Patterns in primes and dynamics on nilmanifolds

Tamar Ziegler

Technion/Hebrew University

EMS Lecture Series

Green-Tao-Z Theorem

Let $\{\psi_i(\vec{x})\}_{i=1}^{k}$ be k affine linear integer forms in n variables
Suppose no 2 forms are affinely dependent. Then

$$\left|\{\vec{x} \in [0,N]^n, \{\psi_i(\vec{x})\}_{i=1}^{k} \subset \mathbb{P}\}\right| \sim \mathfrak{S}(\vec{\psi})\frac{N^n}{(\log N)^k}$$

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Key ingredient: Inverse Theorem for Gowers $U_k[N]$.

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Key ingredient: Inverse Theorem for Gowers $U_k[N]$.

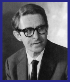
Goal: Describe developments in ergodic theory and arithmetic combinatorics leading to this theorem, as well a some of the ideas in the proof.

Szemerédi's Theorem (1975)

Let $E \subset \mathbb{N}$ of positive density, then E contains arbitrarily long arithmetic progressions.

$$d^*(E) = \limsup \frac{|E \cap [1, N]|}{N}$$

Klaus Roth's Theorem (1953)



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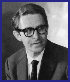

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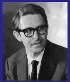

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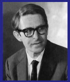

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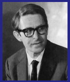

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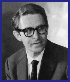

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Fourier analysis: show $1_E - \delta$ has a large Fourier coefficient.

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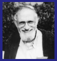

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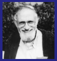

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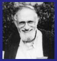

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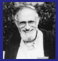

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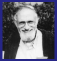

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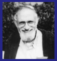

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then
$$E \cap (E - n) \cap \ldots \cap (E - kn) \neq \emptyset$$

New problem: Given a (ergodic) probability measure preserving system (m.p.s.) $\mathbf{X} = (X, \mathscr{B}, \mu, T)$, and a set $A \in \mathscr{B}$ such that $\mu(A) > 0$, find $n > 0$ such that

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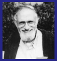

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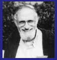

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Idea: study the m.p.s. $\mathbf{X}$ via morphisms to simpler m.p.s.

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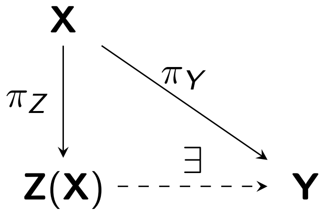
Universal property: if $\mathbf{Y}$ is Krönecker



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$$\lim \frac{1}{N}\sum_{n=1}^{N}\int \pi_* f(z)\pi_* f(z+n\alpha)\pi_* f(z+2n\alpha)d\nu > 0$$

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$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{T} & \mathbf{X} \\ \psi \downarrow & & \downarrow \psi \\ \mathbf{S^1} & \xrightarrow{\cdot\lambda} & \mathbf{S^1} \end{array}$$

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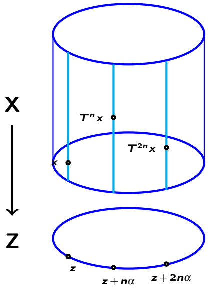

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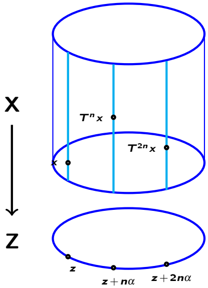

Converse clear: in Abelian groups $z, z+n\alpha$ determine $z+2n\alpha$

Definition (k characteristic factor)

$\pi : \mathbf{X} \to \mathbf{Y}$ is *k-characteristic* if

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- The Kronecker factor $\mathbf{Z}(\mathbf{X})$ is 2-characteristic (Furstenberg).

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Construct a universal sequence of factors

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- Or there is a morphism from $\mathbf{X}$ to a non trivial isometric extension of $\mathbf{Z}_k(\mathbf{X})$.

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$$\mathbf{X} \to \ldots \to \mathbf{Z}_k(\mathbf{X}) \to \mathbf{Z}_{k-1}(\mathbf{X}) \to \ldots \to \mathbf{Z}_1(\mathbf{X}) \to \star$$

• Either $\mathbf{X} \to \mathbf{Z}_k(\mathbf{X})$ is relatively weakly mixing
• Or there is a morphism from $\mathbf{X}$ to a non trivial isometric extension of $\mathbf{Z}_k(\mathbf{X})$.

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- This gives sufficient structure for proving Szemerédi's theorem.

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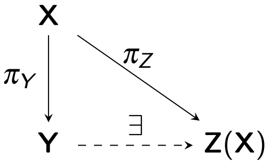

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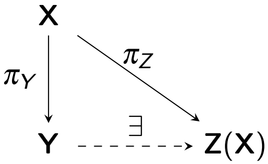

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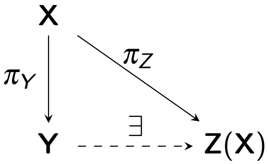

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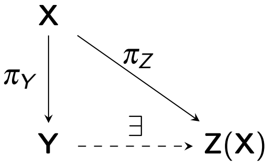

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What are the obstructions on $x, T^n x, \ldots, T^{(k+1)n} x$ preventing them from roming about freely in X ?

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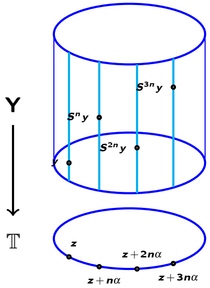

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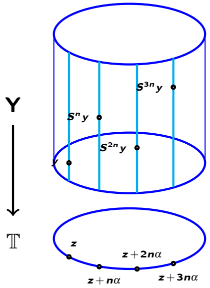

If there is a morphism $\mathbf{X} \to \mathbf{Y}$, these new obstructions will surface.

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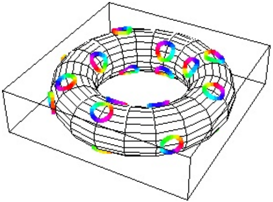

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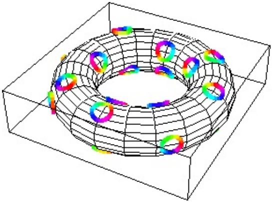

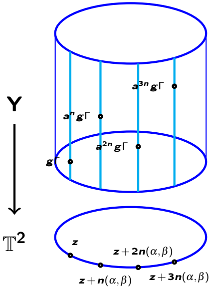

$g\Gamma$ is determined by $a^n g\Gamma, a^{2n} g\Gamma, a^{3n} g\Gamma$.

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- Cheat: $\mathbf{Y}$ is an inverse limit of 2-step nilsystems - a pro-nilsystem.

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Get increased density on a sub progression of size N^s (like Roth).

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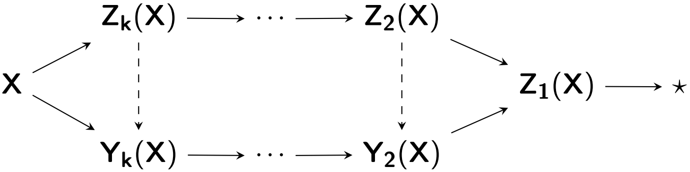

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- Apply Szemerédi's Theorem.

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The role of (pro)-nilsystems in the study of progressions in ergodic theory motivated Green-Tao to conjecture:

Inverse Conjecture for the Gowers norms

Global obstruction (scale N) to Gowers uniformity come from sequences arising from nilsystems.

We want more structure on $\pi_*(\tilde{1}_E)$, bounded is not good enough.

Want an optimal combinatorial factor with $\|\tilde{1}_E - \pi_*(\tilde{1}_E)\|_{U_k}$ is small. Back the inverse question !

When is $\|f\|_{U_k}$ large ???

The role of (pro)-nilsystems in the study of progressions in ergodic theory motivated Green-Tao to conjecture:

Inverse Conjecture for the Gowers norms

Global obstruction (scale N) to Gowers uniformity come from sequences arising from nilsystems.

Compare with

Local theorem for the Gowers norms (Gowers)

Local obstructions (scale N^t) to Gowers U_{k+1} uniformity come from phase polynomials of degree k.

Nilsequence: N/Γ is a k-step nilmanifold. $F : N/\Gamma \to \mathbb{D}$ is a "nice" function. $a \in N$.

$$g(n) = F(a^n\Gamma)$$

is a k-step nilsequence.

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Inverse Conjecture for the Gowers norms

$$\|f\|_{U_{k+1}(N)} \gg 1 \implies |\mathbb{E}_{x \leq N} f(x)\overline{g(x)}| \gg 1$$

for $g(n)$ a "bounded complexity" k-step nilsequence.

Progress:

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- Inverse Theorem for Gowers $U_3[N]$ norms (Green-Tao 05).

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- (modified) Inverse Conjecture for the Gowers norms $U_k[\mathbb{F}_p^n]$ is True ! (Bergelson-Tao-Z (10), Tao-Z (10,12)).

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Theorem (Green-Tao-Z 2012)

Inverse Conjecture for the Gowers $U_k[N]$ norms is true !

Szemerédi (1975)

Arithemtic progressions in sets of integers of positive denisty

Sz 75

Furstenberg 1977

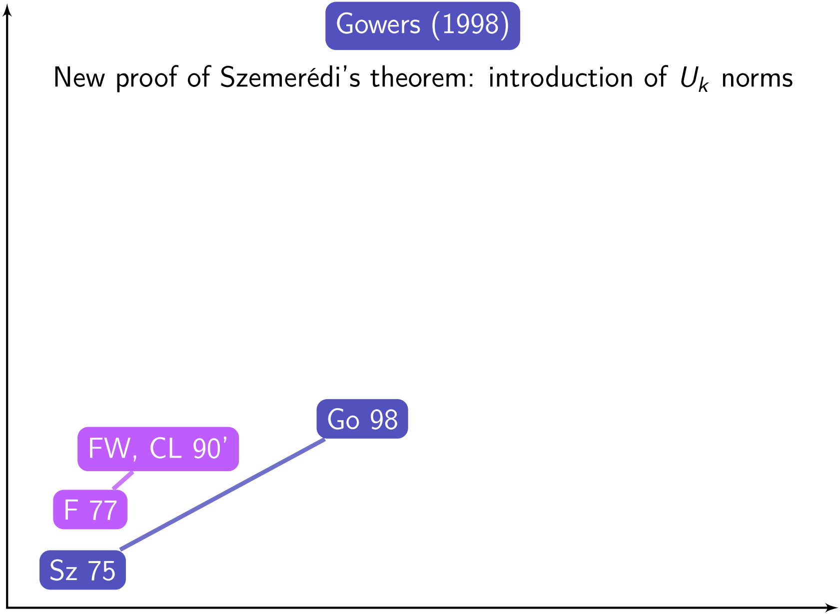
Ergodic thoeretic proof of Szemeredi's theorem

Furstenberg-Weiss, Conze-Lesigne (1990')

Ergodic context: role of 2-step nilpotency in 4 term progressions

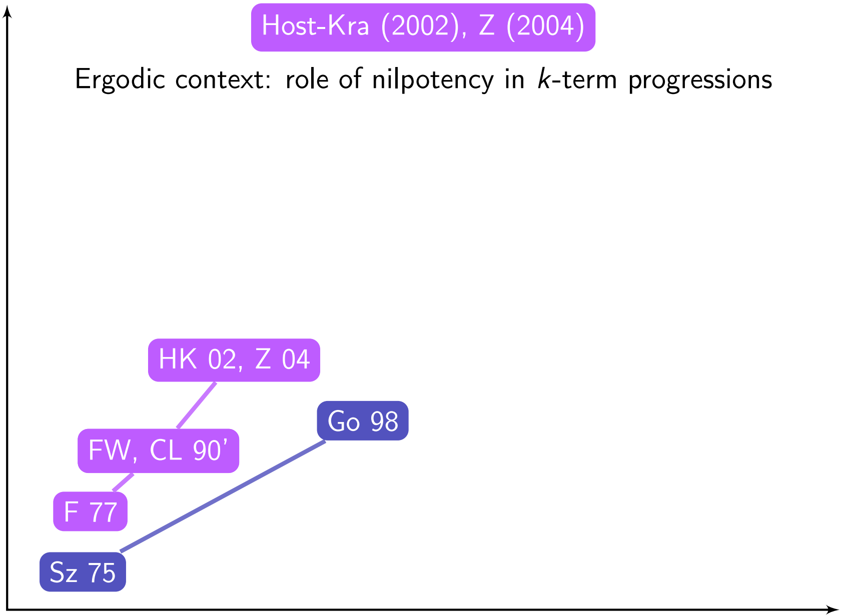
Gowers (1998)

New proof of Szemerédi's theorem: introduction of U_k norms



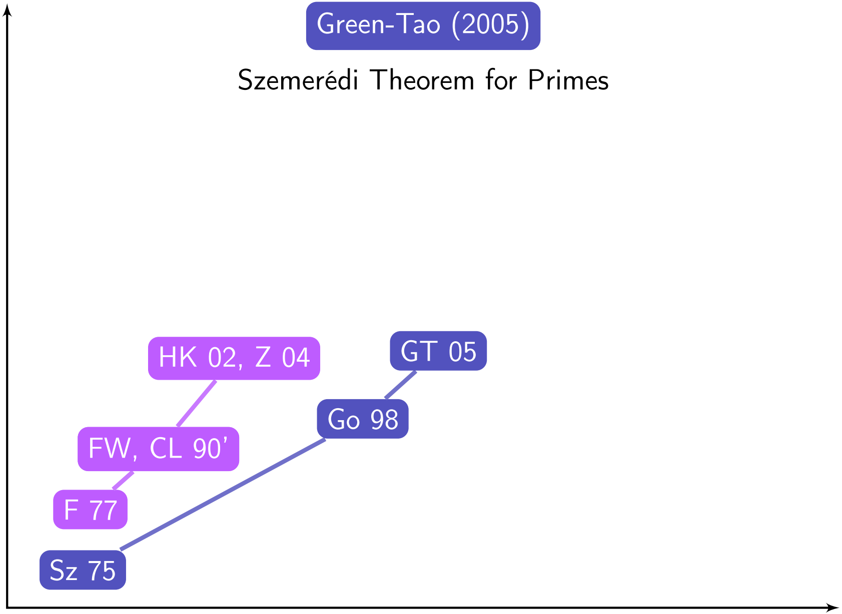
Host-Kra (2002), Z (2004)

Ergodic context: role of nilpotency in k-term progressions



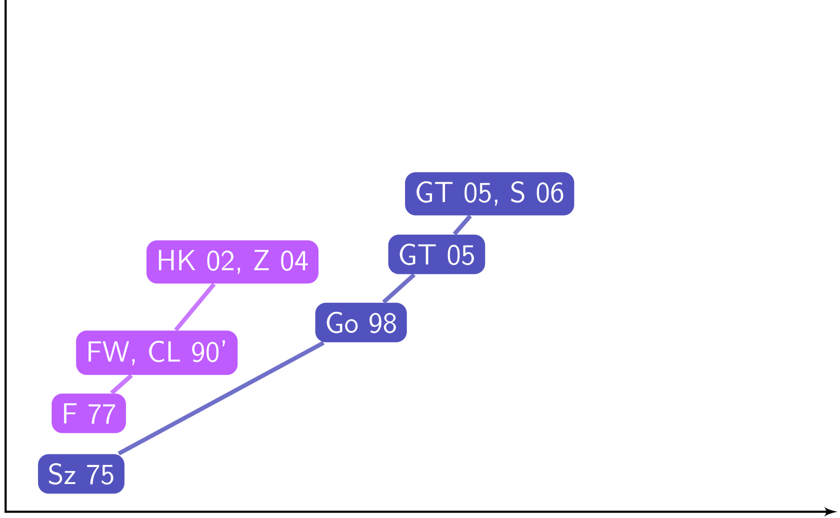
Green-Tao (2005)

Szemerédi Theorem for Primes



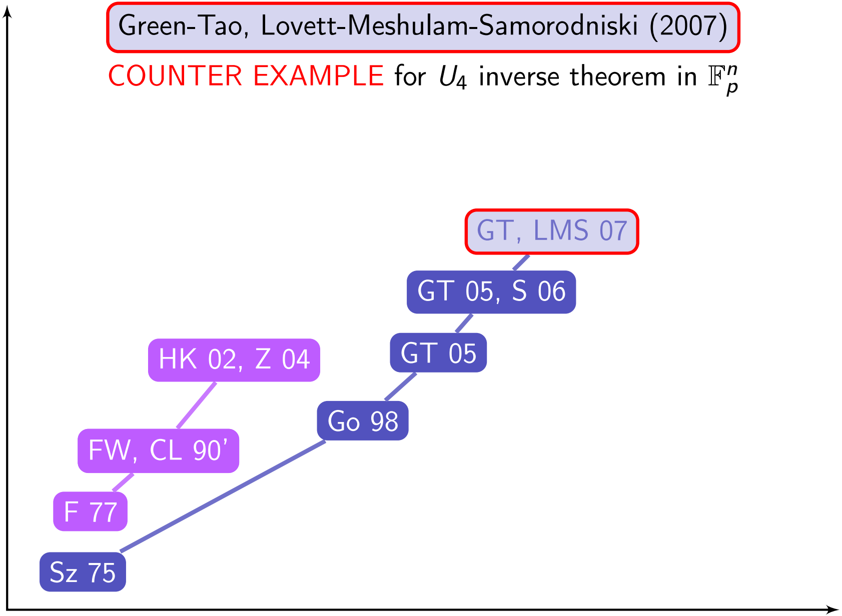
Green-Tao (2005), Samorodniski (2006)

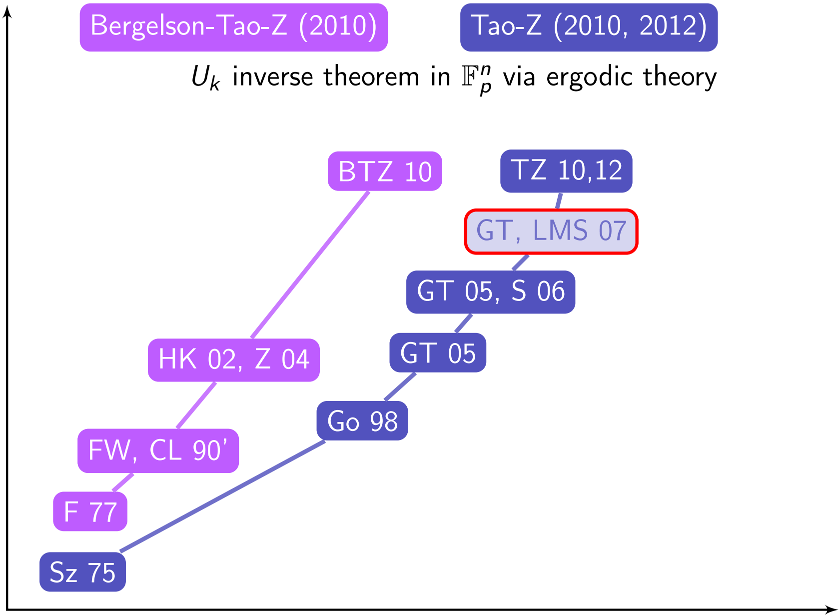
U_3 inverse theorem for $\mathbb{Z}_N$ and $\mathbb{F}_p^n$



Green-Tao, Lovett-Meshulam-Samorodniski (2007)

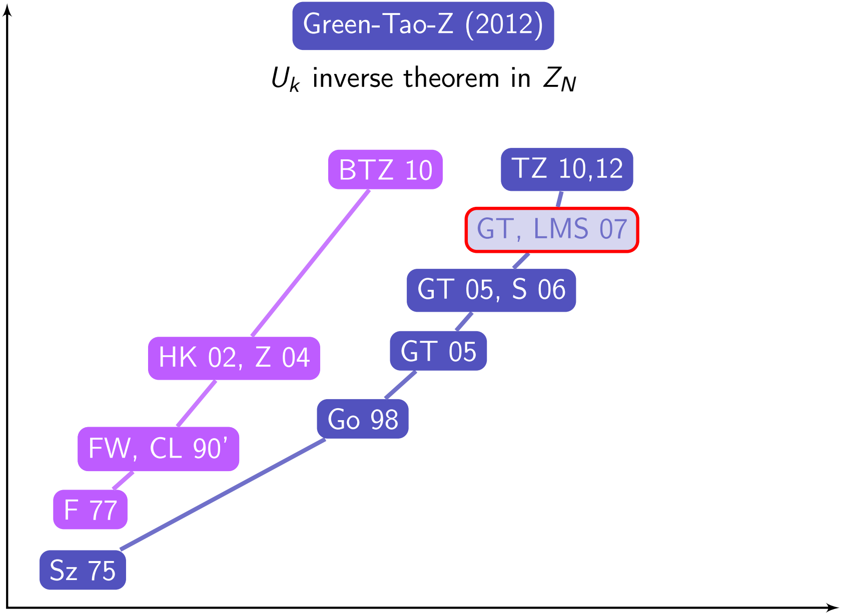
COUNTER EXAMPLE for U_4 inverse theorem in $\mathbb{F}_p^n$




Bergelson-Tao-Z (2010)
Tao-Z (2010, 2012)
U_k inverse theorem in $\mathbb{F}_p^n$ via ergodic theory
BTZ 10
TZ 10,12
GT, LMS 07
GT 05, S 06
GT 05
HK 02, Z 04
Go 98
FW, CL 90'
F 77
Sz 75

Green-Tao-Z (2012)

U_k inverse theorem in Z_N



Thank you !