Yang–Baxter type equations and posets of maximal chains [1]

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Abstract. This paper addresses the problem of constructing higher dimensional versions of the Yang-Baxter equation from a purely combinatorial perspective. The usual Yang–Baxter equation may be viewed as the commutativity constraint on the two-dimensional faces of a permutahedron, a polyhedron which is related to the extension poset of a certain arrangement of hyperplanes and whose vertices are in 1–1 correspondence with maximal chains in the Boolean poset $\mathcal{B}_n$. In this paper, similar constructions are performed in one dimension higher, the associated algebraic relations replacing the Yang–Baxter equation being similar to the permutahedron equation. The geometric structure of the poset of maximal chains in $S_{a_1}\times\cdots\times S_{a_k}$ is discussed in some detail, and cell types are found to be classified by a poset of 'partitions of partitions' in much the same way as those for permutahedra are classified by ordinary partitions.

0: Introduction

Suppose that A is a unital associative algebra. By the (constant quantum) Yang–Baxter equation (YBE) is meant the relation,

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}\ ,$$

amongst elements of $A\otimes A\otimes A$ satisfied by $R\in A\otimes A$, where R_{ij} denotes the element of $A\otimes A\otimes A$ defined by R in the $i^{\rm th}$ and $j^{\rm th}$ factor and $1\in A$ in the third. This equation arises in many different areas of mathematics and physics, such as quantum groups, two–dimensional exactly soluble models in statistical mechanics and knot theory. Just as the existence of, and solutions to, this equation have proved to be of importance in those areas (see, for example, [B], [D1], [Ji] and [Jo]), the search for appropriate higher dimensional analogues is relevant to, and is known to have close connections with, higher dimensional physical problems (see [CS], [MN], [Z1] and [Z2]). However, in the latter case, though many candidates have been proposed, none so far simultaneously satisfy a set of properties analogous to those satisfied by the YBE, and which made it so successful. It is as if the analogue of the YBE in higher dimensions depends upon the use to which it is to be put, a problem reminiscent of the question of the 'right' notion of bicategories or tricategories, in particular with respect to varying degrees of weakness. Currently there are the

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simplex equations [FM], the Zamolodchikov series ([Z1], [BS], [BB]) and at least two types of *permutahedron equations* ([KV], [L1]), which have been developed with varying decrees of connection to physical problems and different levels of solubility; all have reason to be considered as higher dimensional versions of the YBE.

Rather than discussing the requirements of a such an extension of the YBE from a physical perspective, for which the reader is referred to the extensive literature on the subject, this paper will discuss related purely geometric and combinatorial problems. In particular, many of the special properties of the YBE can be construed as combinatorial statements concerning certain posets derived from families of hyperplanes and the paper concentrates on these and their higher versions. Thus, these properties of the YBE are summarised (see also [Z]) in the following statements concerning a special hyperplane arrangement $X^1 = X^1(n, \boldsymbol{\alpha})$, dependent on n distinct, real parameters α_i, considered as forming the vector $\boldsymbol{\alpha}$,

(1A) The intersection poset defining the incidence properties of X^1 is the part of the Boolean poset C^0, of all subsets of $\{1, 2, \ldots, n\}$ with rank ≥ 2, in reverse order.

(1B) The uniform extension poset, P^1, defining the possible real pseudohyperplane extensions of X^1 has elements in 1–1 correspondence with S_n, via an inversion-set map.

(1C) P^1 is a ranked poset with unique minimal and maximal elements.

(1D) There is a geometric realisation of elements of P^1 as the vertices of a convex polyhedron (permutahedron) whose faces are identified, up to translation, by elements of the poset C^1 of partitions of $\{1, 2, \ldots, n\}$.

(1E) If a copy of a vector space, V, is placed at each vertex of the polyhedron of (1D) and maps $R_{ij}: V \to V$ are placed on the edges according to the type $\{i, j\} \in C^1$ of the edge, then the conditions for commutativity of all two-dimensional faces of the polyhedron lead to the YBE, which we will denote $\mathcal{E}^1$,

$$R_{ij} R_{ik} R_{jk} = R_{jk} R_{ik} R_{ij}$$
$$R_{ij} R_{kl} = R_{kl} R_{ij} \; .$$

The orientation here used on edges is induced, using (1B), by the partial order on S_n specified by the standard length function.

At first sight, it might seem that the most central combinatorial object for the YBE is the weak Bruhat order on the symmetric group. Indeed this *is* the structure most directly connected with the YBE. For example, any solution of the YBE leads to representations of the braid groups; see (1E). However, we wish to view this strucure, P^1, as merely a corollary structure from X^1 via the process of forming the uniform extension poset.

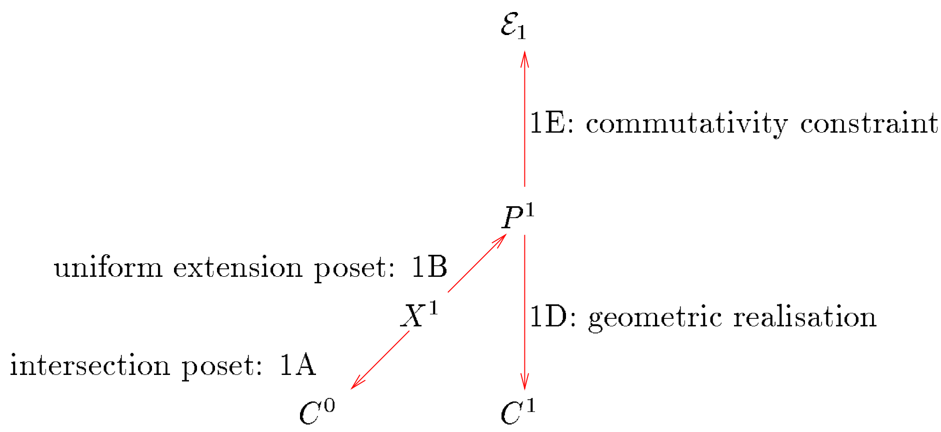


Figure 1: Relations between structures for YBE

The central point of our perspective is what one might call a form of *meta-functoriality.* To a first order of approximation (in fact, precisely to the degree that this statement has a well-defined and unique meaning), a concept relevant at one dimension can be obtained from that at the next lower dimension by a formal procedure of *categorification.* Broadly, this process starts from the axiomatic definition of an algebraic structure and generates another structure by replacing each occurrence of a set by a category, a map by a functor, and of equality of elements by the existence of a morphism; the reader is referred to [L2] and [CF] for some examples.

Thus, the constraints on a system of equations that make it behave in a way analogous to the YBE in higher dimensions for the purpose of applications, can be obtained to the first approximation by categorification of those at the next dimension lower. Assuming that it is possible to find appropriate algebraic structures analogous to the YBE (and that includes whatever vector spaces, tensors and equations may be involved) at each dimension, say $\mathcal{E}^n$, they may then also be acquired, to first order of approximation, from those at the previous dimension by categorification. The practical problem with this approach is that there does not currently exist a precise definition of categorification! However, the aim here is to investigate the extent to which the connections between the various structures of Figure 1, on a combinatorial level, may be extended to higher dimensions, yielding Figure 2. We desire to determine what are X^n, P^n and C^n and the combinatorial constructions given by the horizontal arrows (versions of 'categorification') in that diagram. For example at the level of P^n, it will be seen to be the construction of a poset of maximal chains.

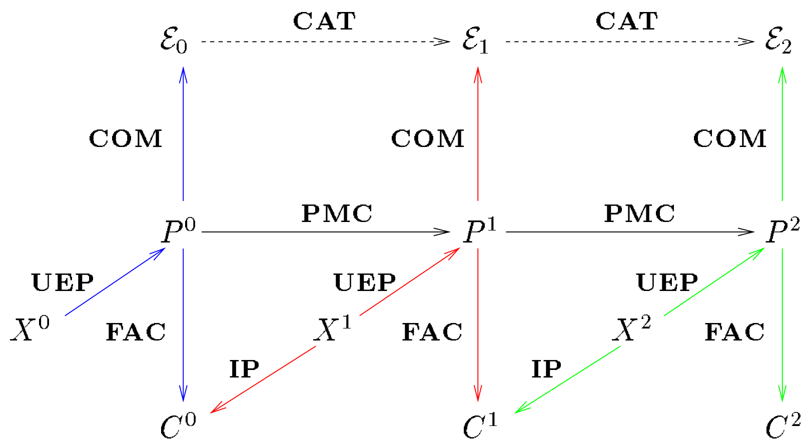


Figure 2: Schematic diagram of connections

In §1, we review the constructions of combinatorial structures from each other which are involved in Figures 1 and 2, namely those of intersection posets (**IP**) and uniform extension posets (**UEP**) along with commutativity constraints (**COM**) and incidence properties (**FAC**) of a permutahedron. These are illustrated in §2 by the explicit computation of the structures in Figure 1, as they arise in (1A)–(1E).

In §3, the third column of Figure 2 is constructed, by analogy with Figure 1, the first element being a hyperplane arrangement X^2 constructed in §3.1 so as to satisfy the higher analogue of Proposition 1A. According to our functorial theory, the poset P^2, should be an extension poset of an appropriate hyperplane arrangement, X^2, whose elements are in 1–1 correspondence with maximal chains in $P^1(n, \boldsymbol{\alpha})$. Analogues of (1A)–(1E) are given in Propositions (2A)–(2E). It is seen that no one form for P^2 suffices for all these statements, but rather that, for different statements, it is necessary to use slightly different notions of extension poset. The definition of the uniform extension poset (as it appears in Proposition 1B) of a hyperplane arrangement, X, is as the collection of subsets of the set of vertices of X for which those elements on any line in X form an initial or final subset of the vertices on that line. Appropriate notions of extensions poset are given in §3.2, generalising that of a uniform extension poset to one for which the order of vertices on a line is only given by a partial order. Propositions 2B and 2C are proved in §3.2. The analogue of the partition poset, C^1, in (1D) is found to be a certain poset of partitions of partitions; the relation between its combinatorics and that of cells in a realisation of P^2 is discussed in §3.3, see Proposition 3.3.7. Geometric depictions of those examples which lead to maximal cells of dimension 3 are discussed in §3.4, and the associated analogues, $\mathcal{E}^2$, of the YBE are given by Proposition 3.5.1 in §3.5.

1: Combinatorial constructions

1.1 From hyperplane arrangements

Throughout this paper, all hyperplane arrangements will be considered over $\mathbf{R}$. The *dimension* of an arrangement refers to the dimension of the ambient space. Recall that an arrangement is said to be *central* if all its hyperplanes pass through the origin. An n–dimensional arrangement is said to be in *general position* if the space of intersections of any r of its elements has codimension r, for all $r \leq n$.

Any n–dimensional arrangement, X, may be considered as related to an $(n+1)$–dimensional central arrangement, by embedding X in the affine space $x_{n+1} = 1$ and then constructing the cone from the origin over it. We will have occasion to use this construction in the reverse direction, to form from a central hyperplane arrangement, X_c, an arrangement, X, in one dimension lower, by taking its intersection with a hyperplane W in general position with respect to X_c. Note that although the incidence properties of X are independent of the choice of W, the orderings of points of intersection of the arrangement in subspaces (such as order of points on a line) will, in general, depend upon the choice of W.

There are two procedures for constructing posets from a hyperplane arrangement which will be used in this paper, namely of forming the *intersection poset* (**IP**) and the *uniform extension poset* (**UEP**).

By the **intersection poset** of an affine arrangement is meant the collection of intersections of hyperplanes in the arrangement, under reverse inclusion. Thus, if X is a hyperplane arrangement, then its intersection poset C will have as elements the distinct intersections $\cap Y$ for subsets $Y \subseteq X$.

If X is a hyperplane arrangement then its **uniform extension poset** P of X, consists of all subsets U of the vertex set of X such that, for all lines $l \in X$, $U \cap l$ is an initial or final subset of the vertices on l. The order on $\mathcal{U}(X)$ is single-step inclusion; that is, $U < T$, for $U, T \in \mathcal{U}(X) \iff \exists U_0, \ldots, U_k \in \mathcal{U}(X)$, such that,

$$U = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_k = T$$

with $|U_i| = |U_{i-1}| + 1$ for $1 \leq i \leq k$. The elements of $\mathcal{U}(X)$ label combinatorially distinct extensions of X by an oriented pseudo-hyperplane (see [BLSWZ], [SZ]).

Example 1.1.1 Let X_c denote the n–dimensional arrangement of hyperplanes consisting of the coordinate hyperplanes, $v_i = 0$. Taking its intersection with the diagonal hyperplane $\sum_i v_i = 1$ yields an $(n-1)$–dimensional arrangement, X, of n hyperplanes in general position. The intersection posets of X_c and X will coincide and will be the Boolean poset of all subsets of $[n]$, under inclusion. Since there are n vertices in X, only two on each line, the uniform extension poset of X in this case will also be the Boolean poset, which may be viewed geometrically as an n–cube.

Example 1.1.2 Let X_c denote the n–dimensional arrangement consisting of diagonal hyperplanes $v_i = v_j$ for all $1 \leq i < j \leq n$. The intersection poset of X_c is the partition lattice whose elements are all partitions of $[n]$, ordered by inclusion (strength of the equivalence relation induced on $[n]$). If one again constructs an arrangement, X, by taking the intersection of X_c with the 'main-diagonal' hyperplane $\sum_i v_i = 1$, one finds that it is central.

1.2 From polyhedra

Starting from a polyhedron, P, endowed with appropriate additional structure, one may construct two other combinatorial structures, a set of equations or algebraic structure $\mathcal{E}$ from *commutativity constraints* (**COM**) and a poset, C, from the *incidence properties of faces* (**FAC**).

First assume that P is an n–dimensional polytope embedded in Euclidean space. By a *facet* of P is meant a non-empty intersection of ($n-1$–dimensional) faces of P; we wish to include vertices and edges in this definition. Define an equivalence relation on facets by which $f \sim f'$ precisely when f and f' represent facets which can be obtained from each other by translation. We assume that P has the property that intersection respects $\sim$, so that it is well-defined on equivalence classes of facets.

Define the **facet poset**, C, to have underlying set consisting of the set of equivalence classes of facets under $\sim$, along with an extra element, $\hat{1}$, which should be geometrically considered as representing the interior of the polyhedron. The relation on C used is that of geometric inclusion. This makes C into a ranked poset, with rank function giving the dimension of the facet, so that it has unique maximal element $\hat{1}$, and unique minimal element $\hat{0}$, the equivalence class consisting of all vertices.

The system of **commutativity constraints**, $\mathcal{E}$, associated to P, may be thought of alternatively as a set of equations, or really as a complete algebraic system. Define a set of symbols labelled by the rank 1 elements of C, that is, by the equivalence classes of edges of P under $\sim$. For each edge equivalence class, choose an orientation on the edges. Then the system of equations, $\mathcal{E}$, is amongst the set of symbols just defined, with each two-dimensional face of P contributing an equation, namely the commutativity constraint of that face. More precisely, to a face with n edges, labelled in order by symbols $x_1, \ldots, x_n$ (not necessarily all distinct) will be associated the equation $x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n} = 1$ where ϵ_i is $+1$ or -1 according as the direction of the arrow on the i^{th} edge, in our enumeration of the edges of the polygon, matches or opposes the order of enumeration. It is clear that the relations obtained are unchanged if we begin enumerating the edges of the face at a different edge, or if we count them in the opposite order.

Example 1.2.1 Suppose that P is an n–dimensional cube. A facet is given by a partition of $[n]$ into three sets, namely those coordinates which are constant at 0, those constant at 1 and those which take the value $\frac{1}{2}$ somewhere in the facet. The equivalence classes of facets are specified by the latter set, so that the facet poset will be the Boolean poset. The commutativity constraints $\mathcal{E}$, will consist of n symbols, $x_1, \ldots, x_n$, representing the n possible edge directions, with relations

$$x_i x_j = x_j x_i \qquad \text{for each } i < j\ ,$$

since the two-dimensional faces are all squares. The algebraic structure is then a free commutative algebra on n generators.

Example 1.2.2 Suppose that P is an $(n-1)$–dimensional permutahedron; that is, the convex hull of the orbit of a generic point in $\mathbf{R}^n$ under the action of S_n which permutes the coordinates. The facet poset, C, is the partition lattice, as will be explaining in more detail in the next section. The equivalence classes of edges are labelled by pairs of elements of $[n]$, so that $\mathcal{E}$ is a system of equations on symbols R_{ij}, say, for $1 \leq i < j \leq n$. The two-dimensional faces are of two types, squares and hexagons, being labelled by elements of C of the form $ij;kl$ and ijk, respectively. The system of equations making up $\mathcal{E}$ is therefore,

$$\left.\begin{aligned} R_{ij}R_{kl} &= R_{kl}R_{ij}\ , \qquad \text{for } i < j \text{ and } k < l\ , \\ R_{ij}R_{ik}R_{jk} &= R_{jk}R_{ik}R_{ij}\ , \qquad \text{for } i < j < k\ . \end{aligned}\right\}$$

These are the *Yang-Baxter equations.* In an appropriate sense, the associate algebraic structure is that of a quantum group.

Example 1.2.3 Suppose that P is a Stasheff polyhedron, that is, its vertices are 'complete' brackettings of an ordered string of n symbols and its edges are defined by joining two strings which differ by a local move $a(bc) \longrightarrow (ab)c$, where a, b and c represent strings of symbols. Equivalently, the vertices are complete triangulations of an $n+1$-gon. The elements of the facet poset can be identified with The symbols in the commutativity constraints are labelled by quadrilaterals inscribed in the $n+1$-gon. The two-dimensional faces are all squares or pentagons and are labelled by pairs of inscribed quadrilaterals or an inscribed pentagon, respectively. The system of equations is the consistency constraints of a collection of associators.

Example 1.2.4 If P is a permutoassociahedron, as defined by Kapranov [K], then the associated algebraic structure, in a suitable sense, should by a quasi-Hopf algebra [D2].

1.3 Combinatorial categorification

The notion of *categorification* is one which transforms a structure relevant in one dimension to one appropriate at the next dimension. It is not currently precisely defined even in the world of algebraic structures (see [CF] and [L2]). As mentioned in the introduction, a strucure A is transformed to a structure $\mathbf{CAT}(A)$ in by a process of weakening. Underlying sets in A are replaced by categories in $\mathbf{CAT}(A)$, operations in A are replaced by functors in $\mathbf{CAT}(A)$, relations/axioms in A are mirrored by the existence of morphisms in $\mathbf{CAT}(A)$. Finally, the axioms in $\mathbf{CAT}(A)$ mirror the relations between the axioms of A.

What we have just described (in an imprecise way) is the horizontal arrows connecting structures at different dimensions on the top line of Figure 2. This induces, via $\mathbf{COM}$, a notion of categorification at the middle level in Figure 2, that is on ranked posets which have a geometric representation.

We start from a ranked poset, P, whose underlying set may be corresponded with vertices of a polyhedron, embedded in Euclidean space with a height function, matching the ranking function. We assume also that P has unique maximal and minimal elements $\hat{0}$ and $\hat{1}$, and that the relation of covering in the poset is identiied with edges in P. Then a new graph may be constructed, $\mathbf{PMC}(P)$, whose underlying set consists of all maximal chains in P,

$$\hat{0} = a_0 < a_1 < \cdots < a_r = \hat{1}\,.$$

Two such maximal chains are connected by an edge precisely when they differ only locally, one section $x = b_0 < b_1 < \cdots < b_i = y$ being replaced by another section $x = c_0 < c_1 < \cdots < c_j = y$ where $x, b_1, \cdots, b_{i-1}, y, c_{j-1}, \cdots, c_1$ is a two-dimensional face of P. In the cases in which we will be interested, it will be possible not only to obtain the structure on $\mathbf{PMC}(P)$ of a graph, but also as a ranked poset and a 'large' piece of it can be viewed as forming a polytope whose dimension is one less than that of P. Just as the edges of $\mathbf{PMC}(P)$ come form the two-dimensional faces of P, one may consider the d–dimensional faces of $\mathbf{PMC}(P)$ to come from $d+1$–dimensional faces of P. It should be noted, however, that in all but the simplest of situations, any appropriate geometric interpretation for the whole of $\mathbf{PMC}(P)$ would be as a union of polytopes of possibly different dimensions, some possibly larger than that of P!

Example 1.3.1 Suppose that $\mathcal{E}$ is the algebraic system which consists of a freely commuting set of generators; that is, symbols $x_1, \ldots, x_n$ with relations $x_i x_j = x_j x_i$. Under $\mathbf{CAT}$, The objects x_i transform into elements X_i of a category. Multiplication of objects turns into a tensor product operation at the category level. The relations making up $\mathcal{E}$ transform to morphisms,

$$R_{ij} : X_i \otimes X_j \longrightarrow X_j \otimes X_i$$

while $\mathbf{CAT}(\mathcal{E})$ contains relations given by the Yang-Baxter equations. The resulting structure is one of braidings bound by compatibility conditions.

Example 1.3.2 The combinatorial version of the previous example starts from P as an n–dimensional cube, $\{0,1\}^n$, considered as a Boolean poset. The height function is the sum of the coordinates and a maximal chain from the minimal element $\hat{0} = (0,\ldots,0)$ to the maximal element $\hat{1} = (1,\ldots,1)$ has length n and is defined by an ordered list $i_1,\ldots,i_n$, which is a permutation of $[n]$. Two maximal chains will be joined by an edge in $\mathbf{PMC}(P)$ precisely when their associated lists differ by the interchange of two adjacent members. Hence $\mathbf{PMC}(P)$ is the permutahedron, which has both the geometric structure of a polyhedron cominf from its definition as the convex hull of points $(i_1,\ldots,i_n)$, and the combinatorial structure of a ranked poset due to the fact that it is the Cayley graph of S_n.

1.4 Notation

Throughout the rest of this paper, $n \in \mathbf{N}$ will be fixed. Then $[n]$ denotes $\{1,2,\ldots,n\}$. Let V denote an n–dimensional, real vector space with basis $\mathbf{e}_1,\ldots,\mathbf{e}_n$ and (v_i) denote coordinates with respect to this basis. Let $\mathbf{e} \in V^*$ denote the sum of the coordinates.

If $\alpha_1,\ldots,\alpha_n$ are distinct real numbers, then $\boldsymbol{\alpha}$ will denote the associated element of V^*. The symbol α stands for the sequence (α_i), up to order preserving maps on $\mathbf{R}$, and $[\boldsymbol{\alpha}]$ denotes the sequence (α_i), up to an equivalence defined by monotonic maps on $\mathbf{R}$. That is $\alpha \in S_n$ while $[\boldsymbol{\alpha}]$ denotes the order of $\{\alpha_i\}$ up to reversal. Similar notation will be used later in this paper for $\boldsymbol{\beta}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$.

All the entries in Figure 2 will be seen to derive from the hyperplane arrangements X^r. These are defined in terms of the affine subspaces H_i, V_r and W_r of V, defined by,

$$H_i\colon\ \mathbf{e}_i \cdot \mathbf{v} = 0$$

$$V_r\colon\ \mathbf{a}_r \cdot \mathbf{v} = 0$$

$$W_r\colon\ \mathbf{a}_r \cdot \mathbf{v} = 1$$

for $i \in [n]$ and $r \geq 0$. The vectors $\mathbf{a}_r$ are chosen with $\mathbf{a}_0 = \mathbf{e}$, $\mathbf{a}_1 = \boldsymbol{\alpha}$ and $\mathbf{a}_2 = \boldsymbol{\beta}$. The hyperplane arrangement X^r is constructed from a central hyperplane arrangement, X^r_c, by taking its intersection with W_r. The first two in the sequence are defined by

$$X^0_c = \{H_i\}_1^n$$

$$X^1_c = \{H_i \cap V_0\}_1^n$$

2: The usual Yang-Baxter equation

In this section we explicitly investigate the central column in Figure 2, that is the connections in Figure 1. The objects arising in the left hand column in Figure 2 are derived from the uniform extension poset of $X^0(n)$, namely $P^0(n) = \{0,1\}^n$. This is the Boolean poset of subsets of $[n]$ with rank function ρ_1 defined by $\rho_1(V) = |V|$. According to Figure 2, the intersection poset of facets of $P^0(n)$ is $C^0([n])$ and hence $C^0([n])$ is also the Boolean poset. Finally $\mathcal{E}_0$ is obtained by the commutativity constraints on $P^0(n)$ and is therefore simply the commutativity of n symbols.

2.1 Hyperplane arrangements

By construction, $X^1_c(n)$ is a central arrangement of n hyperplanes in the $(n-1)$-dimensional space V_0 and $X^1(n,\boldsymbol{\alpha})$ is the associated affine arrangement, combinatorially equivalent to an arrangement of n hyperplanes in $(n-2)$–dimensions, in general position.

Proposition 2.1.1 (1A) *The intersection poset of $X^1(n,\boldsymbol{\alpha})$ is isomorphic to the part of $C^0([n])$ with rank $\rho_1 \geq 2$, in reverse order. Under this correspondence, the dimension of a subspace associated with $u \in C^0([n])$ is $\rho_1(u) - 2$.*

Indeed, an r–dimensional subspace in $X^1(n,\boldsymbol{\alpha})$ is obtained as an intersection of $n-2-r$ hyperplanes. The complement, in $[n]$, of the set of labels defines the associated element of $C^0([n])$.

Example 2.1.2 When α is ordered in ascending order, $X^1(4,\boldsymbol{\alpha})$ has the form shown in Figure 3, an arrangement of 4 lines in general position in $\mathbf{R}^2$.

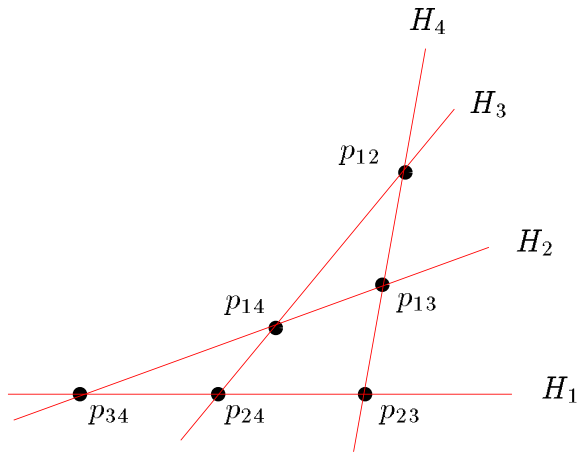


Figure 3: $X^1(4,\boldsymbol{\alpha})$

2.2 Extension posets

The vertices of $X^1(n,\boldsymbol{\alpha})$ are labelled by elements of $C_2^0([n])$, the rank 2 part of $C^0([n])$. The vertex p_{ij}, associated with $\{i,j\} \in C_2^0([n])$ is defined by $\bigcap_{k\neq i,j}(H_k \cap V_0 \cap W_1)$ and therefore has $v_i = -v_j = (\alpha_i - \alpha_j)^{-1}$ while $v_k = 0$ for all $k \neq i,j$. The points p_{ij}, p_{ik} and p_{jk} are collinear and their order is determined by the order of α_i, α_j and α_k in $\mathbf{R}$. We now obtain $P^1(n,[\boldsymbol{\alpha}])$ as the uniform extension poset of $X^1(n,\boldsymbol{\alpha})$. Figure 4 is an illustration of $X^1(4,\boldsymbol{\alpha})$; this diagrammatic representation is due to Zeigler [Z].

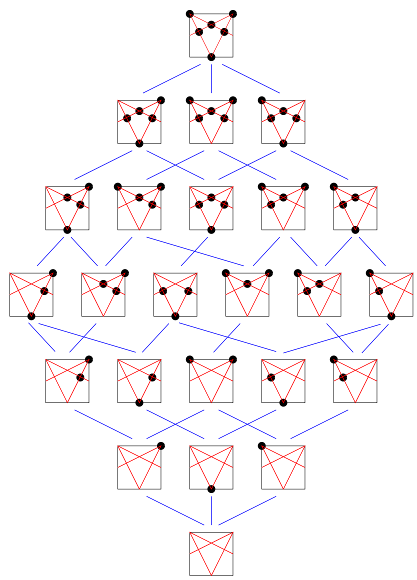


Figure 4: $P^1(4,[\boldsymbol{\alpha}])$

Proposition 2.2.1 (1B) *There is a bijection between S_n, the set of maximal chains in $P^0(n)$, and the vertices of $P^1(n,[\boldsymbol{\alpha}])$ defined by,*

$$\begin{aligned} \mathrm{Inv}_\alpha\colon S_n &\longrightarrow P^1(n,[\boldsymbol{\alpha}]) \\ \sigma &\longmapsto \mathrm{Inv}_\alpha(\sigma) = \{ij \in C_2^0([n]) \mid \alpha_i < \alpha_j,\ \alpha_{\sigma(i)} > \alpha_{\sigma(j)}\}\,. \end{aligned}$$

This is the connection between the first and second columns in Figure 2. If U covers T in $P^1(n,[\boldsymbol{\alpha}])$ with $U \backslash T = \{i,j\}$ then $\sigma = \mathrm{Inv}_\alpha^{-1}(U)$ and $\tau = \mathrm{Inv}_\alpha^{-1}(T)$ are related by $\sigma = \tau \circ (ij)$. By Proposition 2.2.1, the posets $P^1(n,[\boldsymbol{\alpha}])$ are all isomorphic, as $[\boldsymbol{\alpha}]$ ranges over all possible orders, up to reversal, on $\{1,2,\ldots,n\}$, for fixed n.

Proposition 2.2.2 (1C) *$P^1(n,[\boldsymbol{\alpha}])$ is a ranked poset with rank function $|U|$. It has a unique minimal element $\hat{0}=\emptyset$ and maximal element $\hat{1}=C_2^0([n])$.*

2.3 Geometric realisation

Consider $\boldsymbol{\alpha}$ as defining an element of V^*. Put $\mathbf{e}_{ij}={}^{(\mathbf{e}_i-\mathbf{e}_j)}/_{\boldsymbol{\alpha}(\mathbf{e}_i-\mathbf{e}_j)}$, so that $\mathbf{e}_{ij}$ is the position vector of the vertex of $X^1(n,\boldsymbol{\alpha})$ labelled by ij. Define,

$$\begin{aligned}\theta\colon P^1(n,[\boldsymbol{\alpha}]) &\longrightarrow V\\ U &\longmapsto \sum_{ij\in U}\mathbf{e}_{ij}\ .\end{aligned}$$

The image of $P^1(n,[\boldsymbol{\alpha}])$ under θ, forms the vertices of an $(n-1)$-dimensional permutahedron, edges joining points associated with covering elements of $P^1(n,[\boldsymbol{\alpha}])$. Let $C^1([n])$ denote the partition lattice,

$$\bigcup_{r\in\mathbf{N}}\Big\{(U_1,\ldots,U_r)\ \big|\ \text{the } U_i \text{ are disjoint subsets of } [n],\ |U_i|\geq 2\Big\}$$

with rank function $\rho_2(U_1,\ldots,U_r)=\sum_i\big(\rho_1(U_i)-1\big)$.

Proposition 2.3.1 (1D) *θ defines a realisation of the poset $P^1(n,[\boldsymbol{\alpha}])$ as a convex polyhedron, in which elements and covering pairs correspond to vertices and edges of the polyhedron. The k–dimensional faces of $\theta\big(P^1(n,[\boldsymbol{\alpha}])\big)$ are identified, up to translation, by elements $u=(U_1,\ldots,U_r)\in C_k^1([n])$ and are geometrically equivalent to the polyhedra $P^1(|U_1|,[\boldsymbol{\alpha}]_{U_1})\times\cdots\times P^1(|U_r|,[\boldsymbol{\alpha}]_{U_r})$. (Here $[\boldsymbol{\alpha}]_{U_i}$ denotes the restriction of $[\boldsymbol{\alpha}]$ to U_i.)*

For example, there are two types of 2–dimensional face in $P^1(n,[\boldsymbol{\alpha}])$, namely, squares and hexagons, labelled by elements $ij;kl$ and ijk of $C_2^1([n])$, respectively. The edges of P^1 are labelled by $ij\in C_1^1([n])$. This polyhedron defines a groupoid, $G^1(n,[\boldsymbol{\alpha}])$ whose elements are the vertices of the polyhedron and whose arrows are labelled by $\{ij\}$, going from a vertex $\mathbf{v}$ to a vertex $\mathbf{v}+\mathbf{e}_{ij}$.

Proposition 2.3.2 (1E) *A representation of $G^1(n,[\boldsymbol{\alpha}])$, in an algebra A, is determined by elements $R_{ij}\in A$ for which,*

$$\left.\begin{aligned}R_{ij}R_{kl}&=R_{kl}R_{ij}\ ,\\ R_{ij}R_{ik}R_{jk}&=R_{jk}R_{ik}R_{ij}\ ,\qquad \text{for } \alpha_i<\alpha_j<\alpha_k\ .\end{aligned}\right\}$$

For any $x \in C^1$, say $x = (x_1, \ldots, x_r)$, let $[\boldsymbol{\alpha}]$ denote a choice on each set x_i, of an order up to reversal. Define $P^1(x, [\boldsymbol{\alpha}])$ to be the product poset, $P^1(x_1, [\boldsymbol{\alpha}]_1) \times \cdots \times P^1(x_r, [\boldsymbol{\alpha}]_r)$ where $[\boldsymbol{\alpha}]_i$ denotes the restriction of $[\boldsymbol{\alpha}]$ to x_i. By Proposition 1C this poset is ranked, with rank function,

$$U \mapsto \sum_i \left| U \cap \binom{x_i}{2} \right| = |U| \, ,$$

and has unique minimal element $\hat{0}$ and maximal element $\hat{1} = \bigcup_i \binom{x_i}{2}$.

3: Maximal chains in P^1

In this section our aim is to give a geometric realisation of the set of maximal chains in $P^1(n, [\boldsymbol{\alpha}])$ by putting it in bijective correspondence with vertices of a suitable poset $P^2(n, [\boldsymbol{\alpha}], [\boldsymbol{\lambda}])$. By analogy with the previous section, this poset should be constructed from the associated hyperplane arrangement $X^2(\alpha, \beta)$.

Throughout this section, $x = (x_1, \ldots, x_r)$ will denote a fixed element of $C^1([n])$ and $C^1(x)$ will denote the interval $[\hat{0}, x]$ of $C^1([n])$. There is a natural injection $C^0_k \to C^1_{k-1}$ under which $U \mapsto (U)$; elements in the image of this map will henceforth be identified with associated element of C^0_k.

3.1 Hyperplane arrangements

In addition to the set of *distinct* real numbers α_i, we now choose another set of real numbers $\{\beta_i\}_{i=1}^n$. Recall that $X^1_c(n)$ is a central arrangement of hyperplanes, $\{H_i \cap V_0\}$, in which each $ij \in C^1_2([n])$ labels a line H_{ij}. For any $y \in C^1([n])$, define a subspace of V_0 of dimension $\rho_2(y)$ by

$$H_y = (\text{span of lines } H_{ij} \text{ for } ij \leq y) \subset V_0 \, .$$

The hyperplane arrangement X^2 is now defined for $x \in C^1([n])$, by

$$X^2_c(x, \boldsymbol{\alpha}) \equiv \{H_y \cap V_1 \mid y < x \text{ in } C^1_{\rho_2(x)-1}([n])\} \subset H_x \cap V_1$$

while $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ denotes the restriction to W_2.

Proposition 3.1.1 (2A) *For generic $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the intersection poset of $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ contains the part of $C^1(x)$ with rank $\rho_2 \geq 2$, in reverse order. Under this correspondence the dimension of the subspace associated with $u \in C^1(x)$ is $\rho_2(u) - 2$.*

PROOF: For any $u \in C^1(x)$, H_u is a subspace of V of dimension $\rho_2(u)$. The dimension of $H_u \cap V_1 \cap W_2$ is therefore $\rho_2(u) - 2$, assuming that the α's and β's are generic. The result follows since $H_y \subseteq H_z$ when $y \leq z$ in $C^1(x)$. ■

If $u = (U_1, \ldots, U_r) \in C^1(x)$, then H_u is defined by,

$$\left.\begin{aligned} \sum_{j\in U_i} v_j &= 0 \qquad 1 \leq i \leq r \\ v_j &= 0 \qquad \forall j \in (U_1 \cap \cdots \cap U_r) \end{aligned}\right\}$$

The condition for the dimension of $H_u \cap V_1 \cap W_2$ to be $\sum_{i=1}^r (|U_i| - 1) - 2$ is that the coefficient matrix of the associated system,

$$\left.\begin{aligned} &\sum_{j\in U_i} v_j = 0 \qquad 1 \leq i \leq r \\ &\sum_{j\in U} \alpha_j v_j = 0 \\ &\sum_{j\in U} \beta_j v_j = 1 \end{aligned}\right\}$$

has rank $r + 2$. This is ensured by,

$$\left\{ \lambda_{ij} \equiv \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} \,\middle|\, ij \leq x \right\}$$

forming a distinct set of real numbers. In this case a vertex in $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is specified by an element of $C_2^1(x)$, that is, it has a label of the form $ij;kl$ or ijk.

Lemma 3.1.2 *The vertices of $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ associated with elements of $C_2^1(x)$ are all distinct so long as*

(i) for all i, α_j ($j \in x_i$) are distinct;

(ii) λ_{ij} ($ij \in C_1^1(x)$) are distinct;

(iii) $\alpha_i + \alpha_l \neq \alpha_j + \alpha_k$ whenever $ijkl \in C^1(x)$.

PROOF: Observe first that the non-zero coordinates of the vertices labelled by ijk and $ij;kl$ are *precisely* those indexed by elements of $\{i, j, k\}$ and $\{i, j, k, l\}$, respectively. Hence the only vertices which might be coincident are those associated with $ij;kl$ and $ik;jl$ for some $i, j, k, l \in [n]$. This requires $ijkl \in C^1(x)$. However, the values of v_l^{-1} at these two vertices are,

$$(\alpha_i - \alpha_j)(\lambda_{ij} - \lambda_{kl}) \text{ and } (\alpha_i - \alpha_k)(\lambda_{ik} - \lambda_{jl}) \,,$$

respectively. Their difference is $(\alpha_j - \alpha_i + \alpha_k - \alpha_l)(\lambda_{kl} - \lambda_{jl})$. Note that this can only vanish if α_i and α_l are the smallest and largest (in some order) of $\{\alpha_i, \alpha_j, \alpha_k, \alpha_l\}$.■

The conditions of Lemma 3.1.2 on $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbf{R}^{2n}$ define a subset of $\mathbf{R}^{2n}$ within each connected component of which, the order of appearance of vertices along each line in the arrangement, is fixed (see Lemma 3.2.1).

Example 3.1.3 x=123;45;67. $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is shown in Figure 5.

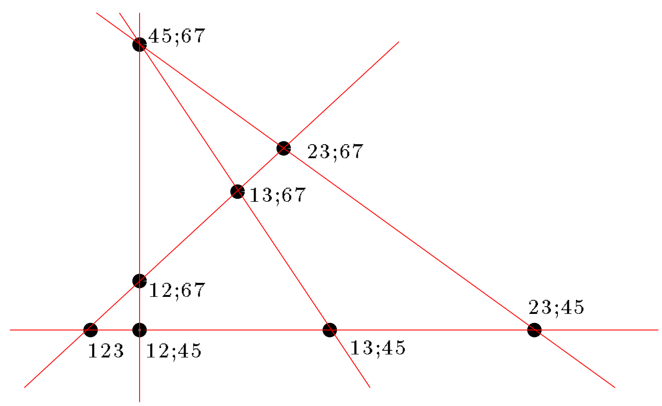


Figure 5: $X^2(123; 45; 67, \boldsymbol{\alpha}, \boldsymbol{\beta})$

Example 3.1.4 x=1234;56. $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is shown in Figure 6. This is a case in which the order of points on the lines is *not* entirely determined by $[\boldsymbol{\alpha}]$ and $[\boldsymbol{\lambda}]$.

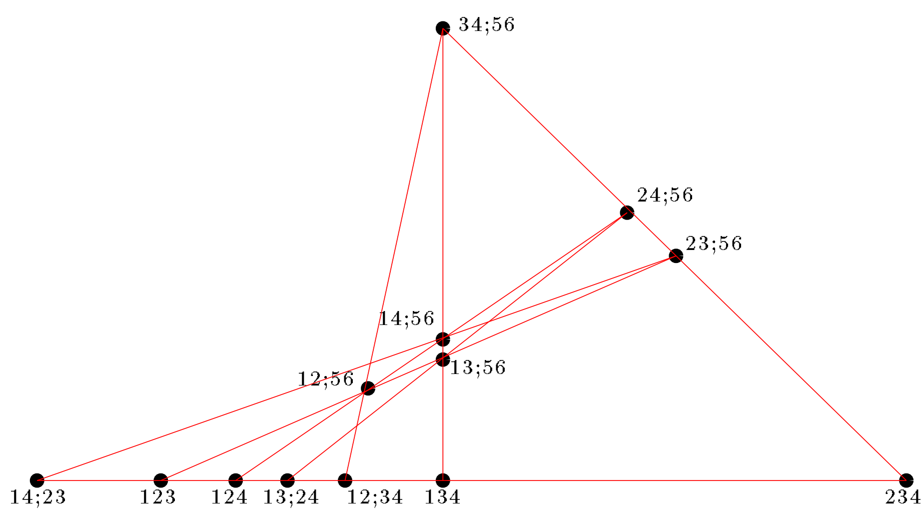


Figure 6: $X^2(1234; 56, \boldsymbol{\alpha}, \boldsymbol{\beta})$

Define $\mathbf{e}_{ij}$, as in §2, by $\mathbf{e}_{ij} = {}^{(\mathbf{e}_i - \mathbf{e}_j)}/_{\boldsymbol{\alpha}(\mathbf{e}_i - \mathbf{e}_j)}$, for $ij \in C_1^1(x)$. If $p, q \in C_1^1(x)$ are distinct, let $p \wedge q$ denote the join of p and q in $C_1^1(x)$, and, when it causes no confusion, we may shorten this to pq. For $p, q \in C_1^1(x)$, define $\{\mathbf{e}_u \mid u \in C_2^1(x)\}$ by setting,

$$\mathbf{e}_{pq} = {}^{(\mathbf{e}_p - \mathbf{e}_q)}/_{\boldsymbol{\beta}(\mathbf{e}_p - \mathbf{e}_q)} .$$

Note that for u's of the form $ij;kl$ only one pair $\{p, q\}$ satisfies $u = pq$. For $u = ijk$, p and q may be chosen as any two elements of $\{ij, ik, jk\}$. That $\mathbf{e}_u$ is well defined follows from the fact that $\mathbf{e}_{ij}$, $\mathbf{e}_{ik}$ and $\mathbf{e}_{jk}$ are collinear. Observe that $\boldsymbol{\beta}(\mathbf{e}_{ij}) = \lambda_{ij}$, so that $\mathbf{e}_u = {}^{(\mathbf{e}_p - \mathbf{e}_q)}/_{(\lambda_p - \lambda_q)}$.

Lemma 3.1.5 *The vertex of $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ with label $u \in C_2^1(x)$ appears at $\mathbf{e}_u$.*

3.2 Extension posets

Let α and λ denote the orders of $\{\alpha_i\}$ and $\{\lambda_{ij}\}$, respectively. That is, α defines an element of $S_{|x_1|} \times \cdots \times S_{|x_r|}$; while λ defines an element of $S_{\sum_i \binom{|x_i|}{2}}$. Denote by $[\boldsymbol{\alpha}]$, the class of α, up to the action of $\mathbf{Z}_2^r$, the i^{th} factor reversing the order of $\{\alpha_j \mid j \in x_i\}$, and by $[\boldsymbol{\lambda}]$, the class of λ, up to the action of $\mathbf{Z}_2$, reversing the entire order. Let $\epsilon_{ij,kl} = \operatorname{sgn}(\alpha_i + \alpha_j - \alpha_k - \alpha_l)$. Note that since $(\alpha_i - \alpha_j)\lambda_{ij} + (\alpha_j - \alpha_k)\lambda_{jk} = (\alpha_i - \alpha_k)\lambda_{ik}$, the order of $\{\lambda_{ij}, \lambda_{ik}, \lambda_{jk}\}$ in $\mathbf{R}$ is determined, up to reversal, by that of $\{\alpha_i, \alpha_j, \alpha_k\}$ while $[\boldsymbol{\lambda}]$ determines $[\boldsymbol{\alpha}]$.

The lines in $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ are labelled by the elements of $C_3^1(x)$ and thus have type $ij;kl;mn$, $ijk;lm$ or $ijkl$.

Lemma 3.2.1 *The order of vertices $\{\mathbf{e}_u \mid u \in C_2^1(y)\}$ on a line labelled by $y \in C_3^1(x)$ in $X^2(x, \boldsymbol{\alpha}, \boldsymbol{\beta})$ depends only upon $[\boldsymbol{\lambda}]$, for $y \notin C_4^0(x)$ while for $y = ijkl$ with $\alpha_i < \alpha_j < \alpha_k < \alpha_l$, it depends only upon $[\boldsymbol{\lambda}]$ and $\epsilon_{il;jk}$.*

PROOF: Observe that for all distinct p, q and $r \in C_1^1(x)$,

$$(\lambda_p - \lambda_q)\mathbf{e}_{pq} + (\lambda_q - \lambda_r)\mathbf{e}_{qr} = (\lambda_p - \lambda_r)\mathbf{e}_{pr} ,$$

while for distinct $p, q, r, s \in C_1^1(x)$ for which $p = ij$, $q = jk$ and $r = ik$, we have,

$$(\alpha_i - \alpha_j)(\lambda_p - \lambda_s)\mathbf{e}_{ps} + (\alpha_j - \alpha_k)(\lambda_q - \lambda_s)\mathbf{e}_{qs} = (\alpha_i - \alpha_k)(\lambda_r - \lambda_s)\mathbf{e}_{rs} .$$

Thus the orders of $\{\alpha_i\}$ and $\{\lambda_{ij}\}$ determine the relative positions of triples of vertices on a line, with labels of the form $\{pq, pr, qr\}$ or $\{ps, qs, rs\}$. This suffices to fix, up to reversal, the order of points on lines labelled by elements of $C_3^1(x)$ of the form $ij;kl;mn$ or $ijk;lm$. To fix the order of the vertices on a line with label $ijkl$, note that,

$$(\alpha_l-\alpha_k)(\lambda_{ij}-\lambda_{kl})\mathbf{e}_{ij;kl}+(\alpha_j-\alpha_l)(\lambda_{ik}-\lambda_{jl})\mathbf{e}_{ik;jl}=(\alpha_i+\alpha_l-\alpha_j-\alpha_k)(\lambda_{ij}-\lambda_{ik})\mathbf{e}_{ijk}\,.$$

■

Remark 3.2.2 Observe from this proof that the value of $\epsilon_{il;jk}$ only affects the relative order of $\mathbf{e}_{ij;kl}$ and $\mathbf{e}_{ik;jl}$. The relation used in this proof enables one to determine those orders $[\boldsymbol{\lambda}]$ for which the vertices on the line $ijkl$ have maximal and minimal elements both of the form $**;**$, a property which will be important in the proof of Lemma 3.2.13. Indeed, assuming i, j, k and l are in the order determined by α, such maximal and minimal elements must be $\mathbf{e}_{ij;kl}$ and $\mathbf{e}_{ik;jl}$. [For, if not, say $\mathbf{e}_{il;jk}$ and $\mathbf{e}_{ij;kl}$ were maximal and minimal. Then $\mathbf{e}_{ijl}$ and $\mathbf{e}_{ikl}$ would both lie between $\mathbf{e}_{il;jk}$ and $\mathbf{e}_{ij;kl}$. However,

$$(\alpha_k-\alpha_j)(\lambda_{il}-\lambda_{jk})\mathbf{e}_{il;jk}+(\alpha_l-\alpha_k)(\lambda_{ij}-\lambda_{kl})\mathbf{e}_{ij;kl}=(\alpha_i+\alpha_k-\alpha_j-\alpha_l)\mathbf{e}_{ijl}$$
$$(\alpha_j-\alpha_k)(\lambda_{il}-\lambda_{jk})\mathbf{e}_{il;jk}+(\alpha_i-\alpha_j)(\lambda_{kl}-\lambda_{ij})\mathbf{e}_{ij;kl}=(\alpha_j+\alpha_l-\alpha_i-\alpha_k)\mathbf{e}_{ikl}$$

and the coefficients of $\mathbf{e}_{il;jk}$ in these two relations have opposite signs, while those of $\mathbf{e}_{ij;kl}$ have the same signs. A similar argument works for $\mathbf{e}_{il;jk}$ and $\mathbf{e}_{ik;jl}$.] This can only occur if, up to reversal, the order $[\boldsymbol{\lambda}]$ restricted to subsets of $\{i,j,k,l\}$ is, ij—kl—il—jl—ik—jk or kl—ij—il—ik—jl—jk. An order on $C_1^1(x)$ which restricts, on some element of C_4^0 to one of the above orders, will be said to be *singular*.

Example 3.2.3 On the line $ij;kl;mn$, the order of the vertices is $ij;kl$, $ij;mn$, $kl;mn$, where λ_{kl} lies between λ_{ij} and λ_{mn}.

Example 3.2.4 For $y=ijk;lm$, assume α_j lies between α_i and α_k. Up to reversal of order and/or interchange of i and k, there are two possible order types for $[\boldsymbol{\lambda}]$, namely,

$$\lambda_{ij}<\lambda_{ik}<\lambda_{jk}<\lambda_{lm}$$
$$\lambda_{ij}<\lambda_{lm}<\lambda_{ik}<\lambda_{jk}\,.$$

The order of vertices on $ijk;lm$ is,

$$ijk,\ ij;lm,\ ik;lm,\ jk;lm$$
$$\text{and}\ \ ij;lm,\ ijk,\ ik;lm,\ jk;lm\,,$$

in these two cases.

Example 3.2.5 For $y = ijkl$ with $\alpha_i < \alpha_j < \alpha_k < \alpha_l$, assume further that $\{\lambda_{ij}\}$ appear in anti-lexicographic order, $\lambda_{ij} < \lambda_{ik} < \lambda_{jk} < \lambda_{il} < \lambda_{jl} < \lambda_{kl}$ (up to reversal). Then the order of the vertices on the line labelled by y is,

$$\begin{aligned} & il;jk,\ jkl,\ ikl,\ ij;kl,\ ik;jl,\ ijl,\ ijk \\ \text{or}\ \ & il;jk,\ jkl,\ ikl,\ ik;jl,\ ij;kl,\ ijl,\ ijk\ , \end{aligned}$$

according as $\alpha_j + \alpha_k \gtrless \alpha_i + \alpha_l$.

In order to define a uniform extension poset, it is necessary to know, for each line in the arrangement, the order of the vertices up to reversal. We now weaken this notion slightly, to deal with only partial orders on the set of vertices.

Definition 3.2.6 *Suppose X is a set. By a total r-order on X is meant a map,*

$$\delta\colon \binom{X}{3} \longrightarrow X$$

such that

(i) for all $T \in \binom{X}{3}$, $\delta(T) \in T$;

(ii) for all $U \in \binom{X}{4}$, as T ranges over $\binom{U}{3}$, $\delta(T)$ takes on just two values, each twice.

Lemma 3.2.7 *The specification of a total r–order on a set X ($|X| \geq 2$) is equivalent to the specification of a total order up to reversal.*

Proof: If $<$ is a total order on X, define a total r–order by defining $\delta(T)$ to be the element of T lying between the other two in the order $<$, for $T \in \binom{X}{3}$. Reversal of $<$ clearly leaves δ unchanged.

Conversely, suppose δ is a total r–order. Pick $x_0, x_1 \in X$. This subdivides $X \backslash \{x_0, x_1\}$ into three disjoint sets.,

$$\begin{aligned} X_0 &= \left\{ x \;\middle|\; \delta(\{x, x_0, x_1\}) = x_0 \right\} \\ X_1 &= \left\{ x \;\middle|\; \delta(\{x, x_0, x_1\}) = x_1 \right\} \\ X_{01} &= \left\{ x \;\middle|\; \delta(\{x, x_0, x_1\}) = x \right\} . \end{aligned}$$

Define a total order in which elements of X_0 are less than those in X_{01}, which are in turn less than those in X_1, while two elements of the same set are compared using,

$$x < y \text{ in } X_0 \text{ iff } \delta\big(\{x, y, x_0\}\big) = y$$
$$x < y \text{ in } X_1 \text{ or } X_{01} \text{ iff } \delta\big(\{x, y, x_0\}\big) = x\ .$$

Transitivity follows from the constraints on δ. ■

By a *partial r–order* is meant the restriction of a total r–order to a subset of $\binom{X}{3}$. We say that $Y \subseteq X$ is *convex* with respect to δ iff $\forall x, y \in Y$, $\Big(\delta\big(\{x, y, z\}\big) = z \Rightarrow z \in Y\Big)$. Any partial order defines a partial r–order, but not all partial r–orders arise in this way. If δ_1 and δ_2 are partial r–orders on a set X, their intersection is defined on the subset of $\binom{X}{3}$ on which $\delta_1 = \delta_2$ and is specified by the common map defined there. It is clear that the set of partial r–orders on X is closed under the taking of intersections.

Definition 3.2.8 *For any $y \in C_3^1(x)$, let $Q^2(y, [\boldsymbol{\lambda}])$ denote the partial r–order which is the intersection of all total r–orders on the vertices lying on the line $X^2(y, \boldsymbol{\alpha}, \boldsymbol{\beta})$ obtained, as $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ vary over all values compatible with $[\boldsymbol{\lambda}]$.*

In fact the partial r–order $Q^2(y, [\boldsymbol{\lambda}])$ always arises from an appropriate partial order which, up to reversal, will be denoted by the same symbol.

Example 3.2.9 When $y = 1234$ and,

$$[\boldsymbol{\lambda}] = (12\text{—}13\text{—}14\text{—}23\text{—}24\text{—}34)\ ,$$

the partial order $Q^2(y, [\boldsymbol{\lambda}])$ is,

```
                      12; 34
                     /      \
14; 23 —— 123 —— 124          134 —— 234.
                     \      /
                      13; 24
```

Note that Q^2 need not have unique maximal and minimal elements. For example, $[\boldsymbol{\lambda}] = (12\text{—}34\text{—}14\text{—}13\text{—}24\text{—}23)$ gives rise to the partial order $Q^2(y, [\boldsymbol{\lambda}])$ as shown below,

```
12; 34
      \
       124 —— 123 —— 14; 23 —— 234 —— 134.
      /
13; 24
```

$Q^2(1234, [\boldsymbol{\lambda}])$ is a total order precisely when $[\boldsymbol{\lambda}]$ is singular (see Remark 3.2.2).

Definition 3.2.10 *$P_s^2(x,[\boldsymbol{\lambda}])$ denotes the set of all subsets $U\subset C_2^1(x)$ for which $U\cap C_2^1(y)$, or its complement in $C_2^1(y)$, is an order ideal in $Q^2(y,[\boldsymbol{\lambda}])$, for all $y\in C_3^1(x)$, under single step inclusion.*

Equivalently, $U\in P_s^2(x,[\boldsymbol{\lambda}])$ precisely when both $U\cap C_2^1(y)$ and $C_2^1(y)\backslash U$ are convex with respect to the partial r–order $Q^2(y,[\boldsymbol{\lambda}])$ on $C_2^1(y)$.

Definition 3.2.11 *$P^2(x,[\boldsymbol{\lambda}])$ denotes the poset of all subsets $U\subset C_2^1(x)$ for which both $U\cap C_2^1(y)$ and $C_2^1(y)\backslash U$ are convex with respect to triples in $C_2^1(y)$ of the form $\{p\vee q, p\vee r, q\vee r\}$, $(p,q,r\in C_1^1(y))$ or $\{ij\vee s, ik\vee s, jk\vee s\}$, for all $y\in C_2^1(x)$. The partial order imposed is that of single step inclusion.*

It is apparent that $P_s^2(x,[\boldsymbol{\lambda}])$ is a subset of $P^2(x,[\boldsymbol{\lambda}])$. Note that for fixed $[\boldsymbol{\alpha}]$, a maximal $\hat{0}$—$\hat{1}$ chain in $P^1(x,[\boldsymbol{\alpha}])$ is specified by a total order on $C_1^1(x)$ by Proposition 2.2.2. Those orders which arise are precisely those for which ik lies between ij and jk whenever j lies between i and k in $[\boldsymbol{\alpha}]$. Let $O^2([\boldsymbol{\alpha}])$ denote the set of such total orders. Thus there is a bijection,

$$\Big\{\text{maximal } \hat{0}\text{—}\hat{1} \text{ chains in } P^1(x,[\boldsymbol{\alpha}])\Big\}\longrightarrow O^2([\boldsymbol{\alpha}])\,.$$

Let $\tilde{O}^2([\boldsymbol{\alpha}])$ be the quotient of $O^2([\boldsymbol{\alpha}])$ by the operation of reversal of the order. Let $O^2_{ns}([\boldsymbol{\alpha}])$ denote the subset of $O^2([\boldsymbol{\alpha}])$ consisting of non-singular orders (see Remark 3.2.2).

Proposition 3.2.12 (2B) *There is a bijection between the set of maximal chains in $P^1(x,[\boldsymbol{\alpha}])$ and the vertices of $P^2(x,[\boldsymbol{\lambda}])$, given whenever $\lambda\in O^2([\boldsymbol{\alpha}])$, by,*

$$\mathrm{Inv}_\lambda(\sigma)=\Big\{u\in C_2^1(x)\;\Big|\; \textit{orders of elements of } C_1^1(u) \textit{ in } \lambda \textit{ and } \sigma \textit{ are reversed}\Big\}\,.$$

Furthermore $P^2(x,[\boldsymbol{\lambda}])$ may be replaced by $P_s^2(x,[\boldsymbol{\lambda}])$ when $\lambda\in O^2_{ns}([\boldsymbol{\alpha}])$.

To prove this proposition we must first verify that the map is well-defined.

Lemma 3.2.13 *Suppose $\lambda,\sigma\in O^2([\boldsymbol{\alpha}])$. Then $\mathrm{Inv}_\lambda(\sigma)\in P^2(x,[\boldsymbol{\lambda}])$. If $\lambda\in O^2_{ns}([\boldsymbol{\alpha}])$, then $\mathrm{Inv}_\lambda(\sigma)\in P_s^2(x,[\boldsymbol{\lambda}])$.*

PROOF: Suppose $y \in C_3^1(x)$. Then we wish to show that $\mathrm{Inv}_\lambda(\sigma) \cap C_2^1(y) = S$ and its complement, are convex in $Q^2(y, [\boldsymbol{\lambda}])$ with respect to suitable triples. However, if $\overline{\sigma}$ denotes the reverse order on $C_1^1(x)$ to that defined by σ, then $\overline{\sigma} \in O^2([\boldsymbol{\alpha}])$, while $\mathrm{Inv}_\lambda(\overline{\sigma}) \cap C_2^1(y) = C_2^1(y) \backslash S$. Thus it suffices to show that S is convex in $Q^2(y, [\boldsymbol{\lambda}])$ for all σ and y. Since S is affected only by the restriction of σ to y, without loss of generality, we may assume $x = y$. In this case, any $\lambda \in O^2([\boldsymbol{\alpha}])$ may be realised by appropriate $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Lemma 3.2.14 *Suppose $u = pq$, $v = qr$, $w = pr$ and $u, v \in S$ with w between u and v in $Q^2(y, [\boldsymbol{\lambda}])$. Then $w \in S$.*

PROOF: For all $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ compatible with $[\boldsymbol{\lambda}]$, $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$ are collinear, with $\mathbf{e}_w$ between $\mathbf{e}_u$ and $\mathbf{e}_v$. Since $(\lambda_p - \lambda_q)\mathbf{e}_u + (\lambda_q - \lambda_r)\mathbf{e}_u = (\lambda_p - \lambda_r)\mathbf{e}_w$, thus λ_q lies between λ_p and λ_r. Since $u, v \in S$,

$$\begin{aligned} p <_\sigma q &\iff p >_\lambda q \\ q <_\sigma r &\iff q >_\lambda r \,. \end{aligned}$$

Thus q lies between p and r in the order σ and so p and r appear in opposite orders in λ and σ. Hence $pr \in S$. ■

Lemma 3.2.15 *Suppose $u = ps$, $v = qs$ and $w = rs$, where $\{p, q, r\} = C_1^1(a)$, some $a \in C_2^1(x)$ and p, q, r and s are distinct. Suppose further that $u, v \in S$ and w lies between u and v in $Q^2(y, [\boldsymbol{\lambda}])$. Then $w \in S$.*

PROOF: Put $a = ijk$ with $p = ij$, $q = jk$ and $r = ik$, say. Since,

$$(\alpha_i - \alpha_j)(\lambda_p - \lambda_s)\mathbf{e}_u + (\alpha_j - \alpha_k)(\lambda_q - \lambda_s)\mathbf{e}_v = (\alpha_i - \alpha_k)(\lambda_r - \lambda_s)\mathbf{e}_w \,.$$

and $\mathbf{e}_w$ lies between $\mathbf{e}_u$ and $\mathbf{e}_v$, hence $(\alpha_i - \alpha_j)(\lambda_p - \lambda_s)$ and $(\alpha_j - \alpha_k)(\lambda_q - \lambda_s)$ have the same sign.

(a) If $\alpha_i - \alpha_j$ and $\alpha_j - \alpha_k$ have the same sign, then α_j lies between α_i and α_k, while λ_p and λ_q are on the same side of λ_s as each other. Since $u, v \in S$, thus

$$(p <_\sigma s \iff \lambda_p > \lambda_s), \; (q <_\sigma s \iff \lambda_q > \lambda_s) \,.$$

Finally, r lies between p and q in both the λ and σ orders, from which it follows that $(r <_\sigma s \iff \lambda_r > \lambda_s)$, so that $w \in S$.

(b) If $\alpha_i - \alpha_j$ and $\alpha_j - \alpha_k$ have opposite signs, then λ_s lies between λ_p and λ_q. Suppose, without loss of generality, that α_k lies between α_i and α_j. Then, since $\lambda, \sigma \in O^2([\boldsymbol{\alpha}])$, hence p lies between q and r in both λ and σ. Since $u, v \in S$, σ_s lies between σ_p and σ_q. The relative order of p, q, r and s is therefore q—s—p—r, in both λ and σ, with the absolute orders opposite in λ and σ. Hence $rs = w \in S$. ■

These two lemmas show that S and S^c are convex with respect to triples of the two types $\{pq, pr, qr\}$ and $\{ps, qs, rs\}$. Hence $\mathrm{Inv}_\lambda(\sigma) \in P^2(x, [\boldsymbol{\lambda}])$.

Next assume that $\lambda \in O^2_{ns}([\boldsymbol{\alpha}])$. The above lemmas deal with all triples of vertices appearing on lines $y \in C^1_3(x)$ of form $ij; kl; mn$ or $ijk; lm$. So suppose $y = ijkl$, with $\alpha_i < \alpha_j < \alpha_k < \alpha_l$, say. Lemmas A and B ensure convexity in respect of triples of vertices containing at most one of the form $**; **$. The only case remaining is where u and v are both of the form $**; **$ and lie in S, while w lies between them in $Q^2([\boldsymbol{\lambda}])$. Repeated use of Lemmas A and B shows that $w \in S$. (For, suppose otherwise that $w \notin S$. Let Q be the part of $Q^2([\boldsymbol{\lambda}])$ consisting of elements comparable with u and v; it has order 6. By the lemmas, S and S^c are convex in both $Q \backslash u$ and $Q \backslash v$. Hence all elements of Q outside the interval between u and v lie in S. By Remark 3.2.2 and since $[\boldsymbol{\lambda}]$ is non-singular, the set of vertices in the interval between u and v is non-empty, say containing t on the opposite side of v to u. Convexity of $Q \backslash v$ gives a contradiction since $u, t \in S$, while w lies between u and v and hence also between u and t.) Hence $\mathrm{Inv}_\lambda(\sigma) \in P^2_s(x, [\boldsymbol{\lambda}])$. ■

When $\lambda \in O^2_{ns}([\boldsymbol{\alpha}])$ the above proof shows that $P^2(x, [\boldsymbol{\lambda}]) = P^2_s(x, [\boldsymbol{\lambda}])$. In fact the case when the restriction of $[\boldsymbol{\lambda}]$ to $\{i, j, k, l\}$ is singular is precisely that in which $Q^2(ijkl, [\boldsymbol{\lambda}])$ is a total r–order. In this case $P^2 \backslash P^2_s$ contains those vertex sets whose restriction to some line $ijkl$ consists of one of $\{ij; kl\}$, or $\{ik; jl\}$, or their complements.

Proof of Proposition 3.2.12: By Lemma 3.2.13, it remains only to verify bijectivity. Suppose that $U \in P^2(x, [\boldsymbol{\lambda}])$, so that $U \subset C^1_2(x)$. For any $p, q \in C^1_1(x)$, say that $p <_\sigma q$ if, and only if,

$$\begin{array}{ll} \text{either} & (\lambda_p < \lambda_q \text{ and } pq \notin U) \\ \text{or} & (\lambda_p > \lambda_q \text{ and } pq \in U). \end{array}$$

This defines the only possible order, σ, for which $\mathrm{Inv}_\lambda(\sigma) = U$. Injectivity of Inv_λ is immediate. For surjectivity we must verify that $<_\sigma$ defines an order and that it lies in $O^2([\boldsymbol{\alpha}])$. Assume that $p <_\sigma q$ and $q <_\sigma r$.

Case (a): $\rho_2(pqr) = 3$. Suppose otherwise that $r <_\sigma p$. Since there is cyclic symmetry, without loss of generality the vertex with label pr lies between those with labels pq and qr on the line pqr. Hence λ_q lies between λ_p and λ_r. If $\lambda_p < \lambda_q < \lambda_r$ then $p <_\sigma q$, $q <_\sigma r$ and $r <_\sigma p$ implies that $pq, qr \notin U$ and $pr \in U$, while if $\lambda_p > \lambda_q > \lambda_r$, it imples that $pq, qr \in U$ and $pr \notin U$. Both contradict the fact that $U \cap C^1_2(pqr)$ and its complement, are convex with respect to $\{pq, pr, qr\}$.

CASE (B): $\rho_2(pqr) = 2$, so that $p = ij$, $q = ik$ and $r = jk$, say. If $ijk \in U$ then $\lambda_p > \lambda_q > \lambda_r$, while if $ijk \notin U$ then $\lambda_p < \lambda_q < \lambda_r$. Either way, $p <_\sigma r$. Also, since $\lambda \in O^2([\boldsymbol{\alpha}])$, α_j must lie between α_i and α_k. Since q lies between p and r this verifies that $\sigma \in O^2([\boldsymbol{\alpha}])$. ■

In terms of the bijection of Proposition 3.2.12, an order σ covers an order τ (in the poset $P^2(x,[\boldsymbol{\lambda}])$) if, and only if, there exists $u \in C^1_2(x)$ such that,

(i) $C^1_1(u)$ are adjacent in σ;

(ii) τ can be obtained from σ by reversing the chain formed by $C^1_1(u) \subset C^1_1(x)$.

Proposition 3.2.16 (2C) *$P^2(x,[\boldsymbol{\lambda}])$ and $P^2_s(x,[\boldsymbol{\lambda}])$ are symmetric ranked posets with rank function $|U|$. They have a (not necessarily unique) minimal element $\hat{0} = \emptyset$ and maximal element $\hat{1} = C^1_2(x)$.*

PROOF: By the definition of $P^2(x,[\boldsymbol{\lambda}])$, with single-step inclusion defining the order, the result follows immediately. The map $U \mapsto C^1_2(x)\backslash U$ defines an involution on $P^2(x,[\boldsymbol{\lambda}])$. Using the correspondence given in Proposition 3.2.12, $U \in P^2(x,[\boldsymbol{\lambda}])$ is minimal if, and only if, the order σ on $C^1_1(x)$ for which $\mathrm{Inv}_\lambda\sigma = U$ is such that for all $u \in U \subset C^1_2(x)$, the elements of $C^1_1(u)$ are not adjacent in σ.

An example for which $P^2(x,[\boldsymbol{\lambda}]) = P^2_s(x,[\boldsymbol{\lambda}])$ has no unique minimal element is provided by $x = 123;456$ with $\lambda = (45 < 12 < 13 < 46 < 56 < 23)$. When $\sigma = (56 < 23 < 13 < 46 < 45 < 12)$,

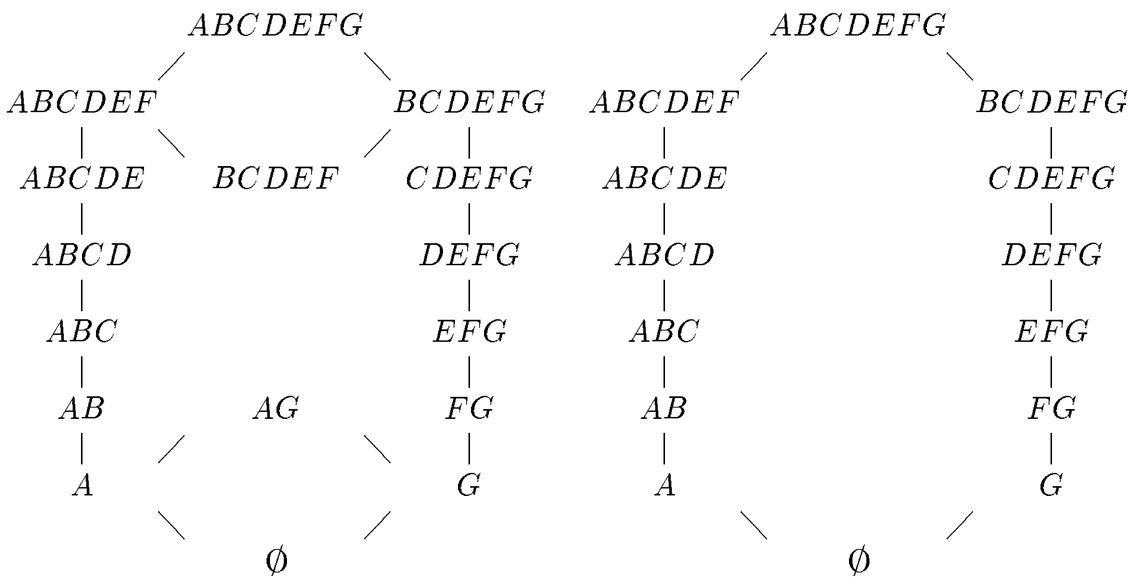
$$\mathrm{Inv}_\lambda\sigma = \{123,\ 12;46,\ 12;56,\ 13;45,\ 13;56,\ 23;45,\ 23;46,\ 456\}$$

is minimal in $P^2(x,[\boldsymbol{\lambda}])$. Similarly, its complement, $\{12;45,\ 13;46,\ 23;56\}$ is maximal in $P^2(x,[\boldsymbol{\lambda}])$. ■

Example 3.2.17 A simpler example, for which P^2 has no unique minimal element, is obtained from $x = 1234$ and $\lambda = (34 < 12 < 14 < 13 < 24 < 23)$, a singular order. In this case $Q^2(x,[\boldsymbol{\lambda}])$ is the total r–order,

$$12;34—134—234—14;23—123—124—13;24.$$

Calling these elements $A, B, \ldots, G$ in order, the posets P^2 and P^2_s are shown below.



3.3 Geometric realisation

Given particular values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, define $\mathbf{e}_p$ for $p \in C_1^1(x)$, and $\mathbf{e}_u$ for $u \in C_2^1(x)$, as in §3.1. Then,

$$\begin{aligned} \boldsymbol{\beta}(\mathbf{e}_p) &= \lambda_p\,, &\qquad \forall p \in C_1^1(x)\,;\\ \boldsymbol{\beta}(\mathbf{e}_u) &= 1\,, &\qquad \forall u \in C_2^1(x)\,. \end{aligned}$$

In a way similar to §2.3, the map,

$$\begin{aligned} \theta\colon P^2\big(x,[\boldsymbol{\lambda}]\big) &\longrightarrow V\\ U &\longmapsto \sum_{u\in U} \mathbf{e}_u \end{aligned}$$

defines a geometric realisation of $P^2\big(x,[\boldsymbol{\lambda}]\big)$. However, as can be seen from Example 3.2.17, $\mathrm{Im}\,(\theta)$ need not form the vertices of a convex polyhedron; indeed the example concluding the proof of Proposition 3.2.16 shows that this is still true when θ is restricted to P_s^2. The rank function is $\boldsymbol{\beta}$.

Definition 3.3.1 *Let $P_r^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ denote the set of extensions of $X^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ by an oriented real hyperplane, two extensions being considered equivalent if the vertex sets determined by the positive sides of the hyperplanes are equivalent. Define a partial order on P_r^2 by single step inclusion. It is clear that $P_r^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ is a subposet of $P_s^2\big(x,[\boldsymbol{\lambda}]\big)$.*

Proposition 3.3.2 *Under the bijection of Proposition 3.2.12, the set* $P_r^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ *maps to the subset of* $O^2([\boldsymbol{\alpha}])$*, consisting of orders on* $C_1^1(x)$ *defined by sequences* $ij \mapsto \frac{\gamma_i-\gamma_j}{\alpha_i-\alpha_j}$ *where* γ_i *are real numbers for which the entries in the sequence are distinct.*

PROOF: By definition, $U \in P_r^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ if, and only if, there exists $\mathbf{v} \in V^*$ and $c \in \mathbf{R}$, such that,

$$U = \left\{u \in C_2^1(x) \mid \mathbf{v}(\mathbf{e}_u) < c\right\}.$$

Using the construction of $\mathbf{e}_u$,

$$\begin{aligned} U &= \left\{p \vee q \;\middle|\; p,q \in C_1^1(x),\, \frac{\mathbf{v}(\mathbf{e}_p)-\mathbf{v}(\mathbf{e}_q)}{\lambda_p-\lambda_q} < c\right\} \\ &= \left\{p \vee q \;\middle|\; p,q \in C_1^1(x), \lambda_p < \lambda_q, \big(\mathbf{v}(\mathbf{e}_p)-\lambda_p c\big) > \big(\mathbf{v}(\mathbf{e}_q)-\lambda_q c\big)\right\} \end{aligned}$$

Under the bijection of Propsition 3.2.12, U is associated with some $\sigma \in O^2([\boldsymbol{\alpha}])$ for which $\mathrm{Inv}_\lambda \sigma = U$. It is clear that, $p <_\sigma q$ if, and only if, $\big(\mathbf{v}(\mathbf{e}_p)-\lambda_p c\big) < \big(\mathbf{v}(\mathbf{e}_q)-\lambda_q c\big)$, so that σ is given by the order of $p \mapsto \mathbf{v}(\mathbf{e}_p)-\lambda_p c$. However, for $p = ij \in C_1^1(x)$,

$$\mathbf{v}(e_p)-\lambda_p c = \frac{\gamma_i-\gamma_j}{\alpha_i-\alpha_j},$$

where $\gamma_i = \mathbf{v}(\mathbf{e}_i) - c\beta_i$. Conversely, given any order determined by such γ_i's, $\mathbf{v}$ and c may be appropriately chosen. ■

Corollary 3.3.3 *The geometric form of* $P_r^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ *is independent of* $\boldsymbol{\beta}$.

Fix $x \in C^1$, say with $x = (x_1,\dots,x_r)$ and $\sum |x_i| = n$. For $ij \in C_1^1(x)$, let $\varphi_{ij} \in V^*$ be defined by,

$$\varphi_{ij}(\mathbf{v}) = \frac{v_i-v_j}{\alpha_i-\alpha_j}.$$

Define hyperplanes, π_u, for $u \in C_2^1(x)$ by, $\pi_{pq}\colon \varphi_p(\mathbf{v}) = \varphi_q(\mathbf{v})$. Then, by Proposition 3.3.2, elements of $P_r^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ are in 1–1 correspondence with the connected components of

$$\mathbf{R}^n \setminus \bigcup_{u \in C_2^1(x)} (\pi_u).$$

Definition 3.3.4 *For $x \in C^1$, define $C^2(x)$ to consist of all unordered sequences $(x_1, \ldots, x_r)$ with $x_i \in C^1(x)$, $\rho_2(x_i) \geq 2$ and*

$$\rho_2(x_1 \wedge \ldots \wedge x_r) = \sum_{i=1}^{r} \bigl(\rho_2(x_i)\bigr) \,.$$

Say that $(x_1, \ldots, x_r) \leq (y_1, \ldots, y_s) \iff \forall i \in [r],\ \exists j \in [s]$ such that $x_i \leq y_j$. The element $(x_1, \ldots, x_r)$ will be denoted $x_1|\cdots|x_r$.

Then $C^2(x)$ is a ranked poset and $\rho_3(x_1|\cdots|x_r) = \sum_{i=1}^{r}\bigl\{\rho_2(x_i) - 1\bigr\}$ defines the rank function. For all $x \in C^1$, x defines an element of $C^2(x)$, namely, the unique maximal element in $C^2(x)$, of rank $\rho_2(x) - 1$.

Example 3.3.5 Take $x = 123;456$. Then 123, $123|456$ and $12;45|13;56$ are elements of $C^2(x)$ of ranks 1, 2 and 2, respectively. Also, $12;45|13;46|23;56 \notin C^2(x)$, since, $\rho_2(12;45) = \rho_2(13;46) = \rho_2(23;56) = 2$, while $12;45 \wedge 13;46 \wedge 23;56 = 123;456$ in C^1, which has rank $4 < 2 + 2 + 2$.

Proposition 3.3.6 *The intersection poset of the arrangement $\bigl\{\pi_u \mid u \in C^1_2(x)\bigr\}$ is isomorphic to $C^2(x)$.*

Proof: Suppose $y_1|\cdots|y_r \in C^2(x)$. For each $k \in [r]$, pick a maximal acyclic graph G_k on $[n]$ such that for any edge ij in G_k we have $ij \leq y_k$. The number of edges in G_k is $\rho_2(y_k)$. Since $\rho_2(y_1 \wedge \cdots \wedge y_r) = \sum \rho_2(y_k)$, the union, T, of the graphs $G_1, \ldots, G_r$ is acyclic. The edges of T are coloured by elements of $[r]$ according to the graph, G_k from which the edge came. Let $k(e)$ denote the colour on the edge $e \in T$. Now consider $\pi_{y_1} \cap \cdots \cap \pi_{y_r}$. This consists of $\mathbf{v} \in V$ for which $\varphi_e(\mathbf{v}) = a_{k(e)}$, for all $e \in T$, some $a_1, \ldots, a_r$. Pick a root on each component of T and orient T away from these roots. On an edge ij, oriented away from i, the above constraint supplies v_j from v_i by,

$$v_j = v_i + a_{k(ij)}(\alpha_j - \alpha_i)\,.$$

Since T is acyclic a point in $\pi_{y_1} \cap \cdots \cap \pi_{y_r}$ is specified by the independent parameters $\{a_1, \ldots, a_r\}$ along with values of v_i at the roots of the components of T (including any singleton elements of $[n]$). The codimension of $\pi_{y_1} \cap \cdots \cap \pi_{y_r}$ in $\mathbf{R}$ is therefore,

$$(\#\text{ edges in } T) - r = \sum \bigl(\rho_2(y_k) - 1\bigr) = \rho_3(y_1 \mid \cdots \mid y_r)\,.$$

It is clear that the order on $C^2(x)$ corresponds to reverse inclusion under the map $y_1 \mid \cdots \mid y_r \mapsto \pi_{y_1} \cap \cdots \cap \pi_{y_r}$. ■

Proposition 3.3.7 (2D) *θ defines a realisation of $P^2_r(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ as a convex polyhedron with rank function defined by $\boldsymbol{\beta} \in V^*$. The k–dimensional faces of $\theta(P^2_r(x,\boldsymbol{\alpha},\boldsymbol{\beta}))$ are identified, up to translation, by elements $u_1 \mid \cdots \mid u_r \in C^2_k(x)$ and are geometrically equivalent to the polyhedra $P^2_r(u_1,\boldsymbol{\alpha},\boldsymbol{\beta}) \times \cdots \times P^2_r(u_r,\boldsymbol{\alpha},\boldsymbol{\beta})$.*

This follows from Proposition 3.3.6 since the interval $[\hat{0}, y_1 \mid \cdots \mid y_r]$ in $C^2(x)$ is $C^2(y_1) \times \cdots \times C^2(y_r)$.

3.4 Examples

The only 3–dimensional polyhedra $P^2_r(x)$ arise from $x \in C^1_4$. There are thus five types coresponding to 12; 34; 56; 78, 123; 45; 67, 123; 456, 1234; 56 and 12345. By the discussion above, the shape of P^2_r is independent of the choice of the parameters $\boldsymbol{\beta}$. The number of vertices in P^2_r for these five polyhedra is given below. Note that the number of vertices in P^2 is given by Stanley's formula [S] to be,

$$\frac{\left(\sum_{k=1}^{r} \binom{|x_k|}{2}\right)!}{\prod_{k=1}^{r} \prod_{j=1}^{|x_k|-1} (2j-1)^{|x_k|-j}}.$$

Table of numbers of vertices			
x	# vertices in $P^2(x)$	# vertices in $P^2_r(x)$	Extra cells in $P^2(x)$
12;34;56;78	24	24	—
123;45;67	40	40	—
123;456	80	76	$P^2(12;45) \times P^2(13;46) \times P^2(23;56)$
1234;56	112	98	$P^2(13;24;56) \times P^2(12;34)$
12345	768	392	*

By Proposition 3.3.6, the number of vertices in $P^2_r(x)$ is given in terms of the Möbius function, μ of $C^2(x)$, as $\sum_{i=0}^{\rho_3(x)} \left| \sum_{y \in C^2_1(x)} \mu(y) \right|$. For the case of $x = 12345$, the poset $C^2(x)$ is schematically represented in Fig. 5. The elements shown are orbit representatives under the action of S_5, the number in parenthesis giving the size of the orbit, while the value of the Möbius function on the orbit is given in square brackets. The number placed against an edge joining vertices labelled u and

v, where u covers v, gives the number of elements in the orbit of v covered by a representative of u. The number of vertices in $P_r^2(12345)$ may be computed in this case to be,

$$1+(15\cdot 1+10\cdot 1)+(60\cdot 1+60\cdot 1+15\cdot 1+10\cdot 3+5\cdot 6)+171=392\,.$$

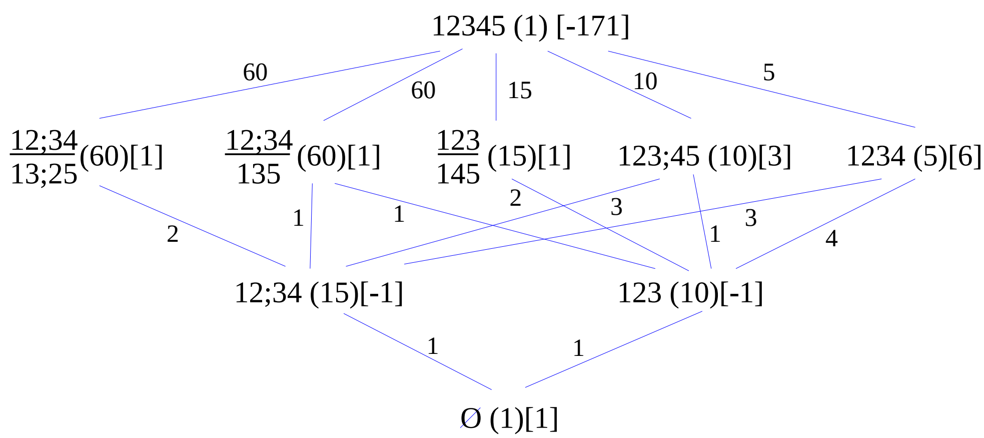


Figure 7

Since the polytopes P_r^2 are centrally symmetric, their structure may be described by an appropriate subdivision of a polygon into polygons, all individually centrally symmetric. Such subdivisions, for the principal cells in the five 3–dimensional polytopes, $P^2(x)$, are given in Figures 8–12.

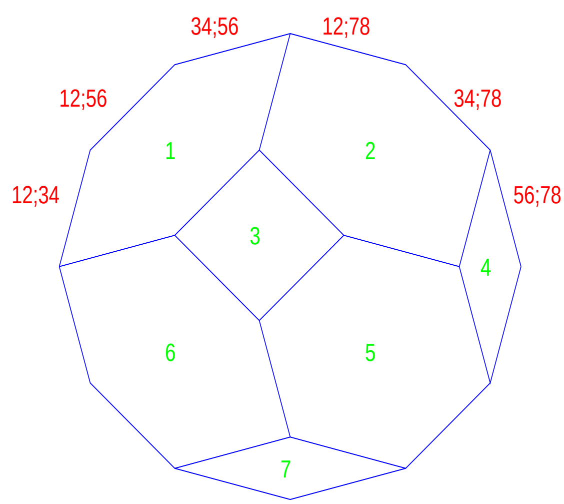


Figure 8: $P^2(12;34;56;78)$

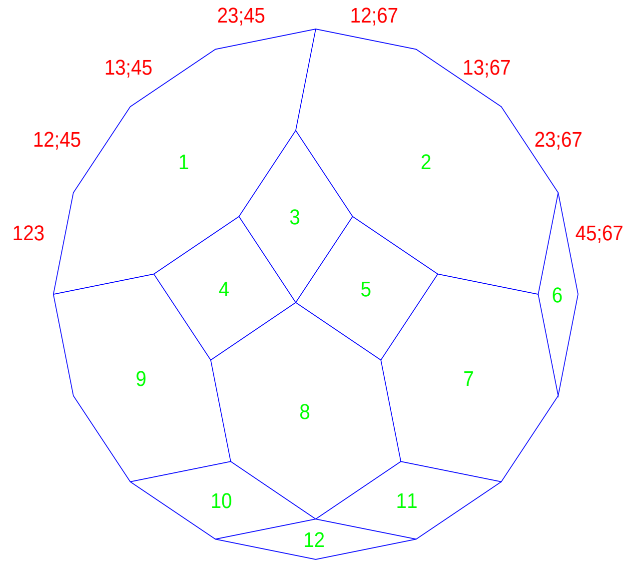


Figure 9: $P^2(123;45;67)$

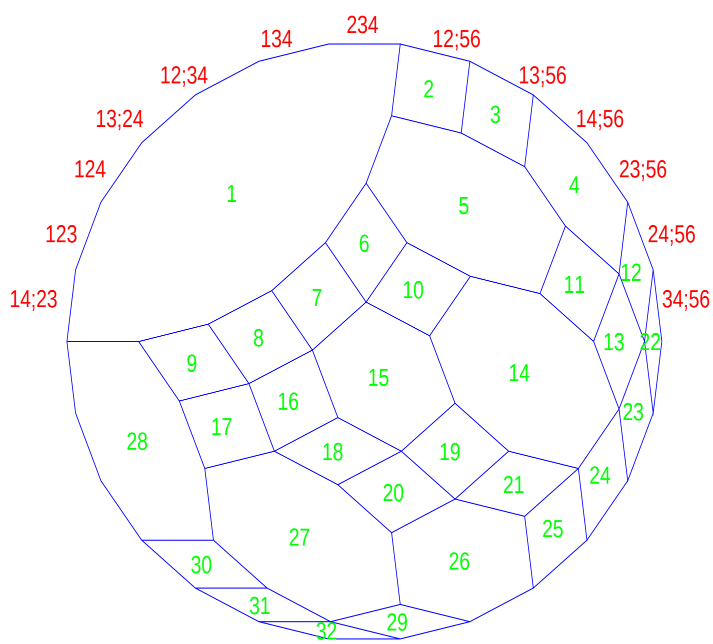


Figure 10: Principal cell in $P^2(1234;56)$

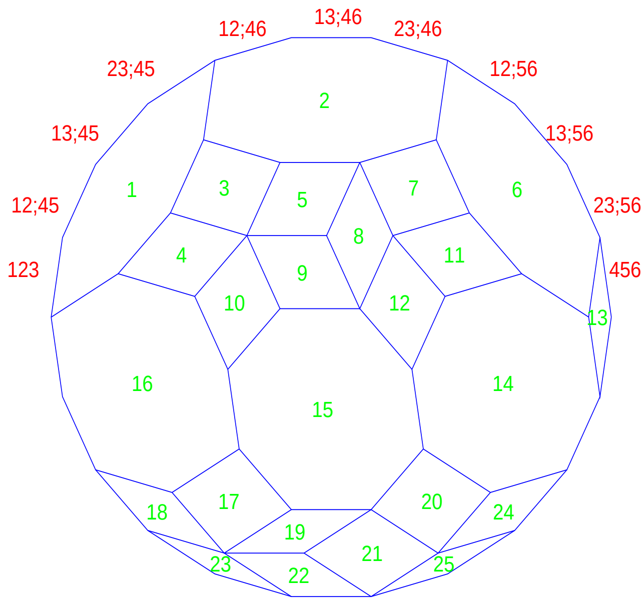


Figure 11: Principal cell in $P^2(123;456)$

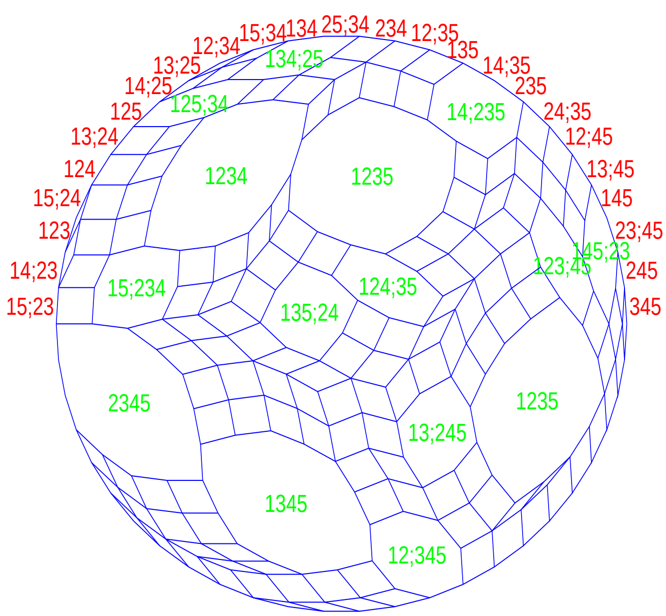


Figure 12: Principal cell in $P^2(12345)$

The hyperplane configuration $\{H_y \mid y \in C^1_{\rho_2(x)-1}(x)\}$ for $x = 12345$ is shown in Figure 13; it is a configuration of lines in $\mathbf{P}^2$, which has been pictured with 1234 chosen as the line at infinity. Observe that, for $x = [n]$, this contains the configuration $\{H_y \mid y \in C^0_{n-1}(x)\}$, which generates the Bruhat order $B(n,2)$ (see [MS 1], [MS 2]). This inclusion defines a map,

$$P^2\big([n], \boldsymbol{\alpha}, \boldsymbol{\beta}\big) \longrightarrow B(n,2)$$

in which $U \mapsto U \cap C^0\big([n]\big)$. Compare the diagram of the principal cell in $P^2(12345)$ in Figure 12 with the 62–vertex diagram of $B(5,2)$ in Figure 14.

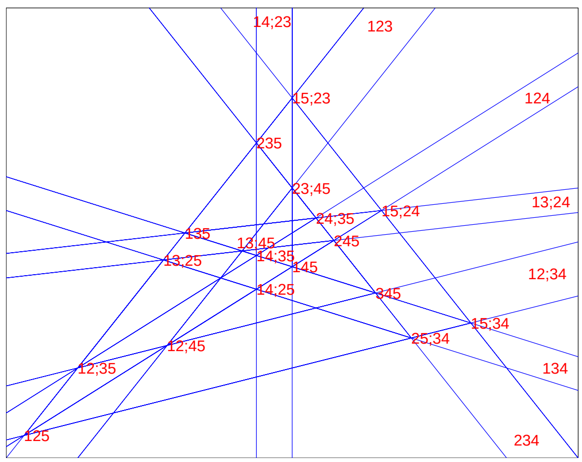


Figure 13: Hyperplane configuration $X^2(12345)$

At this point it may be observed that $P^2_s\big(x, [\boldsymbol{\lambda}]\big)$ (see Definition 3.2.10) has the following geometric description. Let $N = |C^1_1(x)|$ and construct a $2N$–gon, $\Gamma(x)$, with edges labelled by the elements of $C^1_1(x)$ in the order described by $[\boldsymbol{\lambda}]$ and such that opposite edges have the same label. This polygon has two distinguished vertices, say p and p' and the two paths γ and γ' from p to p', around the polygon may be named in such a way that they enumerate $C^1_1(x)$ in the order described by $[\boldsymbol{\lambda}]$ and its reverse, respectively. A maximal chain in $P^2_s\big(x, [\boldsymbol{\lambda}]\big)$ gives a sequence of paths $\gamma_0 = \gamma, \gamma_1, \ldots, \gamma_M = \gamma'$, each from p to p' and contained in the polygon $\Gamma(x)$, such that,

(i) each γ_i contains N line segments, parallel to the appropriate segments in the path γ;

(ii) γ_{i+1} lies in the interior (including the boundary) of the polygon defined by the paths γ_i and γ';

(iii) the interior of the polygon defined by γ_i and γ_{i+1} is connected and is equivalent to $\Gamma(y)$ for some $y \in C^1_2(x)$.

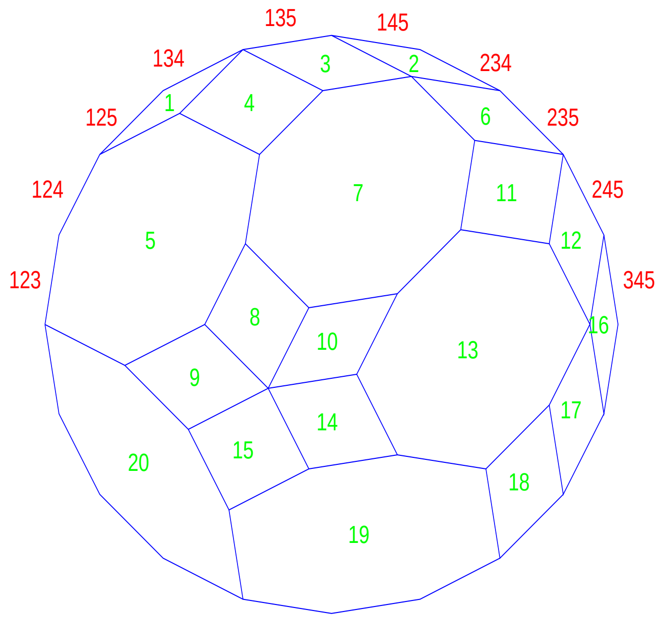


Figure 14: $B(5,2)$

Thus maximal chains in $P^2_s(x,[\boldsymbol{\lambda}])$ may be represented by numbered subdivisions of $\Gamma^2(x)$ into polygons $\Gamma(y)$ with $y \in C^1_2(x)$. The number of polygons (each either a square or a hexagon), M, in any such subdivision is $|C^1_2(x)|$. A particular maximal chain, φ^2 in $P^2_s(x,[\boldsymbol{\lambda}])$ may be obtained by an algorithm in which the polygon Γ_i, adjoined to γ_i to give γ_{i+1}, is the one that minimises the distance from p to the closest point of Γ_i. For $P^2_s(123;456,\ [12 < 13 < 45 < 46 < 23 < 56])$ the result of the application of this algorithm is shown in Figure 15.

When the procedure just outlined is applied one dimension lower, N and M are replaced by $|C^0_1(x)|$ and $|C^0_2(x)|$, respectively, where $x \in C^0$ and $\boldsymbol{\lambda}$ is replaced by $[\boldsymbol{\alpha}]$. The result is a correspondence of maximal chains in the permutahedron poset $P^1(x,[\boldsymbol{\alpha}])$, or equivalently elements of $P^2(x,[\boldsymbol{\alpha}])$, with subdivisions of $\Gamma^1(x)$ into parallelograms. The element φ^1 is identified with a minimal element of P^2. By analogy, one may wish to view φ^2 as the minimal element of a suitable poset P^3 of maximal chains in P^2.

In higher dimensions, the algorithm for the construction of φ may fail, e.g., φ^3 for $x = 12345$. The face labels in Figs 6–9 indicate the chain φ^3 for those elements of C^1_4 for which it is defined. The reader is referred to [E] for a development of connections between zonotopal subdivisions and representation theory.

3.5 Commutativity relations

There are four geometrically distinct types of 2–dimensional face in $P^2_r(x,[\boldsymbol{\lambda}])$, namely squares, hexagons, octagons and 14–gons. These are labelled by elements

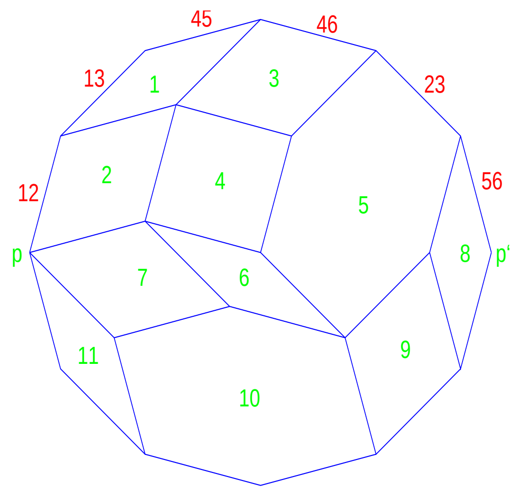


Figure 15: A maximal chain in $P^2(123;456)$

of $C_2^2(x)$, giving,

$$\begin{aligned} \text{squares}: &\quad u|v \text{ with } u,v \in C_2^1(x) \text{ and } \rho_2(u \vee v) = 4 \\ \text{hexagons}: &\quad ij;kl;mn \\ \text{octagons}: &\quad ijk;lm \\ \text{14–gons}: &\quad ijkl\,. \end{aligned}$$

The edges are labelled by elements of $C_2^1(x)$. Let $G^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ denote the groupoid associated with the polyhedron $P_r^2(x,[\boldsymbol{\lambda}])$, defined by analogy with G^1 in §2.3.

Proposition 3.5.1 (2E) *A representation of the groupoid $G^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$ in an algebra A is determined by elements $R_u \in A$ for $u \in C_2^1(x)$, such that,*

$$\begin{aligned} R_uR_v &= R_vR_u \text{ whenever } u|v \in C^2(x)\,, \\ R_{pq}R_{pr}R_{qr} &= R_{qr}R_{pr}R_{pq} \text{ whenever } p,q,r \in C_1^1(x) \text{ are disjoint and } \lambda_p < \lambda_q < \lambda_r\,, \end{aligned}$$

along with two further relations giving the equality of a product with its reverse, the terms $\{R_{ijk}, R_{ij;lm}, R_{ik;lm}, R_{jk;lm}\}$ and $\{R_{ijk}, R_{ijl}, R_{jkl}, R_{ikl}, R_{ij;kl}, R_{ik;jl}, R_{jk;il}\}$ being in orders determined by that of the associated collections of vertices along the lines labelled $ijk;lm$ and $ijkl$, respectively, in $X^2(x,\boldsymbol{\alpha},\boldsymbol{\beta})$.

Remark 3.5.2 Following [MS 2] one may consider the nilpotent completion of the fundamental group of the complement of the complexification of the hyperplanes $\{\pi_u \mid u \in C_2^1(x)\}$ of §3.3. This is found to be generated by degree 1 elements $\{s_u \mid u \in C_2^1(x)\}$ with,

$$\begin{aligned}
[s_u, s_v] &= 0 \qquad \text{for } \rho_2(u \vee v) = 4 \\
[s_{ij;kl}, s_{ij;mn} + s_{kl;mn}] &= 0 \\
[s_{ijk}, s_{ij;lm} + s_{ik;lm} + s_{jk;lm}] &= 0 \\
[s_{jk;lm}, s_{ijk} + s_{ij;lm} + s_{ik;lm}] &= 0 \\
[s_{ijk}, s_{ijk} + s_{ikl} + s_{jkl} + s_{ij;kl} + s_{ik;jl} + s_{il;jk}] &= 0 \\
[s_{ij;kl}, s_{ijk} + s_{ijl} + s_{ikl} + s_{jkl} + s_{ik;jl} + s_{il;jk}] &= 0\,.
\end{aligned}$$

A representation in $V^{\otimes N}$ $(N = |C_1^1(x)|)$ is obtained with $s_{ijk} \mapsto S_{ij,ik,jk}$ and $s_{ij;kl} \mapsto T_{ij,kl}$ so long as,

$$\left.\begin{aligned}
[T_{12}, T_{13} + T_{23}] &= 0 \\
[\overline{S}_{123}, T_{34}] &= 0 \\
[\overline{S}_{123}, \overline{S}_{145} + \overline{S}_{246} + \overline{S}_{356}] &= 0
\end{aligned}\right\}$$

where $\overline{S} = S_{123} - T_{12} - T_{13} - T_{23}$. The first and third relations here are infinitesimal forms of the Yang-Baxter and Zamalodchikov equations (see [K]). The last relation mentioned in Proposition 3.5.1 is a form of the permutahedron relation (see [L1], [KV]). Just as the Yang-Baxter equation plays a central role in knot theory in 3–manifolds, the permutahedron type equations seem to arise from generators of equivalences of 'braid movies' for knots in four dimensions (see [Kh], [CS], [F] and [KT]).

4: Problems and generalisations

In the process of generalisation of Propositions 2.1.1–2.3.2 to the next dimension up, it was seen in §3 that various complications arise. In particular, three different posets, P_s^2, P^2 and P_r^2 were considered, with $P_r^2 \subseteq P_s^2 \subseteq P^2$. The larger poset consists of maximal chains in P^1, while P_r^2 is spherical and arises in connection with C^2, a higher version of the partition lattice. The poset P_s^2 comes as a form of extension poset in which points on a line are only partially ordered, with $P^2 = P_s^2$ for non-singular orders $[\boldsymbol{\lambda}]$ on $C_r^1(x)$. However, even P_s^2 does not generally possess unique minimal and maximal elements. This contrasts with the situation for higher Bruhat orders $B(n,k)$, which may be obtained using the uniform extension poset construction (by pseudohyperplanes) for a cyclic hyperplane arrangement. In [Z] it was shown that for $n-k \le 2$ or $k=2$, $B(n,k)$ is spherical.

Conjecture 4.0.1 *$O^2_g(n) \equiv \{[\boldsymbol{\lambda}] \mid P^2(x,[\boldsymbol{\lambda}])$ has a unique minimal element$\}$ contains both the lexicographic and anti-lexicographic orders $[\boldsymbol{\lambda}]$ on $C^1_1(x)$ (with respect to $[\boldsymbol{\alpha}]$).*

Indeed, it is reasonable to suppose that $O^2_g(x)$ may be defined by a set of 'local' constraints on $[\boldsymbol{\lambda}]$ similar to the non-singularity constraint defining $O^2_{ns}(x)$ (see Remark 3.2.2 and Proposition 3.2.12). As observed in §3.4, P^2_r is in general a proper subset of P^2. The map θ of §3.3 realises P^2 as a set of points whose convex hull is the polytope defined by P^2_r.

Conjecture 4.0.2 *$P^2(x)$ may be expressed as a union of cells of the form $P^2_r(y_1)\times\cdots\times P^2_r(y_r)$ where $y_1,\cdots,y_r \in C^1(x)$ and $\{C^1_1(y_i)\}$ are disjoint, but $y_1 \mid \cdots \mid y_r \notin C^2(x)$.*

Thus, for $x = 123;456$, there are five cells, namely, $P^2_r(123;456)$ along with two copies of $P^2_r(12;45)\times P^2_r(13;46)\times P^2_r(23;56)$ and two copies of $P^2_r(12;56)\times P^2_r(13;46)\times P^2_r(23;45)$ all these being of dimension 3. Note that maximal cells in $P^2(x)$ may have dimension greater than the dimension, $\rho_3(x)$, of the 'big cell' $P^2_r(x)$. For example,

$$P^2_r(12;34)\times P^2_r(13;25)\times P^2_r(14;35)\times P^2_r(15;24)\times P^2_r(23;45)$$

is a 5–dimensional cell in $P^2(12345)$ based at,

$$[34<12<14<35<15<24<13<25<45<23] \in O^2_{ns}(12345)\,,$$

the big cell being $P^2_r(12345)$, of dimension 3.

Some of the constructions of §3 may be recursively extended to higher orders. This gives a configuration of hyperplanes $X^{k+1}(x,\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)},\ldots,\boldsymbol{\alpha}^{(k+1)})$ for $x \in C^k([n])$ and $\boldsymbol{\alpha}^{(1)},\ldots,\boldsymbol{\alpha}^{(k+1)} \in \mathbf{R}^n$, whose intersection poset contains a part of $C^k(x)$. The vertices so defined are labelled by elements of $C^{k-1}_2(x)$ and appear at,

$$\mathbf{e}_{pq} = {(\mathbf{e}_p-\mathbf{e}_q)}/{\boldsymbol{\alpha}^{(k)}(\mathbf{e}_p-\mathbf{e}_q)} \quad \text{for } p,q \in C^{k-1}_2(x) = C^k_1(x)\,.$$

The appropriate definitions of P^{k+1} and P^{k+1}_s are not clear, but P^{k+1}_r is defined as the set of real hyperplane extensions of X^{k+1}. Analogues of Propositions 3.3.2, 3.3.3, 3.3.6 and 3.3.7 hold. Here $C^k(x)$, for $x \in C^{k+1}$, is defined in a way analogous to Definition 3.3.4. However, the complications arising from the choice of the appropriate notion of extension poset become more severe as k increases.

The formal notion of categorification is precisely that required to go from a structure relevant in one dimension, to that appropriate in the next. Special cases are the algebraic (**CAT**) and combinatorial (**PMC**) forms, but also the categorical form which generates n–categories from $(n-1)$–categories. The problems of non-uniqueness of definition and difficulty in finding an object which is both universal and weak is well-known in this form, for which there is perhaps a fundamental combinatorial reason, which also applies in the world of the present paper, as to why the process of construction of P^k and $\mathcal{E}^k$ *cannot* be continued using current notions.

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