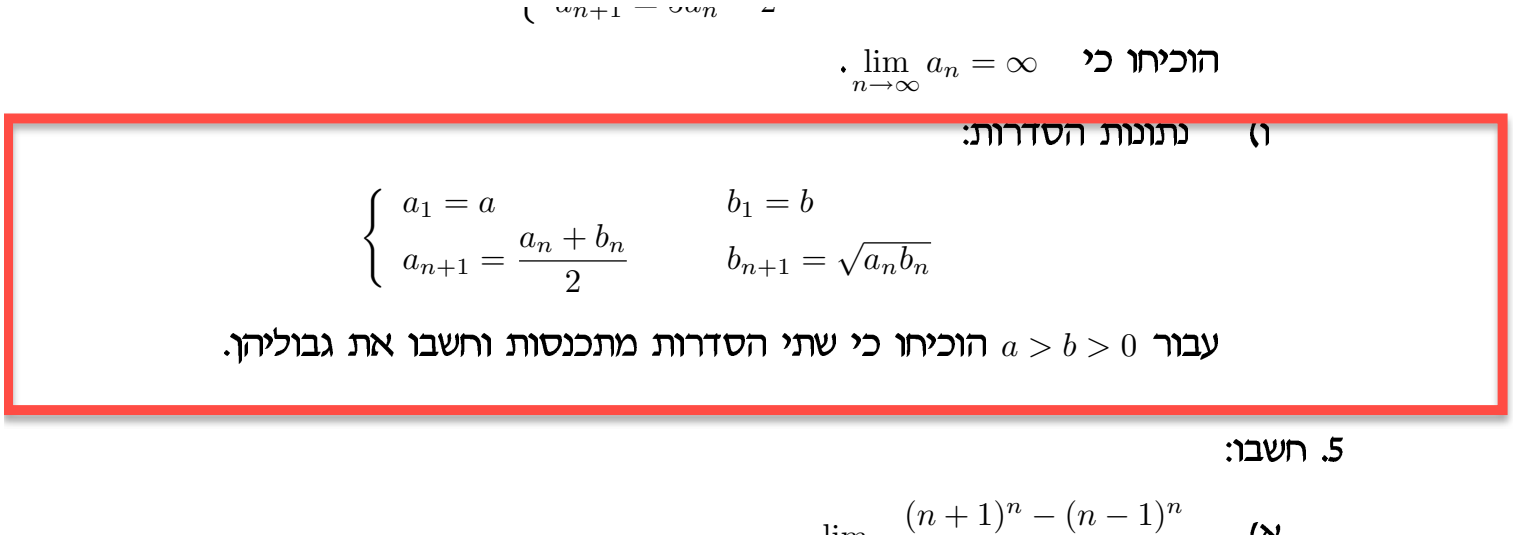
The arithmetic geometric mean (Agm)

Tamar Ziegler

Pictures by David Lehavi

This is part of exercise 5 question 4 in my calculus class Fall 2010:



הוכיחו כי $\lim_{n\to\infty} a_n = \infty$.

ו) נתונות הסדרות:

$$\begin{cases} a_1 = a & b_1 = b \\ a_{n+1} = \dfrac{a_n + b_n}{2} & b_{n+1} = \sqrt{a_n b_n} \end{cases}$$

עבור $a > b > 0$ הוכיחו כי שתי הסדרות מתכנסות וחשבו את גבוליהן.

5. חשבו:

$$\lim \quad (n+1)^n - (n-1)^n$$

Here is the translation (and index change):

$$a_0 = a \qquad b_0 = b$$

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n} \qquad n = 0, 1, \ldots$$

For $a > b > 0$ show that both sequences converge and calculate their limits.

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By the mean inequality:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

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and by induction

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$$a_n \geq b_n.$$

Also

$$a_n \geq \frac{a_n + b_n}{2} = a_{n+1} \geq b_{n+1} = \sqrt{a_n b_n} \geq b_n$$

We have

$$a \geq a_1 \geq \ldots \geq a_n \geq a_{n+1} \geq b_{n+1} \geq b_n \geq \ldots \geq b_1 \geq b$$

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Actually, they converge to a common limit:

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Inductively:

$$0 \leq a_n - b_n \leq \frac{a - b}{2^n}$$

Define $M(a,b)$ - the *arithmetic geometric mean of* a,b to be the common limit of the sequences $\{a_n\},\{b_n\}$

Arithmetic geometric mean

$$M(a,b)=\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n$$

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Simple observations:

$$M(a,b)=M(a_1,b_1)=M(a_2,b_2)=\dots$$

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$$M(a,a)=a$$

Ok then, what about calculating the limit ? Hmm...

$$M(\sqrt{2}, 1) = 1.19814023473559220 74\ldots$$

(accuracy to 19 decimal places).

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After 4 (!!!) iterations we get 19 digits accuracy, the convergence is really really (really) fast. Much better than $\frac{b-a}{2^n} \sim \frac{2}{100}$.

Better calculation:

$$\delta_{n+1} = a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n}$$

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So

$$\frac{\delta_{n+1}}{\delta_n^2} = \frac{1}{2(\sqrt{a_n} + \sqrt{b_n})^2} \xrightarrow[n\to\infty]{} \frac{1}{8M(a,b)}$$

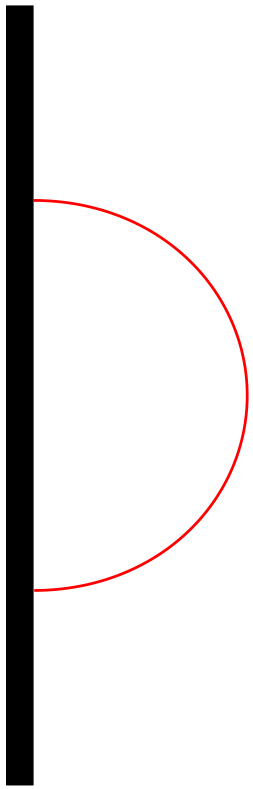
convergence rate

$$\delta_{n+1} \sim C\delta_n^2$$

Indeed, convergence rate is much better - it is quadratic: if at stage n we have k digits accuracy, then at stage $n+1$ our accuracy is $2k$ digits.

The story of the lemniscate

In 1691 - Jacob Bernoulli was working on the following problem:
A thin elastic rod is bent until the two ends are perpendicular to a given line:



Bernoulli showed that the upper half of the curve is given by an equation of the form

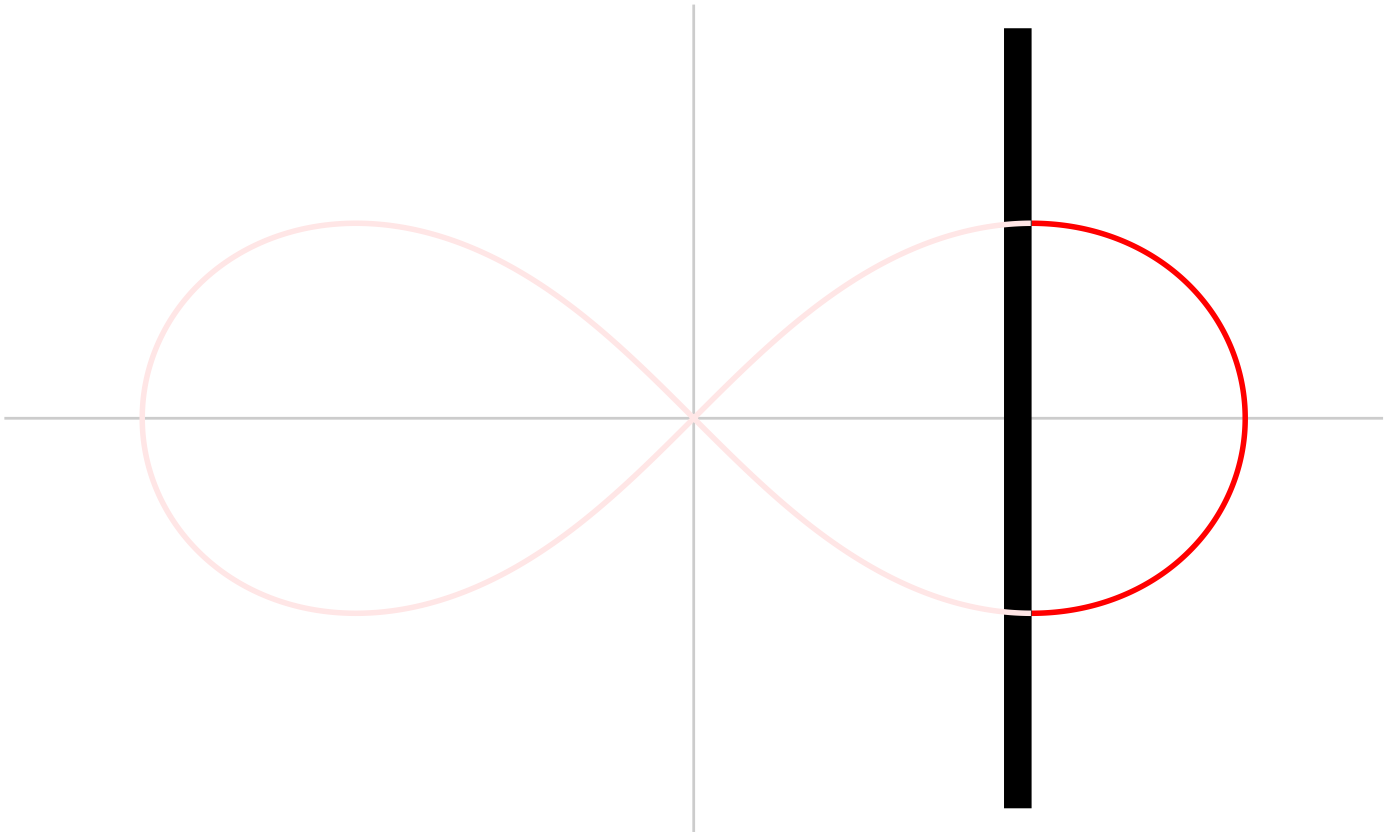
$$y = \int_0^x \frac{z^2}{\sqrt{a^4 - z^4}} dz$$

How is this related to the lemniscate?

Bernoulli sought 'an algebraic curve whose rectification should agree with the rectification of the elastic curve'

In 1694 he found the *lemniscate* (Greek for ribbon):

$$x^2 + y^2 = a\sqrt{x^2 - y^2}$$



1694 - Jacob's younger brother, Johann independently discovered the lemniscate !

Both papers appeared in Acta Eruditorum:
Jacob - September 1694
Johann -October 1694

Anyway, how is this related to agm ???

Stirling (1730):

The length of $1/4$ lemniscate:

$$A = \int_0^1 \frac{1}{\sqrt{1-z^4}} dz = 1.21102877714605987$$

and also:

$$B = \int_0^1 \frac{z^2}{\sqrt{1-z^4}} dz = .5990701173677961 1$$

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Which agrees with $M(\sqrt{2}, 1)$ to 16 decimal places !

Carl Freidrich Gauss

$$M(\sqrt{2},1) = 2 \cdot B = \int_0^1 \frac{z^2}{\sqrt{1-z^4}} dz$$

In 1786 Euler showed

$$A \cdot B = \int_0^1 \frac{1}{\sqrt{1-z^4}} dz \cdot \int_0^1 \frac{z^2}{\sqrt{1-z^4}} dz = \frac{\pi}{4}$$

Coupled with Eulers result we get

$$\tilde{\omega} = 2\int_0^1 \frac{1}{\sqrt{1-z^4}}dz = \frac{\pi}{M(\sqrt{2},1)} = \text{length of } 1/2 \text{ Lemniscate}$$

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Compare with:

$$2\int_0^1 \frac{1}{\sqrt{1-z^2}}dz = \pi = \text{length of } 1/2 \text{ circle}$$

He gave a special name to the function: arclmnsin x

$$\text{arclmnsin } x = \int_0^x \frac{1}{\sqrt{1-z^4}} dz$$

Then $2 \cdot \text{arclmnsin} 1 = \tilde{\omega} = \frac{\pi}{M(\sqrt{2},1)}$ - the period of the lemniscate.

Compare to

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-z^2}} dz$$

with $2 \cdot \arcsin 1 = \pi$ - the period of the circle.

Theorem [Gauss]

$$M(a,b)\cdot\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{a^2(\cos\phi)^2+b^2(\sin\phi)^2}}d\phi=\frac{\pi}{2}$$

(exercise: write the equation of the lemniscare in polar coordinates, then write down the expression for its arclength)

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Proof: Denote

$$I(a,b)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{a^2(\cos\phi)^2+b^2(\sin\phi)^2}}d\phi$$

Key step: Show that

$$I(a,b)=I(a_1,b_1)$$

where

$$a_1=\frac{a+b}{2}\qquad b_1=\sqrt{ab}$$

If we do that, we are done, for then

$$I(a,b) = I(a_1,b_1) = I(a_2,b_2) = \dots$$

Therefore

$$I(a,b) = \lim_{n\to\infty} I(a_n,b_n) = \lim \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{a_n^2(\cos\phi)^2 + b_n^2(\sin\phi)^2}} d\phi$$

The integrand converges uniformly in ϕ to

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{M(a,b)^2(\cos\phi)^2 + M(a,b)^2(\sin\phi)^2}} d\phi = \frac{1}{M(a,b)} \cdot \frac{\pi}{2}$$

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Ingenious change of variables !

$$\sin\phi = \frac{2a\sin\phi'}{a+b+(a-b)(\sin\phi')^2}$$

The range $0 \leq \phi' \leq \frac{\pi}{2}$ corresponds to $0 \leq \phi \leq \frac{\pi}{2}$.

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Gauss then writes 'after the development has been made correctly, it will be seen that'

$$\frac{1}{\sqrt{a^2(\cos\phi)^2 + b^2(\sin\phi)^2}}d\phi = \frac{1}{\sqrt{a_1^2(\cos\phi')^2 + b_1^2(\sin\phi')^2}}d\phi'$$

(Challenge - try to fill in the details ...)

How the **** did Gauss come up with this substitution ???

Lets go back to the integral we are trying to compute:

$$2\int_0^1 \frac{1}{\sqrt{1-z^4}}dz$$

Why is it hard to compute?

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Why is it hard to compute?

Try all you like, the integral

$$2\int_0^x \frac{1}{\sqrt{1-z^4}}dz$$

can not be expressed using elementary functions: polynomials, $\sin, \cos, \exp$ and their inverses.

Compare to

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After a change of variables $z = \sin t$ we get

$$2\int_0^{\arcsin x} \frac{\cos t}{\cos t}dt = 2\int_0^{\arcsin x} 1dt = 2\arcsin x$$

For $x = 1$ we get a period of the function $\sin x$ - namely π.

In fact we can calculate all integral of the form

$$\int \frac{P(\sin t, \cos t)}{Q(\sin t, \cos t)} dt$$

for any polynomials in two variables P, Q like

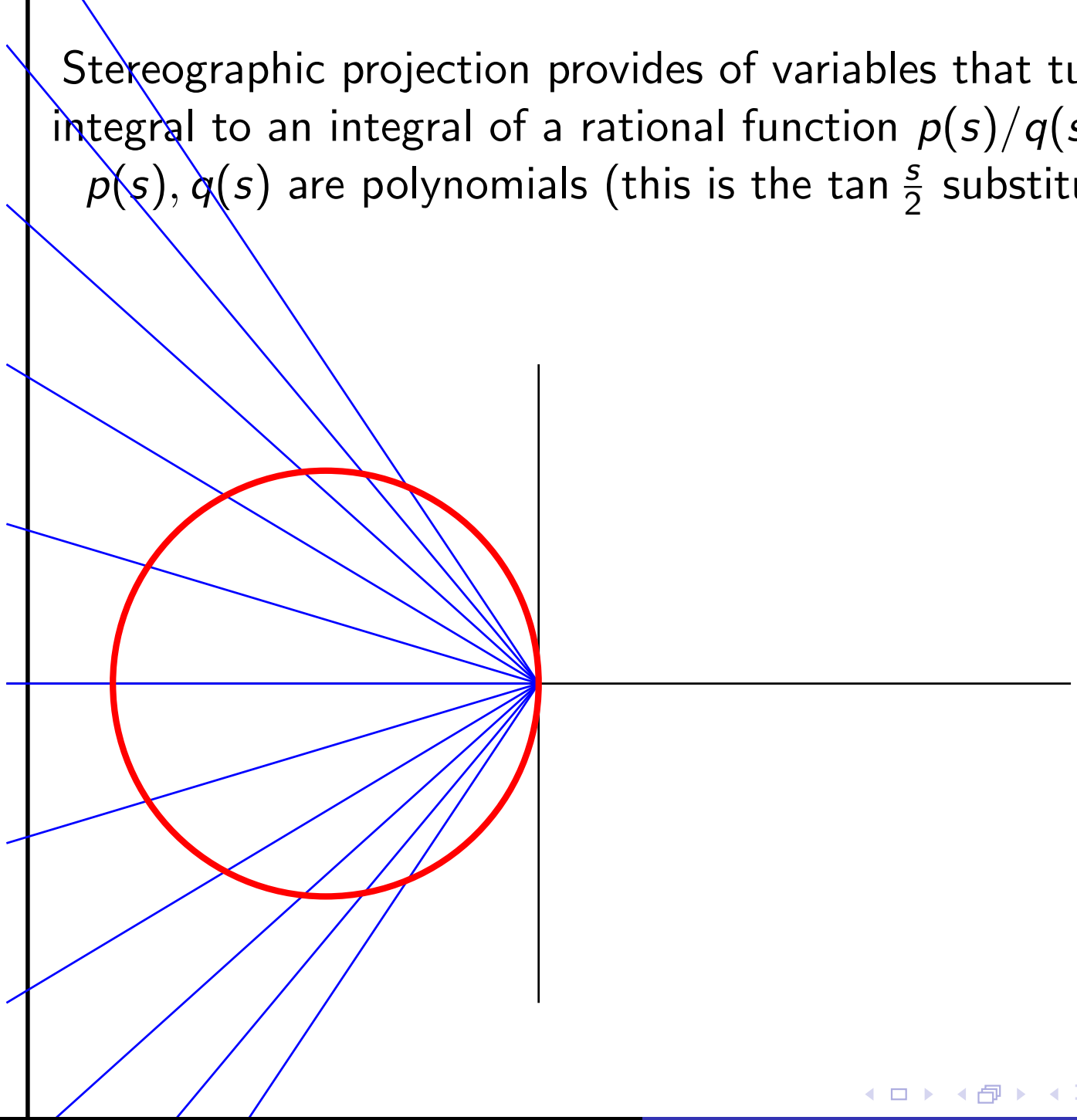
$$\int 1 dt \qquad \text{which is what we got before}$$

but also

$$\int \frac{(\sin t)^2(\cos t)^5 + \sin t}{(\cos t)^{16} + (\sin t)^7(\cos t)^{13}} dt$$

HOW SO?

Stereographic projection provides of variables that turns the integral to an integral of a rational function $p(s)/q(s)$ where $p(s), q(s)$ are polynomials (this is the tan $\frac{s}{2}$ substitution).



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Making a change of variable $z = 1 - 1/x$ we get

$$\int_{-\infty}^0 \frac{1}{\sqrt{(x^2-2x+2)(2x-1)}}dx$$

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Consider the equation

$$y^2 = (x^2-2x+2)(2x-1)$$

This is a special case of

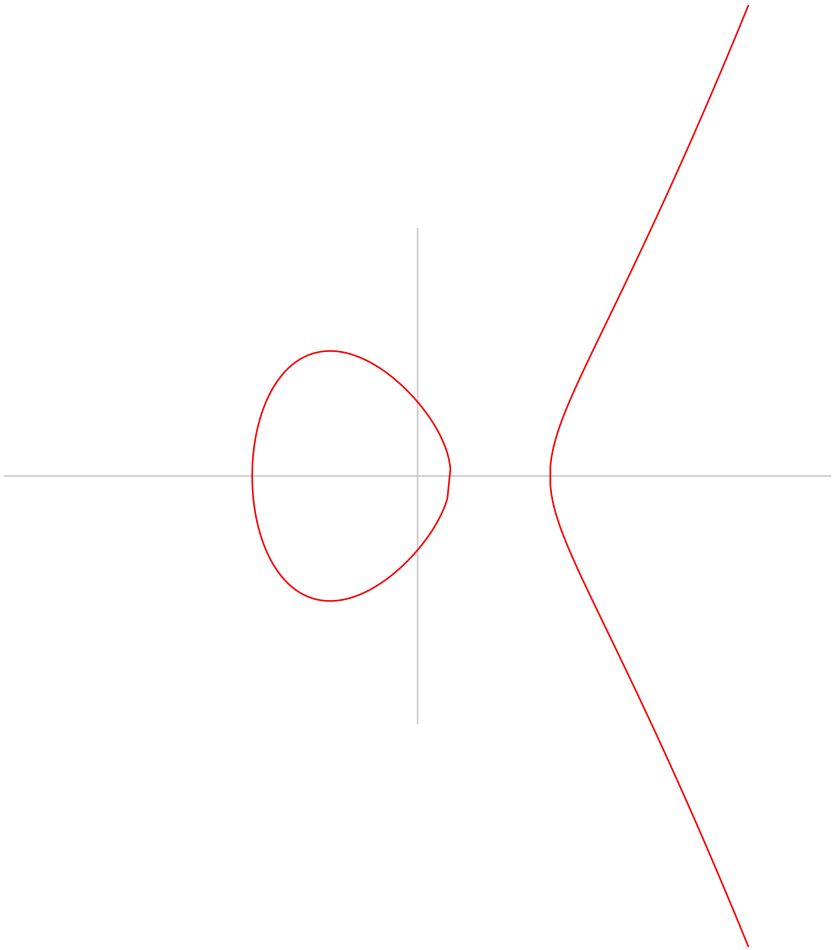
$$y^2 = ax^3 + bx^2 + cx + d$$

The graph describing the solutions to an equation of this type called an *elliptic curve*, and an integral

$$\int \frac{dx}{y(x)}$$

is called an *elliptic integral* (of the first kind).

If all the roots of P are real, then the curve looks like this.



There is no good 'stereographic projection' here (we can only get a map that is $2:1$)

Equations behave much better when we look at them over the complex numbers $\mathbb{C}$.

Recall our equation was

$$y^2 = ax^3 + bx^2 + cx + d$$

and we now think of y, x as taking complex values.

We take one more step - we *projectivize*. We first make the equation homogeneous:

$$\left(\frac{y}{z}\right)^2 = a\left(\frac{x}{z}\right)^3 + b\left(\frac{x}{z}\right)^2 + c\left(\frac{x}{z}\right) + d$$

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multiplying by z^3 we get

$$y^2z = ax^3 + bx^2z + cxz^2 + dz^3$$

(If we substitute $z = 1$ we get our original equation).

We now look at all triples (x, y, z) satisfying the equation,

$$y^2z = ax^3 + bx^2z + cxz^2 + dz^3$$

but we identify two triples

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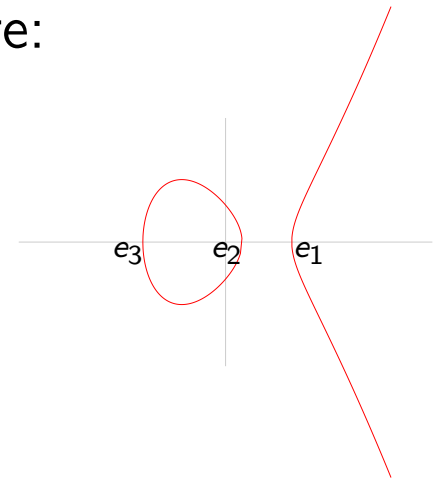
The set of all triple $(x, y, z) \in \mathbb{C}^3$ with this identification is denoted $\mathbb{PC}^3$ (or $\mathbb{P}^2\mathbb{C}$). This is a way to make the set of solutions compact.

DON'T GET LOST JUST YET !

Now we can allow ourselves to take linear transformations of $\mathbb{PC}^3$ (möbius transformations), and actually assume that our elliptic curve has 3 real roots $e_1 \geq e_2 \geq e_3$ and we are looking at the integral

$$\int_{e_3}^{e_2} \frac{1}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} dx$$

Remember the picture:



At last we are ready to describe what the agm substitution does:

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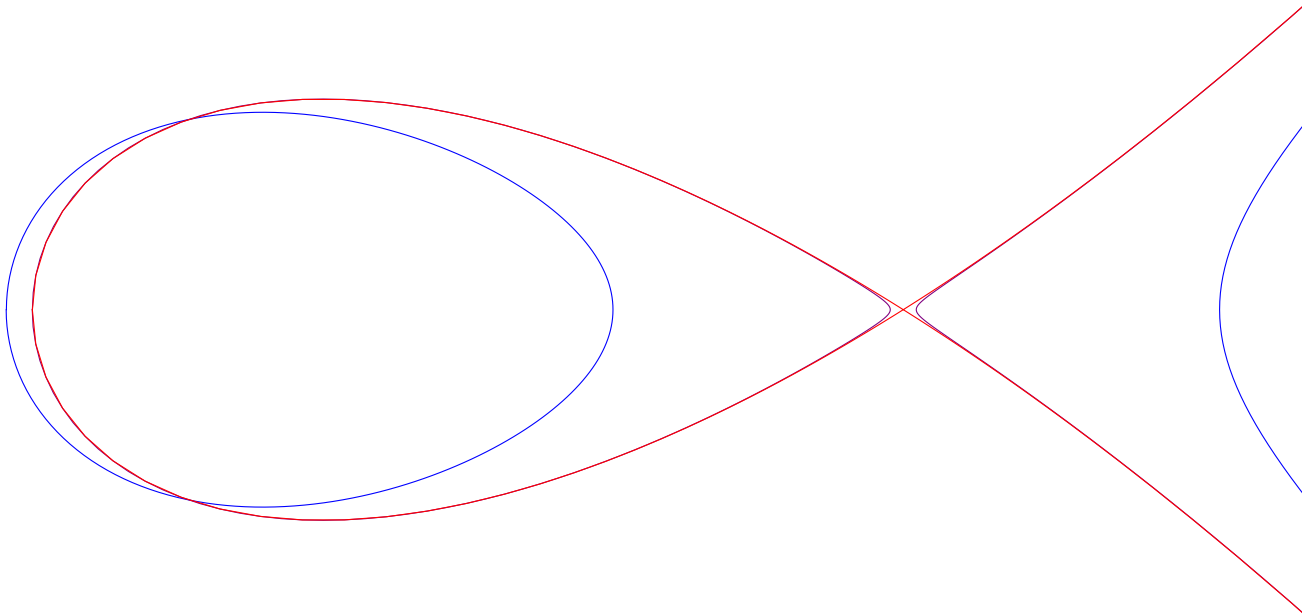
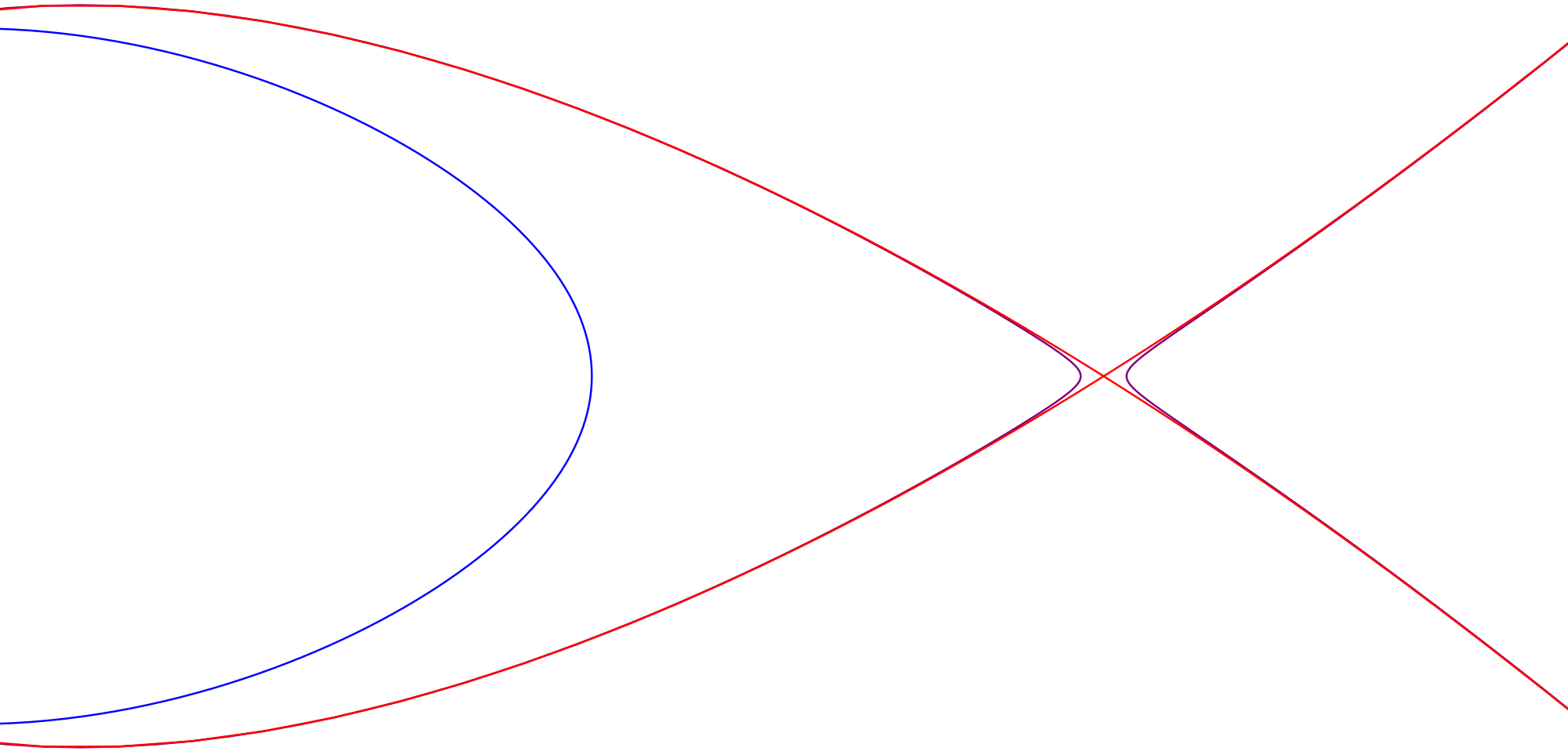
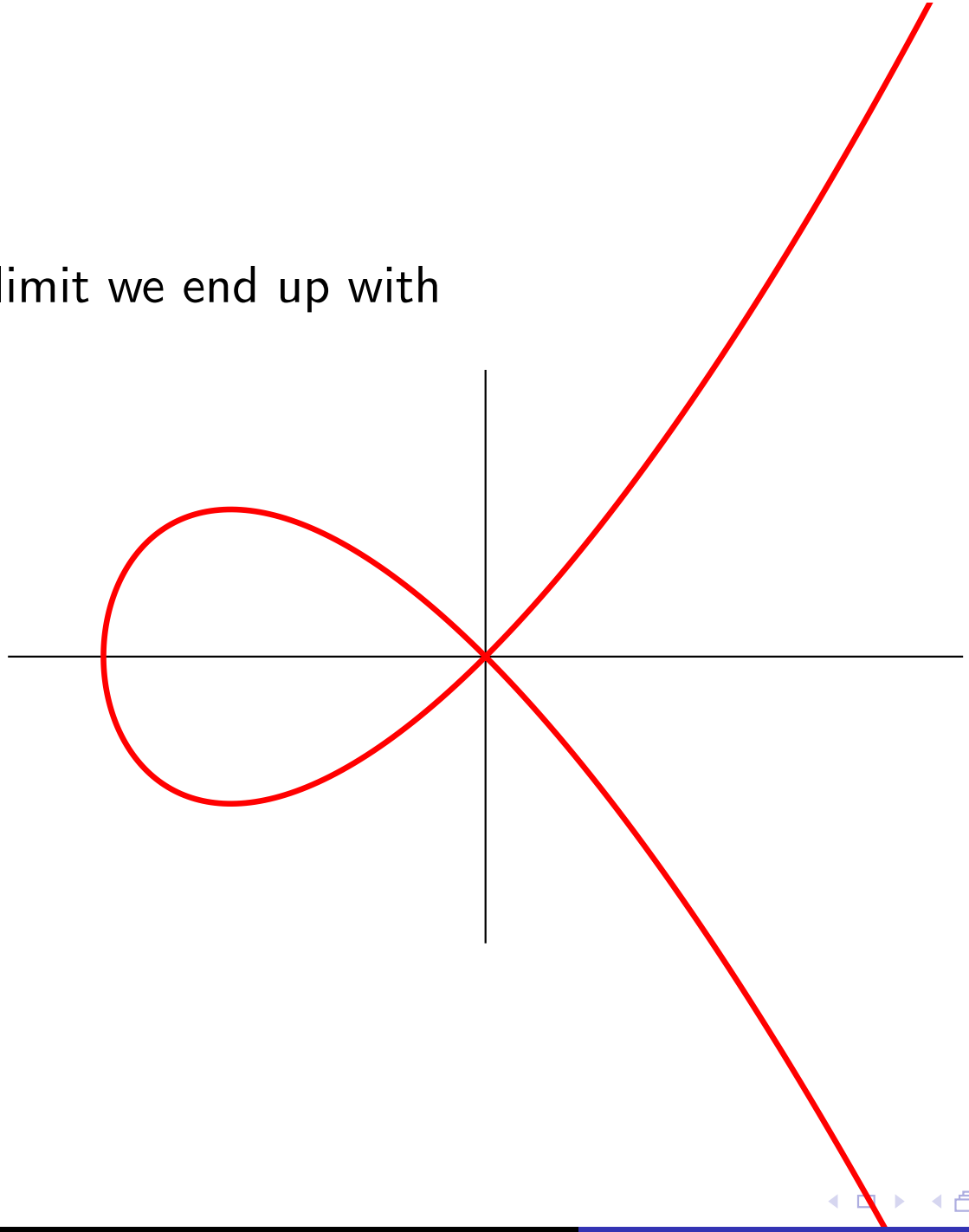


This is a picture of 2 iterations (recall that 2 iterations of the agm give us 4 decimal point precision).

Lets zoom in: e_1, e_2 are getting really close ...



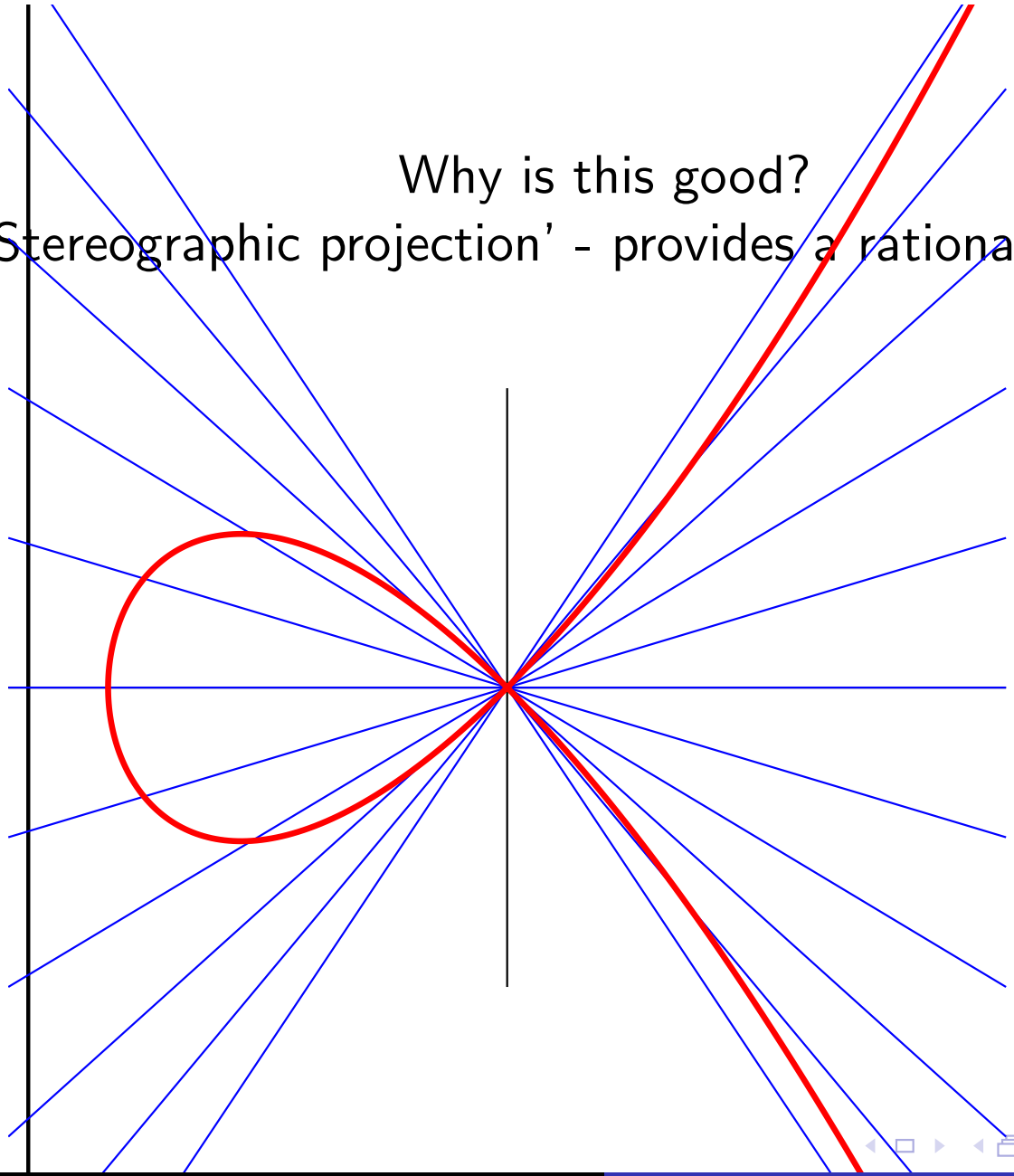
In the limit we end up with



Why is this good?

Why is this good?

'Stereographic projection' - provides a rationalization!!!



At the integral level - this is providing a sequence of approximations ending with something of the form

$$\int_{e_3'}^{e_2'} \frac{1}{\sqrt{(x-e_2')^2(x-e_3')}} dx$$

(these are not the original e_2, e_3 we stated out with)

HOW?

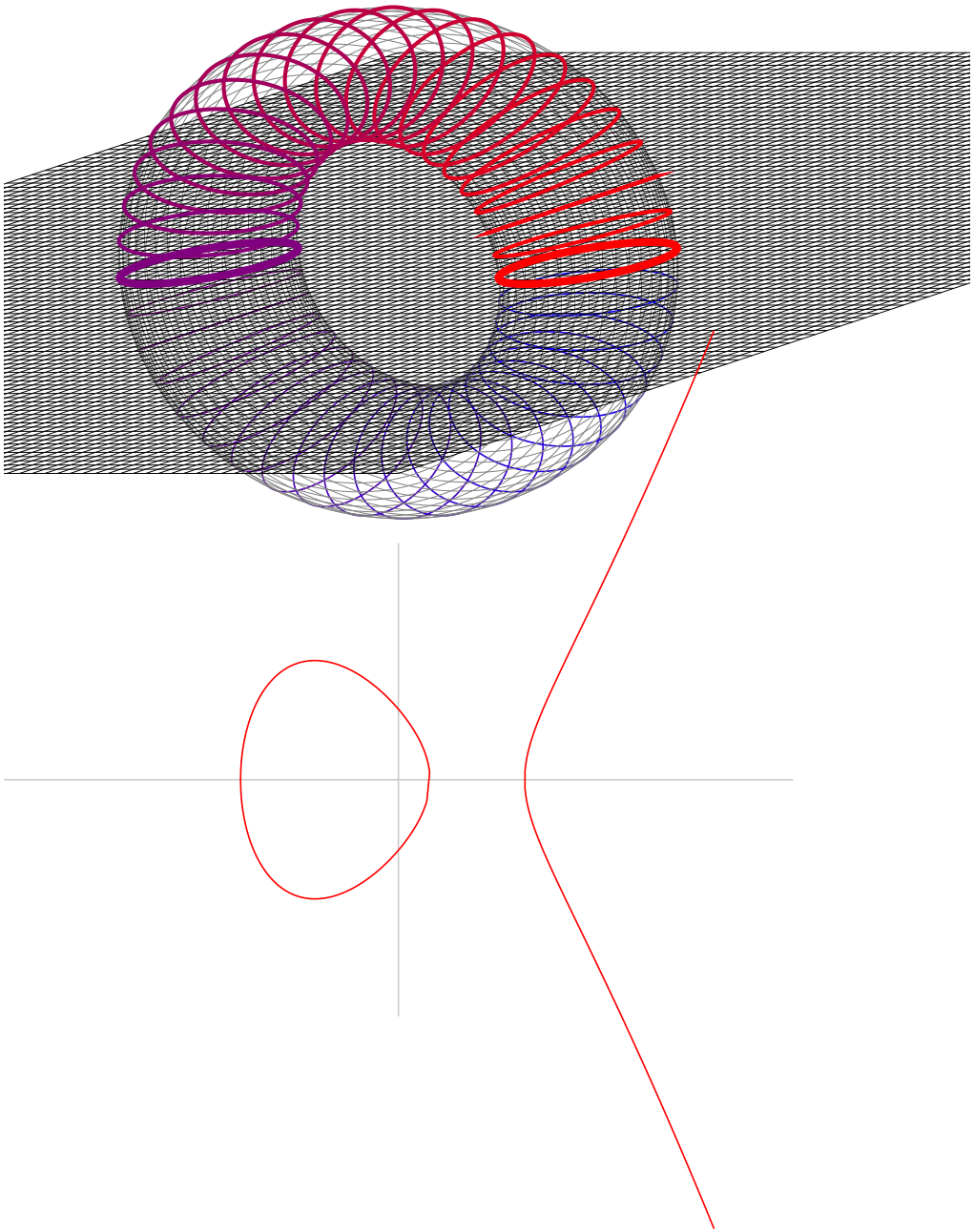
It turns out that a miracle happens and the set of solutions to the equation

$$y^2z = ax^3 + bx^2z + cxz^2 + dz^3$$

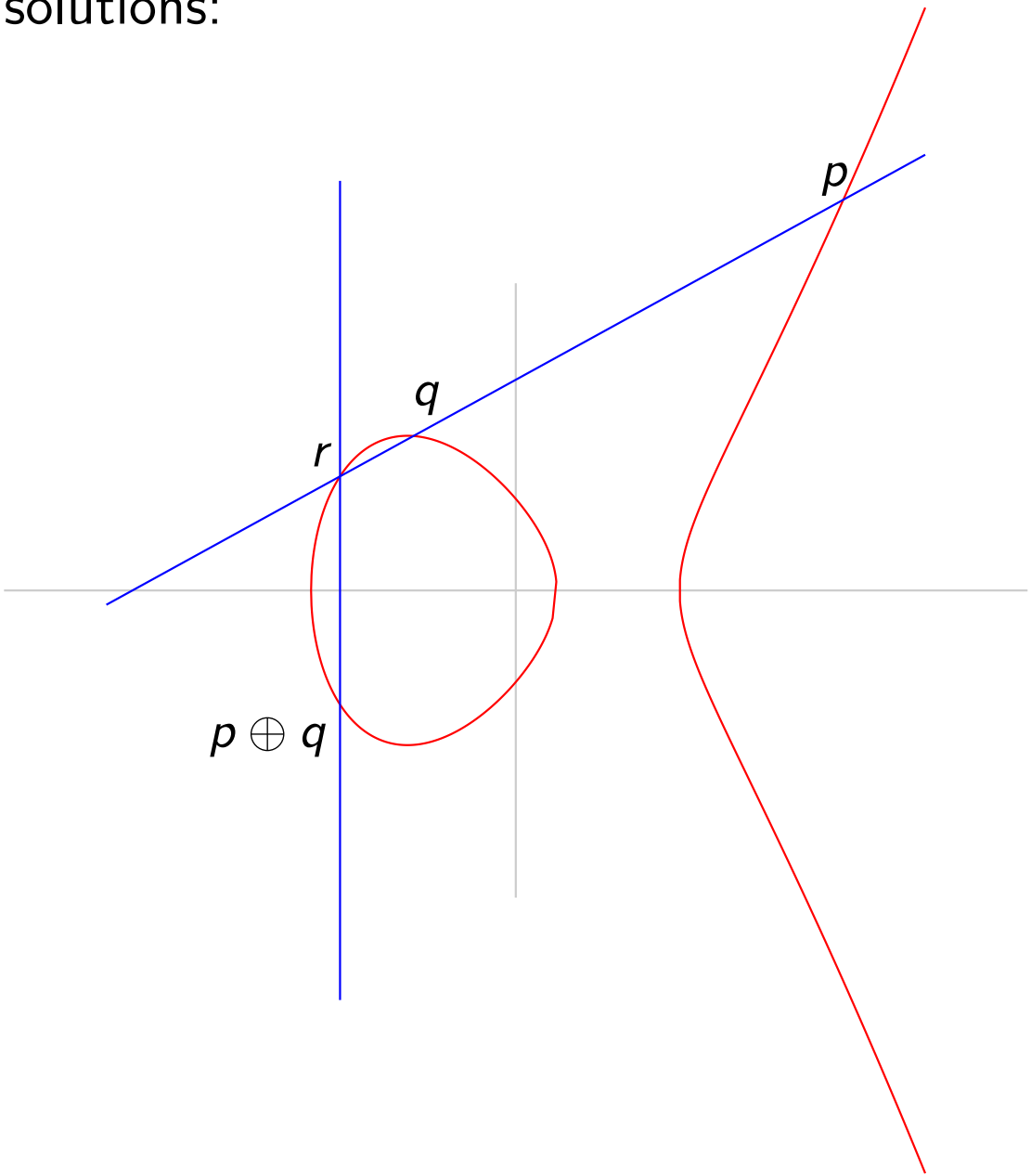
form a group, and this group is isomorphic to $\mathbb{C}/\Lambda$ where

$$\Lambda = \mathbb{Z} + \omega\mathbb{Z}$$

is a lattice.



Here is what the group operation looks like on the subset of real solutions:



Where were we ...

$$F : \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z}) \hookrightarrow \mathbb{P}\mathbb{C}^3$$
$$z \longmapsto [\wp(z) : \wp'(z) : 1]$$
$$0 \longmapsto [0 : 1 : 0]$$

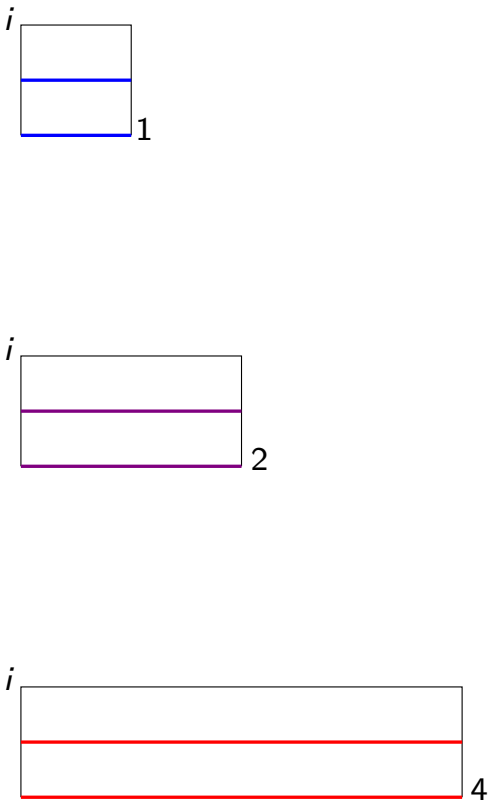
The image of F is an elliptic curve E. (The map from $\mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ onto its image is a homomorphism of (abelian) groups).

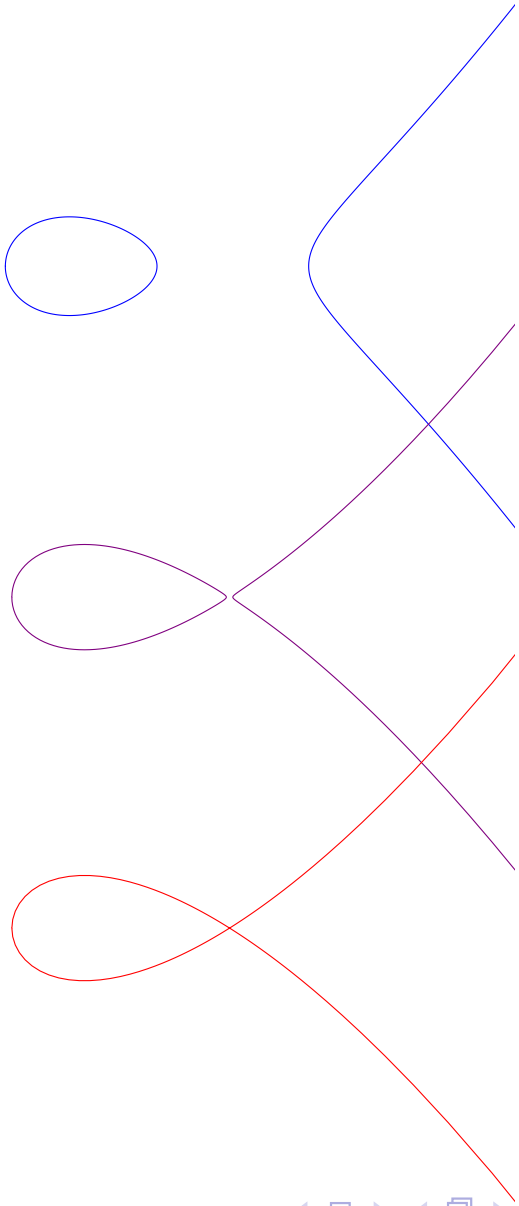
$$x = \wp(z); \quad dx = \wp'(z)dz = ydz$$

Thus

$$\int_{e_3}^{e_2} \frac{dx}{y} = \int_{F^{-1}([e_3,e_2])} 1dz$$

What happens with agm? division of fundamental domain by 2 !





Who is $\wp$? This is the Weierstrass function, it is defined as follows:

Weierstrass $\wp$ function

$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{w\in\Lambda} \frac{1}{(z-w)^2} - \frac{1}{w^2}$$

Recall the map

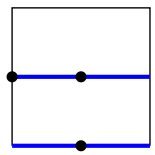
$$\begin{aligned} F : \mathbb{C}/\Lambda &\hookrightarrow \mathbb{P}\mathbb{C}^3 \\ z &\longmapsto [\wp(z,\Lambda) : \wp'(z,\Lambda) : 1] \end{aligned}$$

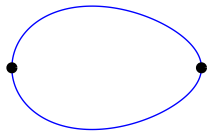
Example 0

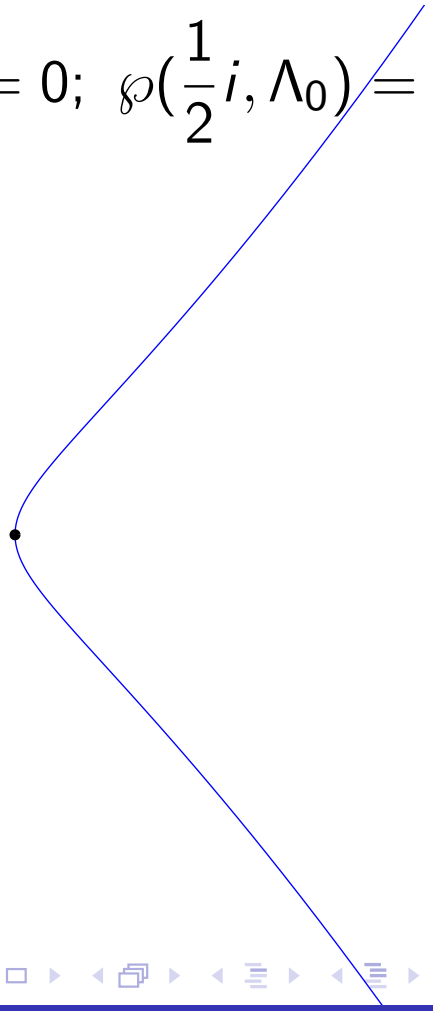
This is the zero iteration of agm:

$$\Lambda_0 = \mathbb{Z} + i\mathbb{Z} \quad \leftrightarrow \quad y_0 = (x-e)x(x+e)$$

$$\wp(0,\Lambda_0) = \infty;\ \wp(\frac{1}{2},\Lambda_0) = e;\ \wp(\frac{1}{2}+i\frac{1}{2},\Lambda_0) = 0;\ \wp(\frac{1}{2}i,\Lambda_0) = -e;$$





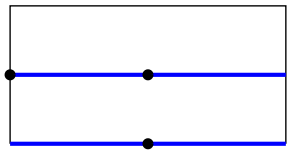


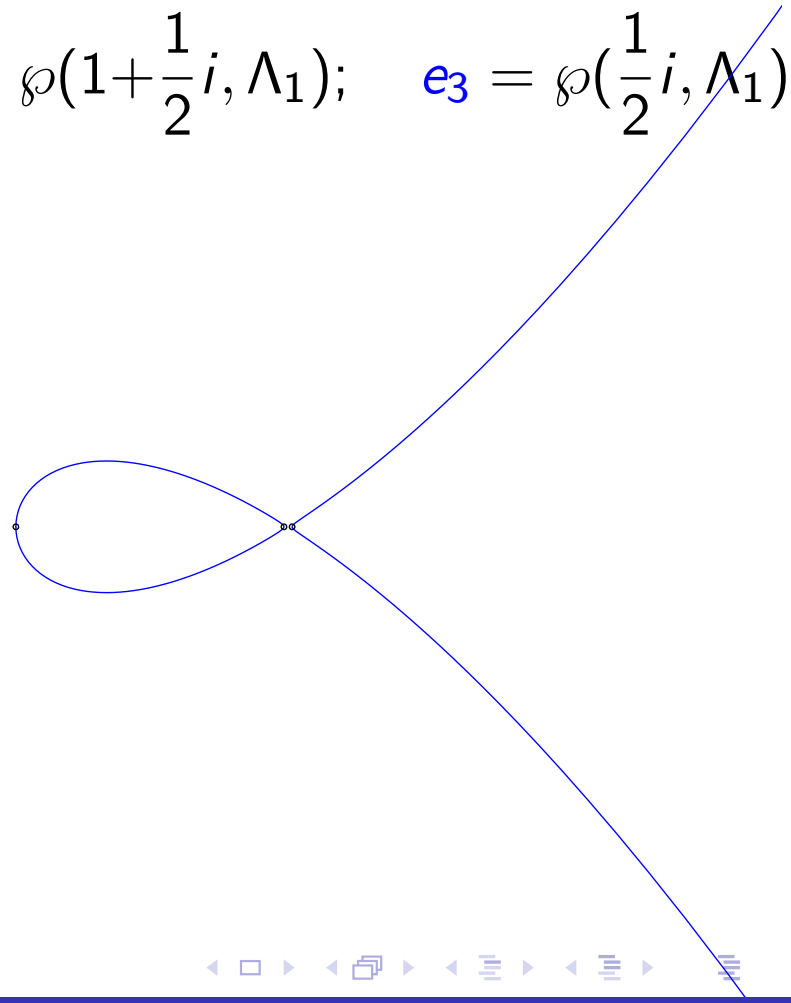
Example 1

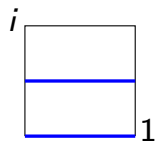
This is first iteration of agm:

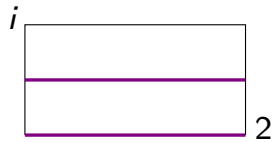
$$\Lambda_1 = 2\mathbb{Z} + i\mathbb{Z} \quad \leftrightarrow \quad y_1 = (x - e_1)(x - e_2)(x - e_3);$$

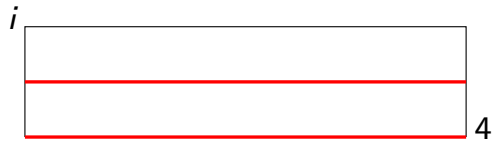
$$\infty = \wp(0, \Lambda_1);\ e_1 = \wp(1, \Lambda_1); \quad e_2 = \wp(1+\frac{1}{2}i, \Lambda_1); \quad e_3 = \wp(\frac{1}{2}i, \Lambda_1)$$

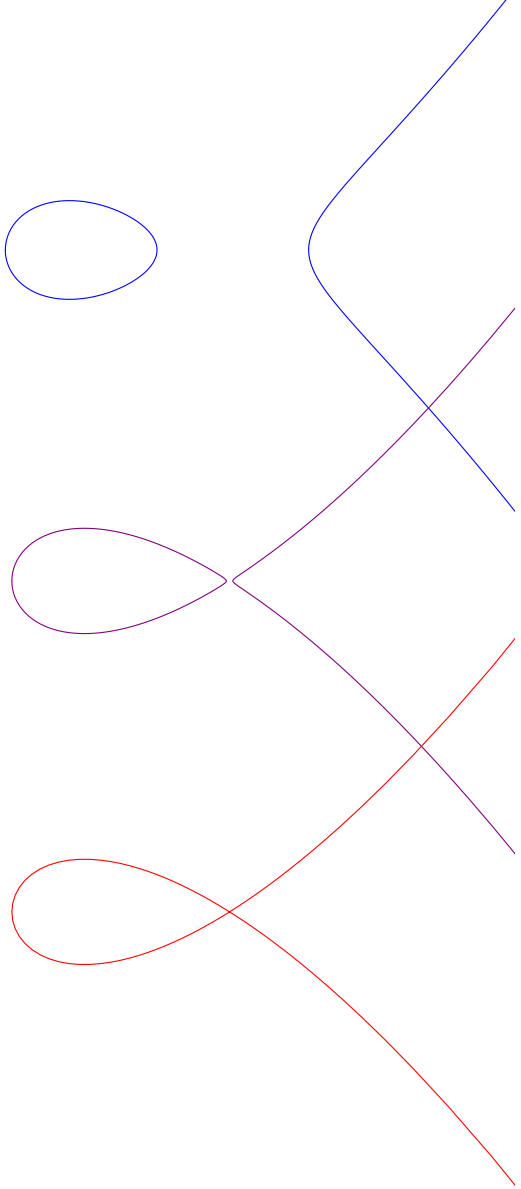





i
1


i
2


i
4



The connection between $a > b$ of agm and the roots $e_1 > e_2 > e_3$ of the elliptic curve :

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{a^2(\cos\phi)^2 + b^2(\sin\phi^2)}} d\phi = \int_{e_3}^{e_2} \frac{1}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}}$$

Where

$$e_1 + e_2 + e_3 = 0; \quad e_1 - e_3 = a; \quad e_2 - e_3 = b$$

Complex case

If a, b are no longer positive real numbers, but two complex numbers then we have a problem ! There is no obvious choice for

$$b_{n+1} = \sqrt{a_n b_n}.$$

We have two choices for each $n \geq 0$, so we get uncountably many possible sequences $\{a_n\}.\{b_n\}$ given a, b.

It turns out all these sequences converge, but only countably many have a non zero limit !

All possible limit values are related !

Theorem (Gauss)

There are special values μ, λ such that all possible limit values μ' are given by

$$\frac{1}{\mu'} = \frac{d}{\mu} + \frac{ic}{\lambda}$$

where d, c are relatively prime and $d \equiv 1 \mod 4$, $c \equiv 0 \mod 4$.

Let $\mathfrak{h} = \{\tau \in \mathbb{C} : \Im\tau > 0\}$ and set $q = e^{\pi i \tau}$

$$p(\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \Theta_3(\tau, 0)$$

This is a holomorphic function of τ. It is one of the Jacobi theta functions.

The set of possible values

$$\left\{ \frac{p(\gamma\tau_0)^2}{a} : \gamma \in \Gamma \right\}$$

i.e. the function $\frac{p(x)^2}{a}$ evaluated on the orbit of the point τ_0 under the group Γ.

$$\Gamma = \{\gamma \in SL_2(\mathbb{Z}); d \equiv 1(4),\ c \equiv 0(4)\}$$

Encryption

How is the agm used for encryption?

We can look at an elliptic curve E over a finite field $\mathbb{F}_q$ instead of over the complex field $\mathbb{C}$. We denote this curve $E(\mathbb{F}_q)$. It is important in this case to know the exact number of points on the curve $\#E(\mathbb{F}_q)$ (this number is $q + O(\sqrt{q})$), that is the number of $x, y \in \mathbb{F}_q$ satisfying the equation

$$y^2 = ax^3 + bx^2 + cx + d.$$

For cryptographic purposes, it is essential to find an elliptic curve E where the number $\#E(\mathbb{F}_q)$ has a large prime factor, so an estimate on its size is not good enough. The agm provides a quick way to evaluate $\#E(\mathbb{F}_q)$ (via a series of approximations to the (canonical lift of the) elliptic curve E).

Some links:

Cox, David A. *The arithmetic-geometric mean of Gauss.*. Enseign. Math. (2) 30 (1984), no. 3-4.

Ritzenthaler, Christophe *AGM for elliptic curves.*