Dynamics and prime solutions to linear equations

Tamar Ziegler

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EMS Lecture Series
Denote by $\mathbb{P}$ the set of prime numbers $\mathbb{P} = \{2, 3, 5, \ldots\}$.

Let $\{\psi_i(\vec{x})\}_{i=1}^k$ be $k$ affine linear forms in $n$ variables

$$
\psi_i(\vec{x}) = \vec{a}_i \cdot \vec{x} + b_i, \quad \vec{a}_i \in \mathbb{Z}^n, b_i \in \mathbb{Z}
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**Question:**
Are there $\vec{x} \in \mathbb{Z}^n$ such $\psi_1(\vec{x}), \ldots, \psi_k(\vec{x}) \in \mathbb{P}$?
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**Question:**

Are there $\vec{x} \in \mathbb{Z}^n$ such $\psi_1(\vec{x}), \ldots, \psi_k(\vec{x}) \in \mathbb{P}$?

Infinitely many such $\vec{x}$? Asymptotics?
\[ \psi(x) = ax + b \]

\[ k = n = 1: \text{ If } a = 1, \ b = 0, \text{ then} \]

\[ \psi(x) = x. \]
$\psi(x) = ax + b$

$k = n = 1$: If $a = 1$, $b = 0$, then

$$\psi(x) = x.$$ 

Theorem (Euclid (~ 300 BC))

There are $\infty$ many primes.
\[\psi(x) = ax + b\]

**k = n = 1:** If \(a = 1, b = 0\), then

\[\psi(x) = x.\]

**Theorem (Euclid (~ 300 BC))**

There are \(\infty\) many primes.

Over 2000 years later ...

**Theorem (Hadamard, de la Vallée-Poussin (1896))**

\[\pi(N) = |\{x \in \mathbb{P}, x \leq N\}| \sim \frac{N}{\log N}\]
More generally:

\[ \psi(x) = ax + b \]
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**Theorem (Johann Peter Gustav Lejeune Dirichlet (1837))**

There are \( \infty \) many primes of the form \( ax + b \)

\[ \iff (a, b) = 1 \] (no local obstructions)
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**Prime number theorem in arithmetic progression**

Each legal arithmetic progression gets its fair share: if \( (a, b) = 1 \), then

\[
\pi(N, a, b) = |\{ x \in \mathbb{P}, x \leq N, x \equiv b \mod a \}| \sim \frac{1}{\phi(a)} \frac{N}{\log N}
\]
$k = 2, n = 1$

\[\psi_1(x) = x \quad \psi_2(x) = x + 2\]
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This is the ”twin prime conjecture”!
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Nothing to report on this...

News Flash: Peter Woit’s blog May 12: Yitang Zhang proved

\[ \psi_1(x) = x \quad \psi_2(x) = x + M \quad M < 70,000,000 \quad !!! \]
$n = 2, \, k$ arbitrary

$\psi_1(x) = x, \quad \psi_2(x) = x + d, \quad \ldots \quad , \psi_k(x) = x + (k - 1)d$

This is an $k$-term arithmetic progression
$n = 2, \ k$ arbitrary

\[ \psi_1(x) = x, \ \psi_2(x) = x + d, \ \ldots \ \psi_k(x) = x + (k - 1)d \]

This is an $k$-term arithmetic progression

**Theorem (Green-Tao (2004))**

The primes contain arbitrarily long arithmetic progressions.

The proof gives a lower bound of the correct order of magnitude \( (C \frac{N^2}{(\log N)^k}) \).
In 2006 Green-Tao proved the following conditional theorem:

Let \( \{ \psi_i(\vec{x}) \}_{k_i=1}^k \) be \( k \) affine linear integer forms in \( n \) variables. Suppose no 2 forms are affinely dependent. Then \( \{ \psi_i(\vec{x}) \}_{k_i=1}^k \subset P \) infinitely often \( \iff \) No local obstructions

\[
\begin{align*}
\left| \{ \vec{x} \in [0, N]^n, \{ \psi_i(\vec{x}) \}_{k_i=1}^k \subset P \} \right| & \sim S(\vec{\psi}, N^{k}) \log N
\end{align*}
\]

Conditioned on 2 conjectures: \( MN(\mathbb{F}_p) \), and \( GI(\mathbb{F}_p) \).

No local obstructions:
- \( \mod p \): for any prime \( p \), there exists \( \vec{x} \in \mathbb{Z}^n \) such that \( \psi_i(\vec{x}) \) is coprime to \( p \) all \( i \).
- \( \mod \infty \): there exist infinitely many \( \vec{x} \in \mathbb{Z}^n \) such that \( \psi_i(\vec{x}) \) is positive.
In 2006 Green-Tao proved the following conditional theorem:

**Conditional Multidimensional Dirichlet**

Let \( \{ \psi_i(\bar{x}) \}_{i=1}^k \) be \( k \) affine linear integer forms in \( n \) variables. Suppose no 2 forms are affinely dependent. Then

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\left| \{\vec{x} \in [0, N]^n, \{\psi_i(\vec{x})\}_{i=1}^k \subset \mathbb{P}\} \right| \sim \mathcal{G}(\vec{\psi}) \frac{N^n}{(\log N)^k}
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- **Some special cases of \( k \) equations \( f(k) \) variables** - Balog (1992) using the HL circle method.
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- **\( k \) equation \( n \geq k + 2 \) variables** - Conditional Multidimensional Dirichlet.
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- **\(k\)** equation **\(n \geq k + 2\)** variables - Conditional Multidimensional Dirichlet. Best possible dependence excluding ”twin prime case”.

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Patterns in primes
Conditional Multidimensional Dirichlet Theorem

Conditioned on $MN(s)$, and $GI(s)$, one can calculate the asymptotic number of prime solutions to any system of $k$ linear equations with integer coefficients in at least $k + 2$ variables.

Remark: The method used by Green and Tao to show the existence of arithmetic progressions in primes can not be used to establish asymptotics or to handle non homogeneous equations, since it relies on Szemeredi’s theorem which is invalid for non homogeneous equations, and can’t provide asymptotics.
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What is $MN(s)$?
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Consider the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k, \text{ where } p_i \text{ are distinct primes;} \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius function is related to the normalized prime counting function $\Lambda(n)$ via an identity arising from the möbius inversion formula.
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$$\left| \sum_{n=1}^{N} \mu(n) \right| \ll A N (\log N)^{-A}$$
Theorem (Davenport (1930s))

For any $\alpha \in [0,1]$

$$\left| \sum_{n=1}^{N} \mu(n)e^{2\pi i n \alpha} \right| \ll_A N \log N$$
Theorem (Davenport (1930s))

For any $\alpha \in [0, 1]$

\[ \left| \sum_{n=1}^{N} \mu(n)e^{2\pi in\alpha} \right| \ll_A N(\log N)^{-A} \]

By the same method: for any polynomial $P$

\[ \left| \sum_{n=1}^{N} \mu(n)e^{2\pi iP(n)} \right| \ll_A N(\log N)^{-A} \]
We will see soon, that for our purposes, we need similar estimates for **bracket polynomials**. Examples:

\[ n^2 \alpha, \ n\alpha \{ n\beta \}, \ n\alpha \{ n^2\beta \}, \ n\alpha \{ \{ n\beta \} n\gamma \} \]
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**Möbius Nilsequence Conjecture (MN(s))**

For any **bracket polynomial** \( P \).

\[
\left| \sum_{n=1}^{N} \mu(n) e^{2\pi i P(n)} \right| \ll_A N(\log N)^{-A}
\]

\( \mu \) does not correlate with bracket polynomial phase functions!
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Who is \( s \)? How is this related to nilsequences (what are they)?
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Who is \(s\)? How is this related to nilsequences (what are they)?

**Theorem (Green-Tao (2007))**

Möbius Nilsequence Conjecture is true.
What is $G^I(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.
What is $GI(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

What are Gowers norms?
GL(s)

What is GL(s)? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

What are Gowers norms? Let $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and let $f : \mathbb{Z}_N \rightarrow \mathbb{D}$.

**Discrete differentiation**

Let $h \in \mathbb{Z}_N$, define the derivative in direction $h$ to be

$$\Delta_h f(n) := f(n + h)f(n)$$
What is $GI(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

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Examples:

- $\Delta_h f(n) \equiv 1$ for all $h \in \mathbb{Z}_N$ if and only if $f(n) \equiv C$. 

What is $Gl(s)$? The Inverse Conjecture for the $U^{s+1}[N]$ Gowers norm.

What are Gowers norms? Let $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and let $f: \mathbb{Z}_N \to \mathbb{D}$.

**Discrete differentiation**

Let $h \in \mathbb{Z}_N$, define the derivative in direction $h$ to be

$$\Delta_h f(n) := f(n + h)\overline{f(n)}$$

Examples:

- $\Delta_h f(n) \equiv 1$ for all $h \in \mathbb{Z}_N$ if and only if $f(n) \equiv C$.
- $\Delta_{h_2} \Delta_{h_1} f(n) \equiv 1$ for all $h_1, h_2 \in \mathbb{Z}_N$ if and only if $f$ is a linear phase polynomial, i.e. $f(n) = e^{2\pi i P(n)}$, where $P$ is a linear polynomial.
Define the Gowers uniformity norms as follows: Let $f : \mathbb{Z}_N \to \mathbb{D}$.

**Gowers norms**

\[
\left\| f \right\|_{U^s[N]}^{2^s} = \frac{1}{N^{s+1}} \sum_{n, h_1, \ldots, h_s \in \mathbb{Z}_N} \Delta h_s \cdots \Delta h_1 f(n)
\]

For $s > 1$ this is a norm.
\[ \| f \|_{U^s[N]}^{2^s} = \frac{1}{N^{s+1}} \sum_{n,h_1,\ldots,h_s \in \mathbb{Z}_N} \Delta_{h_s} \cdots \Delta_{h_1} f(n) \]

Remarks:

- If \( \| f \|_{U^s[N]} = 1 \) then \( \Delta_{h_s} \cdots \Delta_{h_1} f(n) \equiv 1 \) for all \( h_1, \ldots, h_k \in \mathbb{Z}_N \) thus \( f \) is a phase polynomial of degree \( < s \).
\[ \| f \|_{Us[N]}^{2s} = \frac{1}{N^{s+1}} \sum_{\substack{n, h_1, \ldots, h_s \in \mathbb{Z}_N}} \Delta_{h_s} \cdots \Delta_{h_1} f(n) \]

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- Conversely, if \( f \) is a phase polynomial of degree \(< s \), then \( \| f \|_{Us[N]} = 1 \).
∥f∥_{U^s[N]}^{2s} = \frac{1}{N^{s+1}} \sum_{n,h_1,\ldots,h_s \in \mathbb{Z}_N} \Delta_{h_s} \cdots \Delta_{h_1} f(n)

Remarks:

- If \(∥f∥_{U^s[N]} = 1\) then \(\Delta_{h_s} \cdots \Delta_{h_1} f(n) \equiv 1\) for all \(h_1, \ldots, h_k \in \mathbb{Z}_N\) thus \(f\) is a phase polynomial of degree \(< s\).

- Conversely, if \(f\) is a phase polynomial of degree \(< s\), then \(∥f∥_{U^s[N]} = 1\).

- If \(f\) correlates with a polynomial phase function of degree \(< s\) then \(∥f∥_{U^s[N]} \gg \delta 1\).
\[ \|f\|_{U^s[N]}^{2^s} = \frac{1}{N^{s+1}} \sum_{n,h_1,\ldots,h_s \in \mathbb{Z}_N} \Delta_{h_s} \cdots \Delta_{h_1} f(n) \]

Remarks:

- If \( \|f\|_{U^s[N]} = 1 \) then \( \Delta_{h_s} \cdots \Delta_{h_1} f(n) \equiv 1 \) for all \( h_1, \ldots, h_k \in \mathbb{Z}_N \) thus \( f \) is a phase polynomial of degree \( < s \).
- Conversely, if \( f \) is a phase polynomial of degree \( < s \), then \( \|f\|_{U^s[N]} = 1 \).
- If \( f \) correlates with a polynomial phase function of degree \( < s \) then \( \|f\|_{U^s[N]} \gg \delta 1 \).
- If \( f \) is a random function then \( \|f\|_{U^s[N]} = o(1) \).
Recall:

If $f$ correlates with a polynomial phase function of degree $< s$ then $\|f\|_{U^s[N]} \gg \delta_1$.

If $f$ is a random function then $\|f\|_{U^s[N]} = o(1)$.

The inverse question:

What can we say about $f$ if $\|f\|_{U^s[N]} \gg \delta_1$?

Does $f$ correlate with a polynomial phase function?

We will get back to this question soon...
Recall:

- If \( f \) correlates with a polynomial phase function of degree \( < s \) then \( \|f\|_{U^s[N]} \gg \delta \cdot 1 \).
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What can we say about \( f \) if \( \|f\|_{U^s[N]} \gg_\delta 1 \)? Does \( f \) correlate with a polynomial phase function?

We will get back to this question soon ...
Why are Gowers norms important?

Consider the case of 3 term progressions: Let $E \subset \mathbb{Z}_N$ be of size $\delta N$ for some $\delta > 0$. Let's try to count 3 term progressions in $E$.

Let $1_E(x)$ be the characteristic function of the set $E$.

Here is a counting expression:

$$\sum_{x, d \in \mathbb{Z}_N} 1_E(x) 1_E(x + d) 1_E(x + 2d)$$

An observation of Gowers: If $\|1_E - \delta\|_{U^2}[N]$ is small then

$$\sum_{x, d \in \mathbb{Z}_N} 1_E(x) 1_E(x + d) 1_E(x + 2d) \sim \delta^3 N$$

This is the number of 3 term progressions we expect to find in a random subset of $\mathbb{Z}_N$ of size $\delta N$!
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$\|1_E - \delta\|_{U^2[N]}$ small implies that the number of 3 term progressions is as in a random set.
Why are Gowers norms important?

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\[ \sum_{x, d \in \mathbb{Z}_N} 1_E(x)1_E(x + h) \cdots 1_E(x + sh) \sim \delta^{s+1} N^2 \]
Why are Gowers norms important?

Even more generally: Let \( \{ \psi_i(\vec{x}) \}_{i=1}^k \) be \( k \) affine linear integer forms in \( n \) variables. Suppose no 2 forms are affinely dependent.

Then there is some integer \( s \) such that if \( \| 1_E - \delta \| U_s \left[ N \right] \) small then

\[
\sum_{\vec{x} \in \mathbb{Z}^n} 1_E(\psi_1(\vec{x})) 1_E(\psi_2(\vec{x})) \ldots 1_E(\psi_k(\vec{x})) \sim \delta^k \]

So it is REALLY important to find a good way to test whether \( \| f \| U_s \) is small.
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Well ... this brings us back to the inverse question:

The inverse question (GI(s))

What can we say about $f$, if $\|f\|_{U^{s+1}[N]} \gg \delta 1$?
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- $s = 1$: $f$ correlates with a character (a linear phase function), by discrete Fourier analysis.
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You might be tempted to think inductively, that if \( \|f\|_{U^{s+1}} \gg \delta 1 \) then \( f \) correlates with a degree \( \leq s \) phase polynomial (recall that the converse is true).
Well ... this brings us back to the inverse question:

The inverse question \((GL(s))\)

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You might be tempted to think inductively, that if \(\|f\|_{U^{s+1}} \gg \delta 1\) then \(f\) correlates with a degree \(\leq s\) phase polynomial (recall that the converse is true). This turns out to be false ...
A counter example (essentially due to Furstenberg and Weiss): Let

\[
G = \begin{pmatrix}
1 & \mathbb{R} & \mathbb{R} \\
0 & 1 & \mathbb{R} \\
0 & 0 & 1
\end{pmatrix} \quad \Gamma = \begin{pmatrix}
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\[ G = \begin{pmatrix} 1 & R & R \\ 0 & 1 & R \\ 0 & 0 & 1 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 1 & Z & Z \\ 0 & 1 & Z \\ 0 & 0 & 1 \end{pmatrix} \]

- \( G \) is a 2-step nilpotent Lie Group: \([ [G, G], G] = \{1\} \).
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- \( \Gamma \) is a lattice in \( G \), and \( G/\Gamma \) is a nilmanifold.

Let \( x = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Gamma \in G/\Gamma \). Consider the function

\[ F(x) = e^{2\pi i (z - \{x\}y)} \]

\( F \) is well defined on \( G/\Gamma \).
Let $a = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in G$. 

Consider the function $g : \mathbb{Z} \to \mathbb{D}$

$$g(n) = F(a^n \Gamma) = e^{2\pi i \left\{ n \alpha + (n^2)\alpha \beta \right\}}.$$ 

Then $\|g\|_{U^3[\mathbb{N}]} \gg 1$. 

But if $\alpha, \beta$ are irrational then, $g$ does not correlate with any quadratic phase function $e^{2\pi i \gamma}$.
Let \( a = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in G \). Consider the function \( g : \mathbb{Z} \rightarrow \mathbb{D} \)

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Inverse $U^{s+1}[N]$

\[ g(n) = F(a^n \Gamma) \] is an example of a 2-step nilsequence.
Inverse $U^{s+1}[N]$

$g(n) = F(a^n \Gamma)$ is an example of a **2-step nilsequence**.

In general, if $G$ is an $s$-step nilpotent Lie group, $\Gamma$ a lattice, $F$ a ”nice function”, and $a \in G$ then $g(n) = F(a^n \Gamma)$ is an $s$-step nilsequence.
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The inverse conjecture for the Gowers norm asserts that nilsequences are the only obstructions to uniformity:
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The inverse conjecture for the Gowers norm asserts that nilsequences are the **only** obstructions to uniformity:

**The Inverse Conjecture for the Gowers Norms $GI(s)$**

Let $f : \mathbb{Z}_N \rightarrow \mathbb{D}$. Then $\|f\|_{U^{s+1}} \gg 1$ if and only if $f$ correlates with a bounded complexity $s$-step nilsequence.
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Polynomial phase functions of degree $\leq s$ are only a small subset of the set of degree $\leq s$ nilsequences!
$Gl(s)$

Supporting evidence:

- $s = 2$; Green-Tao (2005).

Related ergodic results are true; Host-Kra (2002), Z (2004).

What about the finite field analogue for $U^{s+1}[F^n]$?

- $s = 2$; Green-Tao $p > 2$, Samorodnitsky $p = 2$ (2005).

For $s > 2$, a COUNTER EXAMPLE was given by Green-Tao, and independently by Lovett-Meshulam-Samorodnistky (2007).
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Theorem (Green-Tao-Z (2010))

The Inverse Conjecture for the Gowers Norms is true.

It follows that the (non degenerate) Multidimensional Dirichlet Theorem is true unconditionally!
The Möbius function $\mu$ does not correlate with bounded complexity nilsequences:

$$\frac{1}{N} \sum_{n \leq N} \mu(n) F(a^n \Gamma) \ll A \frac{1}{(\log N)^A}$$
Möbius Nilsequence Theorem - equivalent formulation $MN(s)$

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Inverse Theorem for the Gowers Norms $G^s(\mathbb{N})$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}} \gg 1$ if and only if $f$ correlates with a bounded complexity $s$-step nilsequence.
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Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}} \gg 1$ if and only if $f$ correlates with a bounded complexity $s$-step nilsequence.

Corollary: $\|\mu\|_{U^{s+1}[\mathbb{N}]} = o(1)$. 
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e.g. $\frac{1}{N^2} \sum_{n,d \leq N} \mu(n) \mu(n + d) \mu(n + 2d) \mu(n + 3d) = o(1)$
Sketch of proof of multidimensional Dirichlet:

Consider the von-Mangoldt function

\[ \Lambda(n) = \begin{cases} 
 \log p & \text{if } n = p^k \text{ for some prime } p \text{ and } k > 0; \\
 0 & \text{otherwise.} 
\end{cases} \]

Recall that we are counting prime values of \( \psi_i(\vec{x}) \) for \( i = 1 \) to \( k \). It would be great if we could show that \( \| \Lambda - 1 \|_{U^{s+1}[N]} = o(1) \). For then we would have

\[ \sum_{\vec{x} \in [N]} n \Lambda(\psi_1(\vec{x})) \cdots \Lambda(\psi_k(\vec{x})) \sim N_n, \]

which is what one would expect if one were counting solutions for a random von-Mangoldt function.

This is unfortunately FALSE (small primes are problematic).
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$$\Lambda_{b,W}(n) = \frac{\phi(W)}{W} \Lambda(Wn + b),$$

where $W = \prod_{p \leq w} p$ and we take $w = w(N)$ to be a sufficiently slowly increasing function of $N$. (The normalizations is so that $\frac{1}{N} \sum_{n \leq N} \Lambda_{b,W}(n) \sim 1$).
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**Strategy:** show that if $(b, W) = 1$ then for any $s$

$$\|\Lambda_{b,W} - 1\|_{U^{s+1}[\mathbb{N}]} \to 0$$

We then get:

$$\sum_{\vec{x} \in [\mathbb{N}]^n} \Lambda_{b,W}(\psi_1(\vec{x})) \cdots \Lambda_{b,W}(\psi_k(\vec{x})) \sim N^n$$
By the Inverse Theorem for the Gowers Norms it suffices to show that

\[ \Lambda_{b,W} - 1 \]

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To the rescue comes the Green-Tao transference principle, which allows us to push the inverse theorem from bounded functions to function bounded by a pseudorandom function (this is a whole different story ...).
Don’t miss the next lecture:

Intertwining developments in ergodic theory and arithmetic combinatorics leading to the multidimensional Dirichlet theorem.

Thank you!
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Intertwining developments in
Don’t miss the next lecture:

Intertwining developments in ergodic theory

Thank you!

Tamar Ziegler
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Thank you!
Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if $f$ correlates with a bounded complexity $s$-step nilsequence.
The Inverse Theorem for the Gowers Norms $G^s_l(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if $f$ correlates with a bounded complexity $s$-step nilsequence.

- Inductively for many $h$, $\|\Delta_h f\|_{U^s[N]} \gg 1$ $\iff$ for many $h$, $\Delta_h f$ correlates with an $(s - 1)$-step nilsequence $F_h(a^n_G h/\Gamma_h)$ of bounded complexity.
The Inverse Theorem for the Gowers Norms $GI(s)$

Let $f : \mathbb{Z}_N \to \mathbb{D}$. Then $\|f\|_{U^{s+1}[N]} \gg 1$ if and only if $f$ correlates with a bounded complexity $s$-step nilsequence.

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- Clever CS: For many $h_1 + h_2 = h_3 + h_4$ the orbit of $a_{h_1} \times a_{h_2} \times a_{h_3} \times a_{h_4}$ is not equidistributed in the nilmanifold $G_{h_1}/\Gamma_{h_1} \times G_{h_2}/\Gamma_{h_2} \times G_{h_3}/\Gamma_{h_3} \times G_{h_4}/\Gamma_{h_4}$.
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- Integrate (construction: guess a solution).
The finite field analogue $U^{s+1}[\mathbb{F}_p^n]$ is true in the high characteristic case $p \geq s$; Bergelson-Tao-Z, Tao-Z (2008)
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