alpha < \nu_\alpha < 2b_\alpha \text{ for every } \alpha \in \mathcal{I} \smallsetminus \mathcal{I}_{\mathcal{U}}. \tag{6.7}$$

6.4.4. Remark. Our definition of the perversity p_f is motivated by the observation that in "good" cases, e.g. when $f : X \to Y$ is a dominant generically finite morphism between irreducible schemes of finite type over k, the perversity p_f coincides with the canonical perversity from 6.2.8 (see (6.5)), thus the corresponding t-structure is the !-adapted perverse t-structure (see Lemma 6.2.9).

Theorem 6.4.5. *(a) Let $f : \mathcal{X} \to \mathcal{Y}$ be a locally ind-fp-proper semi-small morphism of ∞-stacks, where $\mathcal{X}$ is topologically smooth, while $\mathcal{Y}$ admits gluing of sheaves. Then the pushforward $K := f_!(\omega_{\mathcal{X}})$ is p_f-perverse.*

(b) Moreover, assume that f is $\mathcal{U}$-small, and let $j : \mathcal{U} \hookrightarrow \mathcal{Y}$ be the open embedding. Then we have an isomorphism $K \simeq j_{!*}j^!(K)$.

Proof. By Proposition 6.2.7 and Lemma 6.3.6, we have to show that we have

$$\eta_\alpha^* K \in {}^p\mathcal{D}^{\leq -\nu_\alpha}(\mathcal{Y}_\alpha) \text{ and } \eta_\alpha^! K \in {}^p\mathcal{D}^{\geq -\nu_\alpha}(\mathcal{Y}_\alpha)$$

for every $\alpha \in \mathcal{I}$, and stronger inclusions

$$\eta_\alpha^* K \in {}^p\mathcal{D}^{\leq -\nu_\alpha - 1}(\mathcal{Y}_\alpha) \text{ and } \eta_\alpha^! K \in {}^p\mathcal{D}^{\geq -\nu_\alpha + 1}(\mathcal{Y}_\alpha).$$

for every $\alpha \in \mathcal{I} \smallsetminus \mathcal{I}_{\mathcal{U}}$. Using (6.6) and (6.7), it thus suffices to show that for every $\alpha \in \mathcal{I}$ we have

$$\eta_\alpha^* K \in {}^p\mathcal{D}^{\leq -2b_\alpha}(\mathcal{Y}_\alpha) \text{ and } \eta_\alpha^! K \in {}^p\mathcal{D}^{\geq -2\delta_\alpha}(\mathcal{Y}_\alpha). \tag{6.8}$$

Since f is locally ind-fp-proper, every f_α is locally ind-fp-proper as well. Moreover, diagram (6.4) gives rise to a natural isomorphism of functors $\eta_\alpha^* f_! \simeq (f_\alpha)_! \widetilde{\eta}_\alpha^*$ (see Corollary 6.1.4) and $\eta_\alpha^! f_! \simeq (f_\alpha)_! \widetilde{\eta}_\alpha^!$ (see Proposition 4.4.3). Therefore we get isomorphisms $\eta_\alpha^! K \simeq (f_\alpha)_!(\omega_{\mathcal{X}_\alpha})$ and $\eta_\alpha^* K \simeq (f_\alpha)_! \widetilde{\eta}_\alpha^*(\omega_{\mathcal{X}})$.

Since $\mathcal{X}$ is topologically smooth, and $\widetilde{\eta}_\alpha : \mathcal{X}_\alpha \to \mathcal{X}$ is fp-locally closed, weakly equidimensional of relative dimension b_α, we conclude from Lemma 5.6.1(e) that $\widetilde{\eta}_\alpha^*(\omega_{\mathcal{X}}) \in {}^p\mathcal{D}_c^{\leq -2b_\alpha}(\mathcal{X}_\alpha)$. Moreover, since f_α is equidimensional, the pullback $f_\alpha^!$ is left t-exact (by Lemma 5.6.1(b)). Therefore by adjunction, we conclude that $(f_\alpha)_!$ is right t-exact, thus

$$\eta_\alpha^* K \simeq (f_\alpha)_!(\widetilde{\eta}_\alpha^*(\omega_{\mathcal{X}})) \in {}^p\mathcal{D}^{\leq -2b_\alpha}(\mathcal{Y}_\alpha),$$

proving the first inclusion in (6.8).

Similarly, since $\omega_{\mathcal{X}_\alpha} \in {}^p\mathcal{D}_c^{\geq 0}(\mathcal{X}_\alpha)$ (by Lemma 5.6.1(a)), and the functor $(f_\alpha)_![-2\delta_\alpha]$ is right t-exact (by Lemma 5.6.4), we deduce that

$$\eta_\alpha^! K \simeq (f_\alpha)_!(\omega_{\mathcal{X}_\alpha}) \in {}^p\mathcal{D}^{\geq -2\delta_\alpha}(\mathcal{Y}_\alpha),$$

proving the second inclusion in (6.8). □

Part 3. **The affine Springer theory**

7. The Goresky-Kottwitz-MacPherson stratification

7.1. **Arc and loop spaces.** We set $\mathcal{O} = k[[t]]$, $K = k((t))$. We recall some basic definitions on arc and loop spaces. Most of the material can be found in [EM, 2-3].

7.1.1. Notation. (a) If X is an $\mathcal{O}$-scheme of finite type, $n \geq 0$, we consider the functor on k-algebras $\mathcal{L}_n^+(X) : A \mapsto X(A[t]/(t^{n+1}))$. It is representable by a k-scheme of finite type, and for every $n \in \mathbb{N}$, the transition maps $\mathcal{L}_{n+1}^+(X) \to \mathcal{L}_n^+(X)$ are affine.

(b) We can consider its arc space $\mathcal{L}^+(X) = \varprojlim_{n\geq 0} \mathcal{L}_n^+(X)$, which is a k-scheme, representing the functor $A \mapsto X(A[[t]])$ (see, for example, [Bh, Cor.1.2]).

(c) Denote by $ev_X : \mathcal{L}^+(X) \to X$ the evaluation map, induced by the projection $A[[t]] \to A$.

(d) For every affine scheme X of finite type over K, we consider its loop space $\mathcal{L}\,X$, representing the functor $A \mapsto X(A((t)))$. It is an ind-affine ind-scheme. If moreover, X has a structure over $\mathcal{O}$, then we have a closed embedding $\mathcal{L}^+(X) \hookrightarrow \mathcal{L}\,X$.

Lemma 7.1.2. *Let $X \to Y$ be an étale map of schemes of finite type over $\mathcal{O}$. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{L}^+(X) & \longrightarrow & \mathcal{L}^+(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is Cartesian. In particular, the induced map $\mathcal{L}^+(X) \to \mathcal{L}^+(Y)$ is étale and finitely presented.

Proof. We have to show that for every ring A the commutative diagram

$$\begin{array}{ccc} X(A[[t]]) & \longrightarrow & Y(A[[t]]) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & Y(A) \end{array}$$

is Cartesian. But this follows from the fact that an étale map is formally étale. □

7.1.3. Remarks. (a) By Lemma 7.1.2, if U is open in X, $\mathcal{L}^+(U)$ is open in $\mathcal{L}^+(X)$. On the other hand, $Z \hookrightarrow X$ is closed, then $\mathcal{L}^+(Z) \hookrightarrow \mathcal{L}^+(X)$ is also a closed embedding, but of infinite type.

(b) If the scheme X is smooth, then the schemes $\mathcal{L}^+_n(X)$ are smooth and transition maps $\mathcal{L}^+_{n+1}(X) \to \mathcal{L}^+_n(X)$ are smooth, surjective. Therefore the arc space $\mathcal{L}^+(X)$ is strongly pro-smooth.

(c) Let H be an algebraic group, and $X \to Y$ be a H-torsor between varieties over k (see 9.2.2). Then the induced morphism $\mathcal{L}^+(X) \to \mathcal{L}^+(Y)$ is an $\mathcal{L}^+(H)$-torsor. Indeed, since H is smooth, there exists an étale covering $Y' \to Y$ such that $X \times_Y Y' \to Y'$ is a trivial H-torsor. Then $\mathcal{L}^+(X) \times_{\mathcal{L}^+(Y)} \mathcal{L}^+(Y') \to \mathcal{L}^+(Y')$ is a trivial $\mathcal{L}^+(H)$-torsor, hence $\mathcal{L}^+(X) \to \mathcal{L}^+(Y)$ is an $\mathcal{L}^+(H)$-torsor (by Lemma 7.1.2). In particular, the natural map $\mathcal{L}^+(H) \setminus \mathcal{L}^+(X) \to \mathcal{L}^+(Y)$ is an equivalence (see 9.2.2(c)).

7.1.4. Stratification by valuation: the $\mathbb{A}^1$-case.

(a) Recall that the arc space $\mathcal{L}^+(\mathbb{A}^1)$ the classifies a functor $A \mapsto A[[t]]$. Thus $\mathcal{L}^+_n(\mathbb{A}^1) \simeq \operatorname{Spec} k[\{a_i\}_{i \in \mathbb{N}}]$.

(b) For every $n \in \mathbb{N}$ let $\mathcal{L}^+(\mathbb{A}^1)_{\geq n} \subset \mathcal{L}^+(\mathbb{A}^1)$ be the closed subscheme, given by equations $a_0 = \ldots = a_{n-1} = 0$, and set $\mathcal{L}^+(\mathbb{A}^1)_{\leq n} := \mathcal{L}^+(\mathbb{A}^1) \smallsetminus \mathcal{L}^+(\mathbb{A}^1)_{\geq n+1}$. Then $\mathcal{L}^+(\mathbb{A}^1)_{\leq n} \subset \mathcal{L}^+(\mathbb{A}^1)$ is an fp-open subscheme, and $\{\mathcal{L}^+(\mathbb{A}^1)_{\leq n}\}_{n \geq 0}$ gives an fp-open covering of $\mathcal{L}^+(\mathbb{A}^1)_\bullet := \mathcal{L}^+(\mathbb{A}^1) \smallsetminus \{0\}$.

(c) For every $n \in \mathbb{N}$, consider the open subscheme $\mathcal{L}^+(\mathbb{A}^1)_n \subset \mathcal{L}^+(\mathbb{A}^1)_{\geq n}$ given by the inequality $a_n \neq 0$. Explicitly, $\mathcal{L}^+(\mathbb{A}^1)_n(A)$ classifies power series $\sum_{i=0}^\infty b_i t^i \in A[[t]]$ such that $b_0 = \ldots = b_{n-1} = 0$ and $b_n \in A^\times$.

(d) By definition, we have $\mathcal{L}^+(\mathbb{A}^1)_n = \mathcal{L}^+(\mathbb{A}^1)_{\geq n} \cap \mathcal{L}^+(\mathbb{A}^1)_{\leq n}$, and $\{\mathcal{L}^+(\mathbb{A}^1)_n\}$ form a bounded constructible stratification (see 6.2.2) of $\mathcal{L}^+(\mathbb{A}^1)_\bullet$.

(e) The open embedding $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ induces an isomorphism $\mathcal{L}^+(\mathbb{G}_m) \xrightarrow{\sim} \mathcal{L}^+(\mathbb{A}^1)_0$ and an embedding (of functors) $\mathcal{L}(\mathbb{G}_m) \hookrightarrow \mathcal{L}(\mathbb{A}^1)$. Moreover, the composition $\mathcal{L}^+(\mathbb{A}^1)_n \hookrightarrow \mathcal{L}^+(\mathbb{A}^1) \hookrightarrow \mathcal{L}(\mathbb{A}^1)$ induces an embedding $\mathcal{L}^+(\mathbb{A}^1)_n \hookrightarrow \mathcal{L}(\mathbb{G}_m)$.

7.1.5. Stratification by valuation: the general case. Let X be an affine scheme over $\mathcal{O}$, and $f \in \mathcal{O}[X]$ a regular function.

(a) Then f induces a morphism $f : \mathcal{L}^+(X) \to \mathcal{L}^+(\mathbb{A}^1)$, and we denote by $\mathcal{L}^+(X)_{(f;\geq n)}$, $\mathcal{L}^+(X)_{(f;\leq n)}$ and $\mathcal{L}^+(X)_{(f;n)}$ the reduced preimages $f^{-1}(\mathcal{L}^+(\mathbb{A}^1)_{\geq n})_{\mathrm{red}}$, $f^{-1}(\mathcal{L}^+(\mathbb{A}^1)_{\leq n})_{\mathrm{red}}$

and $f^{-1}(\mathcal{L}^+(\mathbb{A}^1)_n)_{\rm red}$, respectively. Moreover, if $g \in \mathcal{O}[X]$ is another regular function we can form the reduced intersection $\mathcal{L}^+(X)_{(f;n),(g;m)}$ of $\mathcal{L}^+(X)_{(f;n)}$ and $\mathcal{L}^+(X)_{(g;m)}$.

(b) Note that $\mathcal{L}^+(X)_{(f;\geq n)} \subset \mathcal{L}^+(X)$ is a reduction of an fp-closed subscheme, and $\mathcal{L}^+(X)_{(f;n)} \subset \mathcal{L}^+(X)_{(f;\geq n)}$ is a basic open subscheme given by equation $f^*(a_n) \neq 0$. In particular, both $\mathcal{L}^+(X)_{(f;\geq n)}$ and $\mathcal{L}^+(X)_{(f;n)}$ are affine.

(c) By 7.1.4(d) and Lemma 6.2.4(a), we conclude that $\{\mathcal{L}^+(X)_{(f;n)}\}$ form a bounded constructible stratification of $\mathcal{L}^+(X)_{f\neq 0}$.

(d) Let $X_f \subset X$ be the open subset $f \neq 0$. Using 7.1.4(e), we get an isomorphism $\mathcal{L}^+(X_f)_{\rm red} \xrightarrow{\sim} \mathcal{L}^+(X)_{(f;0)}$ and an embedding $\mathcal{L}^+(X)_{(f;n)} \hookrightarrow \mathcal{L}(X_f)$. In particular, it induces an isomorphism $\mathcal{L}^+(X_f)_{(g;n)} \xrightarrow{\sim} \mathcal{L}^+(X)_{(g;n),(f;0)}$ for every $g \in \mathcal{O}[X]$ and n.

Lemma 7.1.6. *In the situation of 7.1.5, assume that f decomposes as a product $f = \prod_{i=1}^k f_i$. Then $\mathcal{L}^+(X)_{(f;n)}$ decomposes as a disjoint union*

$$\sqcup_{m_1,\ldots,m_k,\sum m_i = n}\mathcal{L}^+(X)_{(f_1;m_1),\ldots,(f_k;m_k)}.$$

Proof. By induction, we reduce to the case $k = 2$. Moreover, by considering morphism $\overline{f} = (f_1, f_2) : X \to \mathbb{A}^2$, we reduce to the case when X is the affine space $\mathbb{A}^2$ with coordinates x, y and $f = xy$. In other words, we have to show that the stratum $\mathcal{L}^+(\mathbb{A}^2)_{(xy;n)}$ decomposes as a disjoint union $\sqcup_{m=0}^n \mathcal{L}^+(\mathbb{A}^2)_{(x;m),(y;n-m)}$, which is straightforward. □

7.1.7. The smooth case. In the situation of 7.1.5, assume that X is smooth over $\mathcal{O}$.

(a) The affine scheme $\mathcal{L}^+(X)$ is pro-smooth (see 7.1.3(b)). Therefore $\mathcal{L}^+(X)_{(f;\leq n)} = f^{-1}(\mathcal{L}^+(\mathbb{A}^1)_{\leq n})$ is a pro-smooth scheme, while $f^{-1}(\mathcal{L}^+(\mathbb{A}^1)_{\geq n})$, and $f^{-1}(\mathcal{L}^+(\mathbb{A}^1)_{\leq n})$ are finitely presented subschemes of $\mathcal{L}^+(X)$. Also $\mathcal{L}^+(X)$ has an open covering $\mathcal{L}^+(X) = \cup_{n\geq 0}\mathcal{L}^+(X)_{(f,\leq n)}$.

(b) By (a) and Lemma 2.1.12, both $f^{-1}(\mathcal{L}^+(\mathbb{A}^1)_{\geq n})$, and $f^{-1}(\mathcal{L}^+(\mathbb{A}^1)_{\leq n})$ are globally placid affine schemes. It now follows from Corollary 2.2.3(b) that $\mathcal{L}^+(X)_{(f;\geq n)}$ and $\mathcal{L}^+(X)_{(f;n)}$ are globally placid affine schemes as well. By 7.1.5(d), we have $\mathcal{L}^+(X)_{(f;0)} \simeq \mathcal{L}^+(X_f)$.

7.2. **Root valuation strata.** In this subsection, we review the results of Goresky-Kottwitz-MacPherson [GKM] and prove Theorem 7.2.5, which is a slight strengthening of [GKM, Thm. 8.2.2(3)].

7.2.1. Basic notation. (a) Let G be a connected reductive group over k, and $\mathfrak{g}$ be the Lie algebra of G, $\mathrm{Ad} : G \to GL(\mathfrak{g})$ the adjoint representation. Let (B, T) be a Borel group and a maximal torus of G, respectively, W its Weyl group, $X_*(T)$ the lattice of cocharacters, and R the set of roots. We also set $\mathfrak{t} := \mathrm{Lie}(T)$ and $\mathfrak{b} := \mathrm{Lie}(B)$. Let $r = \dim(\mathfrak{t})$, we suppose that the characteristic of k is prime to the order of W.

(b) We define the extended affine Weyl group $\widetilde{W} := X_*(T) \rtimes W$.

(c) Let $\mathfrak{c} := \mathfrak{t}/W = \mathrm{Spec}(k[\mathfrak{g}]^G)$ be the Chevalley space of $\mathfrak{g}$. Then we have canonical projections $\chi : \mathfrak{g} \to \mathfrak{c}$ and $\pi : \mathfrak{t} \to \mathfrak{c}$ (compare [Ngo, Thm.1.1.1]). Recall that π is finite, flat and surjective.

(d) Let $\mathfrak{D} := \prod_{\alpha\in R} d\alpha$ be the discriminant function. Then $\mathfrak{D} \in k[\mathfrak{c}] = k[\mathfrak{t}]^W$, and the regular semisimple locus $\mathfrak{c}^{rs} \subset \mathfrak{c}$ is the complement of the locus of zeros of $\mathfrak{D}$. We denote by $\mathfrak{g}^{rs} := \chi^{-1}(\mathfrak{c}^{rs})$ and $\mathfrak{t}^{rs} := \pi^{-1}(\mathfrak{c}^{rs})$ the preimages of $\mathfrak{c}^{rs}$.

(e) Note that the morphism $\chi : \mathfrak{g} \to \mathfrak{c}$ induces a morphism $\chi : \mathcal{L}^+(\mathfrak{g}) \to \mathcal{L}^+(\mathfrak{c})$ between arc spaces.

(f) Let $I := ev_G^{-1}(B) \subset \mathcal{L}^+(G)$ be the Iwahori group scheme, whose Lie algebra is $\mathrm{Lie}(I) = ev_{\mathfrak{g}}^{-1}(\mathfrak{b}) \subset \mathcal{L}^+(\mathfrak{g})$.

7.2.2. Stratification of $\mathcal{L}^+(\mathfrak{t})$.

(a) Let $X = \mathfrak{t}$, and $\mathfrak{D} \in k[\mathfrak{t}]$ be the discriminant function. Then, by 7.1.5(c), we have a bounded stratification of $\mathcal{L}^+(\mathfrak{t})_\bullet := \mathcal{L}^+(\mathfrak{t})_{\mathfrak{D}\neq 0}$ by $\mathcal{L}^+(\mathfrak{t})_{(\mathfrak{D};n)}$.

Since $\mathfrak{D} = \prod_{\alpha\in R} d\alpha$, it follows from Lemma 7.1.6 that each $\mathcal{L}^+(\mathfrak{t})_{(\mathfrak{D};n)}$ decomposes as a disjoint union $\mathcal{L}^+(\mathfrak{t})_{(\mathfrak{D};n)} = \sqcup_{\mathbf{r}}\mathfrak{t}_{\mathbf{r}}$, where $\mathbf{r}$ runs over functions $\mathbf{r} : R \to \mathbb{Z}_{\geq 0}$ such that $d_{\mathbf{r}} := \sum_{a\in R} \mathbf{r}(\alpha)$ equals n.

(b) Explicitly, $\mathfrak{t}_{\mathbf{r}}$ classifies power series $\sum_{i\geq 0} x_i t^i$, where $x_i \in \mathfrak{t}$ for all i such that $\alpha(x_i) = 0$ for $0 \leq i < \mathbf{r}(\alpha)$, and $\alpha(x_i) \neq 0$ for $i = \mathbf{r}(\alpha)$. In other words, $\mathfrak{t}_{\mathbf{r}} \subset \mathfrak{t}$ is given by finitely many equalities of linear functions, and finitely many inequalities. In particular, $\mathfrak{t}_{\mathbf{r}} \subset \mathfrak{t}$ is a connected strongly pro-smooth locally closed affine subscheme.

(c) Note that the natural action of W on $\mathfrak{t}$ induces a W-action on $\mathcal{L}^+(\mathfrak{t})_{(\mathfrak{D},n)}$. Moreover, every $u \in W$ induces an isomorphism $\mathfrak{t}_{\mathbf{r}} \xrightarrow{\sim} \mathfrak{t}_{u(\mathbf{r})}$, where $u(\mathbf{r})$ is defined by the rule $u(\mathbf{r})(\alpha) = \mathbf{r}(u^{-1}(\alpha))$, where $u^{-1}(\alpha)(x) = \alpha(u(x))$ for all $x \in \mathfrak{t}$.)

7.2.3. The twisted version. (a) Let m be the order of W, and set $\mathcal{O}' := k[[t^{1/m}]]$. We choose a primitive m-th root of unity $\xi \in k$, and let $\sigma \in \mathrm{Aut}(\mathcal{O}'/\mathcal{O})$ be the automorphism $\sigma(t^{1/m}) = \xi t^{1/m}$. We set $\mathfrak{t}' := R_{\mathcal{O}'/\mathcal{O}}(\mathfrak{t} \times_{\mathcal{O}} \mathcal{O}')$. In particular, σ defines an automorphism of $\mathcal{L}^+(\mathfrak{t}')$, whose scheme of fixed points is $\mathcal{L}^+(\mathfrak{t})$.

(b) By 7.2.2(a), the space $\mathcal{L}^+(\mathfrak{t}')_{\mathfrak{D}\neq 0}$ has a bounded constructible stratification by $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$ for $n \in \frac{1}{m}\mathbb{Z}_{\geq 0}$, and every $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$ decomposes as a disjoint union $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)} = \sqcup_{\mathbf{r}}\mathfrak{t}'_{\mathbf{r}}$, where $\mathbf{r}$ runs over functions $\mathbf{r} : R \to \frac{1}{m}\mathbb{Z}_{\geq 0}$ such that $d_{\mathbf{r}} = n$.

(c) For each $w \in W$, define $\mathfrak{t}_w$ as the scheme of fixed points of $w\sigma$ in $\mathfrak{t}'$. Then $\mathcal{L}^+(\mathfrak{t}_w)$ is the scheme of fixed points of $w\sigma$ in $\mathcal{L}^+(\mathfrak{t}')$, and $\mathcal{L}^+(\mathfrak{t}_w)_{(\mathfrak{D};n)}$ is the scheme of fixed points of $w\sigma$ in $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$. The decomposition $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)} = \sqcup_{\mathbf{r}}\mathfrak{t}'_{\mathbf{r}}$ from (b) is σ-invariant and induces a decomposition $\mathcal{L}^+(\mathfrak{t}_w)_{(\mathfrak{D};n)} = \sqcup_{\mathbf{r}\,|\,w(\mathbf{r})=\mathbf{r}}\mathfrak{t}_{w,\mathbf{r}}$.

(d) Note that $\mathcal{L}^+(\mathfrak{t}') \simeq \lim_n \mathcal{L}^+_n(\mathfrak{t}')$ is a pro-vector space, and the action of $w\sigma$ on $\mathcal{L}^+(\mathfrak{t}')$ comes from a compatible system of linear actions on vector spaces $\mathcal{L}^+_n(\mathfrak{t}')$.

Therefore the scheme of fixed points $\mathfrak{t}_w = \mathcal{L}^+(\mathfrak{t}')^{\langle w\sigma\rangle}$ is a pro-vector space, thus it is connected and strongly pro-smooth.

Similarly, $\mathfrak{t}_{w,\mathbf{r}}$ is the scheme of fixed points $\mathfrak{t}_{w,\mathbf{r}} = (\mathfrak{t}'_{\mathbf{r}})^{\langle w\sigma\rangle}$, and $\mathfrak{t}'_{\mathbf{r}}$ is an fp-open subscheme of a pro-vector space. Therefore $\mathfrak{t}_{w,\mathbf{r}}$ is an fp-open subscheme of a pro-vector space as well, thus it is connected and strongly pro-smooth as well.

(e) We set $\widetilde{\mathfrak{t}} := \sqcup_{w\in W}\mathfrak{t}_w$. The natural morphism $\widetilde{\mathfrak{t}} \to \mathfrak{t}'$ induces a morphism $\mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$. Moreover, since W acts freely on $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$, the map $\mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$ is a closed embedding. By (c), we have a decomposition $\mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)} = \sqcup_{w,\mathbf{r}}\mathfrak{t}_{w,\mathbf{r}}$.

(f) The W-action on $\mathfrak{t}$ induces W-actions on $\mathfrak{t}'$, $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$ and $\mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)} \subset \mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$. Moreover, the W-action is compatible with the stratification of (e). Namely, for every $u, w \in W$ and $\mathbf{r} : R \to \frac{1}{m}\mathbb{Z}_{\geq 0}$, the u-action induces an isomorphism $u : \mathfrak{t}_{w,\mathbf{r}} \xrightarrow{\sim} \mathfrak{t}_{uwu^{-1},u(\mathbf{r})}$ (compare 7.2.2(c)).

7.2.4. The Chevalley space. Notice that the W-equivariant morphism $\pi : \mathfrak{t} \to \mathfrak{c}$, induces a W-equivariant morphism $\mathfrak{t}' \to \mathfrak{c}'$, which restricts to a W-equivariant $\widetilde{\mathfrak{t}} \to \mathfrak{c}$, hence induces a W-equivariant morphism $\mathcal{L}^+(\widetilde{\mathfrak{t}}) \to \mathcal{L}^+(\mathfrak{c})$. In particular, for every $n \in \frac{1}{m}\mathbb{Z}_{\geq 0}$, we get a W-equivariant morphism $\mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$.

Theorem 7.2.5. *The map $\pi : \mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$ is a W-torsor.*

Proof. To simplify the notation, we set $X := \mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)}$ and $Y := \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$.

Step 1. For every algebraically closed field K/k, the map $\pi : X(K) \to Y(K)$ is surjective, and every fiber is a W-torsor.

Proof. Recall that points of $Y(K)$ are elements y of $\mathfrak{c}(K[[t]]) \cap \mathfrak{c}^{rs}(K((t)))$ such that $v(\mathfrak{D}(y)) = n$, while points of $X(K)$ are elements X of $\mathfrak{t}(K[[t^{1/m}]]) \cap \mathfrak{t}^{rs}(K((t^{1/m})))$ such that $v(\mathfrak{D}(x)) = n$ and $\sigma(x) = w^{-1}(x)$ for some $w \in W$.

Since the $\pi : \mathfrak{t}^{rs} \to \mathfrak{c}^{rs}$ is a W-torsor, every fiber of $X(K) \to Y(K)$ is either a W-torsor or empty. Thus it suffices to show that $\pi : X(K) \to Y(K)$ is surjective. Fix $y \in Y(K)$.

Since $y \in \mathfrak{c}^{rs}(K((t)))$ and $\pi : \mathfrak{t}^{rs} \to \mathfrak{c}^{rs}$ is a W-torsor, there exists a finite Galois extension $M/K((t))$ of degree $m'|m$ such that $y \in \pi(\mathfrak{t}^{rs}(M))$. Since m is invertible in k, we have $M \simeq K((t^{1/m'}))$, thus there exists $x \in \mathfrak{t}^{rs}(K((t^{1/m})))$ such that $\pi(x) = y$. Moreover, $\pi(\sigma(x)) = \sigma(\pi(x)) = \sigma(y) = y$. Thus there exists $w \in W$ such that $\sigma(x) = w^{-1}(x)$. Finally, since $y \in \mathfrak{c}(K[[t]])$ and $\mathfrak{t} \to \mathfrak{c}$ is finite, thus proper, it follows from the valuative criterion that $x \in \mathfrak{t}(K[[t^{1/m}]])$. □

Step 2. It suffices to show that the map $\pi : X \to Y$ is étale and finitely presented.

Proof. Assume that π is étale. Since π is surjective (by Step 1), it is faithfully flat. Thus it suffices to show that the map $a : W \times X \to X \times_Y X : (w, x) \mapsto (wx, x)$ is an isomorphism. Since a is surjective by Step 1, it suffices to show that a is an open embedding.

Since π is étale, the diagonal map $X \to X \times_Y X$ is an open embedding. Therefore the map $a_w : X \to X \times_Y X : x \mapsto (wx, x)$ is an open embedding for all W. Moreover, by Step 1, the images of the a_w's do not intersect. Thus a is an open embedding, and we are done. □

Step 3. It suffices to show that the map $\mathfrak{t}_{\mathbf{r}} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$, induced by π, is étale and finitely presented for all $\mathbf{r} : R \to \mathbb{Z}_{\geq 0}$.

Proof. Since $\mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)}$ is a disjoint union of the $\mathcal{L}^+(\mathfrak{t}_w)_{(\mathfrak{D};n)}$'s, it suffices to show that the map $\mathcal{L}^+(\mathfrak{t}_w)_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$ is étale and finitely presented.

Assume that map $\mathfrak{t}_{\mathbf{r}} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$ is étale and finitely presented for all $\mathbf{r}$. Since $\mathcal{L}^+(\mathfrak{t})_{(\mathfrak{D};n)}$ is a disjoint union of the $\mathfrak{t}_{\mathbf{r}}$'s, we conclude that the map $\mathcal{L}^+(\mathfrak{t})_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$ is étale and finitely presented. Applying this to $\mathfrak{t}'$ instead of $\mathfrak{t}$, we conclude that the map $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{c}')_{(\mathfrak{D};n)}$ is étale and finitely presented.

Finally, since $\mathcal{L}^+(\mathfrak{t}_w)_{(\mathfrak{D};n)}$ (resp. $\mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$) is the scheme of fixed points $w\sigma$ (resp. σ inside $\mathcal{L}^+(\mathfrak{t}')_{(\mathfrak{D};n)}$ (resp. $\mathcal{L}^+(\mathfrak{c}')_{(\mathfrak{D};n)}$), the assertion follows from Lemma 7.2.6 below. □

Lemma 7.2.6. *Let $f : T \to S$ is separated, étale and finitely presented morphism of schemes, and let $\phi_T \in \operatorname{End} T$ and $\phi_S \in \operatorname{End} S$ be endomorphisms such that $f \circ \phi_T = \phi_S \circ f$. Then the induced map between schemes of fixed points $f^{\phi} : T^{\phi_T} \to S^{\phi_S}$ is étale and finitely presented.*

Proof. (compare [GKM, 15.4.2(3)]). Restricting f to the $S^{\phi_S} \subset S$, we may assume that ϕ_S is the identity. Set $\phi := \phi_T$. Then we claim that the embedding $\iota_{\phi} : T^{\phi} \to T$ is an clopen (that is, open and closed). Hence $f^{\iota} = f|_{T^{\phi}}$ is étale and finitely presented, as claimed.

The diagonal map $\Delta : T \to T \times_S T$ is an open embedding, because f is étale, hence a clopen embedding, since f is separated. Taking pullback with respect to $(\mathrm{Id}, \phi) : T \to T \times_S T$, we conclude that the map $T^{\phi} \to T$ is a clopen embedding as well. □

7.2.7. Remark. Since f is formally étale, it is immediate to show that f^{ϕ} is formally étale as well. So the main point of Lemma 7.2.6 was to show that f^{ϕ} finitely presented.

For the rest of the proof, we follow [GKM, 11.1] very closely.

Step 4. We may assume that $\mathfrak{g}$ is semisimple.

Proof. Indeed, we have a decomposition $\mathfrak{g} = \mathfrak{g}_{ss} \times \mathfrak{z}$, where $\mathfrak{g}_{ss}$ is the derived algebra of $\mathfrak{g}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$. Moreover, the morphism $\mathcal{L}^+(\mathfrak{t})_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$

decomposes as a product of the corresponding map for $\mathfrak{g}_{ss}$ and the identity map on $\mathcal{L}^+(\mathfrak{z})$. Thus the assertion for $\mathfrak{g}$ follows from that for $\mathfrak{g}_{ss}$. □

Step 5. It suffices to assume that $\min \mathbf{r} = 0$.

Proof. By Step 4, we can assume that $\mathfrak{g}$ is semisimple. We set $s := \min \mathbf{r}$, and $\mathbf{r}' := \mathbf{r} - s$. Recall that the map $\mathfrak{t} \to \mathfrak{c}$ is $\mathbb{G}_m$-equivariant. Thus the map $\mathcal{L}(\mathfrak{t}) \to \mathcal{L}(\mathfrak{c})$ is $\mathcal{L}(\mathbb{G}_m)$-equivariant. Moreover, element $t^s \in \mathcal{L}(\mathbb{G}_m)$ induces isomorphisms $\mathfrak{t}_{\mathbf{r}'} \xrightarrow{\sim} \mathfrak{t}_{\mathbf{r}}$ and $\mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n-s|R|)} \xrightarrow{\sim} \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$. Therefore the assertion for $\mathbf{r}'$ implies that for $\mathbf{r}$. □

Step 6. Assume now that $\min \mathbf{r} = 0$. Then $R' := \{\alpha \in R \,|\, \mathbf{r}(\alpha) > 0\}$ is a root system of a proper Levi subgroup M of G. Consider the Chevalley space $\mathfrak{c}_M$ of M. Then the discriminant function $\mathfrak{D} \in k[\mathfrak{c}_M]$ decomposes as $\mathfrak{D} = \mathfrak{D}_M \mathfrak{D}^M$, where $\mathfrak{D}_M = \prod_{\alpha \in R'} d\alpha$ and $\mathfrak{D}^M = \prod_{\alpha \in R \smallsetminus R'} d\alpha$. Then the map $\pi : \mathfrak{t}_{\mathbf{r}} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$ decomposes as

$$\mathfrak{t}_{\mathbf{r}} \to \mathcal{L}^+(\mathfrak{c}_M)_{(\mathfrak{D}_M;n),(\mathfrak{D}^M;0)} = \mathcal{L}^+(\mathfrak{c}_M)_{(\mathfrak{D};n),(\mathfrak{D}^M;0)} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}.$$

So it remains to show that both maps are étale and finitely presented. The assertion for the first map follows by induction on $|R|$, so it remains to show the assertion for the second map.

Step 7. Consider the open subscheme $\mathfrak{c}_M^{reg/\mathfrak{g}} := (\mathfrak{c}_M)_{\mathfrak{D}^M} \subset \mathfrak{c}_M$. Thus, by 7.1.4(e), it suffices to show that the map $\mathcal{L}^+(\mathfrak{c}_M^{reg/\mathfrak{g}})_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$ is étale and finitely presented. We claim that the entire map $\mathcal{L}^+(\mathfrak{c}_M^{reg/\mathfrak{g}}) \to \mathcal{L}^+(\mathfrak{c})$ is étale and finitely presented. Namely, the map $\mathfrak{c}_M^{reg/\mathfrak{g}} \to \mathfrak{c}$ is étale, so the assertion follows from Lemma 7.1.2. □

7.2.8. Stratification of $\mathcal{L}^+(\mathfrak{c})$.

(a) By Theorem 7.2.5, the map $\pi : \mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$ is finite étale surjective and finitely presented, and that we have a decomposition $\mathcal{L}^+(\widetilde{\mathfrak{t}})_{(\mathfrak{D};n)} = \sqcup \mathfrak{t}_{w,\mathbf{r}}$, taken over all $(w, \mathbf{r})$ such that $d_{\mathbf{r}} = n$ (see 7.2.3(e)). Therefore for all pairs $(w, \mathbf{r})$, the induced morphism $\pi_{w,\mathbf{r}} : \mathfrak{t}_{w,\mathbf{r}} \to \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};d_{\mathbf{r}})}$ is finite étale.

(b) Since $\mathfrak{t}_{w,\mathbf{r}}$ is connected, $\pi_{w,\mathbf{r}}$ induces a surjective finite étale morphism $\pi_{w,\mathbf{r}} : \mathfrak{t}_{w,\mathbf{r}} \to \mathfrak{c}_{w,\mathbf{r}}$, for a certain connected component $\mathfrak{c}_{w,\mathbf{r}}$ of $\mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};d_{\mathbf{r}})}$.

(c) Let $W_{w,\mathbf{r}}$ be the stabilizer of $(w, \mathbf{r})$ in W via the action $u(w, \mathbf{r}) := (uwu^{-1}, u(\mathbf{r}))$. Then $u \in W$ induces an isomorphism $\mathfrak{t}_{u,\mathbf{r}} \xrightarrow{\sim} \mathfrak{t}_{u(w,\mathbf{r})}$ (see 7.2.3(f)). Since π is a W-torsor, we conclude that for every $u \in W$ we have $\mathfrak{c}_{u(w,\mathbf{r})} = \mathfrak{c}_{w,\mathbf{r}}$, the map $\pi_{w,\mathbf{r}} : \mathfrak{t}_{w,\mathbf{r}} \to \mathfrak{c}_{w,\mathbf{r}}$ is a $W_{w,\mathbf{r}}$-torsor, and $\mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$ decomposes as a disjoint union $\mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)} = \sqcup \mathfrak{c}_{w,\mathbf{r}}$, taken over all representatives of W-orbits of pairs $(w, \mathbf{r})$ such that $d_{\mathbf{r}} = n$.

(d) Since $\mathfrak{c}_{w,\mathbf{r}}$ is a connected component of $\mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};n)}$, we conclude that $\mathfrak{c}_{w,\mathbf{r}}$ is a locally closed finitely presented affine subscheme of $\mathcal{L}^+(\mathfrak{c})$. In particular, $\mathfrak{c}_{w,\mathbf{r}}$ is a connected globally placid affine scheme.

(e) Since $\pi_{w,\mathbf{r}} : \mathfrak{t}_{w,\mathbf{r}} \to \mathfrak{c}_{w,\mathbf{r}}$ is finite étale covering (by (d)), $\mathfrak{t}_{w,\mathbf{r}}$ is strongly pro-smooth (by 7.2.3(d)), while $\mathfrak{c}_{w,\mathbf{r}}$ is a connected globally placid affine scheme (by (e)), we conclude from Corollary 2.1.5 that $\mathfrak{c}_{w,\mathbf{r}}$ is strongly pro-smooth.

7.2.9. Remarks. (a) It is not difficult to write a pro-smooth presentation of each $\mathfrak{c}_{w,\mathbf{r}}$ explicitly.

(b) We don't know whether that the closure of a stratum $\mathfrak{c}_{w,\mathbf{r}}$ is a union of strata.

7.3. **Codimension of strata.**

7.3.1. Notation. (a) Recall that $\mathfrak{t}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{t}_w)$ and $\mathfrak{c}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{c})$ are strongly pro-smooth locally closed finitely presented subschemes (see 7.2.3(d) and 7.2.8(b)(e)). Hence they are of pure codimension (see Lemma 3.3.6), so we can consider codimensions $a_{w,\mathbf{r}} := \operatorname{codim}_{\mathcal{L}^+(\mathfrak{t}_w)}(\mathfrak{t}_{w,\mathbf{r}})$ and $b_{w,\mathbf{r}} := \operatorname{codim}_{\mathcal{L}^+(\mathfrak{c})}(\mathfrak{c}_{w,\mathbf{r}})$.

(b) Recall that r is the rank of G, and $d_{\mathbf{r}} = \sum_{\alpha\in R}\mathbf{r}(\alpha)$. We set $c_w := r - \dim \mathfrak{t}^w$ and $\delta_{w,\mathbf{r}} := \frac{d_{\mathbf{r}}-c_w}{2}$.

The following formula of [GKM, Thm 8.2.2(2)] is crucial for this work.

Proposition 7.3.2. *For every $(w,\boldsymbol{r})$ we have an equality $b_{w,\boldsymbol{r}} = \delta_{w,\boldsymbol{r}} + a_{w,\boldsymbol{r}} + c_w$.*

Corollary 7.3.3. *For every $(w,\boldsymbol{r})$ we have an inequality $b_{w,\boldsymbol{r}} \geq \delta_{w,\boldsymbol{r}}$, and the equality holds if and only if $w = 1$ and $\boldsymbol{r} = 0$.*

Proof. The inequality $b_{w,\mathbf{r}} \geq \delta_{w,\mathbf{r}}$ follows from Proposition 7.3.2 and observation that $a_{w,\mathbf{r}}, c_w \geq 0$. Furthermore, equality holds if and only if $c_w = a_{w,\mathbf{r}} = 0$. Note that equality $c_w = 0$ holds if and only if $w = 1$. In this case, equality $a_{w,\mathbf{r}} = 0$ holds if and only if the subscheme $\mathfrak{t}_{w,\mathbf{r}} = \mathfrak{t}_{\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{t})$ is open, and this happens if and only if $\mathbf{r} = 0$. □

7.3.4. The topologically nilpotent locus.

(a) Consider the closed subscheme $\mathfrak{c}^u := ev_{\mathfrak{c}}^{-1}(0) \subset \mathcal{L}^+(\mathfrak{c})$, where $ev_{\mathfrak{c}} : \mathcal{L}^+(\mathfrak{c}) \to \mathfrak{c}$ is the evaluation map. In particular, $\mathfrak{c}^u \subset \mathcal{L}^+(\mathfrak{c})$ is a strongly pro-smooth connected finitely presented closed subscheme of codimension $\dim \mathfrak{c} = r$.

(b) For every $w \in W$, we denote by $\mathfrak{t}^u_w \subset \mathcal{L}^+(\mathfrak{t}_w)$ the preimage of $\mathfrak{c}^u \subset \mathcal{L}^+(\mathfrak{c})$. Recall that $\mathcal{L}^+(\mathfrak{t}_w)$ classifies power series $\sum_{i=0}^{\infty} x_i t^{i/m}$ such that $w^{-1}(x_i) = \xi^i x_i$ for all i (see 7.2.2(b) and 7.2.3(c)). In particular, we have $x_0 \in \mathfrak{t}^w$.

Under this description, $\mathfrak{t}^u_w \subset \mathcal{L}^+(\mathfrak{t}_w)$ classifies power series with $x_0 = 0$. Therefore $\mathfrak{t}^u_w$ is a connected strongly pro-smooth affine scheme, and $\mathfrak{t}^u_w \subset \mathcal{L}^+(\mathfrak{t}_w)$ is a finitely presented closed subscheme of codimension $\dim \mathfrak{t}^w = r - c_w$.

(c) Notice that we have inclusions $\mathfrak{t}_{w,\mathbf{r}} \subset \mathfrak{t}^u_w$ and $\mathfrak{c}_{w,\mathbf{r}} \subset \mathfrak{c}^u_w$ if $\mathbf{r}(\alpha) > 0$ for all $\alpha \in R$, and $\mathfrak{t}_{w,\mathbf{r}} \subset \mathfrak{t}_w \smallsetminus \mathfrak{t}^u_w$ and $\mathfrak{c}_{w,\mathbf{r}} \subset \mathfrak{c}_w \smallsetminus \mathfrak{c}^u_w$, otherwise. In the first case, we say that $(w,\mathbf{r}) > 0$.

(d) For every $(w,\mathbf{r}) > 0$, we set $b^+_{w,\mathbf{r}} := b_{w,\mathbf{r}} - r$ and $a^+_{w,\mathbf{r}} := \operatorname{codim}_{\mathfrak{t}^u_w}(\mathfrak{t}_{w,\mathbf{r}})$ (use Lemma 3.3.6).

Then we have the following formula

Corollary 7.3.5. *For every $(w, \boldsymbol{r}) > 0$, we have $b^+_{w,\boldsymbol{r}} = \delta_{w,\boldsymbol{r}} + a^+_{w,\boldsymbol{r}}$. In particular, we have an inequality $b^+_{w,\boldsymbol{r}} \geq \delta_{w,\boldsymbol{r}}$, and an equality holds if and only if $\mathfrak{t}_{w,\boldsymbol{r}} \subset \mathfrak{t}^u_w$ is an open stratum.*

Proof. Since $\operatorname{codim}_{\mathcal{L}^+(\mathfrak{t}_w)}(\mathfrak{t}^u_w) = r - c_w$ (see 7.3.4(b)), we conclude that

$$a^+_{w,\mathbf{r}} = a_{w,\mathbf{r}} - (r - c_w) = a_{w,\mathbf{r}} + c_w - r.$$

Therefore by Proposition 7.3.2 we have

$$b^+_{w,\mathbf{r}} = \delta_{w,\mathbf{r}} + a_{w,\mathbf{r}} + c_w - r = \delta_{w,\mathbf{r}} + a^+_{w,\mathbf{r}}.$$

□

7.4. **Flatness assertion.** Next we study the induced GKM stratification on $\mathcal{L}^+(\mathfrak{g})$ and $\operatorname{Lie}(I)$.

7.4.1. Notation. (a) For each $n \in \mathbb{N}$, let $ev_{n,\mathfrak{g}} : \mathcal{L}^+_n(\mathfrak{g}) \to \mathfrak{g}$ be the evaluation map, set $\operatorname{Lie}(I)_n := (ev_{n,\mathfrak{g}})^{-1}(\mathfrak{b})$, and let $v_n : \operatorname{Lie}(I)_n \to \mathcal{L}^+_n(\mathfrak{c})$ be the restriction of $\chi_n : \mathcal{L}^+_n(\mathfrak{g}) \to \mathcal{L}^+_n(\mathfrak{c})$. Note that the isomorphism $\mathcal{L}^+(\mathfrak{g}) \xrightarrow{\sim} \lim_n \mathcal{L}^+_n(\mathfrak{g})$ induces an isomorphism $\operatorname{Lie}(I) \xrightarrow{\sim} \lim_n \operatorname{Lie}(I)_n$.

(b) For every GKM stratum $\mathfrak{c}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{c})$, we denote by $\mathfrak{g}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{c})$ and $\operatorname{Lie}(I)_{w,\mathbf{r}} \subset \operatorname{Lie}(I)$ its preimages.

We have the following result.

Theorem 7.4.2. *For every $n \in \mathbb{N}$, the morphisms $\chi_n : \mathcal{L}^+_n(\mathfrak{g}) \to \mathcal{L}^+_n(\mathfrak{c})$ and $v_n : \operatorname{Lie}(I)_n \to \mathcal{L}^+_n(\mathfrak{c})$ are flat.*

7.4.3. Remarks. (a) The strategy of proof was communicated to us by V. Drinfeld.
(b) In the case when the characteristic of k is zero, a theorem was proven by Mustata-Einsenbud-Frenkel [Mu].

Proof. We prove both assertions at the same time. It suffices to show that there exist faithfully flat morphisms $Z_n \to \mathcal{L}^+_n(\mathfrak{g})$ and $Z_{I,n} \to \operatorname{Lie}(I)_n$ such that both compositions $Z_n \to \mathcal{L}^+_n(\mathfrak{g}) \to \mathcal{L}^+_n(\mathfrak{c})$ and $Z_{I,n} \to \operatorname{Lie}(I)_n \to \mathcal{L}^+_n(\mathfrak{c})$ are flat. We will use a global argument that involves flatness of the Hitchin fibration and its parabolic variant. For convenience of the reader, we will divide our argument into steps.

Step 1. Consider two distinct points $x, \infty \in \mathbb{P}^1(k)$ and an effective divisor D on $\mathbb{P}^1$, supported on $\mathbb{P}^1 \smallsetminus \{x, \infty\}$. We have a $\mathbb{G}_m$-action on $\mathfrak{g}$ by homothety that commutes with adjoint action, thus inducing an $\mathbb{G}_m$-action on $\mathfrak{c}$. Hence we can form the twisted versions $\mathfrak{c}_D := \mathfrak{c} \times^{\mathbb{G}_m} \mathcal{Z}^\times(D)$, where $\mathcal{Z}^\times(D)$ is the $\mathbb{G}_m$-torsor, corresponding to the line bundle $\mathcal{O}(D)$ and similarly $\mathfrak{g}_D := \mathfrak{g} \otimes \mathcal{O}(D)$. Both are vector bundles over $\mathbb{P}^1$, trivialized on $\mathbb{P}^1 \smallsetminus D$. For every G-torsor E on $\mathbb{P}^1$, let $\operatorname{ad}(E)$ be the corresponding vector bundle on $\mathbb{P}^1$. Then the map $\chi : \mathfrak{g} \to \mathfrak{c}$ induces a morphism $H^0(\mathbb{P}^1, \mathrm{ad}(E) \otimes \mathcal{O}(D)) \to H^0(\mathbb{P}^1, \mathfrak{c}_D)$.

We choose D sufficiently big so that the restrictions to the n-th formal neighborhood at x and evaluation at ∞:

$$(ev_x^{(n)}, ev_\infty) : H^0(\mathbb{P}^1, \mathfrak{g}_D) \to H^0(nx \cup \infty, \mathfrak{g}_D) \simeq \mathcal{L}_n^+(\mathfrak{g}) \oplus \mathfrak{g}, \tag{7.1}$$

$$(ev_x^{(n)}, ev_\infty) : H^0(\mathbb{P}^1, \mathfrak{c}_D) \to H^0(nx \cup \infty, \mathfrak{c}_D) \simeq \mathcal{L}_n^+(\mathfrak{c}) \oplus \mathfrak{c}, \tag{7.2}$$

are both surjective.

Step 2. Set $\mathcal{A}_{D,\infty} := \{a \in H^0(\mathbb{P}^1, \mathfrak{c}_D) \mid ev_\infty(a) \in \mathfrak{c}^{rs}\}$, and let $\mathcal{M}_{D,\infty}$ be the corresponding Hitchin total space. More precisely, $\mathcal{M}_{D,\infty}$ classifies pairs (E, ϕ), where E is a G-torsor on $\mathbb{P}^1$ and $\phi \in H^0(\mathbb{P}^1, \mathrm{ad}(E) \otimes \mathcal{O}(D))$ such that $\chi(\phi) \in \mathcal{A}_{D,\infty}$.

From surjectivity of (7.2), we get that

$$\text{the map } ev_x^{(n)} : \mathcal{A}_{D,\infty} \to \mathcal{L}_n^+(\mathfrak{c}) \text{ is smooth and surjective.} \tag{7.3}$$

Step 3. Following Yun (see [Yun1]), we consider the parabolic Hitchin space $\mathcal{M}_{D,\infty}^{par}$, which classifies triples (E, ϕ, E_B) such that

- $(E, \phi) \in \mathcal{M}_{D,\infty}$,
- E_B is a B-reduction of the restriction $E|_x$ such that $ev_x(\phi) \in \mathrm{ad}(E|_x)$ belongs to $\mathrm{ad}(E_B)$.

By [Ngo, 4.16.4], [Yun1, 2.5.2], the fibrations

$$\mathcal{M}_{D,\infty} \to \mathcal{A}_{D,\infty} \text{ and } \mathcal{M}_{D,\infty}^{par} \to \mathcal{A}_{D,\infty}$$

are faithfully flat, so by (7.3), the compositions

$$\mathcal{M}_{D,\infty} \to \mathcal{A}_{D,\infty} \overset{ev_x^{(n)}}{\to} \mathcal{L}_n^+(\mathfrak{c}) \text{ and } \mathcal{M}_{D,\infty}^{par} \to \mathcal{A}_{D,\infty} \overset{ev_x^{(n)}}{\to} \mathcal{L}_n^+(\mathfrak{c}) \tag{7.4}$$

also are.

Step 4. Let $\mathcal{M}_{D,\infty}^{nx} \to \mathcal{M}_{D,\infty}$ be the $\mathcal{L}_n^+(G)$-torsor, classifying trivializations ι of E at the n-th formal neighhourhood at x. Then one has a map

$$\mathrm{res}_n : \mathcal{M}_{D,\infty}^{nx} \to \mathcal{L}_n^+(\mathfrak{g}),$$

which sends triple (E, ϕ, ι) to the image of ϕ under the natural map

$$H^0(\mathbb{P}^1, \mathrm{ad}(E) \otimes \mathcal{O}(D)) \overset{ev_x^{(n)}}{\longrightarrow} H^0(nx, \mathrm{ad}(E)) \overset{\iota}{\longrightarrow} \mathcal{L}_n^+(\mathfrak{g}).$$

Let $Z_n \subset \mathcal{M}_{D,\infty}^{nx}$ be the largest open substack of $\mathcal{M}_{D,\infty}^{nx}$, where res_n is smooth.

Step 5. We claim that the restriction $r_n : Z_n \to \mathcal{L}_n^+(\mathfrak{g})$ of res_n is faithfully flat, and its composition with χ_n is flat.

Proof. First of all, r_n is smooth, by assumption, hence flat. Next, since the first map of (7.4) and the projection $\mathcal{M}_{D,\infty}^{nx} \to \mathcal{M}_{D,\infty}$ are flat, we conclude from the commutativity diagram

$$
\begin{array}{ccccc}
Z_n & \longrightarrow & \mathcal{M}_{D,\infty}^{nx} & \longrightarrow & \mathcal{L}_n^+(\mathfrak{g}) \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \chi_n} \\
 & & \mathcal{A}_{D,\infty} & \longrightarrow & \mathcal{L}_n^+\mathfrak{c}
\end{array}
\tag{7.5}
$$

that the composition $Z_n \to \mathcal{L}_n^+(\mathfrak{g}) \to \mathcal{L}_n^+\mathfrak{c}$ is flat.

We claim that r_n is surjective. More precisely, we claim that the locus of those triples (E, ϕ, ι), where E is trivial, is contained in Z_n and the restriction of res_n to such points is surjective. Let $\widetilde{Z}_n$ be the moduli space of quadruples (E, ϕ, η, ι), where $(E, \eta, \iota) \in \mathcal{M}_{D,\infty}^{nx}$ and η is a trivialization of E, and let $\omega : \widetilde{Z}_n \to \mathcal{M}_{D,\infty}^{nx}$ be the forgetful morphism.

Then the image of ω consists of all triples (E, ϕ, ι), where E is trivial, and it suffices to show that the composition $\mathrm{res}_n \circ\omega$ is smooth and surjective. Indeed, since the morphism $\mathrm{res}_n \circ\omega$ is a smooth morphism between smooth stacks, the differential $d(\mathrm{res}_n \circ\omega)$ is surjective, therefore the differential $d\,\mathrm{res}_n$ is surjective. Since res_n is a morphism between smooth algebraic stacks (by [Ngo, Thm 4.14.1]), this implies that res_n is smooth at each point in the image of ω, and we are done.

Note that $\widetilde{Z}_n$ is canonically isomorphic to the product of $\mathcal{L}_n^+(G)$ and the open subset $H^0(\mathbb{P}^1, \mathfrak{g}_D)_{\infty-rs} \subset H^0(\mathbb{P}^1, \mathfrak{g}_D)$ consisting of $\phi \in H^0(\mathbb{P}^1, \mathfrak{g}_D)$ such that $ev_\infty(\phi) \in \mathfrak{g}^{rs}$. Moreover, under this identification, $\mathrm{res}_n \circ\omega$ is nothing else but composition of the evaluation map $ev_x^{(n)} : \mathcal{L}_n^+(G) \times H^0(\mathbb{P}^1, \mathfrak{g}_D)_{x-rs} \to \mathcal{L}_n^+(G) \times \mathcal{L}_n^+(\mathfrak{g})$ and the action map $\mathcal{L}_n^+(G) \times \mathcal{L}_n^+(\mathfrak{g}) \to \mathcal{L}_n^+(\mathfrak{g})$. Therefore the smoothness and surjectivity of $\mathrm{res}_n \circ\omega : \widetilde{Z}_n \to \mathcal{L}_n^+(\mathfrak{g})$ follows from the surjectivity of (7.1). □

Step 6. Similarly, one considers moduli space $\mathcal{M}_{D,\infty}^{par,nx}$, which classifies quadruples (E, ϕ, E_B, ι) such that

- $(E, \phi, E_B) \in \mathcal{M}_{D,\infty}^{par}$,
- ι is a trivialization of E at the n-th formal neighborhood at x, which induces a trivialization of E_B.

Then we have a Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{M}_{D,\infty}^{par,nx} & \longrightarrow & \mathrm{Lie}(I)_n \\
\big\downarrow & & \big\downarrow \\
\mathcal{M}_{D,\infty}^{nx} & \xrightarrow{\mathrm{res}_n} & \mathcal{L}_n^+(\mathfrak{g})
\end{array}
\tag{7.6}
$$

so the pullback $r_{I,n} : Z_{I,n} \to \mathrm{Lie}(I)_n$ of $r_n : Z_n \to \mathcal{L}_n^+(\mathfrak{g})$ is smooth and surjective, so it remains to show that the composition $Z_{I,n} \to \mathrm{Lie}(I)_n \to \mathcal{L}_n^+\mathfrak{c}$ is flat. But the last composition decomposes as a composition of three flat maps

$$Z_{I,n} \to \mathcal{M}_{D,\infty}^{par,nx} \to \mathcal{M}_{D,\infty}^{par} \to \mathcal{L}_n^+\mathfrak{c},$$

the first of which is an open embedding, the second one is smooth, and the third one the second map of (7.4). Therefore the composition is flat, and the proof is complete. □

Corollary 7.4.4. *The morphisms* $\mathcal{L}_n^+(\chi) : \mathcal{L}^+(\mathfrak{g}) \to \mathcal{L}^+(\mathfrak{c})$ *and* $v : \mathrm{Lie}(I) \to \mathcal{L}^+(\mathfrak{c})$ *are flat.*

Proof. Since property of being flat is preserved by base change and passing to a filtered limit, the assertion follows from Theorem 7.4.2. □

Corollary 7.4.5. *The locally closed subschemes* $\mathrm{Lie}(I)_{w,\boldsymbol{r}} \subset \mathrm{Lie}(I)$ *and* $\mathfrak{g}_{w,\boldsymbol{r}} \subset \mathfrak{g}$ *are of pure codimension* $b_{w,\boldsymbol{r}}$ *(see 7.3.1).*

Proof. Since $\mathcal{L}_n^+(\chi)$ and v are flat morphisms between globally placid affine schemes (by Corollary 7.4.4), they are uo-special (by Lemma 2.1.3) in the sense of 3.2.8(a). Since $\mathfrak{c}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{c})$ is of pure codimension $b_{w,\mathbf{r}}$ (by Proposition 7.3.2), the assertion follows from Corollary 3.3.3. □

8. Geometry of the affine Grothendieck-Springer fibration

8.1. Generalities.

8.1.1. The affine Grothendieck-Springer fibration.

(a) The Chevalley morphism $\chi : \mathfrak{g} \to \mathfrak{c}$ induces a morphism of ind-schemes $\mathcal{L}\,\chi : \mathcal{L}\,\mathfrak{g} \to \mathcal{L}\,\mathfrak{c}$, which we denote by simplicity by χ. We set $\mathfrak{C} := \chi^{-1}(\mathcal{L}^+(\mathfrak{c})) \subset \mathcal{L}\,\mathfrak{g}$.

(b) Since $\mathcal{L}^+(\mathfrak{c}) \subset \mathcal{L}\,\mathfrak{c}$ is an fp-closed subscheme, the preimage $\mathfrak{C} \subset \mathcal{L}\,\mathfrak{g}$ is an fp-closed ind-subscheme of $\mathcal{L}\,\mathfrak{g}$. Since $\mathcal{L}\,\mathfrak{g}$ is an ind-placid scheme (with presentation $\mathcal{L}\,\mathfrak{g} \simeq \mathrm{colim}_i\, t^{-i}\mathcal{L}^+(\mathfrak{g})$), we conclude that $\mathfrak{C}$ is an ind-placid scheme as well.

(c) Set $\widetilde{\mathfrak{C}} := \mathcal{L}\,G \times^I \mathrm{Lie}(I)$, that is, $\widetilde{\mathfrak{C}}$ is a quotient of $[\mathcal{L}\,G \times \mathrm{Lie}(I)]/I$ by the action $h(g,\gamma) := (gh^{-1}, (\mathrm{Ad}\,h)(\gamma))$. Then $\mathcal{L}\,G$ acts on $\widetilde{\mathfrak{C}}$ by the rule $h([g,\gamma]) = ([hg,\gamma])$, and we have a natural equivalence $[\widetilde{\mathfrak{C}}/\mathcal{L}\,G] \simeq [\mathrm{Lie}(I)/I]$.

(d) We have a natural projection map $\mathfrak{p} : \widetilde{\mathfrak{C}} \to \mathfrak{C} : [g,\gamma] \mapsto (\mathrm{Ad}\,g)(\gamma)$, called the *affine Grothendieck-Springer fibration*. The fibers of this map are affine Springer fibers, studied by Kazhdan-Lusztig in [KL].

8.1.2. Remark. The notation $\mathfrak{C}$ comes to indicate that this is the locus of "compact" elements in $\mathcal{L}\,\mathfrak{g}$.

8.1.3. The affine flag variety. (a) Let $\mathfrak{Fl} := \mathcal{L}G/I$ be the affine flag variety. Notice that the map $\iota : \widetilde{\mathfrak{C}} \to \mathfrak{Fl} \times \mathcal{L}\mathfrak{g} : [g,x] \mapsto ([g], (\operatorname{Ad} g)(x))$ identifies $\widetilde{\mathfrak{C}}$ with the closed ind-subscheme of $\mathfrak{Fl} \times \mathfrak{C}$ given by

$$\{([g],\gamma) \in \mathfrak{Fl} \times \mathfrak{C} \,|\, (\operatorname{Ad} g^{-1})(\gamma) \in \operatorname{Lie}(I)\}.$$

Under this identification, the fibration $\mathfrak{p} : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ of 8.1.1 corresponds to the projection to the second factor.

(b) Note that $\mathfrak{Fl}$ has a structure of an ind-projective scheme over k, with a canonical presentation $\mathfrak{Fl} \simeq \operatorname{colim}_i Y_i$ as a colimit of its I-invariant closed projective subschemes.

(c) The presentation of (b) induces a canonical $(I \times I)$-equivariant presentation $\mathcal{L}G \simeq \operatorname{colim}_i \widetilde{Y}_i$ of $\mathcal{L}G$, and a presentation of the ind-scheme $\widetilde{\mathfrak{C}} = \operatorname{colim}_i \widetilde{\mathfrak{C}}_i$ with $\widetilde{\mathfrak{C}}_i := \widetilde{Y}_i \times^I \operatorname{Lie}(I)$.

(d) Notice that each $\widetilde{Y}_i \to Y_i$ is an I-torsor, hence it is pro-smooth. Therefore each $\widetilde{Y}_i$ is a globally placid scheme, thus $\mathcal{L}G \simeq \operatorname{colim}_i \widetilde{Y}_i$ is an ind-placid scheme.

Lemma 8.1.4. *The projection $\mathfrak{p} : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ is ind-fp-proper.*

Proof. Recall that $\mathfrak{p}$ factors as a composition of the closed embedding $\iota : \widetilde{\mathfrak{C}} \hookrightarrow \mathfrak{Fl} \times \mathfrak{C}$ and the projection $p : \mathfrak{Fl} \times \mathfrak{C} \to \mathfrak{C}$. Since $\mathfrak{Fl}$ is ind-projective, the projection p is ind-fp-proper. Thus it remains to show that the closed embedding ι is ind-finitely presented.

Note that the action morphism $a : \mathcal{L}G \times \mathfrak{C} \to \mathcal{L}\mathfrak{g} : a(g,x) = (\operatorname{Ad} g^{-1})(x)$ gives rise to the morphism $\overline{a} : \mathfrak{Fl} \times \mathfrak{C} \to [\mathcal{L}\mathfrak{g}/I]$, and by definition $\widetilde{\mathfrak{C}}$ is the preimage of $[\operatorname{Lie}(I)/I] \subset [\mathcal{L}\mathfrak{g}/I]$. Since the embedding $[\operatorname{Lie}(I)/I] \hookrightarrow [\mathcal{L}\mathfrak{g}/I]$ is ind-finitely presented, the assertion is clear. $\square$

8.1.5. Maximal tori. Recall that there is a natural bijection $w \mapsto T_w$ between conjugacy classes of elements in W and conjugacy classes of maximal tori in G. Notice that the Lie algebra $\mathfrak{t}_w := \operatorname{Lie} T_w$ is canonically isomorphic to the Lie algebra described in 7.2.3(c). In particular, we have an embedding of $\mathfrak{t}_w \hookrightarrow \mathfrak{g}$, unique up to conjugacy.

8.1.6. GKM strata. (a) Recall that we have defined strata $\mathfrak{t}_{w,\mathbf{r}}$ of $\mathcal{L}^+(\mathfrak{t}_w)$ (see 7.2.3(c)) and the corresponding strata $\mathfrak{c}_{w,\mathbf{r}}$ of $\mathcal{L}^+(\mathfrak{c})$ (see 7.2.8(a)). Moreover, every projection $\pi : \mathfrak{t}_{w,\mathbf{r}} \to \mathfrak{c}_{w,\mathbf{r}}$ is a $W_{w,\mathbf{r}}$-torsor (see 7.2.8(c)).

(b) The GKM strata of $\mathcal{L}^+(\mathfrak{c})$ induce the GKM strata $\mathfrak{C}_{w,\mathbf{r}} := \chi^{-1}(\mathfrak{c}_{w,\mathbf{r}}) \subset \mathfrak{C}$ and $\widetilde{\mathfrak{C}}_{w,\mathbf{r}} := \mathfrak{p}^{-1}(\mathfrak{C}_{w,\mathbf{r}}) \subset \widetilde{\mathfrak{C}}$. We have an identification $\widetilde{\mathfrak{C}}_{w,\mathbf{r}} \simeq \mathcal{L}G \times^I \operatorname{Lie}(I)_{w,\mathbf{r}}$, hence $[\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\mathcal{L}G] \simeq [\operatorname{Lie}(I)_{w,\mathbf{r}}/I]$. Projection $\mathfrak{p}$ induces projections $\mathfrak{p}_{w,\mathbf{r}} : \widetilde{\mathfrak{C}}_{w,\mathbf{r}} \to \mathfrak{C}_{w,\mathbf{r}}$ and $\overline{\mathfrak{p}}_{w,\mathbf{r}} : [\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\mathcal{L}G] \to [\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]$.

(c) The embedding of $\mathfrak{t}_w \hookrightarrow \mathfrak{g}$ from 8.1.5 induces an embedding $\mathfrak{t}_{w,\mathbf{r}} \hookrightarrow \mathfrak{C}_{w,\mathbf{r}}$, unique up to conjugacy, and hence induces a canonical morphism $\psi_{w,\mathbf{r}} : \mathfrak{t}_{w,\mathbf{r}} \to [\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}\,G]$. In turn, $\psi_{w,\mathbf{r}}$ induces a morphism $\overline{\psi}_{w,\mathbf{r}} : \mathfrak{c}_{w,\mathbf{r}} = [\mathfrak{t}_{w,\mathbf{r}}/W_{w,\mathbf{r}}] \to [\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}\,G]$.

(d) Set $\mathfrak{C}_{\mathfrak{t},w,\mathbf{r}} := \mathfrak{C}_{w,\mathbf{r}} \times_{\mathfrak{c}_{w,\mathbf{r}}} \mathfrak{t}_{w,\mathbf{r}}$.

8.1.7. Constructible stratification. (a) We set $\mathfrak{c}_\bullet := \mathcal{L}^+(\mathfrak{c})_{\mathfrak{D}\neq 0} \subset \mathcal{L}^+(\mathfrak{c})$, and also $\mathfrak{C}_\bullet := \chi^{-1}(\mathfrak{c}_\bullet) \subset \mathfrak{C}$, $\operatorname{Lie}(I)_\bullet := v^{-1}(\mathfrak{c}_\bullet) \subset \operatorname{Lie}(I)$ and $\widetilde{\mathfrak{C}}_\bullet := \mathfrak{p}^{-1}(\mathfrak{C}_\bullet) \subset \widetilde{\mathfrak{C}}$. In particular, we have a natural identification $\widetilde{\mathfrak{C}}_\bullet \simeq \mathcal{L}\,G \times^I \operatorname{Lie}(I)_\bullet$.

(b) For every $m \in \mathbb{N}$, we set $\mathfrak{c}_{\leq m} := \mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};\leq m)}$ (see 7.1.5), and let $\mathfrak{C}_{\leq m} \subset \mathfrak{C}$ and $\widetilde{\mathfrak{C}}_{\leq m} \subset \widetilde{\mathfrak{C}}$ be the preimages of $\mathfrak{c}_{\leq m}$. Notice that $\mathfrak{c}_{\leq m} \subset \mathcal{L}^+(\mathfrak{c})$ is an fp-open embedding, $\mathfrak{C}_{\leq m} \subset \mathfrak{C}$ is an fp-open embedding, thus $\mathfrak{C}_{\leq m}$ is an ind-placid scheme (by 8.1.1(b)). Note also that $\mathfrak{c}_{\leq 0} = \mathcal{L}^+(\mathfrak{c}^{rs})$ (see 7.1.7).

(c) By definition, we have an fp-open covering $\mathfrak{c}_\bullet = \cup_{m\geq 0}\mathfrak{c}_{\leq m}$, which gives rise to a constructible stratification $\{\mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};m)}\}$ of $\mathfrak{c}_\bullet$ (by 7.1.7), and an fp-open covering $\mathfrak{C}_\bullet = \cup_{m\geq 0}\mathfrak{C}_{\leq m}$.

(d) Since $\mathfrak{c}_{w,\mathbf{r}}$ is a connected component of $\mathcal{L}^+(\mathfrak{c})_{(\mathfrak{D};d_\mathbf{r})}$ for each $(w,\mathbf{r})$ (see 7.2.8(b)), we conclude that $\{\mathfrak{c}_{w,\mathbf{r}}\}_{w,\mathbf{r}}$ form a bounded constructible stratification of $\mathfrak{c}_\bullet$ (use Lemma 6.2.4(c)).

(e) The constructible stratification $\{\mathfrak{c}_{w,\mathbf{r}}\}_{w,\mathbf{r}}$ of $\mathfrak{c}_\bullet$ induces a constructible stratification $\{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}\,G]\}_{w,\mathbf{r}}$ of $[\mathfrak{C}_\bullet/\mathcal{L}\,G]$ (see Lemma 6.2.4(a)).

Lemma 8.1.8. *For every GKM stratum $(w,\mathbf{r})$, have natural isomorphisms*

$$\mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}} \xrightarrow{\sim} \mathfrak{C}_{\mathfrak{t},w,\mathbf{r}} : (g,x) \mapsto ((\operatorname{Ad} g)(x), x) \text{ and}$$

$$\mathcal{L}(G/T_w) \times^{W_{w,\mathbf{r}}} \mathfrak{t}_{w,\mathbf{r}} \xrightarrow{\sim} \mathfrak{C}_{w,\mathbf{r}} : (g,x) \mapsto (\operatorname{Ad} g)(x).$$

Proof. Recall that the map $(g,x) \mapsto ((\operatorname{Ad} g)(x), x)$ induces an isomorphism

$$(G/T_w) \times \mathfrak{t}_w^{rs} \xrightarrow{\sim} \mathfrak{g}^{rs} \times_{\mathfrak{c}^{rs}} \mathfrak{t}_w^{rs}$$

over $\mathfrak{t}_w^{rs}$. Since functor $\mathcal{L}$ preserves limits, it induces an isomorphism

$$\mathcal{L}(G/T_w) \times \mathcal{L}(\mathfrak{t}_w^{rs}) \xrightarrow{\sim} \mathcal{L}(\mathfrak{g}^{rs}) \times_{\mathcal{L}(\mathfrak{c}^{rs})} \mathcal{L}(\mathfrak{t}_w^{rs})$$

over $\mathcal{L}(\mathfrak{t}_w^{rs})$. Restricting it to $\mathfrak{t}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{t}_w)_{(\mathfrak{D},d_\mathbf{r})} \subset \mathcal{L}(\mathfrak{t}_w^{rs})$ (see 7.1.5(d)), we get an isomorphism $(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}} \xrightarrow{\sim} \mathcal{L}(\mathfrak{g}^{rs}) \times_{\mathcal{L}(\mathfrak{c}^{rs})} \mathfrak{t}_{w,\mathbf{r}}$.

Now the first isomorphism of the lemma follows from identifications

$$\mathcal{L}(\mathfrak{g}^{rs}) \times_{\mathcal{L}(\mathfrak{c}^{rs})} \mathfrak{t}_{w,\mathbf{r}} \simeq \mathcal{L}\,\mathfrak{g} \times_{\mathcal{L}\,\mathfrak{c}} \mathfrak{t}_{w,\mathbf{r}} \simeq \mathfrak{C}_{w,\mathbf{r}} \times_{\mathfrak{c}_{w,\mathbf{r}}} \mathfrak{t}_{w,\mathbf{r}},$$

the first of which follows from identification $\mathfrak{g}^{rs} \simeq \mathfrak{g} \times_\mathfrak{c} \mathfrak{c}^{rs}$, thus $\mathcal{L}(\mathfrak{g}^{rs}) \simeq \mathcal{L}\,\mathfrak{g} \times_{\mathcal{L}\,\mathfrak{c}} \mathcal{L}(\mathfrak{c}^{rs})$, while the second one from the identification $\mathfrak{C}_{w,\mathbf{r}} \simeq \mathcal{L}\,\mathfrak{g} \times_{\mathcal{L}\,\mathfrak{c}} \mathfrak{c}_{w,\mathbf{r}}$.

Finally, the second isomorphism of the lemma is obtained from the first one by taking the quotient by $W_{w,\mathbf{r}}$. ∎

The following result will be proven in the next section (see 9.3.2).

Theorem 8.1.9. *For every (not necessary split) maximal torus $S \subset G_K$, the projection $\psi_S : [\mathcal{L} G/(\mathcal{L} S)_{\mathrm{red}}] \to \mathcal{L}(G/S)$ is a topological equivalence.*

Corollary 8.1.10. *The map $[\psi_{w,\boldsymbol{r}}] : [\mathfrak{t}_{w,\boldsymbol{r}}/W_{w,\boldsymbol{r}} \rtimes (\mathcal{L} T_w)_{\mathrm{red}}] \to [\mathfrak{C}_{w,\boldsymbol{r}}/ \mathcal{L} G]$, induced by the map $\psi_{w,\boldsymbol{r}}$ from 8.1.6(c) is a topological equivalence.*

Proof. Since the projection $\psi_{T_w} : [\mathcal{L} G/(\mathcal{L} T_w)_{\mathrm{red}}] \to \mathcal{L}(G/T_w)$ is a topological equivalence by Theorem 8.1.9, it induces an $(\mathcal{L} G \times W_{w,\mathbf{r}})$-equivariant topological equivalence $[\mathcal{L} G/(\mathcal{L} T_w)_{\mathrm{red}}] \times \mathfrak{t}_{w,\mathbf{r}} \to \mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}}$. Taking quotient by $W_{w,\mathbf{r}}$, we deduce from Lemma 8.1.8 and Corollary 2.3.7(c) that the map

$$[\mathcal{L} G/(\mathcal{L} T_w)_{\mathrm{red}}] \times^{W_{w,\mathbf{r}}} \mathfrak{t}_{w,\mathbf{r}} \to \mathcal{L}(G/T_w) \times^{W_{w,\mathbf{r}}} \mathfrak{t}_{w,\mathbf{r}} \simeq \mathfrak{C}_{w,\mathbf{r}} : (g, x) \mapsto (\operatorname{Ad} g)(x) \tag{8.1}$$

is an $\mathcal{L} G$-equivariant topological equivalence. Dividing by $\mathcal{L} G$ and using Corollary 2.3.7(c) again, we get that

$$[\psi_{w,\mathbf{r}}] : [\mathfrak{t}_{w,\mathbf{r}}/W_{w,\mathbf{r}} \rtimes (\mathcal{L} T_w)_{\mathrm{red}}] \to [\mathfrak{C}_{w,r}/ \mathcal{L} G]$$

is a topological equivalence, as claimed. □

Corollary 8.1.11. *The ∞-stack $[\mathfrak{C}_{w,\boldsymbol{r}}/ \mathcal{L} G]$ is topologically placid, and the projection $\psi_{w,\boldsymbol{r}} : \mathfrak{t}_{w,\boldsymbol{r}} \to [\mathfrak{C}_{w,\boldsymbol{r}}/ \mathcal{L} G]$ from 8.1.6(c) is a topologically smooth covering.*

Proof. Recall that $\mathfrak{t}_{w,\mathbf{r}}$ is a placid scheme (see 7.2.3(d)) and $(\mathcal{L} T_w)_{\mathrm{red}}$ is a group scheme, whose neutral connected component is the strongly pro-smooth group $\mathcal{L}^+(T_w)$ (see 9.2.4(b)). Therefore $W_{w,\mathbf{r}} \rtimes (\mathcal{L} T_w)_{\mathrm{red}}$ is a 0-smooth group scheme, thus, by 2.1.14, the quotient $[\mathfrak{t}_{w,\mathbf{r}}/W_{w,\mathbf{r}} \rtimes (\mathcal{L} T_w)_{\mathrm{red}}]$ is a placid stack, and the projection $\mathrm{pr} : \mathfrak{t}_{w,\mathbf{r}} \to [\mathfrak{t}_{w,\mathbf{r}}/W_{w,\mathbf{r}} \rtimes (\mathcal{L} T_w)_{\mathrm{red}}]$ is a smooth covering. On the other hand, by Corollary 8.1.10, the map $[\psi_{w,\mathbf{r}}] : [\mathfrak{t}_{w,\mathbf{r}}/W_{w,\mathbf{r}} \rtimes (\mathcal{L} T_w)_{\mathrm{red}}] \to [\mathfrak{C}_{w,\mathbf{r}}/ \mathcal{L} G]$ is a topological equivalence. This implies that $[\mathfrak{C}_{w,\mathbf{r}}/ \mathcal{L} G]$ is a topologically placid stack, and $\psi_{w,\mathbf{r}} = [\psi_{w,\mathbf{r}}] \circ \mathrm{pr}$ is a topologically smooth covering. □

8.2. **The Affine Springer fibration over a regular stratum.** .

Lemma 8.2.1. *The map $(I/\mathcal{L}^+(T)) \times \mathcal{L}^+(\mathfrak{t}^{rs}) \to \mathrm{Lie}(I)_{\leq 0} : (g, x) \mapsto (\operatorname{Ad} g)(x)$ is an isomorphism.*

Proof. Note that the map $(g, x) \mapsto (\operatorname{Ad} g)(x)$ induces an isomorphism

$$(B/T) \times \mathfrak{t}^{rs} \xrightarrow{\sim} \mathfrak{b}^{rs}. \tag{8.2}$$

The assertion now follows formally. Namely, isomorphism (8.2) extends to an isomorphism $G \times^B (B/T \times \mathfrak{t}^{rs}) \xrightarrow{\sim} G \times^B \mathfrak{b}^{rs}$, which is nothing else but

$$(G/T) \times \mathfrak{t}^{rs} \xrightarrow{\sim} \widetilde{\mathfrak{g}}^{rs}, \tag{8.3}$$

where $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is the Grothendieck-Springer resolution, and $\widetilde{\mathfrak{g}}^{rs}$ is the preimage of $\mathfrak{g}^{rs}$. Applying $\mathcal{L}^+$, we get an isomorphism

$$\mathcal{L}^+(G/T) \times \mathcal{L}^+(\mathfrak{t}^{rs}) \xrightarrow{\sim} \mathcal{L}^+(\widetilde{\mathfrak{g}}^{rs}). \tag{8.4}$$

Taking fiber product of (8.2) and (8.4) over (8.3), we get an isomorphism

$$(\mathcal{L}^+(G/T) \times_{(G/T)} (B/T)) \times \mathcal{L}^+(\mathfrak{t}^{rs}) \xrightarrow{\sim} \mathcal{L}^+(\widetilde{\mathfrak{g}}^{rs}) \times_{\widetilde{\mathfrak{g}}^{rs}} \mathfrak{b}^{rs}. \tag{8.5}$$

Using 7.1.3(c), we get a natural isomorphism

$$\mathcal{L}^+(G/T) \times_{(G/T)} (B/T) \simeq (\mathcal{L}^+(G)/\mathcal{L}^+(T)) \times_{(G/T)} (B/T) \simeq I/\mathcal{L}^+(T).$$

Moreover the projection $\pi : \widetilde{\mathfrak{g}}^{rs} \to \mathfrak{g}$ is étale. Thus, by Lemma 7.1.2, π induces an isomorphism

$$\mathcal{L}^+(\widetilde{\mathfrak{g}}^{rs}) \times_{\widetilde{\mathfrak{g}}^{rs}} \mathfrak{b}^{rs} \simeq \mathcal{L}^+(\mathfrak{g}) \times_{\mathfrak{g}} \mathfrak{b}^{rs} \simeq \mathrm{Lie}(I) \times_{\mathfrak{b}} \mathfrak{b}^{rs} \simeq \mathrm{Lie}(I)_{\leq 0},$$

as claimed. □

Corollary 8.2.2. *We have a natural isomorphism* $[\widetilde{\mathfrak{C}}_{\leq 0}/\mathcal{L}G] \simeq [\mathcal{L}^+(\mathfrak{t}^{rs})/\mathcal{L}^+(T)]$.

Proof. Dividing the isomorphism from Lemma 8.2.1 by the action of I, we get an isomorphism $[\mathcal{L}^+(\mathfrak{t}^{rs})/\mathcal{L}^+(T)] \xrightarrow{\sim} [\mathrm{Lie}(I)_{\leq 0}/I]$. Since $[\widetilde{\mathfrak{C}}_{\leq 0}/\mathcal{L}G] \simeq [\mathrm{Lie}(I)_{\leq 0}/I]$, we are done. □

Corollary 8.2.3. *We have a natural commutative diagram*

$$\begin{array}{ccc} (\mathcal{L}G/\mathcal{L}^+(T)) \times \mathcal{L}^+(\mathfrak{t}^{rs}) & \xrightarrow{\sim} & \widetilde{\mathfrak{C}}_{\leq 0} \\ \mathrm{pr}\downarrow & & \downarrow \mathfrak{p}_{\leq 0} \\ \mathcal{L}(G/T) \times^W \mathcal{L}^+(\mathfrak{t}^{rs}) & \xrightarrow{\sim} & \mathfrak{C}_{\leq 0}. \end{array} \tag{8.6}$$

Proof. The top isomorphism is obtained as a composition

$$(\mathcal{L}G/\mathcal{L}^+(T)) \times \mathcal{L}^+(\mathfrak{t}^{rs}) \simeq \mathcal{L}G \times^I [(I/\mathcal{L}^+(T)) \times \mathcal{L}^+(\mathfrak{t}^{rs})] \xrightarrow{\sim} \mathcal{L}G \times^I \mathrm{Lie}(I)_{\leq 0} \simeq \widetilde{\mathfrak{C}}_{\leq 0},$$

induced by isomorphism of Lemma 8.2.1, while the bottom isomorphism is the isomorphism of Lemma 8.1.8 applied to the open stratum $(w, \mathbf{r}) = (1, 0)$. The fact that the diagram is commutative is straightforward. □

Corollary 8.2.4. *The projection* $(\mathcal{L}G/\mathcal{L}^+(T)) \to \mathcal{L}(G/T)$ *is ind-fp-proper.*

Proof. Since $\mathfrak{p} : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ is ind-fp-proper (by Lemma 8.1.4), its restriction $\mathfrak{p}_{\leq 0} : \widetilde{\mathfrak{C}}_{\leq 0} \to \mathfrak{C}_{\leq 0}$ is ind-fp-proper as well. Using the identification of (8.6), the projection

$$\mathrm{pr} : (\mathcal{L}G/\mathcal{L}^+(T)) \times \mathcal{L}^+(\mathfrak{t}^{rs}) \to \mathcal{L}(G/T) \times^W \mathcal{L}^+(\mathfrak{t}^{rs})$$

is ind-fp-proper as well. Restricting this map to a fibre over any point of $\mathcal{L}^+(\mathfrak{c}^{rs}) = [\mathcal{L}^+(\mathfrak{t}^{rs})/W]$, we conclude that the projection $(\mathcal{L}G/\mathcal{L}^+(T)) \to \mathcal{L}(G/T)$ is ind-fp-proper. □

8.2.5. The $\widetilde{W}$-action on $\widetilde{\mathfrak{C}}_{\leq 0}$.

(a) Let $N := N_G(T)$ be the normalizer. The identification $\widetilde{W} \simeq (\mathcal{L}\,N)_{\mathrm{red}}/\mathcal{L}^+(T)$ gives rise to an action of $\widetilde{W}$ on $(\mathcal{L}\,G/\mathcal{L}^+(T))\times\mathcal{L}^+(\mathfrak{t}^{rs})$ over $\mathcal{L}(G/T)\times^W\mathcal{L}^+(\mathfrak{t}^{rs})$, given by the formula $w(g,x) := (gw^{-1}, w(x))$. Moreover, the quotient of $(\mathcal{L}\,G/\mathcal{L}^+(T)) \times \mathcal{L}^+(\mathfrak{t}^{rs})$ by $\widetilde{W}$ is naturally identified with $(\mathcal{L}\,G/(\mathcal{L}\,T)_{\mathrm{red}}) \times^W \mathcal{L}^+(\mathfrak{t}^{rs})$.

(b) Using the identification (8.6), we obtain from (a) an action of $\widetilde{W}$ on $\widetilde{\mathfrak{C}}_{\leq 0}$ over $\mathfrak{C}_{\leq 0}$, which induces an identification $[\widetilde{\mathfrak{C}}_{\leq 0}/\widetilde{W}] \simeq (\mathcal{L}\,G/(\mathcal{L}\,T)_{\mathrm{red}}) \times^W \mathcal{L}^+(\mathfrak{t}^{rs})$.

Corollary 8.2.6. *The projection $\mathfrak{p}_{\leq 0} : [\widetilde{\mathfrak{C}}_{\leq 0}/\widetilde{W}] \to \mathfrak{C}_{\leq 0}$ is a topological equivalence.*

Proof. The identifications of (8.6) and 8.2.5 identify $\mathfrak{p}_{\leq 0}$ with the topological equivalence (8.1) in the case $w = 1$ and $\mathbf{r} = 0$. □

8.3. **The fibration over a general stratum.** Recall that in 3.2.8(a) we defined a class of uo-equidimensional morphisms between topologically placid ∞-stacks.

Proposition 8.3.1. *The fibration $\overline{\mathfrak{p}}_{w,\boldsymbol{r}} : [\widetilde{\mathfrak{C}}_{w,\boldsymbol{r}}/\,\mathcal{L}\,G] \to [\mathfrak{C}_{w,\boldsymbol{r}}/\,\mathcal{L}\,G]$ is uo-equidimensional.*

Proof. Since the projection $\mathrm{Lie}(I)_{w,\mathbf{r}} \to [\mathrm{Lie}(I)_{w,\mathbf{r}}/I] \simeq [\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\,\mathcal{L}\,G]$ is a smooth covering (see 2.1.14), it suffices to show that the composition

$$\widetilde{\mathfrak{p}}_{w,\mathbf{r}} : \mathrm{Lie}(I)_{w,\mathbf{r}} \to [\mathrm{Lie}(I)_{w,\mathbf{r}}/I] \simeq [\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\,\mathcal{L}\,G] \to [\mathfrak{C}_{w,\mathbf{r}}/\,\mathcal{L}\,G]$$

is uo-equidimensional. Consider the Cartesian diagram

$$\begin{array}{ccccc}
\widetilde{\mathrm{Lie}(I)}_{w,\mathbf{r}} & \xrightarrow{\widetilde{\psi}_{w,\mathbf{r}}} & \mathrm{Lie}(I)_{w,\mathbf{r}} & = & \mathrm{Lie}(I)_{w,\mathbf{r}} \\
{\scriptstyle g_{w,\mathbf{r}}}\downarrow & & {\scriptstyle \widetilde{\mathfrak{p}}_{w,\mathbf{r}}}\downarrow & & {\scriptstyle v_{w,\mathbf{r}}}\downarrow \\
\mathfrak{c}_{w,\mathbf{r}} & \xrightarrow{\overline{\psi}_{w,\mathbf{r}}} & [\mathfrak{C}_{w,\mathbf{r}}/\,\mathcal{L}\,G] & \xrightarrow{\mathrm{pr}} & \mathfrak{c}_{w,\mathbf{r}}.
\end{array}$$

Since the map $\psi_{w,\mathbf{r}} = \overline{\psi}_{w,\mathbf{r}} \circ \pi_{w,\mathbf{r}}$ is a topologically smooth covering (see Corollary 8.1.11, while $\pi_{w,\mathbf{r}} : \mathfrak{t}_{w,\mathbf{r}} \to \mathfrak{c}_{w,\mathbf{r}}$ is a finite étale covering (see 7.2.8(c)), we conclude that $\overline{\psi}_{w,\mathbf{r}}$ is a topologically smooth covering. Therefore it suffices to show that the pullback $g_{w,\mathbf{r}} : \widetilde{\mathrm{Lie}(I)}_{w,\mathbf{r}} \to \mathfrak{c}_{w,\mathbf{r}}$ is uo-equidimensional. Since $\mathrm{pr} \circ \overline{\psi}_{w,\mathbf{r}}$ is the identity, $g_{w,\mathbf{r}}$ decomposes as $v_{w,\mathbf{r}} \circ \widetilde{\psi}_{w,\mathbf{r}}$, so it suffices to show that both $v_{w,\mathbf{r}}$ and $\widetilde{\psi}_{w,\mathbf{r}}$ are uo-equidimensional.

Since $\widetilde{\psi}_{w,\mathbf{r}}$ is a pullback of $\overline{\psi}_{w,\mathbf{r}}$, it is a topologically smooth covering. Thus it is uo-equidimensional. Finally, $v_n : \mathrm{Lie}_n(I) \to \mathcal{L}^+_n(\mathfrak{c})$ is flat morphism between irreducible varieties (by Theorem 7.4.2), we conclude that each v_n is uo-equidimensional (see 3.1.3(b)). Hence its projective limit $v = \lim_n v_n : \mathrm{Lie}(I) \to \mathcal{L}^+(\mathfrak{c})$ is uo-equidimensional as well. Therefore the pullback $v_{w,\mathbf{r}}$ of v to $\mathfrak{c}_{w,\mathbf{r}}$ is uo-equidimensional (see 3.2.8(b)), and the proof is complete. □

8.3.2. The Λ_w-action on $\widetilde{\mathfrak{C}}_{\mathfrak{t},w,\mathbf{r}}$.

Set $\widetilde{\mathfrak{C}}_{\mathfrak{t},w,\mathbf{r}} := \widetilde{\mathfrak{C}}_{w,\mathbf{r}} \times_{\mathfrak{c}_{w,\mathbf{r}}} \mathfrak{t}_{w,\mathbf{r}}$, and $\Lambda_w := \Lambda_{T_w}$ (as in 9.2.4).

(a) Note that the embedding $\iota : \widetilde{\mathfrak{C}} \hookrightarrow \mathfrak{Fl} \times \mathfrak{C}$ of 8.1.3, identifies $\widetilde{\mathfrak{C}}_{w,\mathbf{r}}$ with a closed ind-subscheme

$$\{([g], x) \in \mathfrak{Fl} \times \mathfrak{C}_{w,\mathbf{r}} \,|\, (\operatorname{Ad} g^{-1})(x) \in \operatorname{Lie}(I)\}.$$

(b) Using isomorphism $\mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}} \simeq \mathfrak{C}_{\mathfrak{t},w,\mathbf{r}}$ from Lemma 8.1.8, the ind-scheme $\widetilde{\mathfrak{C}}_{\mathfrak{t},w,\mathbf{r}}$ can be identified with a closed ind-subscheme

$$\{([g], h, x) \in \mathfrak{Fl} \times \mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}} \,|\, \operatorname{Ad}(g^{-1}h)(x) \in \operatorname{Lie}(I)\}.$$

Note that $g \in \mathcal{L}G$ is defined up to a right I-multiplication, thus $g^{-1}h \in \mathcal{L}(G/T_w)$, thus $\operatorname{Ad}(g^{-1}h)(x) \in \mathcal{L}\mathfrak{g}$ is defined up to an $\operatorname{Ad} I$-action, thus the condition $\operatorname{Ad}(g^{-1}h)(x) \in \operatorname{Lie}(I)$ makes sense.

(c) Consider the action of $\mathcal{L}(T_w)$ on $\mathfrak{Fl} \times \mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}}$ over $\mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}}$, defined by the formula $t([g], h, x) := ([(hth^{-1})g], h, x)$. Using the equality

$$g^{-1}(hth^{-1})^{-1}h = g^{-1}ht^{-1}h^{-1}h = g^{-1}ht^{-1},$$

we conclude that the closed ind-subscheme $\widetilde{\mathfrak{C}}_{\mathfrak{t},w,\mathbf{r}} \subset \mathfrak{Fl} \times \mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}}$ from (b) is $\mathcal{L}(T_w)$-invariant. Thus we obtain an action of $\mathcal{L}(T_w)$ on $\widetilde{\mathfrak{C}}_{\mathfrak{t},w,\mathbf{r}}$ over $\mathfrak{C}_{\mathfrak{t},w,\mathbf{r}}$.

(d) Recall that $\Lambda_w = X_*(T_w)^{\Gamma_w}$ is naturally a subgroup of $\mathcal{L}(T_w)$ via the embedding $\lambda \mapsto \lambda(t)$. Therefore the action of $\mathcal{L}(T_w)$ from (c) induces an action of Λ_w on $\widetilde{\mathfrak{C}}_{\mathfrak{t},w,\mathbf{r}}$ over $\mathfrak{C}_{\mathfrak{t},w,\mathbf{r}}$. Thus we can form a quotient $\overline{\mathfrak{C}}_{w,\mathbf{r}} := [\widetilde{\mathfrak{C}}_{\mathfrak{t},w,\mathbf{r}}/\Lambda_w]$.

Recall that in 2.3.8(b) and 2.3.9(a) we defined classes of strongly topologically schematic, locally fp and fp-proper morphisms between topologically placid ∞-stacks. The proof of the following result will be proven in the next section (see 9.3.6).

Theorem 8.3.3. *The projection $\widetilde{\mathfrak{C}}_{\mathfrak{t},w,\boldsymbol{r}} \to \mathfrak{C}_{\mathfrak{t},w,\boldsymbol{r}}$ is strongly topologically schematic, locally fp, and the induced morphism $\overline{\mathfrak{C}}_{w,\boldsymbol{r}} \to \mathfrak{C}_{\mathfrak{t},w,\boldsymbol{r}}$ is strongly topologically fp-proper.*

Corollary 8.3.4. *Consider a Cartesian diagram*

$$\begin{array}{ccc} \widetilde{X}_{w,\boldsymbol{r}} & \xrightarrow{\widetilde{\psi}_{w,\boldsymbol{r}}} & [\widetilde{\mathfrak{C}}_{w,\boldsymbol{r}}/\mathcal{L}_G] \\ {\scriptstyle \widetilde{g}_{w,\boldsymbol{r}}}\downarrow & & {\scriptstyle \overline{\mathfrak{p}}_{w,\boldsymbol{r}}}\downarrow \\ \mathfrak{t}_{w,\boldsymbol{r}} & \xrightarrow{\psi_{w,\boldsymbol{r}}} & [\mathfrak{C}_{w,\boldsymbol{r}}/\mathcal{L}G]. \end{array} \tag{8.7}$$

(a) Then $(\widetilde{X}_{w,\boldsymbol{r}})_{\mathrm{red}}$ is a scheme locally of finite type over $\mathfrak{t}_{w,\boldsymbol{r}}$, and the quotient $[(\widetilde{X}_{w,\boldsymbol{r}})_{\mathrm{red}}/\Lambda_w] = [\widetilde{X}_{w,\boldsymbol{r}}/\Lambda_w]_{\mathrm{red}}$ is an algebraic space, which is fp-proper over $\mathfrak{t}_{w,\boldsymbol{r}}$.

(b) Moreover, the projection $\overline{g}_{w,\boldsymbol{r}} : [\widetilde{X}_{w,\boldsymbol{r}}/\Lambda_w]_{\mathrm{red}} \to \mathfrak{t}_{w,\boldsymbol{r}}$ uo-equidimensional of dimension $\delta_{w,\boldsymbol{r}}$.

(c) The map $\overline{\mathfrak{p}}_{w,\mathbf{r}}$ is topologically locally fp-representable. Moreover, it is uo-equidimensional of relative dimension $\delta_{w,\mathbf{r}}$.

Proof. (a) Since the morphism $\psi_{w,\mathbf{r}}$ factors through $\mathfrak{t}_{w,\mathbf{r}} \to \mathfrak{C}_{\mathfrak{t},w,\mathbf{r}} = \mathfrak{C}_{w,\mathbf{r}} \times_{\mathfrak{c}_{w,\mathbf{r}}} \mathfrak{t}_{w,\mathbf{r}} : x \mapsto (x,x)$, it follows from Theorem 8.3.3 that the projection $\widetilde{X}_{w,\mathbf{r}} \to \mathfrak{t}_{w,\mathbf{r}}$ is strongly topologically schematic, locally fp, and the induced morphism $[\widetilde{X}_{w,\mathbf{r}}/\Lambda_w] \to \mathfrak{t}_{w,\mathbf{r}}$ is strongly topologically fp-proper. Since $[\widetilde{X}_{w,\mathbf{r}}/\Lambda_w]_{\mathrm{red}} \simeq [(\widetilde{X}_{w,\mathbf{r}})_{\mathrm{red}}/\Lambda_w]$, and $\mathfrak{t}_{w,\mathbf{r}}$ is a globally placid affine scheme (by 7.2.3(d)), the assertion now follows from the observation 2.3.9(b).

(b) By Proposition 8.3.1, the projection $\widetilde{g}_{w,\mathbf{r}} : \widetilde{X}_{w,\mathbf{r}} \to \mathfrak{t}_{w,\mathbf{r}}$ is uo-equidimensional, therefore the induced morphism $\overline{g}_{w,\mathbf{r}} : [\widetilde{X}_{w,\mathbf{r}}/\Lambda_w]_{\mathrm{red}} \to \mathfrak{t}_{w,\mathbf{r}}$ is such as well. Next we recall that all fibers of $\widetilde{g}_{w,\mathbf{r}}$ are affine Springer fibers, which are equidimensional of dimension $\delta_{w,\mathbf{r}}$ (see [Be]). Since $\overline{g}_{w,\mathbf{r}}$ is finitely presented, the last assertion follows from Lemma 3.3.2.

(c) For the first assertion, we have to show that for every morphism $\phi : Y \to [\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]$ from an affine scheme Y, the pullback $\overline{\mathfrak{p}}_{w,\mathbf{r}} \times_{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]} Y$ is topologically locally fp-representable. By (a), the assertion holds for $\phi = \psi_{w,\mathbf{r}}$. Since $\psi_{w,\mathbf{r}}$ is surjective in the étale topology, there exists an étale covering $Y' \to Y$ such that the composition $Y' \to Y \to [\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]$ factors through $\psi_{w,\mathbf{r}} \to [\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]$. Since the class of topologically locally fp-representable morphisms is closed under pullbacks and local in the étale topology, we conclude that it contains $\overline{\mathfrak{p}}_{w,\mathbf{r}} \times_{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]} Y'$ and hence also $\overline{\mathfrak{p}}_{w,\mathbf{r}} \times_{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]} Y$.

Similarly, for the second assertion, we have to show that the pullback $\overline{\mathfrak{p}}_{w,\mathbf{r}} \times_{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]} Y$ it is uo-equidimensional of relative dimension $\delta_{w,\mathbf{r}}$ when ϕ is topologically smooth. By (b), the assertion holds for $\phi = \psi_{w,\mathbf{r}}$. Since $\psi_{w,\mathbf{r}}$ is a topologically smooth covering, the assertion about the uo-equidimensionality follows from Lemma 1.4.3, while the assertion about relative dimension follows from Lemma 3.3.2. ☐

8.4. **The perversity of the affine Grothendieck–Springer sheaf.**

8.4.1. The affine Grothendieck-Springer sheaf.

(a) Note that the ind-fp-proper map $\mathfrak{p} : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ (see Lemma 8.1.4) is $\mathcal{L}G$-equivariant, and therefore induces a locally ind-fp-proper map $\overline{\mathfrak{p}} : [\widetilde{\mathfrak{C}}/\mathcal{L}G] \to [\mathfrak{C}/\mathcal{L}G]$ (see 4.4.2).

(b) By Proposition 4.4.3, the pullback $\overline{\mathfrak{p}}^! : \mathcal{D}([\mathfrak{C}/\mathcal{L}G]) \to \mathcal{D}([\widetilde{\mathfrak{C}}/\mathcal{L}G])$ has a left adjoint $\overline{\mathfrak{p}}_! : \mathcal{D}([\widetilde{\mathfrak{C}}/\mathcal{L}G]) \to \mathcal{D}([\mathfrak{C}/\mathcal{L}G])$, satisfying base change.

(c) We set

$$\mathcal{S} := \overline{\mathfrak{p}}_!(\omega_{[\widetilde{\mathfrak{C}}/\mathcal{L}G]}) \in \mathcal{D}([\mathfrak{C}/\mathcal{L}G])$$

and call it the *affine Grothendieck–Springer sheaf.*

(d) We denote by $\mathcal{S}_\bullet \in \mathcal{D}([\mathfrak{C}_\bullet/\mathcal{L}G])$ and $\mathcal{S}_{\leq 0} \in \mathcal{D}([\mathfrak{C}_{\leq 0}/\mathcal{L}G])$ the restrictions of $\mathcal{S}$. Since $\overline{\mathfrak{p}}_!$ admits base change, we have $\mathcal{S}_\bullet \simeq (\overline{\mathfrak{p}}_\bullet)_!(\omega_{[\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]})$, where $\overline{\mathfrak{p}}_\bullet : [\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G] \to [\mathfrak{C}_\bullet/\mathcal{L}G]$ is the restriction of $\overline{\mathfrak{p}}$.

(e) Let $j : [\mathfrak{C}_{\leq 0}/\mathcal{L}G] \hookrightarrow [\mathfrak{C}_\bullet/\mathcal{L}G]$ be the inclusion of the open stratum. By definition, $j^!(\mathcal{S}_\bullet) \simeq \mathcal{S}_{\leq 0}$.

Lemma 8.4.2. *(a) The ∞-stack $[\mathfrak{C}_\bullet/\mathcal{L}G]$ admits gluing of sheaves, the collection $\{[\mathfrak{C}_{w,\boldsymbol{r}}/\mathcal{L}G]\}_{w,\boldsymbol{r}}$ form a constructible stratification of $[\mathfrak{C}_\bullet/\mathcal{L}G]$, and each stratum $[\mathfrak{C}_{w,\boldsymbol{r}}/\mathcal{L}G]$ is topologically placid.*

(b) The ∞-stack $[\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G]$ is smooth, the projection $\overline{\mathfrak{p}}_\bullet : [\widetilde{\mathfrak{C}}_\bullet/\mathcal{L}G] \to [\mathfrak{C}_\bullet/\mathcal{L}G]$ is locally ind-fp-proper, semi-small, and small relative to the open stratum $(1,0)$.

Proof. (a) Recall that since each $\mathfrak{C}_{\leq m}$ is an ind-placid scheme (see 8.1.7(b)), while $\mathcal{L}G$ is an ind-placid group scheme (see 8.1.3(d)), the quotient $[\mathfrak{C}_{\leq m}/\mathcal{L}G]$ admits gluing of sheaves by Proposition 6.1.8. Moreover, since $[\mathfrak{C}_\bullet/\mathcal{L}G]$ is a colimit of the $[\mathfrak{C}_{\leq m}/\mathcal{L}G]$'s, all of whose transition maps are fp-open embeddings (see 8.1.7(c)), it therefore admits gluing of sheaves by Lemma 6.1.5(b). Next $[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]$ is a topologically placid ∞-stack by Corollary 8.1.11. Finally, the fact that $\{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]\}_{w,\mathbf{r}}$ is a constructible stratification was already mentioned in 8.1.7.

(b) The first two assertions follow from the facts that $[\widetilde{\mathfrak{C}}/\mathcal{L}G] \simeq [\mathrm{Lie}(I)/I]$ is smooth, and $\overline{\mathfrak{p}}$ is locally ind-fp-proper (see 8.4.1(a)).

Next, since $\mathrm{Lie}(I)_{w,\mathbf{r}} \subset \mathrm{Lie}(I)$ is a locally closed subscheme of pure codimension $b_{w,\mathbf{r}}$ (see Corollary 7.4.5), we conclude that $[\mathrm{Lie}(I)_{w,\mathbf{r}}/I] \subset [\mathrm{Lie}(I)/I]$ is a locally closed subscheme of pure codimension $b_{w,\mathbf{r}}$ (by 3.3.5(d)), that is, $[\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\mathcal{L}G] \subset [\widetilde{\mathfrak{C}}/\mathcal{L}G]$ is a locally closed subscheme of pure codimension $b_{w,\mathbf{r}}$. Now the smallness assertion follows from Corollary 8.3.4(c) and Corollary 7.3.3. ☐

8.4.3. The perverse t-structure.

(a) By Lemma 8.4.2, $([\mathfrak{C}_\bullet/\mathcal{L}G], \{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]\}_{w,\mathbf{r}})$ is a stratified ∞-stack, which we equip with the canonical perversity $p_\nu = \{\nu_{w,\mathbf{r}}\}$, defined by $\nu_{w,\mathbf{r}} := d_\mathbf{r} + a_{w,\mathbf{r}}$ (see 7.3.1). By Proposition 6.2.7, this perversity p_ν gives rise to the t-structure on $\mathcal{D}([\mathfrak{C}_\bullet/\mathcal{L}G])$, which we call the *perverse t-structure.*

(b) Notice that perversity p_ν coincides with the perversity $p_{\overline{\mathfrak{p}}} := p_{\overline{\mathfrak{p}}_\bullet}$, corresponding to the Grothendieck–Springer fibration $\overline{\mathfrak{p}}_\bullet$ (see 6.4.3). Indeed, by definition perversity $p_{\overline{\mathfrak{p}}}$ is defined by $p_{\overline{\mathfrak{p}}}(w,\mathbf{r}) = b_{w,\mathbf{r}} + \delta_{w,\mathbf{r}}$ (see 7.3.1), so the assertion follows from the Goresky-Kottwitz-MacPherson codimension formula Proposition 7.3.2.

Now we are ready to prove the main result of this work.

Theorem 8.4.4. *(a) The affine Grothendieck–Springer sheaf $\mathcal{S}_\bullet \in \mathcal{D}([\mathfrak{C}_\bullet/\mathcal{L}G])$ is $p_{\overline{\mathfrak{p}}}$-perverse and satisfies $\mathcal{S}_\bullet \simeq j_{!*}(\mathcal{S}_{\leq 0})$.*

(b) There are natural algebra isomorphisms $\mathrm{End}(\mathcal{S}_\bullet) \simeq \mathrm{End}(\mathcal{S}_{\leq 0}) \simeq \overline{\mathbb{Q}}_\ell[\widetilde{W}]$.

8.4.5. Remark. It follows from (b) that $\mathcal{S}_\bullet$ is equipped with a natural action of $\widetilde{W}$. Namely, it follows from the proof (see Proposition 8.4.6) that the action of $\widetilde{W}$ on $\mathcal{S}_{\leq 0}$ is induced by the geometric action of $\widetilde{W}$ on $\widetilde{\mathfrak{C}}_{\leq 0}$ over $\mathfrak{C}_{\leq 0}$, and this action uniquely extends to the action on $\mathcal{S}$.

Proof. (a) is an immediate consequence of the combination of Lemma 8.4.2 and Theorem 6.4.5.

(b) The first isomorphism follows from (a) and Corollary 6.3.5, while the second one is shown in Proposition 8.4.6 below. ☐

Proposition 8.4.6. *We have a natural algebra isomorphism* $\operatorname{End}(\mathcal{S}_{\leq 0}) \simeq \overline{\mathbb{Q}}_\ell[\widetilde{W}]$.

Proof. Note that the topological equivalence $[\widetilde{\mathfrak{C}}_{\leq 0}/\widetilde{W}] \to \mathfrak{C}_{\leq 0}$ from Corollary 8.2.6 induces a topological equivalence $[\widetilde{\mathfrak{C}}_{\leq 0}/\mathcal{L}G \times \widetilde{W}] \to [\mathfrak{C}_{\leq 0}/\mathcal{L}G]$ (by Corollary 2.3.7(c)). Thus the projection $\overline{\mathfrak{p}}_{\leq 0} : [\widetilde{\mathfrak{C}}_{\leq 0}/\mathcal{L}G] \to [\mathfrak{C}_{\leq 0}/\mathcal{L}G]$ satisfies the assumption of Corollary 4.6.8 with $\Gamma := \widetilde{W}$. Since $\mathcal{S}_{\leq 0} \simeq (\overline{\mathfrak{p}}_{\leq 0})_!(\omega_{[\widetilde{\mathfrak{C}}_{\leq 0}/\mathcal{L}G]})$ (see 8.4.1(d)), it suffices to show that $\overline{\mathbb{Q}}_l^{\pi_0([\widetilde{\mathfrak{C}}_{\leq 0}/\mathcal{L}G])} \simeq \overline{\mathbb{Q}}_l$.

Since $[\widetilde{\mathfrak{C}}_{\leq 0}/\mathcal{L}G] \simeq [\mathcal{L}^+(\mathfrak{t}^{rs})/\mathcal{L}^+(T)]$ (see Corollary 8.2.2), and the map $\mathcal{L}^+(\mathfrak{t}^{rs}) \to [\mathcal{L}^+(\mathfrak{t}^{rs})/\mathcal{L}^+(T)]$ is surjective, while $\mathcal{L}^+(\mathfrak{t}^{rs})$ is a connected, the assertion follows from Corollary 4.6.5. ☐

8.4.7. The affine Springer sheaf.

(a) Recall (see 7.3.4(a)) that $\mathfrak{c}^u \subset \mathcal{L}^+(\mathfrak{c})$ is a closed finitely presented subscheme, and let $\mathfrak{c}^u_\bullet \subset \mathfrak{c}_\bullet$, $\mathfrak{C}^u_\bullet \subset \mathfrak{C}_\bullet$, $\operatorname{Lie}(I)^u \subset \operatorname{Lie} I$, etc. be the preimage of $\mathfrak{c}^u \subset \mathcal{L}^+(\mathfrak{c})$. In particular, $i^u : [\mathfrak{C}^u_\bullet/\mathcal{L}G] \to [\mathfrak{C}_\bullet/\mathcal{L}G]$ is a finitely presented closed embedding.

(b) Recall (see 7.3.4(c)) that $\mathfrak{c}^u_\bullet \subset \mathfrak{c}_\bullet$ (and hence also $\mathfrak{C}^u_\bullet \subset \mathfrak{C}_\bullet$ and $\operatorname{Lie}(I)^u_\bullet \subset \operatorname{Lie}(I)_\bullet$) is a union of all strata $\mathfrak{c}_{w,\mathbf{r}}$ such that $(w,\mathbf{r}) > 0$. Therefore $\{[\mathfrak{C}_{w,\mathbf{r}}/\mathcal{L}G]\}_{(w,\mathbf{r})>0}$ is a constructible stratification of $[\mathfrak{C}^u_\bullet/\mathcal{L}G]$. As in Lemma 8.4.2, we therefore conclude that $[\mathfrak{C}^u_\bullet/\mathcal{L}G]$ admits gluing of sheaves (by Proposition 6.1.8 and Lemma 6.1.5(b)).

(c) Let $\overline{\mathfrak{p}}^u_\bullet : [\mathcal{X}^u_\bullet/\mathcal{L}G] \to [\mathfrak{C}^u_\bullet/\mathcal{L}G]$ be the restriction of $\overline{\mathfrak{p}}$.

(d) We set $\mathcal{S}^u := (i^u)^!\mathcal{S} \in \mathcal{D}([\mathfrak{C}^u/\mathcal{L}G])$, and call it the *affine Springer sheaf*. We also let $\mathcal{S}^u_\bullet \in \mathcal{D}([\mathfrak{C}^u_\bullet/\mathcal{L}G])$ be the !-pullback of $\mathcal{S}^u$.

Lemma 8.4.8. *The ∞-stack $[\widetilde{\mathfrak{C}}^u_\bullet/\mathcal{L}G]$ is smooth, the affine Springer fibration $\overline{\mathfrak{p}}^u_\bullet$: $[\widetilde{\mathfrak{C}}^u_\bullet/\mathcal{L}G] \to [\mathfrak{C}^u_\bullet/\mathcal{L}G]$ is locally fp-proper, and semi-small, and the corresponding perversity $p^u_{\overline{\mathfrak{p}}} := p^u_{\overline{\mathfrak{p}}_\bullet}$ satisfies $p^u_{\overline{\mathfrak{p}}}(w,\boldsymbol{r}) = (b_{w,\boldsymbol{r}} - r) + \delta_{w,\boldsymbol{r}}$ (compare 8.4.3(a)).*

Proof. The argument is almost identical to that of Lemma 8.4.2. The smoothness assertion follows from the isomorphism $[\widetilde{\mathfrak{C}}^u_\bullet/\mathcal{L}G] \simeq [\operatorname{Lie}(I)^u_\bullet/\operatorname{Lie}(I)]$ and smoothness $\operatorname{Lie}(I)^u$. The locally ind-fp-properness of $\overline{\mathfrak{p}}^u_\bullet$ follows from that for $\overline{\mathfrak{p}}_\bullet$. Next, since

$\mathrm{Lie}(I)^u \subset \mathrm{Lie}(I)$ is a closed subscheme of pure codimension r, we conclude that (as in Lemma 8.4.2) that $[\widetilde{\mathfrak{C}}_{w,\mathbf{r}}/\mathcal{L}G] \subset [\widetilde{\mathfrak{C}}^u_\bullet/\mathcal{L}G]$ is a locally closed subscheme of pure codimension $b_{w,\mathbf{r}} - r$. The remaining assertions now follow from Corollary 8.3.4(c) and Corollary 7.3.5. □

Theorem 8.4.9. *The affine Springer sheaf $\mathcal{S}^u_\bullet$ is $p^u_{\mathfrak{p}}$-perverse and satisfies $\mathcal{S}^u_\bullet \simeq (\overline{\mathfrak{p}}^u_\bullet)_!(\omega_{[\mathrm{Lie}(I)^u_\bullet/I]})$.*

Proof. Since $\overline{\mathfrak{p}}_\bullet$ is locally ind-fp-proper, the base change morphism $(\overline{\mathfrak{p}}^u_\bullet)_!(\widetilde{i}^u)^! \to (i^u)^!(\overline{\mathfrak{p}}_\bullet)_!$, corresponding to the Cartesian diagram

$$\begin{array}{ccc} [\mathrm{Lie}(I)^u_\bullet/I] & \xrightarrow{\widetilde{i}^u} & [\mathrm{Lie}(I)_\bullet/I] \\ {\scriptstyle \overline{\mathfrak{p}}^u_\bullet}\downarrow & & {\scriptstyle \overline{\mathfrak{p}}_\bullet}\downarrow \\ [\mathfrak{C}^u_\bullet/\mathcal{L}G] & \xrightarrow{i^u} & [\mathfrak{C}_\bullet/\mathcal{L}G]. \end{array}$$

is an isomorphism (by Proposition 4.4.3). Therefore we get an isomorphism $\mathcal{S}^u_\bullet \simeq (\overline{\mathfrak{p}}^u_\bullet)_!(\omega_{[\mathrm{Lie}(I)^u_\bullet/I]})$. The assertion now follows from a combination of Lemma 8.4.8 and Theorem 6.4.5. □

One can ask whether $\mathcal{S}^u_\bullet$ is a intermediate extension of its restriction to a suitable fp-open substack, and what is the minimal subset satisfying this property.

8.4.10. Conjecture. (a) Note that for every $w \in W$ the affine scheme $\mathfrak{t}^u_w$ is irreducible. Therefore there exists a function $\mathbf{r}^+_w : R \to \mathbb{Q}_{\geq 0}$ such that $\mathfrak{t}_{w,\mathbf{r}^+_w} \subset \mathfrak{t}^u_w$ is an open stratum.

(b) We conjecture that the union $\cup_w \mathfrak{c}_{w,\mathbf{r}^+_w}$ is open in $\mathfrak{c}^u$. More precisely, $\{\mathfrak{c}_{w,\mathbf{r}^+_w}\}_{w\in W}$ gives a constructible stratification of a certain fp-open subscheme $\mathfrak{c}^{u,+} \subset \mathfrak{c}^u_\bullet$.

8.4.11. Remark. Assume that conjecture 8.4.10(b) holds, and let $\mathfrak{C}^{u,+} \subset \mathfrak{C}^u_\bullet$ be the preimage of $\mathfrak{c}^{u,+} \subset \mathfrak{c}^u_\bullet$. Then $[\mathfrak{C}^{u,+}/\mathcal{L}G] \subset [\mathfrak{C}^u_\bullet/\mathcal{L}G]$ is an open union of strata. Then Lemma 8.4.8, Corollary 7.3.5 and Theorem 6.4.5 would imply that $\mathcal{S}^u_\bullet$ is the intermediate extension of its restriction to $[\mathfrak{C}^{u,+}/\mathcal{L}G]$. Moreover, this is the largest open union of strata, satisfying this property.

8.4.12. Example. Assume that $G = \mathrm{SL}_2$. In this case, $\mathfrak{c} \simeq \mathbb{A}^1$, $\mathfrak{c}^u \subset \mathcal{L}^+(\mathfrak{c})$ is the locus $\mathcal{L}^+(\mathbb{A}^1)_{\geq 1}$, and $\mathfrak{c}^{u,+} \subset \mathfrak{c}^u_\bullet$ is the locus $\mathcal{L}^+(\mathbb{A}^1)_{\geq 1} \cap \mathcal{L}^+(\mathbb{A}^1)_{\leq 2}$, that is, the union of two strata $\mathcal{L}^+(\mathbb{A}^1)_1$ and $\mathcal{L}^+(\mathbb{A}^1)_2$. In particular, our conjecture holds in this case.

8.5. **Perverse t-structure on $[(\mathcal{L}\mathfrak{g})_\bullet/\mathcal{L}G]$.**

8.5.1. The $\mathbb{G}_m$-action. (a) Recall that the natural $\mathbb{G}_m$-action $(a, x) \mapsto ax$ on $\mathfrak{g}$ commutes with the adjoint action of G. Thus it induces the $\mathbb{G}_m$-action on $\mathfrak{c} = \mathfrak{g}//G$ such that the projection $\chi : \mathfrak{g} \to \mathfrak{c}$ is $\mathbb{G}_m$-equivariant. In particular, the induced map $\pi : \mathfrak{t} \to \mathfrak{c}$ is $\mathbb{G}_m$-equivariant.

(b) Furthermore, there exists a (noncanonical) isomorphism $\mathfrak{c} \xrightarrow{\sim} \mathbb{A}^r$ under which the $\mathbb{G}_m$-action on $\mathfrak{c}$ corresponds to a $\mathbb{G}_m$-action on $\mathbb{A}^1$, given by $a(x_1, \ldots, x_r) = (a^{d_1}x_1, \ldots, a^{d_r}x_r)$ for certain positive integers $d_1, \ldots, d_r$.

(c) The $\mathbb{G}_m$-actions on $\mathfrak{t}$ and $\mathfrak{c}$ induce $\mathcal{L}\,\mathbb{G}_m$-actions on $\mathcal{L}\,\mathfrak{g}$, $\mathcal{L}\,\mathfrak{t}_w$ and $\mathcal{L}\,\mathfrak{c}$ such that the induced maps $\chi : \mathcal{L}\,\mathfrak{g} \to \mathcal{L}\,\mathfrak{c}$ and $\pi : \mathcal{L}\,\mathfrak{t}_w \to \mathcal{L}\,\mathfrak{c}$ are $\mathcal{L}\,\mathbb{G}_m$-equivariant.

8.5.2. Constructible stratification of $(\mathcal{L}\,\mathfrak{c})_\bullet$.

(a) By definition, for every GKM-stratum $\mathfrak{t}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{t}_w)$ and every $n \geq 0$ the action of element $t^n \in \mathcal{L}\,\mathbb{G}_m$ on $\mathcal{L}\,\mathfrak{t}_w$ from 8.5.1(c) induces an isomorphism $\mathfrak{t}_{w,\mathbf{r}} \xrightarrow{\sim} \mathfrak{t}_{w,\mathbf{r}+n}$, that is, $\mathfrak{t}_{w,\mathbf{r}+n} = t^n\mathfrak{t}_{w,\mathbf{r}}$.

(b) Since the GKM stratum $\mathfrak{c}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{c})$ is defined to be the image $\pi(\mathfrak{t}_{w,\mathbf{r}})$, and π is $\mathcal{L}\,\mathbb{G}_m$-equivariant, we conclude that the action of element $t^n \in \mathcal{L}\,\mathbb{G}_m$ induces an isomorphism $\mathfrak{c}_{w,\mathbf{r}} \xrightarrow{\sim} \mathfrak{c}_{w,\mathbf{r}+n}$, that is, $\mathfrak{c}_{w,\mathbf{r}+n} = t^n\mathfrak{c}_{w,\mathbf{r}}$.

(c) For every pair $(w, \mathbf{r})$, where $w \in W$ and $\mathbf{r} : R \to \mathbb{Q}$, we choose $n \geq 0$ such that $\mathbf{r} + n \geq 0$. To this data we can associate an fp-locally closed subscheme $\mathfrak{c}_{w,\mathbf{r}+n} \subset \mathcal{L}^+(\mathfrak{c}) \subset \mathcal{L}\,\mathfrak{c}$ (see 7.2.8), so we can consider another fp-locally closed subscheme $\mathfrak{c}_{w,\mathbf{r}} := t^{-n}\mathfrak{c}_{w,\mathbf{r}+n} \subset \mathcal{L}\,\mathfrak{c}$. Moreover, using observation of (b) one sees that $\mathfrak{c}_{w,\mathbf{r}}$ is independent of the choice of n (hence coincides with that of 7.2.8 when $\mathbf{r} \geq 0$).

(d) We claim that the collection $\{\mathfrak{c}_{w,\mathbf{r}}\}_{w,\mathbf{r}}$ form a constructible stratification of $(\mathcal{L}\,\mathfrak{c})_\bullet := (\mathcal{L}\,\mathfrak{c})_{\mathfrak{D}\neq 0}$.

First of all, since $\mathcal{L}^+(\mathfrak{c}) \subset \mathcal{L}\,\mathfrak{c}$ is an fp-closed subscheme, the same is true for each $t^{-n}\mathcal{L}^+(\mathfrak{c}) \subset \mathcal{L}\,\mathfrak{c}$. Moreover, using isomorphism $\mathcal{L}(\mathbb{A}^1) \simeq \operatorname{colim}_n t^{-n}\mathcal{L}^+(\mathbb{A}^1)$ and 8.5.1(b), we conclude that we have presentation $\mathcal{L}\,\mathfrak{c} \simeq \operatorname{colim}_n t^{-n}\mathcal{L}^+(\mathfrak{c})$ as a filtered colimit of its fp-closed subschemes, hence a similar presentation $(\mathcal{L}\,\mathfrak{c})_\bullet \simeq \operatorname{colim}_n t^{-n}\mathfrak{c}_\bullet$.

Next we notice that we have $\mathfrak{c}_{w,\mathbf{r}} \subset t^{-n}\mathcal{L}^+(\mathfrak{c})$ if and only if $\mathfrak{c}_{w,\mathbf{r}+n} = t^n\mathfrak{c}_{w,\mathbf{r}} \subset \mathcal{L}^+(\mathfrak{c})$. Since $\{\mathfrak{c}_{w,\mathbf{r}}\}_{w,\mathbf{r}\geq 0}$ form a bounded constructible stratification of $\mathfrak{c}_\bullet$ (see 8.1.7(d)), we thus conclude that $\{\mathfrak{c}_{w,\mathbf{r}}\}_{w,\mathbf{r}\geq -n}$ form a bounded constructible stratification of $t^{-n}\mathfrak{c}_\bullet$, hence $\{\mathfrak{c}_{w,\mathbf{r}}\}_{w,\mathbf{r}}$ form a constructible stratification of $(\mathcal{L}\,\mathfrak{c})_\bullet := (\mathcal{L}\,\mathfrak{c})_{\mathfrak{D}\neq 0}$.

8.5.3. The perverse t-structure. Set $(\mathcal{L}\,\mathfrak{g})_\bullet := \chi^{-1}((\mathcal{L}\,\mathfrak{c})_\bullet) \subset \mathcal{L}\,\mathfrak{g}$.

(a) For every $(w, \mathbf{r})$ as in 8.5.2(c), the the preimage $(\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}} := \chi^{-1}(\mathfrak{c}_{w,\mathbf{r}}) \subset \mathcal{L}\,\mathfrak{g}$ is a fp-locally closed ind-subscheme, and $\{(\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}}\}_{w,\mathbf{r}}$ form a constructible stratification of $(\mathcal{L}\,\mathfrak{g})_\bullet$ (by 8.5.2(d)). Therefore the quotient ∞-stack $[(\mathcal{L}\,\mathfrak{g})_\bullet/\,\mathcal{L}\,G]$ is equipped with a constructible stratification $\{[(\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}}/\,\mathcal{L}\,G]\}_{w,\mathbf{r}}$.

(b) Since χ is $\mathbb{G}_m$-equivariant, the action of $t^n \in \mathcal{L}\,\mathbb{G}_m$ on $\mathcal{L}\,\mathfrak{g}$ induces an isomorphism $(\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}} \xrightarrow{\sim} (\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}+n}$, hence $[(\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}}/\,\mathcal{L}\,G] \xrightarrow{\sim} [(\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}+n}/\,\mathcal{L}\,G]$. Using equality $(\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}} = \mathfrak{C}_{w,\mathbf{r}}$ for all $\mathbf{r} \geq 0$, we thus conclude from Corollary 8.1.11 that each $[(\mathcal{L}\,\mathfrak{g})_{w,\mathbf{r}}/\,\mathcal{L}\,G]$ is topologically placid.

(c) By (a) and (b), $[(\mathcal{L}\,\mathfrak{g})_\bullet/\,\mathcal{L}\,G]$ is a stratified ∞-stack. Moreover, arguing as in Lemma 8.4.2, it follows from Proposition 6.1.8 that $[(\mathcal{L}\,\mathfrak{g})_\bullet/\,\mathcal{L}\,G]$ admits gluing of sheaves.

(d) Notice that for every GKM stratum $(w,\mathbf{r})$ from 8.4.3 and every $n\geq 0$, the expression $\nu_{w,\mathbf{r}}$ satisfies $\nu_{w,\mathbf{r}+n}=\nu_{w,\mathbf{r}}+n\dim G$.

(e) For an arbitrary $(w,\mathbf{r})$, we choose $n\geq 0$ such that $\mathbf{r}+n\geq 0$. In this case, $\nu_{w,\mathbf{r}+n}$ was defined in 8.4.3, and we set $\nu_{w,\mathbf{r}}:=\nu_{w,\mathbf{r}+n}-n\dim G$. By (d), $\nu_{w,\mathbf{r}}$ is independent of n and coincides with that of 8.4.3 when $\mathbf{r}\geq 0$.

(f) By Proposition 6.2.7, the perversity $p_\nu:=\{\nu_{w,\mathbf{r}}\}$ gives rise to the t-structure on $\mathcal{D}([(\mathcal{L}\mathfrak{g})_\bullet/\mathcal{L}G])$, which we call the *perverse t-structure.*

9. Completion of proofs.

9.1. **Quotients of ind-schemes.**

9.1.1. Let a group Δ act on an ind-scheme Z over an ind-scheme Y.

(a) We say that Δ acts *discretely*, if for every qcqs fp-closed subscheme $Z'\subset Z$, the set of $\delta\in\Delta$ such that $\delta(Z')\cap Z'\neq\emptyset$ is finite.

(b) We say that Δ acts *freely*, if the action map $a:\Delta\times Z\to Z\times Z:(\delta,x)\mapsto(\delta(x),x)$ is injective.

Proposition 9.1.2. *Let $h:Z\to Y$ be an ind-fp-proper morphism between ind-schemes, and let Δ be a group acting on Z over Y, freely and discretely. Then,*

(a) The quotient $\overline{Z}:=[Z/\Delta]$ is an ind-algebraic space, ind-fp-proper over Y.

(b) Assume that for every fp-closed qcqs subscheme $Y'\subset Y$ there exists a closed qcqs fp-subscheme $Z'\subset Z$ such that $h^{-1}(Y'(K))\subset\bigcup_{\delta\in\Delta}\delta(Z'(K))$ for every algebraically closed field K.

Then h is strongly topologically schematic, locally fp, and the induced morphism $\overline{h}:\overline{Z}\to Y$ is strongly topologically fp-proper.

Proof. Note that any presentation $Y\simeq\operatorname{colim}_i Y_i$ of Y induces a presentation $Z\simeq\operatorname{colim}_i Z_i$ with $Z_i:=Z\times_Y Y_i$. Since all assertion for $h:Z\to Y$ formally follow from corresponding assertion for $h_i:Z_i\to Y_i$, we can replace h by h_i, thus assuming that Y is a qcqs scheme.

Let $Z'\subset Z$ be an fp-closed qcqs subscheme. For every finite subset $D\subset\Delta$, we denote by $Z'_D:=\cup_{\delta\in D}\delta(Z')\subset Z$ the smallest closed subscheme of Z, containing each $\delta(Z')$. Then for every subset $\Delta'\subset\Delta$, we set $Z'_{\Delta'}:=\operatorname{colim}_{D\subset\Delta'}Z'_D$. Since $Z'\to Y$ is fp-proper, we conclude that $Z'_D\to Y$ is fp-proper for every D, thus $Z'_{\Delta'}\to Y$ is ind-fp-proper.

We claim that the inclusion $Z'_{\Delta'}\hookrightarrow Z$ is a fp-closed embedding. For this we have to show that for every fp-closed qcqs subscheme $Z''\subset Z$, the intersection $Z'_{\Delta'}\cap Z''\subset Z''$ is an fp-closed subscheme. Since homotopy colimits commute with pullbacks, we have $Z'_{\Delta'}\cap Z''=\operatorname{colim}_D(Z'_D\cap Z'')$, and each $Z'_D\cap Z''\subset Z''$ is a closed subscheme, it suffices to show that the family $\{Z'_D\cap Z''\}_D$ stabilizes. Since the action of Δ on Z is discrete, the set of $\delta\in\Delta$ such that $\delta(Z')\cap Z''\neq\emptyset$ is finite, so the stabilization follows.

Note that $Z'_\Delta := \operatorname{colim}_D Z'_D$ is the smallest Δ-invariant closed ind-subscheme of Z, containing Z'. We form the quotient $\overline{Z}' := [Z'_\Delta/\Delta]$.

Claim 9.1.3. *(a) The ind-scheme Z'_Δ is a scheme, locally fp over Y.*

(b) The quotient $\overline{Z}'$ is an algebraic space fp-proper on Y, and the projection $Z' \to \overline{Z}'$ is surjective.

(c) For every pair of fp-closed qcqs subschemes $Z' \subset Z''$, the induced map $\overline{Z}' \to \overline{Z}''$ is an fp-closed embedding.

We now complete the proof of Proposition 9.1.2, assuming the claim.

(a) Choose a presentation $Z \simeq \operatorname{colim}_\alpha Z_\alpha$ of Z. Since $Z_\alpha \subset Z_{\alpha,\Delta} \subset Z$, we get an isomorphism $Z \simeq \operatorname{colim}_\alpha Z_{\alpha,\Delta}$. Taking the quotient by Δ, we get an isomorphism $\overline{Z} \simeq \operatorname{colim}_\alpha \overline{Z}_\alpha$. Since every $\overline{Z}_\alpha$ is an algebraic space, fp-proper over Y (by Claim 9.1.3(b)), and each $\overline{Z}_\alpha \to \overline{Z}_\beta$ is an fp-closed embedding by (by Claim 9.1.3(c)), and the assertion follows.

(b) By our assumption, there exists α such that $Z_\Delta(K) = Z_{\Delta,\alpha}(K)$ and $\overline{Z}(K) = \overline{Z}_\alpha(K)$ for all algebraically closed fields K. In particular, for all $Z_\beta \supseteq Z_\alpha$, the fp-closed embeddings $Z_{\Delta,\alpha} \to Z_{\Delta,\beta}$ and $\overline{Z}_\alpha \to \overline{Z}_\beta$ induce bijections on K-points. Hence the induced maps $(Z_{\Delta,\alpha})_{\mathrm{red}} \to (Z_{\Delta,\beta})_{\mathrm{red}}$ and $(\overline{Z}_\alpha)_{\mathrm{red}} \to (\overline{Z}_\beta)_{\mathrm{red}}$ are isomorphisms. Therefore the maps $(Z_{\Delta,\alpha})_{\mathrm{red}} \to Z_{\mathrm{red}}$ and $(\overline{Z}_\alpha)_{\mathrm{red}} \to \overline{Z}_{\mathrm{red}}$ are isomorphisms as well. Since $Z_{\Delta,\alpha} \to Y$ is schematic, locally fp (by Claim 9.1.3(a)), while $\overline{Z}_\alpha \to Y$ is fp-proper (by Claim 9.1.3(a)), the assertion follows. $\square$

It remains to show Claim 9.1.3.

Proof of Claim 9.1.3. (a) We have to show that every point $x \in Z'_\Delta$ has an open neighbourhood, which is a scheme finitely presented over Y. Since every point of Z'_Δ is a Δ-translate of a point of Z', it is sufficient to prove it for $x \in Z'$. We claim that the whole Z' has such a neighborhood. Let $\Sigma := \{\delta \in \Delta \mid \delta(Z') \cap Z' \neq \emptyset\}$. By assumption, it is a finite set.

Then $Z'_{\Delta\smallsetminus\Sigma} \subset Z$ is a closed subfunctor, hence $U := Z'_\Delta \smallsetminus Z'_{\Delta\smallsetminus\Sigma} \subset Z$ is an open subfunctor. Since $Z' \cap Z'_{\Delta\smallsetminus\Sigma} = \emptyset$ by the definition of Σ, we have $Z' \subset U$, and clearly $U \subset Z'_\Sigma$. Thus U is an open subscheme of Z'_Σ, hence it is a scheme, locally of finite presentation over Y, as claimed.

(b) As Δ acts freely, it defines an étale equivalence relation on Z'_Δ, thus $\overline{Z}' = [Z'_\Delta/\Delta]$ is an algebraic space (see [St, Tag. 0264]), locally of finite presentation over Y.

Moreover, since Z'_Δ is a filtered colimit $\operatorname{colim}_D Z'_D$ with Z'_D proper (thus separated) over Y, we conclude that Z'_Δ is separated over Y. Next we claim that the map $\overline{Z}' \to Y$ is separated.

We have to show that the map $a : \Delta \times Z'_\Delta \to Z'_\Delta \times_Y Z'_\Delta$ is proper. It suffices to show the properness of the restriction of a to the inverse image of $Z'_D \times_Y Z'_D$ for every finite subset $D \subset \Delta$. But this inverse image is the disjoint union of maps $a_{D,\delta} : Z'_D \cap \delta^{-1}(Z'_D) \to Z'_D \times_Y Z'_D$. As the action is discrete, this union is finite. So one has to prove that each $a_{D,\delta}$ is a closed embedding. But each $a_{D,\delta}$ is the restriction of the graph of $\delta : Z'_\Delta \to Z'_\Delta$, which is a closed embedding, as $Z'_\Delta \to Y$ is separated.

Finally, we claim that $\overline{Z}' \to Y$ is fp-proper. Indeed, since $Z' \to \overline{Z}'$ is surjective, Z' is fp-proper over Y and $\overline{Z}'$ separated over Y, we conclude that $\overline{Z}' \to Y$ is proper by [St, Tag 08AJ]. As it is both locally of finite presentation and proper, it is finitely presented.

(c) Since $Z' \subset Z''$ is a closed subfunctor, we conclude that $Z'_\Delta \subset Z''_\Delta$ and hence also $\overline{Z}' \subset \overline{Z}''$ is a closed subfunctor. It is finitely presented by [St, Tag 02FV], because both $\overline{Z}'$ and $\overline{Z}''$ are fp-proper over Y . □

9.1.4. Remark. In the situation of Proposition 9.1.2, assume that for every algebraically closed field K, the map $Z(K) \to Y(K)$ is a Δ-torsor. Then the assumption of Proposition 9.1.2(b) is equivalent to the assumption that for every fp-closed qcqs subscheme $Y' \subset Y$ there exists an fp-closed qcqs subscheme $Z' \subset Z$ such that $f(Z'(K)) \supseteq Y'(K)$ for every algebraically closed field K.

9.2. **Passing to tame Galois invariants.**

Lemma 9.2.1. *Let $h : Z \to Y$ and Δ that satisfy the assumptions of Proposition 9.1.2, and let Γ be a finite group, acting on Z, Y and by group automorphisms on Δ such that the map h and the action map $\Delta \times Z \to Z$ are Γ-equivariant. Then the induced map $h^\Gamma : Z^\Gamma \to Y^\Gamma$ between ind-schemes of invariants is ind-fp-proper, and the action of Δ^Γ on Z^Γ satisfies all the assumptions Proposition 9.1.2 as well.*

Proof. Though the assertion is a straightforward generalization of the argument of [KL], we sketch the argument for the convenience of the reader.

First of all, the assertion that the action of Δ^Γ on Z^Γ is a free and discrete follows from the corresponding assertion for Δ and Z. Next, replacing $Z \to Y$ by its pullback to Y^Γ, we can assume that Γ acts trivially on Y. As in Proposition 9.1.2, it is sufficient to check the assertion after pullback by fp-closed qcqs subscheme, so we assume that Y is a qcqs scheme. Choose a presentation $Z = \operatorname{colim}_\alpha Z_\alpha$. Then each $Z_i \to Y$ is fp-proper, thus the scheme of Γ-fixed points $Z_\alpha^\Gamma \subset Z_\alpha$ is a closed finitely-presented subscheme (by [GKM, Lem 15.2.1]). Hence it is fp-proper over Y as well, therefore $Z^\Gamma = \operatorname{colim}_\alpha Z_\alpha^\Gamma$ is ind-fp-proper over Y.

It remains to show that if h satisfies the assumption of Proposition 9.1.2(b), then h^Γ satisfies the assumption of Proposition 9.1.2(b) as well. By assumption, there exists a closed qcqs fp-subscheme $Z' \subset Z$ such that $Z(K) = \bigcup_{\delta\in\Delta} \delta(Z'(K))$ for all algebraically closed fields K. We want to construct a qcqs fp-closed subscheme

$Z'' \subset Z^\Gamma$ such that

$$Z^\Gamma(K) \subset \bigcup_{\delta \in \Delta^\Gamma} \delta(Z''(K)). \tag{9.1}$$

For every $\gamma \in \Gamma$, let $D_\gamma \subset \Delta$ be the set of all $\delta \in \Delta$ such that $\delta({}^\gamma Z') \cap Z' \neq \emptyset$. By assumption, every D_γ is finite. Consider the map $\phi : \Delta \to \prod_{\gamma \in \Gamma} \Delta$, defined as $\phi(\delta) = \{\delta^{-1}{}^\gamma\delta\}_{\gamma \in \Gamma}$, and set $D := \phi(\Delta) \cap \prod_{\gamma \in \Gamma} D_\gamma$. Then D is finite, hence there exists a finite subset $C \subset \Delta$ such that $\phi(C) = D$.

We claim that $Z'' := \bigcup_{\delta \in C}(\delta(Z') \cap Z^\Gamma)$ satisfied the required property. Since $Z' \subset Z$ is an fp-closed qcqs subscheme and C is finite, we conclude that $Z'' \subset Z^\Gamma$ is an fp-closed qcqs subscheme as well. It suffices to show that Z'' satisfies (9.1), or ,equivalently, that

$$Z^\Gamma(K) \subset \bigcup_{\delta \in \Delta^\Gamma \cdot C} \delta(Z'(K)).$$

But this is straightforward. Indeed, set $\Delta' := \phi^{-1}(D) \subset \Delta$. First we claim that

$$Z^\Gamma(K) \subset \bigcup_{\delta \in \Delta'} \delta(Z'(K)). \tag{9.2}$$

Indeed, for every $z \in Z^\Gamma(K) \subset Z(K) = \bigcup_{\delta \in \Delta} \delta(Z'(K))$ there exist $\delta \in \Delta$ and $z' \in Z'(K)$ such that $z = \delta(z')$. We want to show that $\delta \in \Delta'$. Since $z \in Z^\Gamma(K)$, for every $\gamma \in \Gamma$ we have $\delta(z') = {}^\gamma(\delta(z')) = {}^\gamma\delta({}^\gamma z')$, therefore $\delta^{-1}{}^\gamma\delta({}^\gamma z') = z'$. In particular, we have $\delta^{-1}{}^\gamma\delta \in D_\gamma$ for every $\gamma \in \Gamma$. Thus $\phi(\delta) \in D$, hence $\delta \in \Delta'$.

It suffices to show that

$$\Delta' = \Delta^\Gamma \cdot C. \tag{9.3}$$

For every $\delta \in \Delta'$ choose $c \in C$ such that $\phi(\delta) = \phi(c)$, and set $\delta' := \delta c^{-1}$. We claim that $\delta' \in \Delta^\Gamma$. Indeed, for every $\gamma \in \Gamma$, we have $\delta^{-1}{}^\gamma\delta = c^{-1}{}^\gamma c$, hence $\delta c^{-1} = {}^\gamma\delta{}^\gamma c^{-1}$. Thus $\delta' = {}^\gamma\delta'$, as claimed. $\square$

9.2.2. H-torsors. (a) Let H be a group ind-scheme over k acting on an ∞ stack $\mathcal{X}$. In this case, we can form the quotient $[\mathcal{X}/H] \in \mathrm{St}_k$.

(b) We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of ∞-stacks is an *H-torsor*, if f is surjective in the étale topology, and the natural map $a : H \times \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} : (h, x) \mapsto (h(x), x)$ is an isomorphism.

(c) As in the classical case, if $f : \mathcal{X} \to \mathcal{Y}$ is a H-torsor, then the morphism $\overline{f} : [\mathcal{X}/H] \to \mathcal{Y}$ induced by f is an isomorphism. Indeed the isomorphism $a : H \times \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is $(H \times H)$-equivariant, and taking quotient by $H \times H$, we get an isomorphism $[\mathcal{X}/H] \to [\mathcal{X}/H] \times_{\mathcal{Y}} [\mathcal{X}/H]$, which implies that $\overline{f}$ is an embedding, that is, $[\mathcal{X}/H](U) \to \mathcal{Y}(U)$ is an embedding of spaces for each $U \in \mathrm{Aff}_k$. On the other hand, since f is a surjective map of sheaves, we conclude that $\overline{f}$ is surjective, thus an isomorphism.

(d) As in the classical case, the quotient map $\mathcal{X} \to [\mathcal{X}/H]$ is an H-torsor, and has the property that for every $U \in \mathrm{Aff}_k$, the ∞-space $[\mathcal{X}/H](U)$ classifies pairs $(\widetilde{U}, \widetilde{\phi})$, where $\widetilde{U} \to U$ is an H-torsor, and $\widetilde{\phi} : \widetilde{U} \to \mathcal{X}$ is an H-equivariant map.

Indeed, consider $\mathcal{Y} \in \mathrm{PreSt}_k$ such that $\mathcal{Y}(U)$ satisfies pairs $(\widetilde{U}, \widetilde{\phi})$ as above. Then we have natural map $f : \mathcal{X} \to \mathcal{Y}$, which maps $h : U \to \mathcal{X}$ to $H \times U \to \mathcal{X} : (h, u) \mapsto h(\phi(u))$. One can check that $\mathcal{Y}$ is actually an ∞-stack, and f is an H-torsor. Then, by (c), the induced map $[\mathcal{X}/H] \to \mathcal{Y}$ is an isomorphism, and the assertion is proven.

Lemma 9.2.3. *Let $h : Z \to Y$ be an H-torsor between ind-schemes, and let Γ be a finite group that acts on Z, Y and by group automorphisms on H such that the map h and the action map $H \times Z \to Z$ are Γ-equivariant.*

Assume that

(1) *H can be written as a filtered limit of algebraic groups $H \simeq \lim_i H_i$, each H_i is smooth and connected, and each projection $H_{i+1} \to H_i$ is a surjective map, whose kernel K_i is a vector group.*
(2) *The action of Γ on H extends to an action of Γ on the projective system $\{H_i\}$, and the order of Γ is prime to the characteristic of k.*
(3) *The induced map $f^\Gamma : Z^\Gamma \to Y^\Gamma$ is surjective.*

Then f^Γ is an H^Γ-torsor.

Proof. By definition, the action $a : H \times Z \to Z \times_Y Z$ is an isomorphism. Therefore the induced map $H^\Gamma \times Z^\Gamma \to Z^\Gamma \times_{Y^\Gamma} Z^\Gamma$ is an isomorphism. It suffices to show that the projection $Z^\Gamma \to Y^\Gamma$ is surjective in the étale topology.

Taking base change with respect to a morphism $S \to Y^\Gamma$, where S is affine, we can assume that Y is affine and Γ acts trivially on Y.

For every i, we set $Z_i := Z \times^H H_i$. Since $Z \to Y$ is an H-torsor, and $H \simeq \lim_i H_i$, we conclude that each $Z_i \to Y$ is an H_i-torsor, and $Z \simeq \lim_i Z_i$. Taking Γ-invariants, we conclude that the action map $H_i^\Gamma \times Z_i^\Gamma \to Z_i^\Gamma \times_Y Z_i^\Gamma$ is an isomorphism.

Since $Z_i \to Y$ is a H_i-torsor, while H_i is smooth, we conclude that the projection $Z_i \to Y$ is smooth. As the order of Γ is prime to the characteristic of k, the projection $Z_i^\Gamma \to Y$ is smooth (by [GKM, 15.4.2]). Moreover, it is surjective by assumption, thus an H_i-torsor.

By construction, $Z_{i+1} \to Z_i$ is a K_i-torsor. Since $|\Gamma|$ is prime to the characteristic of k, we conclude that $H^1(\Gamma, K_i) = 0$. Hence $Z_{i+1}^\Gamma \to Z_i^\Gamma$ is surjective, so by the previous paragraph, it is a K_i^Γ-torsor.

Since K_i is a vector group, we conclude that K_i^Γ is a vector group as well, thus every K_i^Γ-torsor between affine schemes is trivial. Therefore each projection $Z_{i+1}^\Gamma \to Z_i^\Gamma$

has a section, hence the same is also holds for $Z^\Gamma \to Z_i^\Gamma$. Since $Z_i^\Gamma \to Y$ is surjective in the étale topology, the same holds for $Z^\Gamma \to Y$, and the proof is complete. $\square$

9.2.4. Loop groups on tame tori. (a) Recall that every torus S over F has a natural structure of a smooth group scheme $S_{\mathcal{O}}$ over $\mathcal{O}$, also known as the Neron model. Moreover, when S is tame, that is, split over a tamely ramified extension F'/F, it has the following explicit description. Namely, let F'/F be the splitting field of S with Galois group $\Gamma := \mathrm{Gal}(F'/F)$. Then the torus $S' := S_{F'}$ is split, thus has a natural structure $S'_{\mathcal{O}}$ over $\mathcal{O}_{F'}$, and we set $S_{\mathcal{O}} := (S'_{\mathcal{O}})^\Gamma$. Then we can define the arc group $\mathcal{L}^+(S) := \mathcal{L}^+(S_{\mathcal{O}})$.

(b) Set $\Lambda_S := X_*(S)^\Gamma$. We claim that we have a natural isomorphism

$$(\mathcal{L}\, S)_{\mathrm{red}} \simeq \mathcal{L}^+(S) \times \Lambda_S.$$

Indeed, when S is split, the assertion reduces to the case of $S = \mathbb{G}_m$, which is easy. In the general case, let $S' := S_{F'}$ be as in (a). Then $(\mathcal{L}\, S')_{\mathrm{red}} \simeq \mathcal{L}^+(S') \times \Lambda_{S'}$, by the split case. Thus, taking Γ-invariants, we get $((\mathcal{L}\, S')_{\mathrm{red}})^\Gamma \simeq \mathcal{L}^+(S')^\Gamma \times (\Lambda_{S'})^\Gamma$.

Since $\Lambda_S = (\Lambda_{S'})^\Gamma$, by definition, and $\mathcal{L}^+(S')^\Gamma \simeq \mathcal{L}^+(S)$ and $\mathcal{L}(S')^\Gamma \simeq \mathcal{L}\, S$, because loop and arc-functors commute with limits, it suffices to show that $((\mathcal{L}\, S')_{\mathrm{red}})^\Gamma \simeq ((\mathcal{L}\, S')^\Gamma)_{\mathrm{red}}$. Since $((\mathcal{L}\, S')^\Gamma)_{\mathrm{red}} \subset ((\mathcal{L}\, S')_{\mathrm{red}})^\Gamma \subset \mathcal{L}\, S'^\Gamma$, it suffices to show that $((\mathcal{L}\, S')_{\mathrm{red}})^\Gamma \simeq \mathcal{L}^+(S')^\Gamma \times (\Lambda_{S'})^\Gamma$ is reduced, or what is the same, that $\mathcal{L}^+(S')^\Gamma$ is reduced. Since $\mathcal{L}^+(S')^\Gamma \simeq \lim \mathcal{L}_n^+(S')^\Gamma$, and $|\Gamma|$ is prime to the characteristic of k, each $\mathcal{L}_n^+(S')^\Gamma$ is smooth (see [GKM, 15.4.2]), thus reduced.

9.3. **Proof of Theorem 8.1.9 and Theorem 8.3.3.** Our proof of Theorem 8.1.9 will be based on the following simple criterion.

Lemma 9.3.1. *(a) If $f : \mathcal{X} \to \mathcal{Y}$ is a topological equivalence of ∞-stacks, then $f(K) : \mathcal{X}(K) \to \mathcal{Y}(K)$ is an equivalence for every algebraically closed K.*

(b) Conversely, let $f : \mathcal{X} \to \mathcal{Y}$ be a topologically proper (see 2.3.8) morphism of ∞-stacks such that $f(K) : \mathcal{X}(K) \to \mathcal{Y}(K)$ is an equivalence for every algebraically closed field K. Then f is a topological equivalence.

Proof. (a) Since an algebraically closed field K is perfect, the assertion follows from Corollary 2.3.7(d).

(b) By Corollary 2.3.7(b), it order to show that f is a topological equivalence, it suffices to show that the induced map $f_U : U \times_{\mathcal{Y}} \mathcal{X} \to U$ is a topological equivalence. Since condition (i) for f implies that for f_U, we can assume that $\mathcal{Y} = U$ is affine. Next, since $\mathcal{X}_{\mathrm{perf}} \to \mathcal{X}$ is a topological equivalence, we can assume that $\mathcal{X}$ is an algebraic space, and f is proper. In this case, our assumption that $f(K)$ is a bijection for all K implies that f is a universal homeomorphism. In particular, f is quasi-finite, thus it follows from the Zariski Main theorem (see [St, Tag. 082K]) that f is finite. Therefore $\mathcal{X}$ is affine as well, hence f is a topological equivalence by definition. $\square$

9.3.2. *Proof of Theorem 8.1.9.* By Lemma 9.3.1, it suffices to show that ψ_S is topologically proper and induces a bijection on K-points for every algebraically closed field K. For the assertion about K-points, it suffices to show that the projection $[\mathcal{L}\,G(K)/\,\mathcal{L}\,G(K)] \to \mathcal{L}(G/S)(K)$ is bijective. Since this projection can be rewritten as $G(K((t)))/S(K((t))) \to (G/S)(K((t)))$, it is bijective, because $H^1(K((t)), S) = 1$. Thus it remains to show that ψ_S is topologically proper.

By 9.2.4(b), we have the natural isomorphism between $\Lambda_S \simeq [(\mathcal{L}\,S)_{\mathrm{red}}/\mathcal{L}^+(S)]$. In particular, the group Λ_S acts naturally on $[\mathcal{L}\,G/\mathcal{L}^+(S)]$ and we have a natural isomorphism $[\mathcal{L}\,G/(\mathcal{L}\,S)_{\mathrm{red}}] \simeq [[\mathcal{L}\,G/\mathcal{L}^+(S)]/\Lambda_S]$. Therefore Theorem 8.1.9 immediately follows from a combination of Proposition 9.1.2 and Claim 9.3.3 below. □

Claim 9.3.3. *(a) The projection $[\mathcal{L}\,G/\mathcal{L}^+(S)] \to \mathcal{L}(G/S)$ is ind-fp-proper.*

(b) The action of Λ_S on $[\mathcal{L}\,G/\mathcal{L}^+(S)]$ is free and discrete.

(c) The action of Λ_S satisfies the assumption of Proposition 9.1.2(b).

Proof. First we a going to show all assertions in the split case, and than in general.

Split case. Assume first that S is split. Replacing S by its conjugate, we can assume that $S = T$. In this case, assertion (a) was shown in Corollary 8.2.4. Next, Λ_T acts freely on $[\mathcal{L}\,G/\mathcal{L}^+(T)]$, because $\Lambda_T \cap \mathcal{L}^+(T) = \{1\}$.

Note that the presentation $\mathcal{L}\,G \simeq \mathrm{colim}_i\,\widetilde{Y}_i$ of $\mathcal{L}\,G$ (see 8.1.3(c)) induces a presentation $\mathcal{L}\,G/\mathcal{L}^+(T) \simeq \mathrm{colim}_i(\widetilde{Y}_i/\mathcal{L}^+(T))$ of $\mathcal{L}\,G/\mathcal{L}^+(T)$. Thus to show that the action is discrete, we have to check that for every i the set of $\lambda \in \Lambda$ such that $\widetilde{Y}_i \cdot \lambda \cap \widetilde{Y}_i \neq \emptyset$ is finite. This is an assertion about K-points, and is standard.

Finally, by remark 9.1.4, we have to show that for every fp-closed subscheme $Z \subset \mathcal{L}(G/T)$ there exists i such that the projection $p : \mathcal{L}\,G \to \mathcal{L}(G/T)$ satisfies $p(\widetilde{Y}_i(K)) \supseteq Z(K)$ for every K.

First, choose an fp-closed subscheme $Z' \subset \mathcal{L}(G/T)$ such that the action map $a : \mathcal{L}\,G \times \mathcal{L}(G/T) \to \mathcal{L}(G/T)$ satisfies $a(\mathcal{L}^+G \times Z) \subset Z'$. Next, let $U \subset G$ be a maximal unipotent subgroup, normalized by T. Then the map $p' := p|_U : U \to G/T$ is a closed embedding, thus $Z'' := p'^{-1}(Z') \subset \mathcal{L}\,U$ is an fp-closed qcqs subscheme. Finally, choose i such that the $\mathcal{L}^+G \cdot Z'' \subset \widetilde{Y}_i$.

We want to show that $p(\widetilde{Y}_i(K)) \supseteq Z(K)$ for every K. Choose a point

$$z \in Z(K) \subset \mathcal{L}(G/T)(K) = \mathcal{L}\,G(K)/\,\mathcal{L}\,T(K).$$

Using decomposition $\mathcal{L}\,G(K) = \mathcal{L}^+G(K)\cdot\mathcal{L}\,U(K)\cdot\mathcal{L}\,T(K)$, there exist $g \in \mathcal{L}^+(G)(K)$ and $u \in \mathcal{L}\,U(K)$ such that $p(gu) = z$. Then $p(u) = g^{-1}z \in a(\mathcal{L}^+(G) \times Z) \subset Z'$. Thus $u \in Z''$, and $gu \in \mathcal{L}^+(G) \cdot Z'' \subset \widetilde{Y}_i$.

This completes the proof in the split case.

General case. To prove the assertion in general, we set $S' := R_{F'/F}S_{F'}$ and $G' := R_{F'/F}G_{F'}$. By the split case, the map $p : [\mathcal{L}\,G'/\mathcal{L}^+(S')] \to \mathcal{L}(G'/S')$ is ind-proper, and the action of $\Lambda_{S'}$ on $[\mathcal{L}G'/\mathcal{L}^+(S')]$ satisfies all the assumptions of Proposition 9.1.2. Then the assertion follows from a combination of Lemma 9.2.1 proven above and Lemma 9.3.4 below. □

Lemma 9.3.4. *The natural embeddings* $\mathcal{L}(G/S) \to \mathcal{L}(G'/S')^\Gamma$ *and* $\mathcal{L}G/\mathcal{L}^+(S) \to (\mathcal{L}G'/\mathcal{L}^+(S'))^\Gamma$ *are isomorphisms.*

Proof. Note that $G' \to G'/S'$ is an S'-torsor. Since S' is smooth, and Γ is prime to the characteristic of k, we conclude from Lemma 9.2.3 that the induced map $(G')^\Gamma \to (G'/S')^\Gamma$ is an $(S')^\Gamma$-torsor, thus the natural map $(G')^\Gamma/(S')^\Gamma \to (G'/S')^\Gamma$ is an isomorphism. Since $G = (G')^\Gamma$ and $S = (S')^\Gamma$ we conclude that the natural map $G/S \to (G'/S')^\Gamma$ is a isomorphism. Since loop functor $\mathcal{L}$ commute with all limits, we conclude that the maps $\mathcal{L}(G/S) \to \mathcal{L}(G'/S')^\Gamma$ and $\mathcal{L}G \to \mathcal{L}(G')^\Gamma$ are isomorphisms.

The proof of the second assertion is similar. Namely, the map $\mathcal{L}G' \to \mathcal{L}G'/\mathcal{L}^+(S')$ is an $\mathcal{L}^+(S')$-torsor, and $\mathcal{L}^+(S') \simeq \lim_n \mathcal{L}_n^+(S')$. Now Γ naturally acts on every $\mathcal{L}_n^+(S')$, and Lemma 9.2.3 applies. Therefore we conclude that the map $(\mathcal{L}G')^\Gamma \to (\mathcal{L}G'/\mathcal{L}^+(S'))^\Gamma$ is a $\mathcal{L}^+(S')^\Gamma$-torsor. Hence the map $(\mathcal{L}G')^\Gamma/\mathcal{L}^+(S')^\Gamma \to (\mathcal{L}G'/\mathcal{L}^+(S'))^\Gamma$ is an isomorphism, thus the map $\mathcal{L}G/\mathcal{L}^+(S) \to (\mathcal{L}G'/\mathcal{L}^+(S'))^\Gamma$ is an isomorphism, as claimed. □

Corollary 9.3.5. *For every fp-closed qcqs subscheme* $Z \subset \mathcal{L}(G/S)$*, there exists* i *such that for every algebraically closed field* K *the projection* $\mathrm{pr}_S : \mathcal{L}G \to \mathcal{L}(G/S)$ *satisfies* $\mathrm{pr}_S^{-1}(Z)(K) \subset \widetilde{Y}_i(K) \cdot \Lambda_S$*, thus* $\mathrm{pr}_S(\widetilde{Y}_i(K)) \supseteq Z(K)$*.*

Proof. Note that pr_S decomposes as $\mathcal{L}G \overset{\alpha}{\to} \mathcal{L}G/\mathcal{L}^+S \overset{\beta}{\to} \mathcal{L}(G/S)$. By Claim 9.3.3(c), there exists an fp-closed qcqs subscheme $Z' \subset \mathcal{L}G/\mathcal{L}^+S$ such that $\beta^{-1}(Z(K)) \subset \Lambda_S \cdot Z'(K)$ for every K. Set $Z'' := \alpha^{-1}(Z') \subset \mathcal{L}G$. Then Z'' is an fp-closed qcqs subscheme, thus $Z'' \subset \widetilde{Y}_i$ for some i. By construction, we have

$$\mathrm{pr}_S^{-1}(Z(K)) = \alpha^{-1}\beta^{-1}(Z(K)) \subset \alpha^{-1}(\Lambda_S \cdot Z'(K)) = \Lambda_S \cdot Z''(K) \subset \Lambda_S \cdot \widetilde{Y}_i(K).$$

□

9.3.6. *Proof of Theorem 8.3.3.* We are going to show that all assumptions of Proposition 9.1.2 are satisfied. Recall that the projection $\mathfrak{p} : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ is ind-fp-proper (by Lemma 8.1.4). Taking pullback to $\mathfrak{C}_{\mathrm{t},w,\mathbf{r}}$, we conclude that the map $\mathfrak{p}_{\mathrm{t},w,\mathbf{r}} : \widetilde{\mathfrak{C}}_{\mathrm{t},w,\mathbf{r}} \to \mathfrak{C}_{\mathrm{t},w,\mathbf{r}}$ is ind-fp-proper.

Next we claim that the action of Λ_w on $\widetilde{\mathfrak{C}}_{\mathrm{t},w,\mathbf{r}}$ is discrete. Note that the presentation $\mathfrak{Fl} = \mathrm{colim}_i Y_i$ from 8.1.3(b) gives rise to a presentation $\widetilde{\mathfrak{C}}_{\mathrm{t},w,\mathbf{r}} = \mathrm{colim}_i \widetilde{\mathfrak{C}}_{\mathrm{t},w,\mathbf{r},i}$, where $\widetilde{\mathfrak{C}}_{\mathrm{t},w,\mathbf{r},i} \subset \widetilde{\mathfrak{C}}_{\mathrm{t},w,\mathbf{r}}$ consists of triples (g, h, x) such that $g \in Y_i$. Thus it suffices to show that for every i, the set $\lambda \in \Lambda_w$ such that $\lambda(\widetilde{\mathfrak{C}}_{\mathrm{t},w,\mathbf{r},i}) \cap \widetilde{\mathfrak{C}}_{\mathrm{t},w,\mathbf{r},i} \neq \emptyset$ is finite.

By definition, for every such λ there exist $g \in Y_i$ and $h \in \mathcal{L}(G/T_w)$ such that $g' := (h\lambda h^{-1})g \in Y_i$. Choose representatives $\widetilde{g} \in \widetilde{Y}_i$ of g and $\widetilde{h} \in \mathcal{L}G$ of h. Then

$\widetilde{g}' := (\widetilde{h}\lambda\widetilde{h}^{-1})\widetilde{g} \in \widetilde{Y}_i$, thus $\widetilde{h}\lambda\widetilde{h}^{-1} = \widetilde{g}'\widetilde{g}^{-1} \in \widetilde{Y}_i\widetilde{Y}_i^{-1}$. Then the conjugacy class of such λ's in $\mathcal{L}G$ is bounded, thus the set of such λ's is finite.

To show that that Λ_w acts freely, notice that λ has a fixed point if and only if $h\lambda h^{-1}g \in gI$, that is, $\lambda \in (h^{-1}g)I(h^{-1}g)^{-1}$, that is, $\lambda \in \Lambda_w \cap (h^{-1}g)I(h^{-1}g)^{-1}$. But the latter intersection is torsion free, discrete and bounded, thus trivial.

Thus, the conditions of Proposition 9.1.2 are satisfied, hence $\overline{\mathfrak{C}}_{w,\mathbf{r}}$ is an ind-algebraic space, ind-fp-proper over $\mathfrak{C}_{\mathfrak{t},w,\mathbf{r}}$.

To show that it is topologically proper, we have to check that the condition of Proposition 9.1.2(b) is satisfied as well, that is, for every fp-closed subscheme Z of $\mathfrak{C}_{\mathfrak{t},w,\mathbf{r}} \simeq \mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}}$, there exists i such that

(9.4)

$$\text{for each } (g,h,x) \in \mathfrak{f}_{\mathfrak{t},w,\mathbf{r}}^{-1}(Z)(K) \text{ there exists } \lambda \in \Lambda_w \text{ such that } (h\lambda h^{-1})g \in Y_i.$$

Recall (see Lemma 8.1.8(a)) that the action map $(g,x) \mapsto (\operatorname{Ad} g)(x)$ induces a finite map $a : \mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}} \simeq \mathfrak{C}_{\mathfrak{t},w,\mathbf{r}} \to \mathfrak{C}_{w,\mathbf{r}}$. Therefore $a^{-1}(\operatorname{Lie} I) \subset \mathcal{L}(G/T_w) \times \mathfrak{t}_{w,\mathbf{r}}$ is an fp-closed qcqs subscheme, thus there exists fp-closed qcqs subschemes $Z_1, Z_2 \subset \mathcal{L}(G/T_w)$ such that $Z \subset Z_1 \times \mathfrak{t}_{w,\mathbf{r}}$ and $a^{-1}(\operatorname{Lie} I) \subset Z_2 \times \mathfrak{t}_{w,\mathbf{r}}$.

By Corollary 9.3.5, there exist indexes i_1, i_2 such that $\operatorname{pr}_{T_w}(\widetilde{Y}_{i_1}(K)) \supseteq Z_1(K)$ and $\operatorname{pr}_{T_w}^{-1}(Z_2(K)) \subset \widetilde{Y}_{i_2}(K) \cdot \Lambda_w$. We claim that every index i such that $\widetilde{Y}_{i_1} \cdot \widetilde{Y}_{i_2}^{-1} \subset \widetilde{Y}_i$ satisfies (9.4).

Indeed, choose a representative $\widetilde{g} \in \mathcal{L}G(K)$ of g. Since $(h,x) \in Z \subset Z_1 \times \mathfrak{t}_{w,\mathbf{r}}$, there exists a representative $\widetilde{h} \in \widetilde{Y}_{i_1}(K)$ of h. Since $(g,h,x) \in \widetilde{\mathfrak{C}}$, we have $\operatorname{Ad}(\widetilde{g}^{-1}\widetilde{h})(x) \in \operatorname{Lie}(I)$. Hence $\operatorname{pr}_{T_w}(\widetilde{g}^{-1}\widetilde{h}) \in Z_2$, therefore there exists $\lambda \in \Lambda_w$ such that $\widetilde{g}^{-1}\widetilde{h}\lambda^{-1} \in \widetilde{Y}_{i_2}$. Then $(\widetilde{h}\lambda\widetilde{h}^{-1})\widetilde{g} = \widetilde{h}(\widetilde{g}^{-1}\widetilde{h}\lambda^{-1})^{-1}$ belongs to $\widetilde{Y}_{i_1}\widetilde{Y}_{i_2}^{-1} \subset \widetilde{Y}_i$, hence $(h\lambda h^{-1})g \in Y_i$, as claimed. □

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