Dynamics and prime solutions to linear equations

Tamar Ziegler

Technion/Hebrew University

EMS Lecture Series

Denote by $\mathbb{P}$ the set of prime numbers $\mathbb{P} = \{2, 3, 5, \ldots\}$.

Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear forms in n variables

$$\psi_i(\vec{x}) = \vec{a}_i \cdot \vec{x} + b_i, \qquad \vec{a}_i \in \mathbb{Z}^n, b_i \in \mathbb{Z}$$

Denote by $\mathbb{P}$ the set of prime numbers $\mathbb{P} = \{2, 3, 5, \ldots\}$.

Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear forms in n variables

$$\psi_i(\vec{x}) = \vec{a}_i \cdot \vec{x} + b_i, \qquad \vec{a}_i \in \mathbb{Z}^n, b_i \in \mathbb{Z}$$

Question:

Are there $\vec{x} \in \mathbb{Z}^n$ such $\psi_1(\vec{x}), \ldots, \psi_k(\vec{x}) \in \mathbb{P}$?

Denote by $\mathbb{P}$ the set of prime numbers $\mathbb{P} = \{2, 3, 5, \ldots\}$.

Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear forms in n variables

$$\psi_i(\vec{x}) = \vec{a}_i \cdot \vec{x} + b_i, \qquad \vec{a}_i \in \mathbb{Z}^n, b_i \in \mathbb{Z}$$

Question:

Are there $\vec{x} \in \mathbb{Z}^n$ such $\psi_1(\vec{x}), \ldots, \psi_k(\vec{x}) \in \mathbb{P}$?

Infinitely many such $\vec{x}$? Asymptotics?

$\psi(x) = ax + b$

$k = n = 1$: If $a = 1, b = 0$, then

$$\psi(x) = x.$$

$\psi(x) = ax + b$

$k = n = 1$: If $a = 1, b = 0$, then

$$\psi(x) = x.$$

Theorem (Euclid ($\sim$ 300 BC))

There are ∞ many primes.

$\psi(x) = ax + b$

$k = n = 1$: If $a = 1, b = 0$, then

$$\psi(x) = x.$$

Theorem (Euclid ($\sim$ 300 BC)) 

There are ∞ many primes.

Over 2000 years later ...

Theorem (Hadamard, de la Vallée-Poussin (1896))  

$$\pi(N) = |\{x \in \mathbb{P}, x \leq N\}| \sim \frac{N}{\log N}$$

More generally:

$$\psi(x) = ax + b$$

More generally:

$$\psi(x) = ax + b$$

Theorem (Johann Peter Gustav Lejeune Dirichlet (1837))

There are ∞ many primes of the form $ax + b$
$\iff (a, b) = 1$ (no local obstructions)

More generally:

$$\psi(x) = ax + b$$

Theorem (Johann Peter Gustav Lejeune Dirichlet (1837))

There are ∞ many primes of the form $ax + b$ $\iff$ $(a, b) = 1$ (no local obstructions)

Prime number theorem in arithemtic progression

Each legal arithmetic progression gets its fair share: if $(a, b) = 1$, then

$$\pi(N, a, b) = |\{x \in \mathbb{P}, x \leq N, x \equiv b \mod a\}| \sim \frac{1}{\phi(a)} \frac{N}{\log N}$$

$k = 2, n = 1$

$$\psi_1(x) = x \qquad \psi_2(x) = x + 2$$

$k = 2, n = 1$

$$\psi_1(x) = x \qquad \psi_2(x) = x + 2$$

This is the "twin prime conjecture" !

$k = 2, n = 1$

$$\psi_1(x) = x \qquad \psi_2(x) = x + 2$$

This is the "twin prime conjecture" !

Nothing to report on this ...

$k = 2, n = 1$

$$\psi_1(x) = x \qquad \psi_2(x) = x + 2$$

This is the "twin prime conjecture" !

~~Nothing to report on this~~ ...

$k = 2, n = 1$

$$\psi_1(x) = x \qquad \psi_2(x) = x + 2$$

This is the "twin prime conjecture" !

~~Nothing to report on this~~ ...

News Flash: Peter Woit's blog May 12: Yitang Zhang proved

$$\psi_1(x) = x \qquad \psi_2(x) = x + M \qquad M < 70,000,000 \quad !!!$$

arithmetic progressions

$n = 2$, k arbitrary

$$\psi_1(x) = x, \quad \psi_2(x) = x + d, \quad \dots \quad , \psi_k(x) = x + (k-1)d$$

This is an k-term arithmetic progression

$n = 2,\ k$ arbitrary

$$\psi_1(x) = x, \quad \psi_2(x) = x + d, \quad \ldots \quad , \psi_k(x) = x + (k-1)d$$

This is an k-term arithmetic progression

Theorem (Green-Tao (2004))

The primes contain arbitrarily long arithmetic progressions.

The proof gives a lower bound of the correct order of magnitude $(C\frac{N^2}{(\log N)^k})$.

In 2006 Green-Tao proved the following conditional theorem:

In 2006 Green-Tao proved the following conditional theorem:

Conditional Multidimentional Dirichlet

Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear integer forms in n variables
Suppose no 2 forms are affinely dependent. Then

$$\{\psi_i(\vec{x})\}_{i=1}^{k} \subset \mathbb{P} \quad \text{infinitely often} \iff \text{No local obstructions}$$

In 2006 Green-Tao proved the following conditional theorem:

Conditional Multidimentional Dirichlet

Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear integer forms in n variables
Suppose no 2 forms are affinely dependent. Then

$$\{\psi_i(\vec{x})\}_{i=1}^{k} \subset \mathbb{P} \quad \text{infinitely often} \iff \text{No local obstructions}$$

$$\left|\{\vec{x} \in [0, N]^n, \{\psi_i(\vec{x})\}_{i=1}^{k} \subset \mathbb{P}\}\right| \sim \mathfrak{S}(\vec{\psi})\frac{N^n}{(\log N)^k}$$

In 2006 Green-Tao proved the following conditional theorem:

Conditional Multidimentional Dirichlet

Let $\{\psi_i(\vec{x})\}_{i=1}^k$ be k affine linear integer forms in n variables
Suppose no 2 forms are affinely dependent. Then

$$\{\psi_i(\vec{x})\}_{i=1}^k \subset \mathbb{P} \quad \text{infinitely often} \iff \text{No local obstructions}$$

$$\left|\{\vec{x} \in [0,N]^n, \{\psi_i(\vec{x})\}_{i=1}^k \subset \mathbb{P}\}\right| \sim \mathfrak{S}(\vec{\psi})\frac{N^n}{(\log N)^k}$$

Conditioned on 2 conjectures: $MN(s)$, and $GI(s)$.

In 2006 Green-Tao proved the following conditional theorem:

Conditional Multidimentional Dirichlet

Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear integer forms in n variables Suppose no 2 forms are affinely dependent. Then

$$\{\psi_i(\vec{x})\}_{i=1}^{k} \subset \mathbb{P} \quad \text{infinitely often} \iff \text{No local obstructions}$$

$$\left|\{\vec{x} \in [0,N]^n, \{\psi_i(\vec{x})\}_{i=1}^{k} \subset \mathbb{P}\}\right| \sim \mathfrak{S}(\vec{\psi}) \frac{N^n}{(\log N)^k}$$

Conditioned on 2 conjectures: $MN(s)$, and $GI(s)$.

No local obstructions:

- mod p: for any prime p, there exists $\vec{x} \in \mathbb{Z}^n$ such that $\psi_i(\vec{x})$ is coprime to p all i.

In 2006 Green-Tao proved the following conditional theorem:

Conditional Multidimentional Dirichlet

Let $\{\psi_i(\vec{x})\}_{i=1}^k$ be k affine linear integer forms in n variables
Suppose no 2 forms are affinely dependent. Then

$$\{\psi_i(\vec{x})\}_{i=1}^k \subset \mathbb{P} \quad \text{infinitely often} \iff \text{No local obstructions}$$

$$\left|\{\vec{x} \in [0,N]^n, \{\psi_i(\vec{x})\}_{i=1}^k \subset \mathbb{P}\}\right| \sim \mathfrak{S}(\vec{\psi})\frac{N^n}{(\log N)^k}$$

Conditioned on 2 conjectures: $MN(s)$, and $GI(s)$.

No local obstructions:

- mod p: for any prime p, there exists $\vec{x} \in \mathbb{Z}^n$ such that $\psi_i(\vec{x})$ is coprime to p all i.
- mod ∞: there exist infinitely many $\vec{x} \in \mathbb{Z}^n$ such that $\psi_i(\vec{x})$ is positive.

In 2006 Green-Tao proved the following conditional theorem:

Conditional Multidimentional Dirichlet

Let $\{\psi_i(\vec{x})\}_{i=1}^k$ be k affine linear integer forms in n variables
Suppose no 2 forms are affinely dependent. Then

$$\{\psi_i(\vec{x})\}_{i=1}^k \subset \mathbb{P} \quad \text{infinitely often} \iff \text{No local obstructions}$$

$$\left|\{\vec{x} \in [0,N]^n, \{\psi_i(\vec{x})\}_{i=1}^k \subset \mathbb{P}\}\right| \sim \mathfrak{S}(\vec{\psi})\frac{N^n}{(\log N)^k}$$

Conditioned on 2 conjectures: $MN(s)$, and $GI(s)$.

No local obstructions:

- mod p: for any prime p, there exists $\vec{x} \in \mathbb{Z}^n$ such that $\psi_i(\vec{x})$ is coprime to p all i.
- mod ∞: there exist infinitely many $\vec{x} \in \mathbb{Z}^n$ such that $\psi_i(\vec{x})$ is positive.

Remark: In solving linear equations in primes variables, the name of the game is

$\#$ equations versus $\#$ variables

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.
- 1 equation 3 variables, e.g the Ternary Goldbach Theorem: every odd number > 5 is a sum of 3 primes:

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.
- 1 equation 3 variables, e.g the Ternary Goldbach Theorem: every odd number > 5 is a sum of 3 primes: proved by Vinogradov (1937) for large enough numbers,

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.
- 1 equation 3 variables, e.g the Ternary Goldbach Theorem: every odd number > 5 is a sum of 3 primes: proved by Vinogradov (1937) for large enough numbers, very (very) recently proved by Helfgott (13 May 2013) !

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.
- 1 equation 3 variables, e.g the Ternary Goldbach Theorem: every odd number > 5 is a sum of 3 primes: proved by Vinogradov (1937) for large enough numbers, very (very) recently proved by Helfgott (13 May 2013) ! or $x_1 + x_3 = 2x_2$ (3 term progressions) proved by Van der Corput (1939). Both proved using the Hardy Littlewood Circle Method.

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.
- 1 equation 3 variables, e.g the Ternary Goldbach Theorem: every odd number > 5 is a sum of 3 primes: proved by Vinogradov (1937) for large enough numbers, very (very) recently proved by Helfgott (13 May 2013) ! or $x_1 + x_3 = 2x_2$ (3 term progressions) proved by Van der Corput (1939). Both proved using the Hardy Littlewood Circle Method.
- Some special cases of k equations $f(k)$ variables - Balog (1992) using the HL circle method.

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.
- 1 equation 3 variables, e.g the Ternary Goldbach Theorem: every odd number > 5 is a sum of 3 primes: proved by Vinogradov (1937) for large enough numbers, very (very) recently proved by Helfgott (13 May 2013) ! or $x_1 + x_3 = 2x_2$ (3 term progressions) proved by Van der Corput (1939). Both proved using the Hardy Littlewood Circle Method.
- Some special cases of k equations $f(k)$ variables - Balog (1992) using the HL circle method.
- 2 equation 4 variables, e.g $x_1 + x_3 = 2x_2$; $x_2 + x_4 = 2x_3$. This is 4 term progressions - Green-Tao (2006).

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.
- 1 equation 3 variables, e.g the Ternary Goldbach Theorem: every odd number > 5 is a sum of 3 primes: proved by Vinogradov (1937) for large enough numbers, very (very) recently proved by Helfgott (13 May 2013) ! or $x_1 + x_3 = 2x_2$ (3 term progressions) proved by Van der Corput (1939). Both proved using the Hardy Littlewood Circle Method.
- Some special cases of k equations $f(k)$ variables - Balog (1992) using the HL circle method.
- 2 equation 4 variables, e.g $x_1 + x_3 = 2x_2$; $x_2 + x_4 = 2x_3$. This is 4 term progressions - Green-Tao (2006).
- k equation $n \geq k + 2$ variables - Conditional Multidimentional Dirichlet.

Remark: In solving linear equations in primes variables, the name of the game is

equations versus # variables

The more variables the easier the question is.

- 1 equation 2 variables, e.g $x_1 - x_2 = 2$. Twin primes.
- 1 equation 3 variables, e.g the Ternary Goldbach Theorem: every odd number > 5 is a sum of 3 primes: proved by Vinogradov (1937) for large enough numbers, very (very) recently proved by Helfgott (13 May 2013) ! or $x_1 + x_3 = 2x_2$ (3 term progressions) proved by Van der Corput (1939). Both proved using the Hardy Littlewood Circle Method.
- Some special cases of k equations $f(k)$ variables - Balog (1992) using the HL circle method.
- 2 equation 4 variables, e.g $x_1 + x_3 = 2x_2$; $x_2 + x_4 = 2x_3$. This is 4 term progressions - Green-Tao (2006).
- k equation $n \geq k + 2$ variables - Conditional Multidimentional Dirichlet. Best possible dependence excluding "twin prime case".

Conditional Multidimentional Dirichlet Theorem

Conditioned on $MN(s)$, and $GI(s)$, one can calculate the asymptotic number of prime solutions to any system of k linear equations with integer coefficients in at least $k+2$ variables.

Conditional Multidimentional Dirichlet Theorem

Conditioned on $MN(s)$, and $GI(s)$, one can calculate the asymptotic number of prime solutions to any system of k linear equations with integer coefficients in at least $k+2$ variables.

Remark: The method used by Green and Tao to show the existence of arithmetic progressions in primes can not be used to establish asymptotics or to handle non homogeneous equations, since it relies on Szemeredi's theorem which is invalid for non homogeneous equations, and can't provide asymptotics.

$MN(s)$

What is $MN(s)$?

$MN(s)$

What is $MN(s)$? The Möbius Nilsequences conjecture.

$MN(s)$

What is $MN(s)$? The Möbius Nilsequences conjecture.

Consider the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{, where } p_i \text{ are distinct primes;} \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius function is related to the normalized prime counting function $\Lambda(n)$ via an identity arising from the möbius inversion formula.

$MN(s)$

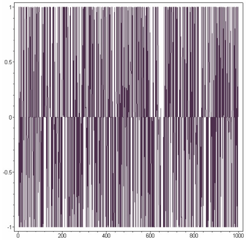
What is $MN(s)$? The Möbius Nilsequences conjecture.

Consider the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{, where } p_i \text{ are distinct primes;} \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius function is related to the normalized prime counting function $\Lambda(n)$ via an identity arising from the möbius inversion formula.

$$\left| \sum_{n=1}^{N} \mu(n) \right| \ll_A N(\log N)^{-A}$$



$MN(s)$

Theorem (Davenport (1930s))

For any $\alpha \in [0,1]$

$$\left|\sum_{n=1}^{N} \mu(n) e^{2\pi i n\alpha}\right| \ll_A N(\log N)^{-A}$$

$MN(s)$

Theorem (Davenport (1930s))

For any $\alpha \in [0,1]$

$$\left| \sum_{n=1}^{N} \mu(n) e^{2\pi i n \alpha} \right| \ll_A N(\log N)^{-A}$$

By the same method: for any polynomial P

$$\left| \sum_{n=1}^{N} \mu(n) e^{2\pi i P(n)} \right| \ll_A N(\log N)^{-A}$$

We will see soon, that for our purposes, we need similar estimates for bracket polynomials. Examples:

$$n^2\alpha,\ n\alpha\{n\beta\},\ n\alpha\{n^2\beta\},\ n\alpha\{\{n\beta\}n\gamma\}$$

We will see soon, that for our purposes, we need similar estimates for bracket polynomials. Examples:

$$n^2\alpha,\ n\alpha\{n\beta\},\ n\alpha\{n^2\beta\},\ n\alpha\{\{n\beta\}n\gamma\}$$

Möbius Nilsequence Conjecture ($MN(s)$)

For any bracket polynomial P.

$$\left|\sum_{n=1}^{N} \mu(n) e^{2\pi i P(n)}\right| \ll_A N(\log N)^{-A}$$

μ does not correlate with bracket polynomial phase functions !

We will see soon, that for our purposes, we need similar estimates for bracket polynomials. Examples:

$$n^2\alpha,\ n\alpha\{n\beta\},\ n\alpha\{n^2\beta\},\ n\alpha\{\{n\beta\}n\gamma\}$$

Möbius Nilsequence Conjecture ($MN(s)$)

For any bracket polynomial P.

$$\left|\sum_{n=1}^{N}\mu(n)e^{2\pi i P(n)}\right| \ll_A N(\log N)^{-A}$$

μ does not correlate with bracket polynomial phase functions !

Who is s ? How is this related to nilsequences (what are they)?

We will see soon, that for our purposes, we need similar estimates for bracket polynomials. Examples:

$$n^2\alpha,\ n\alpha\{n\beta\},\ n\alpha\{n^2\beta\},\ n\alpha\{\{n\beta\}n\gamma\}$$

Möbius Nilsequence Conjecture ($MN(s)$)

For any bracket polynomial P.

$$\left|\sum_{n=1}^{N} \mu(n) e^{2\pi i P(n)}\right| \ll_A N(\log N)^{-A}$$

μ does not correlate with bracket polynomial phase functions !

Who is s ? How is this related to nilsequences (what are they)?

Theorem (Green-Tao (2007))

Möbius Nilsequence Conjecture is true.

$GI(s)$

What is $GI(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

$GI(s)$

What is $GI(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

What are Gowers norms?

$GI(s)$

What is $GI(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

What are Gowers norms? Let $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and let $f : \mathbb{Z}_N \to \mathbb{D}$.

Discrete differentiation

Let $h \in \mathbb{Z}_N$, define the derivative in direction h to be

$$\Delta_h f(n) := f(n+h)\overline{f(n)}$$

$GI(s)$

What is $GI(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

What are Gowers norms? Let $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and let $f : \mathbb{Z}_N \to \mathbb{D}$.

Discrete differentiation

Let $h \in \mathbb{Z}_N$, define the <u>derivative in direction h</u> to be

$$\Delta_h f(n) := f(n+h)\overline{f(n)}$$

Examples:

- $\Delta_h f(n) \equiv 1$ for all $h \in \mathbb{Z}_N$ if and only if $f(n) \equiv C$.

$GI(s)$

What is $GI(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

What are Gowers norms? Let $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and let $f : \mathbb{Z}_N \to \mathbb{D}$.

Discrete differentiation

Let $h \in \mathbb{Z}_N$, define the derivative in direction h to be

$$\Delta_h f(n) := f(n+h)\overline{f(n)}$$

Examples:

- $\Delta_h f(n) \equiv 1$ for all $h \in \mathbb{Z}_N$ if and only if $f(n) \equiv C$.
- $\Delta_{h_2}\Delta_{h_1} f(n) \equiv 1$ for all $h_1, h_2 \in \mathbb{Z}_N$ if and only if f is a linear phase polynomial, i.e. $f(n) = e^{2\pi i P(n)}$, where P is a linear polynomial.

$GI(s)$

Define the Gowers uniformity norms as follows: Let $f : \mathbb{Z}_N \to \mathbb{D}$.

Gowers norms

$$\|f\|_{U^s[N]}^{2^s} = \frac{1}{N^{s+1}} \sum_{n,h_1,\ldots,h_s \in \mathbb{Z}_N} \Delta_{h_s} \cdots \Delta_{h_1} f(n)$$

For $s > 1$ this is a norm.

$$\|f\|_{U^s[N]}^{2^s} = \frac{1}{N^{s+1}} \sum_{n,h_1,\ldots,h_s \in \mathbb{Z}_N} \Delta_{h_s} \cdots \Delta_{h_1} f(n)$$

Remarks:

- If $\|f\|_{U^s[N]} = 1$ then $\Delta_{h_s} \cdots \Delta_{h_1} f(n) \equiv 1$ for all $h_1, \ldots, h_k \in \mathbb{Z}_N$ thus f is a phase polynomial of degree $< s$.

$$\|f\|_{U^s[N]}^{2^s} = \frac{1}{N^{s+1}} \sum_{n,h_1,\ldots,h_s \in \mathbb{Z}_N} \Delta_{h_s} \cdots \Delta_{h_1} f(n)$$

Remarks:

- If $\|f\|_{U^s[N]} = 1$ then $\Delta_{h_s} \cdots \Delta_{h_1} f(n) \equiv 1$ for all $h_1, \ldots, h_k \in \mathbb{Z}_N$ thus f is a phase polynomial of degree $< s$.
- Conversely, if f is a phase polynomial of degree $< s$, then $\|f\|_{U^s[N]} = 1$.

$$\|f\|_{U^s[N]}^{2^s} = \frac{1}{N^{s+1}} \sum_{n,h_1,\ldots,h_s \in \mathbb{Z}_N} \Delta_{h_s} \cdots \Delta_{h_1} f(n)$$

Remarks:

- If $\|f\|_{U^s[N]} = 1$ then $\Delta_{h_s} \cdots \Delta_{h_1} f(n) \equiv 1$ for all $h_1, \ldots, h_k \in \mathbb{Z}_N$ thus f is a phase polynomial of degree $< s$.
- Conversely, if f is a phase polynomial of degree $< s$, then $\|f\|_{U^s[N]} = 1$.
- If f correlates with a polynomial phase function of degree $< s$ then $\|f\|_{U^s[N]} \gg_\delta 1$.

$$\|f\|_{U^s[N]}^{2^s} = \frac{1}{N^{s+1}} \sum_{n,h_1,\ldots,h_s \in \mathbb{Z}_N} \Delta_{h_s} \cdots \Delta_{h_1} f(n)$$

Remarks:

- If $\|f\|_{U^s[N]} = 1$ then $\Delta_{h_s} \cdots \Delta_{h_1} f(n) \equiv 1$ for all $h_1, \ldots, h_k \in \mathbb{Z}_N$ thus f is a phase polynomial of degree $< s$.
- Conversely, if f is a phase polynomial of degree $< s$, then $\|f\|_{U^s[N]} = 1$.
- If f correlates with a polynomial phase function of degree $< s$ then $\|f\|_{U^s[N]} \gg_\delta 1$.
- If f is a random function then $\|f\|_{U^s[N]} = o(1)$.

Recall:

Recall:

- If f correlates with a polynomial phase function of degree $< s$ then $\|f\|_{U^s[N]} \gg_\delta 1$.
- If f is a random function then $\|f\|_{U^s[N]} = o(1)$.

Recall:

- If f correlates with a polynomial phase function of degree $< s$ then $\|f\|_{U^s[N]} \gg_\delta 1$.
- If f is a random function then $\|f\|_{U^s[N]} = o(1)$.

The inverse question

What can we say about f if $\|f\|_{U^s[N]} \gg_\delta 1$?

Recall:

- If f correlates with a polynomial phase function of degree $< s$ then $\|f\|_{U^s[N]} \gg_\delta 1$.
- If f is a random function then $\|f\|_{U^s[N]} = o(1)$.

The inverse question

What can we say about f if $\|f\|_{U^s[N]} \gg_\delta 1$? Does f correlate with a polynomial phase function?

Recall:

- If f correlates with a polynomial phase function of degree $< s$ then $\|f\|_{U^s[N]} \gg_\delta 1$.
- If f is a random function then $\|f\|_{U^s[N]} = o(1)$.

The inverse question

What can we say about f if $\|f\|_{U^s[N]} \gg_\delta 1$? Does f correlate with a polynomial phase function?

We will get back to this question soon ...

Why are Gowers norms important?

Why are Gowers norms important?

Consider the case of 3 term progressions: Let $E \subset \mathbb{Z}_N$ be of size δN for some $\delta > 0$. Let's try to count 3 term progressions in E. Let $1_E(x)$ be the characteristic function of the set E.

Why are Gowers norms important?

Consider the case of 3 term progressions: Let $E \subset \mathbb{Z}_N$ be of size δN for some $\delta > 0$. Let's try to count 3 term progressions in E. Let $1_E(x)$ be the characteristic function of the set E.

Here is a counting expression

$$\sum_{x,d \in \mathbb{Z}_N} 1_E(x) 1_E(x+d) 1_E(x+2d)$$

Why are Gowers norms important?

Consider the case of 3 term progressions: Let $E \subset \mathbb{Z}_N$ be of size δN for some $\delta > 0$. Let's try to count 3 term progressions in E. Let $1_E(x)$ be the characteristic function of the set E.

Here is a counting expression

$$\sum_{x,d \in \mathbb{Z}_N} 1_E(x) 1_E(x+d) 1_E(x+2d)$$

An observation of Gowers: If $\|1_E - \delta\|_{U_2[N]}$ is small then

$$\sum_{x,d \in \mathbb{Z}_N} 1_E(x) 1_E(x+d) 1_E(x+2d) \sim \delta^3 N^2$$

Why are Gowers norms important?

Consider the case of 3 term progressions: Let $E \subset \mathbb{Z}_N$ be of size δN for some $\delta > 0$. Let's try to count 3 term progressions in E. Let $1_E(x)$ be the characteristic function of the set E.

Here is a counting expression

$$\sum_{x,d \in \mathbb{Z}_N} 1_E(x) 1_E(x+d) 1_E(x+2d)$$

An observation of Gowers: If $\|1_E - \delta\|_{U_2[N]}$ is small then

$$\sum_{x,d \in \mathbb{Z}_N} 1_E(x) 1_E(x+d) 1_E(x+2d) \sim \delta^3 N^2$$

This is the number of 3 term progressions we expect to find in a random subset of $\mathbb{Z}_N$ of size δN !

Why are Gowers norms important?

$\|1_E - \delta\|_{U^2[N]}$ small implies that the number of 3 term progressions is as in a random set.

Why are Gowers norms important?

$\|1_E - \delta\|_{U^2[N]}$ small implies that the number of 3 term progressions is as in a random set.

There is a similar relation between $U^s[N]$ and $s+1$ term progressions.

Why are Gowers norms important?

$\|1_E - \delta\|_{U^2[N]}$ small implies that the number of 3 term progressions is as in a random set.

There is a similar relation between $U^s[N]$ and $s+1$ term progressions.

$\|1_E - \delta\|_{U^s[N]}$ small implies that the number of s+1 term progressions is as in a random set.

Why are Gowers norms important?

$\|1_E - \delta\|_{U^2[N]}$ small implies that the number of 3 term progressions is as in a random set.

There is a similar relation between $U^s[N]$ and $s+1$ term progressions.

$\|1_E - \delta\|_{U^s[N]}$ small implies that the number of s+1 term progressions is as in a random set.

$$\sum_{x,d \in \mathbb{Z}_N} 1_E(x) 1_E(x+h) \ldots 1_E(x+sh) \sim \delta^{s+1} N^2$$

Why are Gowers norms important?

Even more generally: Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear integer forms in n variables Suppose no 2 forms are affinely dependent.

Why are Gowers norms important?

Even more generally: Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear integer forms in n variables Suppose no 2 forms are affinely dependent.

Then there is some integer s such that if $\|1_E - \delta\|_{U^s[N]}$ small then

$$\sum_{\vec{x}\in\mathbb{Z}_N^n} 1_E(\psi_1(\vec{x}))1_E(\psi_2(\vec{x}))\dots 1_E(\psi_k(\vec{x})) \sim \delta^k N^n$$

Why are Gowers norms important?

Even more generally: Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear integer forms in n variables Suppose no 2 forms are affinely dependent.

Then there is some integer s such that if $\|1_E - \delta\|_{U^s[N]}$ small then

$$\sum_{\vec{x}\in\mathbb{Z}_N^n} 1_E(\psi_1(\vec{x}))1_E(\psi_2(\vec{x}))\dots 1_E(\psi_k(\vec{x})) \sim \delta^k N^n$$

So it is REALLY important to find a good way to test whether $\|f\|_{U^s}$ is small.

The inverse question

Well ... this brings us back to the inverse question:

The inverse question ($GI(s)$)

What can we say about f, if $\|f\|_{U^{s+1}[N]} \gg_\delta 1$?

The inverse question

Well ... this brings us back to the inverse question:

The inverse question ($GI(s)$)

What can we say about f, if $\|f\|_{U^{s+1}[N]} \gg_\delta 1$?

Suppose $\|f\|_{U^{s+1}[N]} \gg_\delta 1$

- $s = 0$: f correlates with a constant function, trivial.

The inverse question

Well ... this brings us back to the inverse question:

The inverse question ($GI(s)$)

What can we say about f, if $\|f\|_{U^{s+1}[N]} \gg_\delta 1$?

Suppose $\|f\|_{U^{s+1}[N]} \gg_\delta 1$

- $s = 0$: f correlates with a constant function, trivial.
- $s = 1$: f correlates with a character (a linear phase function), by discrete Fourier analysis.

The inverse question

Well ... this brings us back to the inverse question:

The inverse question ($GI(s)$)

What can we say about f, if $\|f\|_{U^{s+1}[N]} \gg_\delta 1$?

Suppose $\|f\|_{U^{s+1}[N]} \gg_\delta 1$

- $s = 0$: f correlates with a constant function, trivial.
- $s = 1$: f correlates with a character (a linear phase function), by discrete Fourier analysis.

You might be tempted to think inductively, that If $\|f\|_{U^{s+1}} \gg_\delta 1$ then f correlates with a degree $\leq s$ phase polynomial (recall that the converse is true).

The inverse question

Well ... this brings us back to the inverse question:

The inverse question ($GI(s)$)

What can we say about f, if $\|f\|_{U^{s+1}[N]} \gg_\delta 1$?

Suppose $\|f\|_{U^{s+1}[N]} \gg_\delta 1$

- $s = 0$: f correlates with a constant function, trivial.
- $s = 1$: f correlates with a character (a linear phase function), by discrete Fourier analysis.

You might be tempted to think inductively, that If $\|f\|_{U^{s+1}} \gg_\delta 1$ then f correlates with a degree $\leq s$ phase polynomial (recall that the converse is true). This turns out to be false ...

A counter example (essentially due to Furstenberg and Weiss): Let

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

A counter example (essentially due to Furstenberg and Weiss): Let

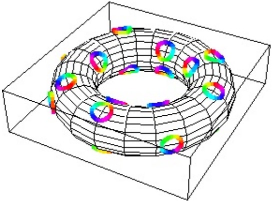
$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

- G is a 2-step nilpotent Lie Group: $[[G, G], G] = \{1\}$.

A counter example (essentially due to Furstenberg and Weiss): Let

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

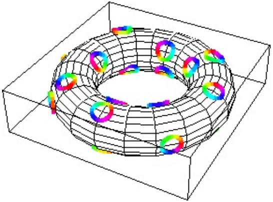
- G is a 2-step nilpotent Lie Group: $[[G, G], G] = \{1\}$.
- Γ is a lattice in G, and G/Γ is a nilmanifold.



A counter example (essentially due to Furstenberg and Weiss): Let

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

- G is a 2-step nilpotent Lie Group: $[[G,G],G] = \{1\}$.
- Γ is a lattice in G, and G/Γ is a nilmanifold.



- Let $\mathbf{x} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Gamma \in G/\Gamma$. Consider the function

$$F(\mathbf{x}) = e^{2\pi i (z - \{x\} y)}$$

F is well defined on G/Γ.

Let $a = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in G$.

Let $a = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in G$. Consider the function $g : \mathbb{Z} \to \mathbb{D}$

$$g(n) = F(a^n\Gamma) = e^{2\pi i(\{n\alpha\}n\beta + \binom{n}{2}\alpha\beta)}$$

Let $a = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in G$. Consider the function $g : \mathbb{Z} \to \mathbb{D}$

$$g(n) = F(a^n\Gamma) = e^{2\pi i(\{n\alpha\}n\beta + \binom{n}{2}\alpha\beta)}$$

Then

$$\|g\|_{U^3[N]} \gg 1$$

Let $a = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in G$. Consider the function $g : \mathbb{Z} \to \mathbb{D}$

$$g(n) = F(a^n\Gamma) = e^{2\pi i(\{n\alpha\}n\beta + \binom{n}{2}\alpha\beta)}$$

Then

$$\|g\|_{U^3[N]} \gg 1$$

But if α, β are irrational then, g does not correlate with any quadratic phase function $e^{2\pi i n^2\gamma}$!

Inverse $U^{s+1}[N]$

$g(n) = F(a^n\Gamma)$ is an example of a 2-step nilsequence.

Inverse $U^{s+1}[N]$

$g(n) = F(a^n\Gamma)$ is an example of a 2-step nilsequence.

In general, if G is an s-step nilpotent Lie group , Γ a lattice, F a "nice function", and $a \in G$ then $g(n) = F(a^n\Gamma)$ is an s-step nilsequence.

Inverse $U^{s+1}[N]$

$g(n) = F(a^n\Gamma)$ is an example of a 2-step nilsequence.

In general, if G is an s-step nilpotent Lie group , Γ a lattice, F a "nice function", and $a \in G$ then $g(n) = F(a^n\Gamma)$ is an s-step nilsequence.

The inverse conjecture for the Gowers norm asserts that nilsequences are the only obstructions to uniformity:

Inverse $U^{s+1}[N]$

$g(n) = F(a^n\Gamma)$ is an example of a 2-step nilsequence.

In general, if G is an s-step nilpotent Lie group , Γ a lattice, F a "nice function", and $a \in G$ then $g(n) = F(a^n\Gamma)$ is an s-step nilsequence.

The inverse conjecture for the Gowers norm asserts that nilsequences are the only obstructions to uniformity:

The Inverse Conjecture for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

Inverse $U^{s+1}[N]$

$g(n) = F(a^n\Gamma)$ is an example of a 2-step nilsequence.

In general, if G is an s-step nilpotent Lie group , Γ a lattice, F a "nice function", and $a \in G$ then $g(n) = F(a^n\Gamma)$ is an s-step nilsequence.

The inverse conjecture for the Gowers norm asserts that nilsequences are the only obstructions to uniformity:

The Inverse Conjecture for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

Polynomial phase functions of degree $\leq s$ are only a small subset of the set of degree $\leq s$ nilsequences !

$GI(s)$

Supporting evidence:

- $s = 2$; Green-Tao (2005).

$GI(s)$

Supporting evidence:

- $s = 2$; Green-Tao (2005).
- If $\|f\|_{U^{s+1}[N]}$ is sufficiently close to 1, then the conjecture is true; Alon-Kaufman-Krivelevich-Litsyn and Ron (2005).

$GI(s)$

Supporting evidence:

- $s = 2$; Green-Tao (2005).
- If $\|f\|_{U^{s+1}[N]}$ is sufficiently close to 1, then the conjecture is true; Alon-Kaufman-Krivelevich-Litsyn and Ron (2005).
- Related ergodic results are true; Host-Kra (2002), Z (2004).

$GI(s)$

Supporting evidence:

- $s = 2$; Green-Tao (2005).
- If $\|f\|_{U^{s+1}[N]}$ is sufficiently close to 1, then the conjecture is true; Alon-Kaufman-Krivelevich-Litsyn and Ron (2005).
- Related ergodic results are true; Host-Kra (2002), Z (2004).

What about the finite field analogue for $U^{s+1}[\mathbb{F}_p^n]$?

$Gl(s)$

Supporting evidence:

- $s = 2$; Green-Tao (2005).
- If $\|f\|_{U^{s+1}[N]}$ is sufficiently close to 1, then the conjecture is true; Alon-Kaufman-Krivelevich-Litsyn and Ron (2005).
- Related ergodic results are true; Host-Kra (2002), Z (2004).

What about the finite field analogue for $U^{s+1}[\mathbb{F}_p^n]$?

- $s = 2$; Green-Tao $p > 2$, Samorodnitsky $p = 2$ (2005).

$Gl(s)$

Supporting evidence:

- $s = 2$; Green-Tao (2005).
- If $\|f\|_{U^{s+1}[N]}$ is sufficiently close to 1, then the conjecture is true; Alon-Kaufman-Krivelevich-Litsyn and Ron (2005).
- Related ergodic results are true; Host-Kra (2002), Z (2004).

What about the finite field analogue for $U^{s+1}[\mathbb{F}_p^n]$?

- $s = 2$; Green-Tao $p > 2$, Samorodnitsky $p = 2$ (2005).
- For $s > 2$, a COUNTER EXAMPLE was given by Green-Tao, and independently by Lovett-Meshulam-Samorodnistky (2007).

$Gl(s)$

Theorem (Green-Tao-Z (2010))

The Inverse Conjecture for the Gowers Norms is true.

It follows that the (non degenerate) Multidimentional Dirichlet Theorem is true unconditionally !

Möbius Nilsequence Theorem - equivalent formulation $MN(s)$

The Möbius function μ does not correlate with bounded complexity nilsequences:

$$\frac{1}{N}\sum_{n\leq N}\mu(n)F(a^n\Gamma) \ll_A \frac{1}{(\log N)^A}$$

Möbius Nilsequence Theorem - equivalent formulation $MN(s)$

The Möbius function μ does not correlate with bounded complexity nilsequences:

$$\frac{1}{N}\sum_{n\leq N}\mu(n)F(a^n\Gamma) \ll_A \frac{1}{(\log N)^A}$$

Inverse Theorem for the Gowers Norms $GI(s)$

Let $f:\mathbb{Z}_N\to\mathbb{D}$. Then $\|f\|_{U^{s+1}}\gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

Möbius Nilsequence Theorem - equivalent formulation $MN(s)$

The Möbius function μ does not correlate with bounded complexity nilsequences:

$$\frac{1}{N}\sum_{n\leq N}\mu(n)F(a^n\Gamma) \ll_A \frac{1}{(\log N)^A}$$

Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

Corollary: $\|\mu\|_{U^{s+1}[N]} = o(1)$.

Möbius Nilsequence Theorem - equivalent formulation $MN(s)$

The Möbius function μ does not correlate with bounded complexity nilsequences:

$$\frac{1}{N}\sum_{n\leq N}\mu(n)F(a^n\Gamma) \ll_A \frac{1}{(\log N)^A}$$

Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

Corollary: $\|\mu\|_{U^{s+1}[N]} = o(1)$.

$$e.g. \quad \frac{1}{N^2}\sum_{n,d\leq N}\mu(n)\mu(n+d)\mu(n+2d)\mu(n+3d) = o(1)$$

Sketch of proof of multidimensional Dirichlet:

Consider the von-Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and } k > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Sketch of proof of multidimensional Dirichlet:

Consider the von-Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and } k > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we are counting prime values of $\{\psi_i(\vec{x})\}_{i=1}^{k}$

Sketch of proof of multidimensional Dirichlet:

Consider the von-Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and } k > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we are counting prime values of $\{\psi_i(\vec{x})\}_{i=1}^k$ It would be great if we could show that $\|\Lambda - 1\|_{U^{s+1}[N]} = o(1)$.

Sketch of proof of multidimensional Dirichlet:

Consider the von-Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and } k > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we are counting prime values of $\{\psi_i(\vec{x})\}_{i=1}^k$ It would be great if we could show that $\|\Lambda - 1\|_{U^{s+1}[N]} = o(1)$. For then we would have

$$\sum_{\vec{x} \in [N]^n} \Lambda(\psi_1(\vec{x})) \cdots \Lambda(\psi_k(\vec{x})) \sim N^n,$$

which is what one would expect if one were counting solutions for a random von-Mangold function.

Sketch of proof of multidimensional Dirichlet:

Consider the von-Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and } k > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we are counting prime values of $\{\psi_i(\vec{x})\}_{i=1}^{k}$ It would be great if we could show that $\|\Lambda - 1\|_{U^{s+1}[N]} = o(1)$. For then we would have

$$\sum_{\vec{x} \in [N]^n} \Lambda(\psi_1(\vec{x})) \cdots \Lambda(\psi_k(\vec{x})) \sim N^n,$$

which is what one would expect if one were counting solutions for a random von-Mangold function.

This is unfortunately FALSE (small primes are problematic).

But ...

But ... there is a chance if one looks at Λ along arithmetic progressions with slowly increasing gap:

But ... there is a chance if one looks at Λ along arithmetic progressions with slowly increasing gap:

$$\Lambda_{b,W}(n) = \frac{\phi(W)}{W}\Lambda(Wn + b),$$

where $W = \prod_{p \leq w} p$ and we take $w = w(N)$ to be a sufficiently slowly increasing function of N. (The normalizations is so that $\frac{1}{N}\sum_{n \leq N} \Lambda_{b,W}(n) \sim 1$).

But ... there is a chance if one looks at Λ along arithmetic progressions with slowly increasing gap:

$$\Lambda_{b,W}(n) = \frac{\phi(W)}{W}\Lambda(Wn+b),$$

where $W = \prod_{p\leq w} p$ and we take $w = w(N)$ to be a sufficiently slowly increasing function of N. (The normalizations is so that $\frac{1}{N}\sum_{n\leq N}\Lambda_{b,W}(n) \sim 1$).

Strategy: show that if $(b, W) = 1$ then for any s

$$\|\Lambda_{b,W} - 1\|_{U^{s+1}[N]} \to 0$$

We then get:

$$\sum_{\vec{x}\in[N]^n} \Lambda_{b,W}(\psi_1(\vec{x}))\cdots\Lambda_{b,W}(\psi_k(\vec{x})) \sim N^n$$

By the Inverse Theorem for the Gowers Norms it suffices to show that

$$\Lambda_{b,W} - 1$$

does not correlate with nilsequences, but this follows from the Möbius Nilsequence Theorem.

By the Inverse Theorem for the Gowers Norms it suffices to show that

$$\Lambda_{b,W} - 1$$

does not correlate with nilsequences, but this follows from the Möbius Nilsequence Theorem.

(Not so small) Cheat: Inverse theorem applies to bounded function, $\Lambda_{b,W}$ isn't bounded !

By the Inverse Theorem for the Gowers Norms it suffices to show that

$$\Lambda_{b,W} - 1$$

does not correlate with nilsequences, but this follows from the Möbius Nilsequence Theorem.

(Not so small) Cheat: Inverse theorem applies to bounded function, $\Lambda_{b,W}$ isn't bounded !

To the rescue comes the Green-Tao transference principle, which allows us to push the inverse theorem from bounded functions to function bounded by a pseudorandom function (this is a whole different story ...).

Don't miss the next lecture:

Don't miss the next lecture:

Intertwining developments in

Don’t miss the next lecture:

Intertwining developments in ergodic theory

Don't miss the next lecture:

Intertwining developments in ergodic theory and arithmetic combinatorics

Don't miss the next lecture:

Intertwining developments in ergodic theory and arithmetic combinatorics leading to the multidimensional Dirichlet theorem.

Don't miss the next lecture:

Intertwining developments in ergodic theory and arithmetic combinatorics leading to the multidimensional Dirichlet theorem.

Thank you !

The Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

The Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

- Inductively for many h, $\|\Delta_h f\|_{U^s[N]} \gg 1 \implies$ for many h, $\Delta_h f$ correlates with an $(s-1)$-step nilsequence $F_h(a_h^n G_h/\Gamma_h)$ of bounded complexity.

The Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

- Inductively for many h, $\|\Delta_h f\|_{U^s[N]} \gg 1 \implies$ for many h, $\Delta_h f$ correlates with an $(s-1)$-step nilsequence $F_h(a_h^n G_h/\Gamma_h)$ of bounded complexity.
- Clever CS: For many $h_1 + h_2 = h_3 + h_4$ the orbit of $a_{h_1} \times a_{h_2} \times a_{h_3} \times a_{h_4}$ is not equidtributed in the nilmanifold $G_{h_1}/\Gamma_{h_1} \times G_{h_2}/\Gamma_{h_2} \times G_{h_3}/\Gamma_{h_3} \times G_{h_4}/\Gamma_{h_4}$.

The Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

- Inductively for many h, $\|\Delta_h f\|_{U^s[N]} \gg 1 \implies$ for many h, $\Delta_h f$ correlates with an $(s-1)$-step nilsequence $F_h(a_h^n G_h/\Gamma_h)$ of bounded complexity.
- Clever CS: For many $h_1 + h_2 = h_3 + h_4$ the orbit of $a_{h_1} \times a_{h_2} \times a_{h_3} \times a_{h_4}$ is not equidtributed in the nilmanifold $G_{h_1}/\Gamma_{h_1} \times G_{h_2}/\Gamma_{h_2} \times G_{h_3}/\Gamma_{h_3} \times G_{h_4}/\Gamma_{h_4}$.
- Orbit closures in nilmanifolds are algebraic. Use to find some form of an algebraic relation between the a_h (for many h).

The Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

- Inductively for many h, $\|\Delta_h f\|_{U^s[N]} \gg 1 \implies$ for many h, $\Delta_h f$ correlates with an $(s-1)$-step nilsequence $F_h(a_h^n G_h/\Gamma_h)$ of bounded complexity.
- Clever CS: For many $h_1 + h_2 = h_3 + h_4$ the orbit of $a_{h_1} \times a_{h_2} \times a_{h_3} \times a_{h_4}$ is not equidtributed in the nilmanifold $G_{h_1}/\Gamma_{h_1} \times G_{h_2}/\Gamma_{h_2} \times G_{h_3}/\Gamma_{h_3} \times G_{h_4}/\Gamma_{h_4}$.
- Orbit closures in nilmanifolds are algebraic. Use to find some form of an algebraic relation between the a_h (for many h).
- Use additive combinatorics to obtain bracket linearity in the parameter h (for many h)

The Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

- Inductively for many h, $\|\Delta_h f\|_{U^s[N]} \gg 1 \implies$ for many h, $\Delta_h f$ correlates with an $(s-1)$-step nilsequence $F_h(a_h^n G_h/\Gamma_h)$ of bounded complexity.
- Clever CS: For many $h_1 + h_2 = h_3 + h_4$ the orbit of $a_{h_1} \times a_{h_2} \times a_{h_3} \times a_{h_4}$ is not equidtributed in the nilmanifold $G_{h_1}/\Gamma_{h_1} \times G_{h_2}/\Gamma_{h_2} \times G_{h_3}/\Gamma_{h_3} \times G_{h_4}/\Gamma_{h_4}$.
- Orbit closures in nilmanifolds are algebraic. Use to find some form of an algebraic relation between the a_h (for many h).
- Use additive combinatorics to obtain bracket linearity in the parameter h (for many h)
- Use the fact that $\Delta_h \Delta_k f = \Delta_k \Delta_h f$ (and quantitative equidistribution) to obtain some symmetry in n, h

The Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if f correlates with a bounded complexity s-step nilsequence.

- Inductively for many h, $\|\Delta_h f\|_{U^s[N]} \gg 1 \implies$ for many h, $\Delta_h f$ correlates with an $(s-1)$-step nilsequence $F_h(a_h^n G_h/\Gamma_h)$ of bounded complexity.
- Clever CS: For many $h_1 + h_2 = h_3 + h_4$ the orbit of $a_{h_1} \times a_{h_2} \times a_{h_3} \times a_{h_4}$ is not equidtributed in the nilmanifold $G_{h_1}/\Gamma_{h_1} \times G_{h_2}/\Gamma_{h_2} \times G_{h_3}/\Gamma_{h_3} \times G_{h_4}/\Gamma_{h_4}$.
- Orbit closures in nilmanifolds are algebraic. Use to find some form of an algebraic relation between the a_h (for many h).
- Use additive combinatorics to obtain bracket linearity in the parameter h (for many h)
- Use the fact that $\Delta_h \Delta_k f = \Delta_k \Delta_h f$ (and quantitative equidistribution) to obtain some symmetry in n, h
- Integrate (construction: guess a solution).

U^s in finite field geometry

- The finite field analogue $U^{s+1}[\mathbb{F}_p^n]$ is true in the high characteristic case $p \geq s$; Bergelson-Tao-Z, Tao-Z (2008)

U^s in finite field geometry

- The finite field analogue $U^{s+1}[\mathbb{F}_p^n]$ is true in the high characteristic case $p \geq s$; Bergelson-Tao-Z, Tao-Z (2008)
- The "correct" analogue is true in low characteristic Tao-Z (2011).

U^s in finite field geometry

- The finite field analogue $U^{s+1}[\mathbb{F}_p^n]$ is true in the high characteristic case $p \geq s$; Bergelson-Tao-Z, Tao-Z (2008)
- The "correct" analogue is true in low characteristic Tao-Z (2011).