Witten-Reshetikhin-Turaev invariants of Seifert manifolds

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Abstract. For Seifert homology spheres, we derive a holomorphic function of K whose value at integer K is the $\mathfrak{sl}_2$ Witten-Reshetikhin-Turaev invariant, Z_K, at $q{=}\exp 2\pi i/K$. This function is expressed as a sum of terms, which can be naturally corresponded to the contributions of flat connections in the stationary phase expansion of the Witten-Chern-Simons path integral. The trivial connection contribution is found to have an asymptotic expansion in powers of K^{-1} which, for K an odd prime power, converges K–adically to the exact total value of the invariant Z_K at that root of unity. Evaluations at rational K are also discussed. Using similar techniques, an expression for the coloured Jones polynomial of a torus knot is obtained, providing a trivial connection contribution which is an analytic function of the colour. This demonstrates that the stationary phase expansion of the Chern-Simons-Witten theory is exact for Seifert manifolds and for torus knots in S^3. The possibility of generalising such results is also discussed.

1. Introduction and main results

Suppose that M is a compact oriented 3–manifold without boundary. In [Wi], Witten formally defined a topological invariant $Z_{k+\check{c}_{\mathfrak{g}}}(M)$, dependent on some additional data, namely a choice of a Lie algebra $\mathfrak{g}$ (with dual Coxeter number $\check{c}_{\mathfrak{g}}$) and of a level $k\in\mathbf{Z}$, in the form of a functional integral,

$$Z_{k+\check{c}_{\mathfrak{g}}}(M)=\int_{\mathcal{A}/\mathcal{G}}\mathrm{e}^{\frac{ik}{4\pi}\int_M\langle A,dA+{}^1\!/\!_3[A,A]\rangle\,d\tau}\mathcal{D}A\,, \tag{1.1}$$

over a quotient of the space of G–connections on M by an appropriate gauge group, $\mathcal{G}$. For the integrand to be well-defined, that is invariant under $\mathcal{G}$, one needs k to be an integer. Although many attempts have been made to give a direct and calculable meaning to this Feynman integral, it remains only a formal expression from which valid results can be derived when the functional integral is manipulated according to certain rules; see for example [A2], [AS1], [AS2], [B], [BN1], [R1], [R2] and [R4]. The approaches which are closest in spirit to that of (1.1) employ the notion of a topological field theory (see [A1]) whose definition is based on Segal's mathematical definition of conformal field theory. From this perspective, $Z_K(M)$ should be viewed as a special case of invariants of pairs (L,M), of a link (coloured by representations of G) contained in a 3-manifold M, the associated form of (1.1) containing additional factors, a *Wilson loop* corresponding to each component of the link L. When $M=S^3$, the invariant $Z_K(S^3,L)$ is known as a *quantum link invariant* and is usually considered as a polynomial in $q^{\pm\frac{1}{2}}$ where $q=\mathrm{e}^{\frac{2\pi i}{K}}$; they can be obtained independently using the quantum group $U_q\mathfrak{g}$.

In this paper, we consider only the cases $G=SU(2)$ and $G=SO(3)$; the associated invariants will be denoted Z_K and Z'_K respectively. Many alternative and

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completely rigorous formulations of $Z_K(M)$ have been obtained, primarily using the description of a compact, connected, orientable 3–manifold M, without boundary, as obtained by Dehn surgery around a suitable link L_M, in S^3. Reshetikhin and Turaev [RT] found $Z_K(M)$ as a combination of the quantum invariants of L obtained from all possible choices of irreducible representations attached to the components of L. This sum will only be finite when q is a root of unity. It is still something of a mystery that while quantum invariants of links in S^3 are defined for all values of q, being polynomials, this happy state of affairs is not true of any of the definitions so far known for $Z_K(M,L)$ when $M\neq S^3$; however, see [L3].

From the formulation of [RT], it is seen that $Z_K(M,\emptyset)$ can be defined for all roots of unity q, rather than just those of the form $\mathbf{e}^{\frac{2\pi i}{K}}$. Very few concrete computations of $Z_K(M,\emptyset)$, as a function of the order, K, of the root of unity q, have been carried out — see [FG], [J], [KL1], [KM1], [KM2] and [N] for some such computations. It follows quickly from its definition that, for fixed order K, $Z_K(M,\emptyset)$ can be written as an algebraic function of q, with rational coefficients. In the normalisation for which the invariant for S^3 is 1, denote the invariant for the pair $(M,\emptyset)$, as an algebraic function of q at $K^{\rm th}$ roots of unity, by $Z_K(M)$. Kirby and Melvin [KM2] derived a symmetry principle for terms in the sum arising in $Z_K(M)$ and thereby obtained a slightly finer invariant, which is just the associated $SO(3)$–invariant, $Z'_K(M)$. We now describe some of the known results on the forms of these functions of $h=q-1$.

Theorem ([M1], [M2], [O1], [O2], [O3]) *Suppose that K is an odd prime and M is an oriented $\mathbf{Z}/K\mathbf{Z}$–homology sphere. Let $H=|H_1(M,\mathbf{Z})|$, so that $K\not| H$.*

(a) As a function of q, $Z'_K(M)\in\mathbf{Z}[h]$, so that for some $a_{m,K}(M)\in\mathbf{Z}$, one has $Z'_K(M)=\sum_m a_{m,K}(M)h^m$. For $0\leq m\leq K-2$, $a_{m,K}(M)$ is uniquely determined by this condition as an element of $\mathbf{Z}/K\mathbf{Z}$.

(b) There exist rational numbers $\lambda_m(M)\in\mathbf{Z}[\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{2m+1},\frac{1}{H}]$ such that, for any prime $K\geq 2m+3$, $a_{m,K}(M)\equiv\left(\frac{H}{K}\right)\lambda_m(M)$ as elements of $\mathbf{Z}/K\mathbf{Z}$, where $\left(\frac{H}{K}\right)$ denotes the quadratic residue while

$$\lambda_0(M)=H^{-1}\,,\qquad \lambda_1(M)=6H^{-1}\lambda(M)\,,$$

where $\lambda(M)$ denotes the Casson–Walker [Wa] invariant of M in Casson's normalisation.

As a result of part (b) of this Theorem, Ohtsuki defines a formal power series

$$Z_\infty(M) = \sum_{m=0}^{\infty} \lambda_m(M) h^m ,$$

with rational coefficients, which is an invariant of rational homology 3–spheres, M. The coefficients $\lambda_m(M)$ in this series were computed explicitly in [L1] for the case of arbitrary surgery around $(2,n)$ torus knots, and found to be expressible in terms of Bernoulli (or Euler) numbers. Furthermore, it was found in [L1] that this formal power series can be viewed as the asymptotic expansion of a holomorphic function of K, defined by an integral expression convergent on a half plane; see §4.8 for the form of these integrals.

Theorem ([L1], [L2]) *Suppose that M is a rational homology sphere obtained by integer surgery around a $(2,n)$ torus knot, with arbitrary framing and that K is an odd prime power coprime to $H = |H^1(M,\mathbf{Z})|$. Then*

(a) $Z_\infty \in \mathbf{Z}[\frac{1}{2},\frac{1}{H}][[h]]$;

(b) the formal power series $Z_\infty(M)$ converges K-adically to $\widetilde{Z}_K(M)$; that is, there is an equality between $Z_\infty(M)$ and $\widetilde{Z}_K(M)$ as elements of $\mathbf{Z}_K[[h]]/(\phi_K(1+h))$, where $\mathbf{Z}_K$ denote those rationals with denominators coprime to K, and $\phi_K(x)$ denotes the K^{th} cyclotomic polynomial, $\prod_{(n,K)=1}\left(x - \mathbf{e}^{\frac{2\pi i n}{K}}\right)$.

Similar results are conjectured to hold for arbitrary rational homology spheres. In particular, they are known from [R6] for K prime whenever M is either a Seifert manifold or obtained by surgery on S^3 around an arbitrary knot. The property of K–adic convergence gives an infinite set of congruences modulo powers of K satisfied by the terms $\lambda_m(M)$, for each K, very much stronger than those given in Ohtsuki's result.

In the current paper we provide a complete and self-contained analysis of WRT invariants of Seifert-fibred homology spheres (§4) and of torus knots in S^3 (see §5). The basic notation used in the paper is introduced in §2, and the construction of WRT invariants via both integer and rational surgery is summarised in §3. The main technique used in this paper is the Lemma of §4.2, a simple consequence of Cauchy's residue theorem, which enables sums to be transformed into integrals, thereby rewriting a sum only meaningful for integer values of some parameter in a form analytically extendable.

For Seifert manifolds, the initial evaluation of $Z_K(M)$ in §4.1 is as a sum of terms, the number of terms being proportional to K. This is tranformed into an integral in §4.3 for the case of $K \in \mathbf{Z}$ and in §4.4 for $K \in \mathbf{Q}$. The reformulation of $Z_K(M)$ is now as a sum of terms, some integrals and some residues, and these are seen to be able to be corresponded with the contributions in a stationary phase expansion of (1.1) from conjugacy classes of flat connections derived in [R3]. The precise form of the terms is discussed in §4.5, the integral term corresponding to the trivial connection and reducible connections, while the residue terms come from irreducible connections. From the form of the terms, it is also seen that each can be naturally extended to a holomorphic function of K.

The form of the trivial connection contribution, $Z^0_K(M)$, as a simple integral enables an asymptotic expansion in powers of $h=q-1$ to be carried out, leading to direct verification of the integrality properties of the coefficients, some new formulae for the second and third order coefficients and new conjectures concerning their divisibility properties in general (see §4.6). It also enables a new proof of K–adic convergence in §4.7. Some numerical data is given for the contributions from various connections in §4.9, both for the case of primitive roots of unity ($K\in\mathbf{Z}$) and other roots of unity (rational K). Finally, the same techniques are applied to torus knots and some more general conjectures are given in §6.

2. Notation

2.1 Some elementary number theory

Suppose that M and N are integers with M odd. We use the *Jacobi symbol* $\left(\frac{N}{M}\right)$. When M is prime it is defined to be 0, -1 or 1 according as N is divisible by M, is not a quadratic residue modulo M, or otherwise. It is extended to arbitrary integers M by multiplicativity with respect to M, with $\left(\frac{N}{2}\right)=(-1)^N$.

Throughout this paper, whenever K is an integer, all expressions are to be understood algebraically as functions of $q=\exp\frac{2\pi i}{K}$, that is, as representing elements of $\mathbf{Q}[A]$, where $A=\exp\frac{\pi i}{2K}$ is a fourth root of q, which is chosen so that it has order precisely $4K$. By this means, all the expressions take on a meaning in $\mathbf{Q}[A]$, when $q=\exp\frac{2m\pi i}{K}$, where m is an integer coprime to $4K$, now not necessarily 1. Despite this, we will use a (consistent) notation in which some expressions will not at first sight appear to be elements of $\mathbf{Q}[A]$. In particular,

$$
\begin{aligned}
i &\equiv A^K\,,\\
\sin\frac{\pi\alpha}{2K} &\equiv \frac{1}{2i}(A^\alpha - A^{-\alpha})\quad\text{for }\alpha\in\mathbf{Z}\,,\\
\sqrt{K} &\equiv \frac{1}{2(1+i)}\sum_{s=0}^{4K-1}A^{s^2}\,.
\end{aligned}
$$

This notation is strictly valid, when evaluated in $\mathbf{C}$ only when $m=1$ and in other cases signs may be introduced; for the particular expressions just discussed these signs are given below.

$$
\begin{aligned}
i &\longrightarrow (-1)^{\frac{m-1}{2}} i \\
\sin \frac{\pi\alpha}{2K} &\longrightarrow (-1)^{\frac{m-1}{2}} \sin \frac{\pi m \alpha}{2K} \\
\sqrt{K} &\longrightarrow \left(\frac{K}{m}\right) \sqrt{K}
\end{aligned}
$$

Observe in particular that the scaling factor in the transformation of $\sqrt{K}$ is multiplicative in K.

In the normalisation of manifold invariants discussed in the next section, the quantities G_0, G_+ and G_- will enter, where

$$
\begin{aligned}
G_0^{-1} &= \sqrt{\frac{2}{K}} \sin \frac{\pi}{K}\,, \\
G_+ &= \frac{A^{-3}}{2(A^2 - A^{-2})} \sum_{s=1}^{4K} (-A)^{s^2} = \frac{(-i)^K \, \mathrm{e}^{-\pi i/4} \sqrt{K/2}}{A^3 \sin \frac{\pi}{K}}\,,
\end{aligned}
$$

and G_- is given by exactly the same form as G_+, except that A is everywhere replaced by A^{-1}. Also denote by $G_{m,n}$ the Gauss sum $\sum_{s=0}^{n-1} \mathrm{e}^{\frac{2\pi i m s^2}{n}}$.

Suppose that P and Q are coprime integers. The *Dedekind sum* is defined by

$$
s(P,Q) = \frac{1}{4Q} \sum_{j=1}^{Q-1} \cot\left(\frac{\pi j}{Q}\right) \cot\left(\frac{\pi P j}{Q}\right)\,,
$$

for $Q > 0$, with $s(P,-Q) = -s(P,Q)$. For any matrix $\Lambda = \begin{pmatrix} P & R \\ Q & S \end{pmatrix} \in SL(2,\mathbf{Z})$, the *Rademacher function* is defined by

$$
\Phi(\Lambda) = \frac{P+S}{Q} - 12 s(P,Q) \in \mathbf{Z}\,;
$$

when $Q = 0$, one sets $\Phi(\Lambda) = \frac{R}{S}$.

2.2 Manifold notation

Suppose that L is a (framed) link, embedded in three manifold M, whose components are labelled with elements of $SL(2,\mathbf{Z})$, providing a string of matrices $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_c)$. This data enables one to construct another 3-manifold, denoted $M_{L,\mathbf{\Lambda}}$, by rational Dehn surgery. That is, $M_{L,\mathbf{\Lambda}}$ is obtained as the result of gluing N, a disjoint union of tubular neighbourhoods N_j of the components L_j of L, to

$M\backslash N$ with the identification described by $\mathbf{\Lambda}$. The $j^{\rm th}$ components of the common boundary are identified according to $\Lambda_j = \begin{pmatrix} P_j & R_j \\ Q_j & S_j \end{pmatrix}$, considered as an element of the mapping class group of the torus. That is, the gluing identifies the meridian on ∂N_j to the curve on the $j^{\rm th}$ component of $\partial M\backslash N$ homotopic to, P_j times a meridian plus Q_j times a longitude, and similarly for the longitude. The resulting manifold depends only on P_j and Q_j, but it will be convenient to specify Λ_j. Let

$$\Phi(L,\mathbf{\Lambda}) = \sum_j \Phi(\Lambda_j) - 3\sigma(L)\ ,$$

where $\sigma(L)$ denotes the signature of the link L, that is, the signature of the linking matrix of L whose diagonal entries are $\frac{P_j}{Q_j}$.

There is a $(K-1)$–dimensional representation, ρ_K, of $PSL(2,\mathbf{Z})$ in which the standard generators $\mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ map to

$$\begin{aligned} \big(\rho_K(\mathbf{S})\big)_{\alpha,\beta} &= \sqrt{\frac{2}{K}} \sin\frac{\pi\alpha\beta}{K}\ , \\ \big(\rho_K(\mathbf{T})\big)_{\alpha,\beta} &= \mathbf{e}^{\frac{\pi i}{4}} A^{\alpha^2} \delta_{\alpha=\beta}\ . \end{aligned}$$

This representation has matrix entries lying in $\mathbf{Q}[A]$, when K is even, but only $\mathbf{Q}[\sqrt{A}]$ when K is odd. Let d^Λ_α denote the matrix element $\big(\rho_K(\Lambda)\big)_{\alpha,1}$ and let $\mathbf{d}^\Lambda$ denote the vector whose $\alpha^{\rm th}$ component is d^Λ_α. In the phenomenology of topological field theory, at level $K-2$, the vector space associated with a torus has dimension $K-1$ and ρ_K gives the action of the mapping class group of the torus. In this sense, $\mathbf{d}^\Lambda$ is interpreted as the vector associated with a solid torus whose boundary has been twisted by Λ. The following explicit formula was obtained in [J], for $\Lambda = \begin{pmatrix} P & R \\ Q & S \end{pmatrix}$ when $Q \neq 0$,

$$d^\Lambda_\alpha = \frac{i.\mathrm{sign}Q}{\sqrt{2K|Q|}}\, \mathbf{e}^{-\pi i\Phi(\Lambda)/4} \sum_{n=0}^{q-1} \sum_{\mu=\pm1} \mu\, \mathbf{e}^{\frac{\pi i}{2KQ}\left(P\alpha^2 - 2(2Kn+\mu)\alpha + S(2Kn+\mu)^2\right)}\ .$$

In the case of $\Lambda = \mathbf{S}$, one finds that

$$d^{\mathbf{S}}_\alpha = \frac{A^{2\alpha} - A^{-2\alpha}}{A^2 - A^{-2}} = d_\alpha\ ,$$

the quantum dimension of the α-dimensional representation of $U_q\mathfrak{sl}_2$.

3. Witten-Reshetikhin-Turaev invariants

3.1 Integer surgery presentation

Suppose that M is a 3–manifold obtained by surgery around the framed link L in S^3. Represent L by a link diagram, $\mathcal{D}$, with the blackboard framing. The $\mathfrak{sl}_2$ Witten-Reshetikhin-Turaev invariant of the empty link in M, at the root of unity q, will be denoted $Z_K(M)$. It can be computed generally as a suitably normalised version of the partition function of a certain state model, whose states are allowed assignments of an element of $\{0,1,\ldots,K-2\}$ to each of the components of L, as well as to each of the regions into which $\mathcal{D}$ divides the plane. Such an assignment is said to be *allowed* if the infinite region is labelled 0 and, for each edge of $\mathcal{D}$, the triple of integers assigned to the two adjacent regions and the component containing the edge, form a q–admissible triple, that is, they have form (a,b,c) satisfying the Clebsch-Gordon conditions

$$a+b+c\in 2\mathbf{Z}, \qquad |a-b|\le c\le a+b, \qquad a+b+c\le 2(K-2)\,.$$

The weight, $W_{\mathcal{D}}$, assigned to a state, σ, is defined as the product of local weights associated with each vertex, edge, face and component of $\mathcal{D}$, each of which is, up to sign and powers of q, a sum of certain ratios of products of q-factorials, namely quantum dimensions, theta nets and quantum $6j$ symbols.

The invariant $Z_K(M)$ is now obtained from the partition function of this state model by renormalisation, so that

$$Z_K(M)=G_+^{-n_+}G_-^{-n_-}\sum_{\text{states }\sigma}W_{\mathcal{D}}(\sigma)\,,$$

where n_+ and n_- are the numbers of positive and negative eigenvalues, respectively, of the linking matrix defined by the framed link L. Also G_+ and G_- denote the partition function evaluations on an unknot with framings 1 and -1 respectively; they take the form of Gauss sums.

The WRT invariant can be alternatively computed from the generalised Jones polynomial of a link presentation. Whenever a link L is *coloured* by placing non-negative integers $(\alpha_j)_{j=1}^c=\boldsymbol{\alpha}$, one on each component of L, one may compute the coloured Jones polynomial $J_{\boldsymbol{\alpha}}(L)$, as a polynomial in A^2 and A^{-2}, where we use the normalisation in which the value on the unlink is $\prod_j d_{\alpha_j}$. Suppose now that M is a three-manifold obtained by integer Dehn surgery around a framed link L. Then the Witten-Reshetikhin-Turaev invariant, normalised to be 1 for S^3 is computed by

$$Z_K(M)=G_+^{-n_+}G_-^{-n_-}G_0^{-n_0}\sum_{\boldsymbol{\alpha}}J_{\boldsymbol{\alpha}}(L)\prod_{j=1}^{|L|}d_{\alpha_j}\,, \tag{3.1}$$

where the sum is over $\boldsymbol{\alpha}$ for which $1 \le \alpha_j \le K-1$ for all j and n_0 is the rank of $H_1(M,\mathbf{Z})$. Indeed $J_{\boldsymbol{\alpha}}(L)$ is nothing but the state sum in the state model mentioned above, over a restricted set of states constrained by the condition that the label on the $j^{\rm th}$ component is α_j and the local contributions to the weight come from edges, regions and vertices, omitting components. When L is an unknot, this gives

$$Z_K(S^2\times S^1) = {G_0}^{-1}\sum_{\alpha=1}^{K-1} d_\alpha^2 = G_0\ .$$

3.2 Rational surgery presentation

In this section we assume that M is presented as rational surgery around a link L, with surgery data given by a string $\boldsymbol{\Lambda}$ of matrices in $SL(2,\mathbf{Z})$.

Suppose now that we are given a representation of each $\Lambda_j = \begin{pmatrix} P_j & R_j \\ Q_j & S_j \end{pmatrix}$ as a word in $\mathbf{S}$ and $\mathbf{T}$; equivalently, pick a continued fraction expansion of $\frac{P_j}{Q_j}$. One can construct a framed link $L(\boldsymbol{\Lambda})$ by adjoining to each component of L a simply linked chain of unknots, the length of each chain being the length of the corresponding continued fraction and the framing on the components being determined by the terms in the expansion. Then $L(\boldsymbol{\Lambda})$ has the property that $M = S^3_{L,\boldsymbol{\Lambda}}$ is equivalently expressed as integer surgery $S^3_{L(\boldsymbol{\Lambda})}$. One may now compute $Z_K(M)$ from (3.1) with $L(\boldsymbol{\Lambda})$ in place of L. In [J], the following formula was derived for $Z_K(M)$, directly in terms of L and $\boldsymbol{\Lambda}$,

$$ {G_0}^{-1} Z_K(M) = \mathbf{e}^{\frac{\pi i}{4}\frac{K-2}{K}\Phi(L,\boldsymbol{\Lambda})} \sum_{\boldsymbol{\alpha}} J_{\boldsymbol{\alpha}}(L,K)\prod_{j=1}^{|L|} d_{\alpha_j}^{\Lambda_j}\ . \tag{3.2}$$

where the sum is over $\boldsymbol{\alpha}$ for which $1\le\alpha_j\le K-1$ for all j.

Consider a Hopf link L. Perform rational $\frac{P}{Q}$ surgery around L_1 on the complement of a neighbourhood, N_2, of L_2. Choose $\Lambda = \begin{pmatrix} P & R \\ Q & S \end{pmatrix} \in SL(2,\mathbf{Z})$. The resulting surgery operation is equivalent to twisting the boundary of N_2 by $\mathbf{S}\Lambda$, which has ratio $-\frac{Q}{P}$. In the process a 2-framing correction is acquired, so that

$$\mathbf{e}^{\frac{\pi i}{4}\frac{K-2}{K}\Phi(L,\boldsymbol{\Lambda})}\sum_{\alpha=1}^{K-1}\big(\rho_K(\mathbf{S})\big)_{\beta,\alpha} d_\alpha^\Lambda = \mathbf{e}^{\frac{\pi i}{4}\frac{K-2}{K}\Phi(L_2,\Lambda)} d_\beta^{\mathbf{S}\Lambda}\ ,$$

where $\boldsymbol{\Lambda} = (\mathbf{S},\Lambda)$. Since $\sigma(L)=0$, this reduces to

$$\mathbf{e}^{\frac{\pi i}{4}\frac{K-2}{K}\Phi(\Lambda)}\sum_{\alpha=1}^{K-1} d_\alpha^\Lambda \sin\frac{\pi\alpha\beta}{K} = \sqrt{\frac{K}{2}}\,\mathbf{e}^{\frac{\pi i}{4}\frac{K-2}{K}\left(\Phi(\mathbf{S}\Lambda)+3\,\mathrm{sign}\frac{Q}{P}\right)} d_\beta^{\mathbf{S}\Lambda}\ . \tag{3.3}$$

This formula will be used later.

4. Seifert manifolds

Suppose that P_j and Q_j, for $j = 1, \ldots, N$, are non-zero integers for which P_j is coprime to Q_j for all j, while the P_j's are pairwise coprime. Construct a link, L, with $(N+1)$ components obtained by adjoining to an N–component unlink, a single unknotted component (which we count as the 0^{th} component) whose linking number with each of the components of the unlink is one. Let $\mathbf{\Lambda}$ denote rational surgery data on L whose ratios are $\frac{P_j}{Q_j}$ for each of the components of the unlink and 0 on the final component. Then $S^3_{L,\mathbf{\Lambda}}$ is the N-fibred Seifert manifold which is usually denoted $X(\frac{P_1}{Q_1}, \ldots, \frac{P_n}{Q_N})$, and which we shall denote by M. Throughout this section we put

$$P = \prod_{j=1}^{N} P_j\,, \qquad H = P\sum_{j=1}^{N} \frac{Q_j}{P_j} = |H_1(X, \mathbf{Z})|\,,$$

so that H and P are coprime integers. The signature of the link L can be calculated using

$$\sigma(L) = \sum_{j=1}^{N} \mathrm{sign}\left(\frac{Q_j}{P_j}\right) - \mathrm{sign}\left(\frac{H}{P}\right).$$

Also, it will be convenient to introduce

$$\begin{aligned} \phi &= 3\,\mathrm{sign}\left(\frac{H}{P}\right) + \sum_{j=1}^{N}\left(12 s(Q_j, P_j) - \frac{Q_j}{P_j}\right)\,, \\ B &= -\frac{\mathrm{sign} P}{4\sqrt{|P|}}\, \mathbf{e}^{\frac{3\pi i}{4}\mathrm{sign}\left(\frac{H}{P}\right)}\,. \end{aligned}$$

The quantity ϕ is related to the Casson–Walker invariant (in Casson's normalisation) $\lambda(M)$ by

$$-24\lambda(M) = \phi + \frac{P}{H}(N - 2 - \sum_j P_j^{-2})\,. \tag{4.1}$$

4.1 Computation of invariants

Using (3.2), the invariant $Z_K(M)$ may be computed. By [Wi],

$$J_{\boldsymbol{\alpha}}(L) = \frac{\prod_{j=1}^{N} \sin\frac{\pi\beta\alpha_j}{K}}{\left(\sin\frac{\pi}{K}\right)\left(\sin\frac{\pi\beta}{K}\right)^{N-1}}\,.$$

Thus, by (3.2),

$$Z_K(M) = \mathbf{e}^{\frac{\pi i}{4}\frac{K-2}{K}\Phi(L,\boldsymbol{\Lambda})} \sum_\beta \frac{d_\beta^{\mathbf{S}}}{\left(\sin\frac{\pi}{K}\right)\left(\sin\frac{\pi\beta}{K}\right)^{N-1}} \prod_{j=1}^{N}\sum_{\alpha_j=1}^{K-1} d_{\alpha_j}^{\Lambda_j}\sin\frac{\pi\beta\alpha_j}{K}\,.$$

Applying (3.3) to evaluate the sums over α_j's for $j \neq 0$, and recalling the definition of $\Phi(L,\boldsymbol{\Lambda})$ and the form of $\sigma(L)$, one obtains

$$\begin{aligned} Z_K(M) &= \mathbf{e}^{\frac{\pi i}{4}\frac{K-2}{K}\left(3\,\mathrm{sign}\left(\frac{H}{P}\right)+\sum\Phi(\mathbf{S}\Lambda_j)\right)} \sum_{\beta=1}^{K-1} \frac{\left(\frac{K}{2}\right)^{N/2} d_\beta^{\mathbf{S}}}{\left(\sin\frac{\pi\beta}{K}\right)^{N-1}} \prod_{j=1}^{N} d_\beta^{\mathbf{S}\Lambda_j} \\ &= \frac{\mathrm{sign}P}{\sqrt{|P|}}\,\mathbf{e}^{\frac{3\pi i}{4}\frac{K-2}{K}\mathrm{sign}\left(\frac{H}{P}\right)}\sqrt{\frac{2}{K}}\left(\sin\frac{\pi}{K}\right)^{-1}\left(\mathbf{e}^{-\frac{\pi i}{2K}\sum\Phi(\mathbf{S}\Lambda_j)}\right)\cdot\Sigma \end{aligned}$$

by substituting for d_α^Λ, where

$$\Sigma = \sum_{\beta=1}^{K-1}\left(\sin\frac{\pi\beta}{K}\right)^{2-N}\prod_{j=1}^{N}\sum_{n_j=0}^{P_j-1}\frac{i}{2}\sum_{\mu_j=\pm}\mu_j\,\mathbf{e}^{\frac{\pi i}{2KP_j}\left(-Q_j\beta^2-2(2Kn_j+\mu_j)\beta+R_j(2Kn_j+\mu_j)^2\right)}.$$

Observe that Σ may be considered as a sum over $\beta\in[1,K-1]$, $\mathbf{n}\in\prod_{j=1}^{N}[0,P_j-1]$ and $\boldsymbol{\mu}\in\{-1,1\}^N$, of a signed exponential which may be considered as the product of N terms. The j^{th} term is invariant under the two changes

(i) $(\beta,n_j,\mu_j)\longrightarrow(\beta+2K,n_j+Q_j,\mu_j)$

(ii) $(\beta,n_j,\mu_j)\longrightarrow(\beta,n_j+P_j,\mu_j)$

Therefore, for each j_0, the complete summand is invariant under the change of variables,

$$\beta\longrightarrow\beta+2K\frac{P}{P_{j_0}}\,,\qquad n_j\longrightarrow n_j+\frac{Q_{j_0}P}{P_{j_0}}\delta_{j=j_0}\,,\qquad \boldsymbol{\mu}\longrightarrow\boldsymbol{\mu}\,.$$

Since P_j is coprime to Q_j, for each j while all the P_j's are coprime to each other, it is possible to perform the sum over $\mathbf{n}$ and β, for fixed $\boldsymbol{\mu}$, by fixing $\mathbf{n}=\mathbf{0}$ and letting β range over

$$\left\{c+2Kd\;\middle|\;1\le c\le K-1, 0\le d<P\right\}.$$

Performing the sum over $\boldsymbol{\mu}$ one now obtains,

$$\Sigma = \mathbf{e}^{\frac{\pi i}{2K}\sum\frac{R_j}{P_j}}\sum_\beta\left(\sin\frac{\pi\beta}{K}\right)^{2-N}\;\mathbf{e}^{-\frac{\pi i}{2K}\frac{H}{P}\beta^2}\prod_{j=1}^{N}\sin\frac{\pi\beta}{KP_j}\,,$$

where the sum is over the set of $(K-1)P$ values in $[0, 2KP-1]$ just described. Since the summand is now an even function of β which is periodic with period $2KP$, by the addition of a factor of $\frac{1}{2}$, the range of summation may be replaced by all integers in $[-KP, KP]$ which are not divisible by K. Putting this form for Σ back into the above expression for $Z_K(M)$ now leaves the result,

$$Z_K(M) = \frac{BG_0}{K}\,\mathrm{e}^{-\frac{\pi i}{2K}\phi} \sum_{\substack{\beta=-PK \\ K\nmid\beta}}^{PK} \mathrm{e}^{-\frac{\pi i}{2K}\frac{H}{P}\beta^2} \frac{\prod_{j=1}^{N}\left(\mathrm{e}^{\frac{\pi i\beta}{KP_j}} - \mathrm{e}^{-\frac{\pi i\beta}{KP_j}}\right)}{\left(\mathrm{e}^{\frac{\pi i\beta}{K}} - \mathrm{e}^{-\frac{\pi i\beta}{K}}\right)^{N-2}} . \tag{4.2}$$

4.2 Integral representations of sums

The aim of this section is to reformulate the sum of (4.2) into the form of a holomorphic function; in particular it will take the form of a sum of two terms, one an integral whose integrand is similar to the summand in (4.2), and the other term being a sum of a number of rational functions of exponentials, the number of terms being independent of K. For large K, such an expression is more easily computable than the sum of $2P(K-1)$ terms in (4.2), while its behaviour and asymptotic expansions can be more easily determined.

We start by defining functions $h_N(\beta, x)$ and $f_N(\beta, x)$ by

$$h_N(\beta, x) = \frac{\mathrm{e}^{-\frac{\pi i}{2K}\frac{H}{P}\beta^2}\,\mathrm{e}^{\frac{2\pi i\beta x}{K}}}{(\mathrm{e}^{\frac{\pi i\beta}{K}} - \mathrm{e}^{-\frac{\pi i\beta}{K}})^{N-2}(1-\mathrm{e}^{-2\pi i\beta})} = \frac{f_N(\beta, x)}{1-\mathrm{e}^{-2\pi i\beta}} .$$

Let C denote a contour in the complex plane which follows a line through the origin from $(-1+i)\infty$ to $(1-i)\infty$, except for a deviation close to the origin around a clockwise semicircle below the line. If $\frac{H}{P}$ is negative, then we modify the definition of C by rotating it through $\pi/2$ in a clockwise direction. Set

$$\Theta_N(x) = \int_C h_N(\beta, x)\, d\beta .$$

Observe that

(i) $h_N(\beta + 2PK, x) = h_N(\beta, x - HK)\,\mathrm{e}^{4\pi i P x}$;

(ii) $h_N(\beta, x) \longrightarrow 0$ exponentially quickly as $\beta \longrightarrow \infty$ on $C + x$ for any $x \in \mathbf{R}$;

(iii) $h_N(\beta, x)$ has poles at integer points on the real line;

(iv) $\mathrm{Res}\big(h_N(\beta, x), \beta = n\big) = (2\pi i)^{-1} f_N(n, x)$ when $n \notin K\mathbf{Z}$.

We deduce that

$$\begin{aligned}\Theta_N(x) &= \int_C h_N(\beta,x)\,d\beta \\ &= \int_{C+2PK} h_N(\beta,x)\,d\beta + 2\pi i \sum_{n=0}^{2PK-1} \mathrm{Res}\big(h_N(\beta,x),\beta=n\big) \\ &= \Theta_N(x-HK)\,\mathrm{e}^{4\pi i Px} + 2\pi i \sum_{m=0}^{2P-1} \mathrm{Res}\big(h_N(\beta,x),\beta=mK\big) + \sum_{\substack{n=0\\ K\nmid n}}^{2PK-1} f_N(n,x)\end{aligned}$$

and therefore that when $2Px \in \mathbf{Z}$,

$$\sum_{\substack{n=0\\ K\nmid n}}^{2PK-1} f_N(n,x) = \Theta_N(x) - \Theta_N(x-HK) - 2\pi i \sum_{m=0}^{2P-1} \mathrm{Res}\big(h_N(\beta,x),\beta=mK\big)\,. \quad (4.3)$$

Since $f_N(n,x)$ is a periodic function of n with period $2PK$, the sum on the left hand side of this equation may be replaced by one over any other period. The difference of the two values of Θ_N can be written as a single integral,

$$\begin{aligned}\Theta_N(x) - \Theta_N(x-HK) &= \int_C f_N(\beta,x)\big(1+\mathrm{e}^{-2\pi i\beta}+\cdots+\mathrm{e}^{-2\pi i\beta(H-1)}\big)\,d\beta \\ &= \int_C \big(f_N(\beta,x)+\cdots+f_N(\beta,x-(H-1)K)\big)\,d\beta\,,\end{aligned}$$

the integrand being a sum of H terms. Note that the left hand side of (4.3) is unaltered under the replacement $x \longrightarrow x+K$, whereas all terms in the right hand side will change. The end result is the following.

Lemma *Sums and integrals of f_N are related by*

$$\begin{aligned}&\sum_{\substack{n=-PK\\ K\nmid n}}^{PK} f_N(n,x) + 2\pi i \sum_{m=0}^{2P-1} \mathrm{Res}\left(\frac{f_N(\beta,x)}{1-\mathrm{e}^{-2\pi i\beta}},\beta=mK\right) \\ &\qquad = \int_C f_N(\beta,x)\big(1+\mathrm{e}^{-2\pi i\beta}+\cdots+\mathrm{e}^{-2\pi i\beta(H-1)}\big)\,d\beta\end{aligned} \quad (4.4)$$

whenever $2Px \in \mathbf{Z}$.

4.3 Holomorphic representation of invariants

To apply the result of the previous section to the computation of the sum in (4.2), observe that the summand is a combination of terms of the form of $f_N(\beta, x)$. Indeed, the sum may be written as

$$\sum_{\substack{\beta=0 \\ K \not| \beta}}^{2PK-1} \sum_{\epsilon \in \{+,-\}^N} \left(\prod_{j=1}^{N} \epsilon(j) \right) f_N(\beta, x_\epsilon) \,,$$

where $x_\epsilon = \frac{1}{2} \sum_{j=1}^{N} \frac{\epsilon(j)}{P_j}$. Since $2Px_\epsilon \in \mathbf{Z}$ for all ϵ, thus (4.4), being a relation linear in f_N also holds when $f_N(\beta, x)$ is replaced by the summand, $f(\beta)$, in (4.2).

Theorem 1 *For an arbitrary Seifert manifold $M = X(\frac{P_1}{Q_1}, \ldots, \frac{P_N}{Q_N})$, the $\mathfrak{sl}_2$ WRT invariants at the K^{th} root of unity is given by*

$$Z_K(M) = \frac{BG_0}{Kq^{\frac{\phi}{4}}} \left(\sum_{t=0}^{H-1} \int_C f(\beta)\, \mathrm{e}^{-2\pi i t\beta}\, d\beta - 2\pi i \sum_{m=0}^{2P-1} \mathrm{Res}\Big(\frac{f(\beta)}{1-\, \mathrm{e}^{-2\pi i\beta}}, \beta = mK \Big) \right) \tag{4.5}$$

where $f(\beta) = q^{-\frac{H\beta^2}{4P}} \left(q^{\frac{\beta}{2}} - q^{-\frac{\beta}{2}} \right)^{2-N} \prod_{j=1}^{N} \left(q^{\frac{\beta}{2P_j}} - q^{-\frac{\beta}{2P_j}} \right)$ and C is a diagonal line contour through the origin, passing from $(-1+i)\infty$ to $(1-i)\infty$ for $\frac{P}{H} > 0$, or rotated clockwise through $\frac{\pi}{2}$ for $\frac{P}{H} < 0$.

Special case of $N = 3$

In the special case of 3-fibred Seifert homology spheres, one has $N = 3$ and the residues appearing in (4.5) may be explicitly evaluated. Indeed the residue at $\beta = mK$ in (4.5) is

$$\mathcal{R}_m = \frac{2}{\pi} HK(-1)^m\, \mathrm{e}^{-\frac{HK\pi i}{2P}m^2} \left(\frac{m}{P} - \frac{1}{H} + \frac{i}{HK} \sum_{j=1}^{3} \frac{1}{P_j} \cot \frac{\pi m}{P_j} \right) \prod_j \sin \frac{\pi m}{P_j} \,.$$

Meanwhile, let $\mathcal{S}_m$ denote the corresponding residue of the integrand appearing in the first term of (4.5),

$$\begin{aligned} \mathcal{S}_m &= \mathrm{Res}\Big(f(\beta) \cdot \frac{1-\, \mathrm{e}^{-2\pi i H\beta}}{1-\, \mathrm{e}^{-2\pi i\beta}}, \beta = mK \Big) \\ &= -\frac{4}{\pi} HK(-1)^m\, \mathrm{e}^{-\frac{HK\pi i}{2P}m^2} \prod_j \sin \frac{\pi m}{P_j} \,. \end{aligned}$$

It is fairly easy to see, from explicit calculations, that

$$\mathcal{R}_{m+P} = (-1)^{P(1+\sum \frac{1}{P_j})-HK(m+P/2)}\mathcal{R}_m - \frac{1}{2}\mathcal{S}_{m+P}\,,$$
$$\mathcal{S}_{m+P} = (-1)^{P(1+\sum \frac{1}{P_j})-HK(m+P/2)}\mathcal{S}_m\,,$$

while $\mathcal{R}_{-m} = \mathcal{R}_m - H^{-1}\mathcal{S}_m$ and $\mathcal{S}_{-m} = -\mathcal{S}_m$, from which it follows that

$$\sum_{m=0}^{2P} \mathcal{R}_m = \frac{1}{2}\sum_{m=0}^{P} \mathcal{S}_m$$

when P is even. When P is odd, this holds when $HK \equiv 2$ modulo 4, but not in general otherwise.

Thus, for the particular case of 3-fibred Seifert homology 3-spheres with $P_1 = 2$, (4.5) may be rewritten solely in terms of the integrand $g(\beta) = f(\beta)\cdot\frac{1-\mathbf{e}^{-2\pi i H\beta}}{1-\mathbf{e}^{-2\pi i\beta}}$ as

$$Z_K(M) = \frac{BG_0}{Kq^{\frac{\phi}{4}}}\left(\int_C g(\beta)\,d\beta - \pi i\sum_{m=0}^{P-1}\mathrm{Res}\big(g(\beta),\beta = mK\big)\right) = \frac{BG_0}{Kq^{\frac{\phi}{4}}}\int_D g(\beta)\,d\beta$$

where the contour D consists of a union of line segments, being identical to C above the real axis, a translated version $C + PK$ below the real axis, along with P open line segments along the real axis joining points jK to $(j+1)K$ for $j = 0, 1, \ldots, P-1$. That is, $Z_K(M)$ is, up to a simple factor, the principal value of the integral of $g(\beta)$ over a Z-shaped contour. To remove the dependence on K of the contour, one may rescale the variables, leading to the following theorem.

Theorem 2 *Consider a three-fibred Seifert manifold $M = X(\frac{P_1}{Q_1}, \frac{P_2}{Q_2}, \frac{P_3}{Q_3})$. Let $H = |H^1(M, \mathbf{Z})|$, $P = P_1P_2P_3$ while ϕ and B are as defined at the start of §4, all constants dependent only on the manifold M. In either of the cases $P_1 = 2$ or $HK \equiv 2$ (4), the $S^2\times S^1$-normalised Witten-Reshetikhin-Turaev invariant of M at the $K^{\rm th}$ root of unity, may be expressed as an integral*

$$Z_K(M) = \frac{BG_0}{2\pi i}q^{-\frac{\phi}{4}}\sum_{t=0}^{H-1}\int \mathbf{e}^{\frac{iHKy^2}{8\pi P} - tKy}\,\frac{\prod_{j=1}^{3}\left(\mathbf{e}^{\frac{y}{2P_j}} - \mathbf{e}^{-\frac{y}{2P_j}}\right)}{\left(\mathbf{e}^{\frac{y}{2}} - \mathbf{e}^{-\frac{y}{2}}\right)}dy$$

where the integral is the principal value of that taken around a Z-shaped contour running diagonally across the y-plane, from $-(1\pm i)\infty$ into the origin, then up the imaginary axis to $2\pi iP$ and finally out to $(1\pm i)\infty$, the sign being that of P.

4.4 Rational K

All the calculations so far in this paper have assumed that K is an integer, since they have used the interpretation in terms of braiding and fusing matrices from conformal field theory which only directly makes sense in this case. However $Z_K(M) \in \mathbf{Z}[q]$, so that $Z_K(M) = f_K(q)$ for some polynomial f_K with integer coefficients dependent on K, where $q = \mathbf{e}^{\frac{2\pi i}{K}}$. One may therefore define

$$Z_{\frac{r}{s}}(M) = f_r\left(\mathbf{e}^{\frac{2\pi i s}{r}}\right) ,$$

whenever $r, s \in \mathbf{Z}$ are coprime, to extend $Z_K(M)$ to rational values of K. Indeed the formulation of $Z_K(M)$ in terms of representations of quantum groups (see Reshetikhin-Turaev [RT] and Kauffman-Lins [KL2]) is really in terms of a parameter $q = \mathbf{e}^{\frac{2\pi i}{K}}$ (which need only be a root of unity) rather than as a function of K. In this section we will derive an analogous integral expression for $Z_K(M)$ to that found for the case of integer K in the previous section, to give the values of the invariant at these other roots of unity.

In (4.2), the right-hand side contains expressions which all lie in $\mathbf{Q}[x]$ where $x = \mathbf{e}^{\frac{\pi i}{2KP}}$ is a $4KP^{\text{th}}$ root of unity and a $4P^{\text{th}}$ root of q. Indeed, it can be rewritten as

$$Z_K(M) = \frac{\epsilon}{2\sqrt{2K|P|}} \frac{x^{P\phi}}{x^{2P} - x^{-2P}} \sum_{\substack{\beta=-PK \\ K\not|\beta}}^{PK} x^{-H\beta^2} \frac{\prod_{j=1}^{N}\left(x^{\frac{\beta P}{P_j}} - x^{-\frac{\beta P}{P_j}}\right)}{\left(x^{2P\beta} - x^{-2P\beta}\right)^{N-2}} , \tag{4.6}$$

where it may be noted that $P\phi$ is an integer and $\epsilon = \mathbf{e}^{\frac{\pi i}{4}\operatorname{sign}\frac{H}{P}}$ is an eighth root of unity. Since it is known that the left-hand side has the form $f_K(q)$ for some polynomial f_K with integer coefficients, the values at other K^{th} roots of unity may be computed by transforming the right hand side by the Galois action. Indeed, if x is replaced by another primitive $4KP^{\text{th}}$ root of unity, $\mathbf{e}^{\frac{\pi i s}{2KP}}$, where s is coprime to $4KP$, then by §2.1, the extra scaling introduced can be computed. For any s coprime to K, it is possible to choose to replace s by another element of its residue class modulo K, in such a way that it remain coprime to $4KP$ while having residue 1 mod 4. In this case, the scaling factor in (4.6) will come only from the square-root and ϵ, namely the sign $(-1)^{\frac{s-1}{4}}\left(\frac{s}{2K|P|}\right)$. Thus, for rational $K = \frac{r}{s}$, the expression for $Z_K(M)$ is exactly as in (4.2), except for the insertion of an extra factor and a change in limits,

$$Z_K(M) = \frac{(-1)^{\frac{s-1}{4}}}{\sqrt{s}}\left(\frac{s}{2r|P|}\right)\frac{BG_0}{K}\,\mathbf{e}^{-\frac{\pi i}{2K}\phi} \sum_{\substack{\beta=-Pr \\ r\not|\beta}}^{Pr} \mathbf{e}^{-\frac{\pi i}{2K}\frac{H}{P}\beta^2} \frac{\prod_{j=1}^{N}\left(\mathbf{e}^{\frac{\pi i\beta}{KP_j}} - \mathbf{e}^{-\frac{\pi i\beta}{KP_j}}\right)}{\left(\mathbf{e}^{\frac{\pi i\beta}{K}} - \mathbf{e}^{-\frac{\pi i\beta}{K}}\right)^{N-2}} . \tag{4.7}$$

Note that in this expression G_0 is the function of K defined in §2.1, and *not* the ratio of the values of the invariant of $S^1\times S^2$ and of S^3, at the particular root of unity described by K. The argument of §4.2 that changes a sum into an integral as used in §4.3 for integer K, works in the same way here, with the translation now being $\beta\longrightarrow\beta+2Pr$. There are now poles of the function at points in $\mathbf{Z}\cup K\mathbf{Z}$. Those integers not in $K\mathbf{Z}$ are precisely those not divisible by r and this leaves

$$Z_K(M)=D\left[\sum_{t=0}^{Hs-1}\int_C f(\beta)\,\mathrm{e}^{-2\pi it\beta}\,d\beta-2\pi i\sum_{m=0}^{2Ps-1}\mathrm{Res}\left(\frac{f(\beta)}{1-\mathrm{e}^{-2\pi i\beta}},\beta=mK\right)\right].$$

where D is the term appearing before the sum in (4.7).

For three-fibred Seifert homology spheres under the same conditions as in the previous section, the part of the residue sum contributed by those terms with $m\in s\mathbf{Z}$ is precisely half the sum of residues of the integrand above at $\beta=mr$ for $0<m<P$. These terms are therefore included in the integral by modifying the integral to be the principal value around a contour obtained from C by shifting the part in the lower half-plane by Pr and inserting the line segment from the origin to Pr. The remaining terms, when m is not divisible by s, have a zero sum in this case, there being an antisymmetry in the summand under $m\longrightarrow m+Ps$. This leaves the following expression for $Z_K(M)$ for a three-fibred Seifert manifold at $K=\frac{r}{s}$, that is the WRT invariant of M at the root of unity $q=\mathrm{e}^{2\pi i\frac{s}{r}}$,

$$Z_{\frac{r}{s}}(M)=\pm\frac{iBq^{-\frac{\phi}{4}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\sqrt{\frac{2}{r}}\int\frac{1-\mathrm{e}^{-2\pi iHs\beta}}{1-\mathrm{e}^{-2\pi i\beta}}\frac{\prod_{j=1}^{3}\left(q^{\frac{\beta}{2P_j}}-q^{-\frac{\beta}{2P_j}}\right)}{q^{\frac{\beta}{2}}-q^{-\frac{\beta}{2}}}q^{-\frac{H\beta^2}{4P}}\,d\beta$$

where the sign is that found in (4.7), and $s\equiv1$ modulo 4 is chosen to be coprime to $4Pr$. The conditions required are that either P be even, or $Hr\equiv2$ modulo 4. The path of integration is a Z-shaped curve, as in the previous section, with middle segment from 0 to Pr and with outer segments pointing in the direction of $\pm1-i$.

At first sight it may seem that the expression just given for $Z_K(M)$ when K is rational (not integer) depends on r in an essential way, and not only on K. In fact, the specific expression for K as a rational, that is in terms of r or s individually, enters thrice, once by the factor $\sqrt{\frac{2}{r}}$, the second time via the contour of integration, and finally via the exponent of s in the integrand. However, we claim these are all 'natural' in the following sense. Introduce a new parameter N, considered as a large integer. Then

$$\lim_{N\longrightarrow\infty}\frac{1}{N}\sum_{a=1}^{N}(-1)^a q^{\frac{a^2}{4}}\sim\sqrt{\frac{1}{2r}}$$

up to multiplication by an eighth root of unity, while the integral over a Z-shaped contour with centre section from 0 to NP, will take the same value as for $N=r$, for any odd multiple N of s. Finally, since the value of the integral is unchanged under a translation of the contour through $2Pr$, one may equivalently change variables $\beta \longrightarrow \beta + 2Pr$ which brings an extra factor of $\mathbf{e}^{-2\pi i H s\beta}$ into the integrand; this shows that the integral will scale by a factor of N whenever such a factor is introduced into the exponent in the numerator of the first term in the integrand. Combining these facts gives $Z_{\frac{r}{s}}(M)$ as a limit as $N \longrightarrow \infty$ amongst odd multiples of s, of an expression identical to that given above except that

(a) the appearance of s in the integrand is replaced by N;

(b) the contour is chosen to have the same Z-shaped form, with centre piece from 0 to NKP;

(c) the term $\sqrt{\frac{2}{r}}$ is replaced by the reciprocal of $\sum_{a=0}^{NK}(-1)^a q^{\frac{a^2}{4}}$.

4.5 Asymptotic expansions

In all the discussions so far, K has been either an integer or rational, and indeed the WRT invariant which we were initially computing is only defined at roots of unity. We now consider the formula (4.5) to define a function of a complex-valued parameter K. Some care must be taken, however, since the shape of the residue terms changes abruptly when K becomes non-integer; indeed, we do *not* want to continue using a sum of residues away from integer K, but rather use the (unique) 'simple' holomorphic function of K which agrees with these values at integers. It is also necessary to choose the contour C appropriately. The existence of such a holomorphic function is demonstrated by the first approach below, where an explicit computation of the residues is performed.

Since the asymptotic behaviour of a holomorphic function possessing an asymptotic series is entirely defined by its behaviour at integer values, it is unnecessary in this section to compute with any non-integer values. In the following, we will equivalently consider $Z_K(M)$ as a (multi-valued) function of $q = \mathbf{e}^{\frac{2\pi i}{K}} = 1 + h$.

Analytic structure

We will first discuss the analytic behaviour of $Z_K(M)$, for which it is most convenient to transform the variable in (4.5) from β to $y = \frac{2\pi i\beta}{K}$, leaving the result

$$Z_K(M) = \frac{BG_0}{q^{\frac{\phi}{4}}}\left[\frac{1}{2\pi i}\sum_{t=0}^{H-1}\int_{C'} F(y)\,\mathbf{e}^{Kg_t(y)}dy - \sum_{m=1}^{2P-1}\mathrm{Res}\left(\frac{F(y)\,\mathbf{e}^{Kg_0(y)}}{1-\mathbf{e}^{-Ky}}, y = 2\pi i m\right)\right] \tag{4.8}$$

where $F(y) = \left(\mathbf{e}^{\frac{y}{2}} - \mathbf{e}^{-\frac{y}{2}}\right)^{2-N}\prod_{j=1}^{N}\left(\mathbf{e}^{\frac{y}{2P_j}} - \mathbf{e}^{-\frac{y}{2P_j}}\right)$ and $g_t(y) = \frac{iHy^2}{8\pi P} - ty$, $C' = iC$.

We proceed to discuss the form of the two terms in (4.8). The second may be explicitly evaluated. Start by defining generalised Bernoulli-type numbers, B^n_m, by their generating function

$$\sum_{m=0}^{\infty} B^n_m z^m = \frac{z^n}{(\mathbf{e}^{\frac{z}{2}} - \mathbf{e}^{-\frac{z}{2}})^n}$$

so that $B^n_0 = 1$, $B^n_1 = 0$ and $B^n_2 = -\frac{n}{24}$. By expanding $F(y)$ into a sum of terms

$$F(y) = \left(\mathbf{e}^{\frac{y}{2}} - \mathbf{e}^{-\frac{y}{2}}\right)^{2-N} \sum_{\epsilon:[N]\longrightarrow\{\pm\}} \epsilon\, \mathbf{e}^{\sum \frac{\epsilon(j)}{2P_j} y},$$

we may calculate the residue in (4.8) to be

$$\mathrm{Res}\left(\frac{F(y)\,\mathbf{e}^{KG_0(y)}}{1-\mathbf{e}^{-Ky}}, y = 2\pi i m\right) = \sum_{\epsilon} \epsilon(-1)^{m(N-1)}\, \mathbf{e}^{\pi i \sum \frac{\epsilon(j)}{P_j} m - \frac{\pi i H K}{2P} m^2}$$
$$\sum_{\substack{r,s,t\geq 0\\ r+s+2t\leq N-2}} \frac{B^{N-2}_r B^1_s \left(\frac{iH}{8\pi P}\right)^t K^{s+t-1}}{t!(N-2-r-s-2t)!} \left(\frac{1}{2} + \sum \frac{\epsilon(j)}{2P_j} - \frac{mHK}{2P}\right)^{N-2-r-s-2t}.$$

It may be checked that the coefficient of K^{-1} in this expression always vanishes, so that the dependence of the contribution of the residues to $Z_K(M)$ upon K has the form

$$Z_K^{\mathrm{res}}(M) = G_0 q^{-\frac{\phi}{4}} \sum_{m=1}^{2P-1} \mathbf{e}^{-\frac{\pi i H m^2}{2P} K} H_m(K) \tag{4.9}$$

where $H_m(K)$ is a polynomial in K of degree at most $N-3$. There is therefore defined an extension of $Z_K^{\mathrm{res}}(M)$ to a holomorphic function of K away from the origin, only using combinations of rational and exponential functions.

For the first term in (4.8), it is sufficient to note that its integrand is an analytic function of y and K away from $K=0$ and from $y \in \mathbf{Z}$, with a double zero at $P\mathbf{Z}$. In order to keep the integrand convergent, with Gaussian type behaviour along the contour of integration, we choose this contour to be a line through the origin in the direction of $\sqrt{\frac{iP}{HK}}$. Combining the two terms, we now obtain a (double-valued) holomorphic function of K, away from the origin, whose value at integers coincides with the WRT invariant.

Flat connection contributions

From the physical perspective with respect to the Feynman path integral formulation of $Z_K(M)$ in (1.1), it is expected that a stationary phase expansion will yield an asymptotic expansion for large K consisting of a sum of contributions labelled by equivalence classes of flat connections.

Starting from (4.8), and translating the contours in the individual terms so as to form paths of steepest descent, we pick up extra residues,

$$Z_K(M) = \frac{BG_0}{q^{\frac{\phi}{4}}}\left[\frac{1}{2\pi i}\int_{C'} F(y)\,\mathrm{e}^{Kg_0(y)}dy + \sum_{t=1}^{H-1}\frac{1}{2\pi i}\int_{C'_t} F(y)\,\mathrm{e}^{Kg_t(y)}dy\right.$$
$$\left. - \sum_{m=1}^{2P-1}\mathrm{Res}\left(\frac{F(y)\,\mathrm{e}^{Kg_0(y)}}{1-\mathrm{e}^{-Ky}}, y = 2\pi i m\right) - \sum_{t=1}^{H-1}\sum_{m=1}^{\left[\frac{2Pt}{H}\right]}\mathrm{Res}\Big(F(y)\,\mathrm{e}^{Kg_t(y)}, y = -2\pi i m\Big)\right],$$

where C'_t is a contour parallel to C' and passing through the stationary phase point of the integrand, namely $y = -4\pi i\frac{P}{H}t$. The first term, denoted $Z_K^0(M)$ in what follows, is the trivial connection contribution. The second term is a sum of contributions from reducible flat connections, while the sum of the last two terms give the contribution from irreducible flat connections.

In the notation of [R3], the reducible connections are labelled by integers m_j $(0 \le j \le N)$ with $0 \le m_j \le P_j - 1$ for which $0 < \beta_{st} \equiv \frac{2KP}{H}\left(m_0 - \sum \frac{m_j}{P_j}\right) < K$. The precise correspondence with the contributions to the second term above labelled by integers t with $1 \le t \le H-1$ is

$$t = \frac{H}{P}s + c\,,$$

where $s \in \mathbf{Z}$ and $|c| < \frac{H}{2P}$ making $c = \sum \frac{m_j}{P_j} - m_0$ and $m_j \equiv -sQ_j \bmod P_j$. (It may be noted that the normalisation used in [R3] differs from that here by a factor of G_0, since here we normalise with respect to S^3, rather than $S^1 \times S^2$.)

The irreducible connections in [R3] are labelled by (m_j, l) with $l = 0, 1$ and $0 \le m_j \le \frac{P_j}{2}$ while $m_j \in \mathbf{Z} + \frac{1}{2}Q_j l$ for $1 \le j \le N$. The correspondence with the labels m and (m,t) in the last two terms in the above expression for $Z_K(M)$ is via $m \equiv l \bmod 2$, while

$$t - \frac{Hm}{2P} = -m_0 + \sum \pm\frac{m_j}{P_j},\ \text{for some } m_0 \in \mathbf{Z}\,,$$

for the last term, the same expression being used with t omitted for the third term.

Observe that this makes the trivial connection contribution to $Z_K(M)$, namely

$$Z_K^0(M)=\frac{BG_0}{Kq^{\phi/4}}\int_C q^{-\frac{H\beta^2}{4P}}\frac{\prod\left(q^{\frac{\beta}{2P_j}}-q^{-\frac{\beta}{2P_j}}\right)}{(q^{\frac{\beta}{2}}-q^{-\frac{\beta}{2}})^{N-2}}\,d\beta\,, \tag{4.10}$$

into a holomorphic single-valued function of K away from the origin, there being two occurrences of a choice of square-root, one in G_0 and the other in the direction of the contour. For the case of rational K, a similar shifting of contours may be carried out, so as to pass through the relevant stationary phase points, and then the first term will, up to sign, have exactly the same analytic expression $Z_K^0(M)$ as for integer K, where G_0 is now defined by $G_0^{-1}=\sqrt{\frac{2}{r}}\sin\frac{\pi}{K}$.

Theorem 3 *For Seifert manifolds, there is a natural (double-valued) holomorphic function of K, defined away from the origin, whose value at integers coincides with the WRT invariant. It is a sum of polynomial multiples of exponential functions (residues) and integrals, obtained from the expression in Theorem 1 by translating contours so as to pass through stationary points. The terms can be identified with the contributions from all flat connections (trivial, reducible and irreducible) appearing from a stationary phase expansion of (1.1). Up to correct accounting of signs, the holomorphic extension, $Z_K^0(M)$, of the trivial connection contribution is also valid for rational K.*

Thus the *stationary phase approximation is exact* for Seifert manifolds, giving *exactly* the value of $Z_K(M)$, not just asymptotically for large K, but also exactly for small (finite) K.

Asymptotic structure of trivial connection contribution

To find the asymptotic expansion of the trivial and reducible connection contributions in (4.8), one deforms the contour in each individual integral, so as to pass through the stationary point of g_t, namely $y=y_t=\frac{4\pi P}{iH}t$. Now applying Laplace's method gives a contribution from the t^{th} term (for $0\le t<H$) of

$$Z_K^t(M)=\frac{2Bq^{-\frac{\phi}{4}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\sqrt{\frac{iP}{\pi H}}\,\mathrm{e}^{2\pi i\frac{P}{H}Kt^2}\sum_{n=1}^{\infty}a_{n,t}\Gamma\Big(\lambda_{n,t}+\frac{1}{2}\Big)K^{-\lambda_{n,t}} \tag{4.11}$$

where $a_{n,t}$ are the coefficients in an asymptotic expansion of $F(y)$ about $y=y_t$,

$$F\Big(g_t^{-1}\big(g_t(y_t)-x\big)\Big)\sim\sum_{n=1}^{\infty}a_{n,t}x^{\lambda_{n,t}}\,.$$

Here, for each t, the sequence $\{\lambda_{n,t}\}$ must be strictly increasing, with $\lambda_{0,t} > -1$. For $t = 0$, $F(y)$ has a double zero at the origin and thus $\lambda_{n,0} = n$, while for $t \neq 0$, $\lambda_{n,t} = n - 1$. When the imaginary part of K has the same sign as $\frac{H}{P}$, the reducible connection contribution will be exponentially smaller than that of the trivial connection. The trivial connection contribution is

$$Z_K^0(M) \sim \frac{2Bq^{-\frac{\phi}{4}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sqrt{\frac{iP}{H}} \sum_{n=1}^{\infty} \frac{F^{(2n)}(0)}{n!} \left(\frac{2\pi iP}{HK}\right)^n .$$

This result may be compared with equation (2.28) in [R6]. However, $F(z)$ is a meromorphic function with poles at $2\pi im$ for $m \in \mathbf{Z}\backslash\{0\}$, so we may write

$$F^{(2n)}(0) = -(2n)! \sum_{m\in\mathbf{Z}\backslash\{0\}} \mathrm{Res}\left(\frac{F(z)}{z^{2n+1}}; z = 2\pi im\right) .$$

The ratio of coefficients of $\left(\frac{2\pi i}{K}\right)^n$ and $\left(\frac{2\pi i}{K}\right)^{n-1}$ (or equivalently, of h^n and h^{n-1}) in the expansion of $Z_K(M)$, is thus asymptotically for n large,

$$\frac{P}{H}\frac{(2n+1)(2n+2)}{n+1}\left(\frac{1}{2\pi i}\right)^2 \sim -\frac{P}{\pi^2 H} n .$$

4.6 Computing coefficients and integrality properties

In order to obtain more precise properties of the coefficients appearing in the asymptotic expansion of the trivial connection contribution, $Z_K^0(M)$, in powers of $h = q - 1$, rather than K^{-1}, one may return to (4.10). Expand

$$\frac{\prod_{j=1}^{N} \left(q^{\frac{1}{2P_j}} - q^{-\frac{1}{2P_j}}\right)}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{N-2}} = \sum_{m=2}^{\infty} c_m h^m , \tag{4.12}$$

observing that the function on the left hand side has a double zero at $q = 1$. Then we compute

$$\int_C f(\beta)\, d\beta = \sum_{m=2}^{\infty} c_m \int_C q^{-\frac{H\beta^2}{4P}} (q^\beta - 1)^m \, d\beta .$$

Expanding the integrand into a sum of $m + 1$ Gaussians, we may evaluate them, obtaining, as an asymptotic series

$$\begin{aligned} Z_K^0(M) &\sim \frac{BG_0}{Kq^{\frac{\phi}{4}}} \sum_{m=2}^{\infty} c_m \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \sqrt{\frac{2KP}{iH}} q^{\frac{Ps^2}{H}} \\ &= \frac{q^{\frac{1}{2}-\frac{\phi}{4}}}{2h\sqrt{H}} \sum_{m=2}^{\infty} c_m \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} q^{\frac{Ps^2}{H}} \end{aligned}$$

Let $\widetilde{Z}_\infty(M) = H^{\frac{3}{2}} q^{-6\lambda(M)} Z^0_K(M)$ be a normalised version of the trivial connection contribution, so that

$$\widetilde{Z}_\infty(M) = \frac{H q^{\frac{1}{2}-\frac{\phi}{4}-6\lambda(M)}}{2h} \sum_{m=2}^{\infty} c_m \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} q^{\frac{P_s{}^2}{H}} \tag{4.13}$$

considered as a formal power series in h. To see that this makes sense, the coefficient of each h^n in $\widetilde{Z}_\infty(M)$ being a finite sum of terms, observe that

$$\sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} s^a = \frac{d^a}{dx^a}\Big|_{x=0} (\mathbf{e}^x - 1)^m ,$$

which therefore vanishes for $a < m$. Hence, for a fixed value of m, the coefficient of h^n in the contribution to $\widetilde{Z}_\infty(M)$ will vanish for $2(n+1) < m$. The expansion

$$\widetilde{Z}_\infty(M) = \sum_{n=0}^{\infty} L_n(M) h^n = H q^{-6\lambda(M)} \sum_{n=0}^{\infty} \lambda_n(M) h^n$$

therefore defines invariants $L_n(M)$ closely related to the invariants $\lambda_N(M)$ whose existence was shown by Ohtsuki in [O3]; however, it will be seen that L_n obey more natural properties than do λ_n.

The coefficient of h^0 comes only from the term $m = 2$ and is

$$L_0(M) = H\lambda_0 = \frac{H}{2} c_2 \frac{2P}{H} = 1$$

since $c_2 = P^{-1}$. The coefficient of h^1 in $\widetilde{Z}_\infty(M)$ can be computed to be

$$L_1(M) = \frac{H}{2}\left[\frac{P}{H}\left(\frac{7P}{H} - \frac{\phi}{2}\right) c_2 + \frac{P^2}{H^2}(18c_3 + 12c_4)\right] - 6\lambda(M) = 0$$

using (4.1) with $c_2 = P^{-1}$, $c_3 = -P^{-1}$ and $c_4 = \frac{1}{24P}\left(\sum_j P_j^{-2} + 24 - N\right)$, which is in accordance with the work of Murakami who showed that $\lambda_1(M) = 6H^{-1}\lambda(M)$. Later coefficients have more complicated forms and may be computed using *Mathematica* [Wo].

Theorem 4 *The normalisation $\widetilde{Z}_\infty(M) = q^{-6\lambda(M)} H^{\frac{3}{2}} Z^0_K(M)$ of the trivial connection contribution has an asymptotic expansion in powers of $h = q-1$, $\widetilde{Z}_\infty(M) = \sum_{n=0}^{\infty} L_n(M) h^n$ in which $L_0(M) = 1$ and $L_1(M) = 0$, while for Seifert manifolds,*

$$L_2(M) = -\frac{1}{24} + \frac{P^2}{24H^2}\left[\sum_{j<k} P_j^{-2} P_k^{-2} + (2-N)\sum_j P_j^{-2} + \binom{2-N}{2}\right] ,$$

$$L_2(M) + L_3(M) = \frac{P^3}{72H^3} \sum_{i<j<k} (P_i^{-2} - 1)(P_j^{-2} - 1)(P_k^{-2} - 1) .$$

Observe that the denominator of $L_2(M)$ is a divisor of a power of $2H$. When $H=1$ so that we have an integer homology sphere, it is also possible to check that $\lambda_2(M)$ is an integer divisible by 3, in agreement with [LW].

Corollary *For any integer homology sphere M appearing as a Seifert fibred manifold, $L_2(M)\in 6\mathbf{Z}$ and $L_2(M)+L_3(M)\in 48\mathbf{Z}$.*

Also note that when M is replaced by its mirror image $\overline{M}$, the effect on $Z_K(M)$ is to replace q by q^{-1}, and this has non-trivial effects on the coefficients $\lambda_n(M)$ and $L_n(M)$, so that

$$\begin{aligned}\lambda_1(\overline{M}) &= -\lambda_1(M)\,,\\ \lambda_2(\overline{M}) &= \lambda_1(M)+\lambda_2(M)\,,\\ \lambda_3(\overline{M}) &= -\lambda_1(M)-2\lambda_2(M)-\lambda_3(M)\,,\end{aligned}$$

while in terms of L_n, we have that $L_1(M)\equiv 0$ for all M, with $L_2(\overline{M})=L_2(M)$ and $L_2(M)+L_3(M)$ reversing sign under change of orientation. This may be explicitly verified in this case. For computations of higher order Ohtsuki invariants, see [L1] and [R6], the latter for surgeries around the knots 4_1 and 6_1. On the basis of these computations we make the following conjecture. In terms of the Ohtsuki invariants, $L_2=H\lambda_2+\frac{1}{2}H\lambda_1(1-H\lambda_1)$ while $L_2+L_3=H\lambda_3+H(1-H\lambda_1)(\lambda_2-\frac{H^2}{3}\lambda_1^2+\frac{H}{6}\lambda_1)$.

Conjecture $L_2(M)\in 6\mathbf{Z}$ *and* $L_2(M)+L_3(M)\in 48\mathbf{Z}$ *for any integer homology sphere M.*

Integrality properties

The coefficients c_m in (4.12) lie in $\mathbf{Z}[\frac{1}{2P}]$. Thus, looking at (4.13), it may be seen that $\lambda_m(M)\in\mathbf{Z}[\frac{1}{2PH}]$. We now use a variant of the argument of [R5] to show that in fact $\widetilde{Z}_\infty\in\mathbf{Z}[\frac{1}{H}][[h]]$. Just as we computed the expansion for $Z_K(M)$ in (4.13) by integrating Gaussian terms coming from the expansion (4.12), one may start from an expansion of the same product with one less term, say the i^{th}, with

$$\frac{\prod_{j\neq i}\left(q^{\frac{1}{P_j}}-1\right)}{(q-1)^{N-2}}=\sum_{m=2}^{\infty}c'_m h^m\,.$$

Now observe that $f(\beta)$ is an even function and thus $\int_c f(\beta)\,d\beta$ can be computed as $-2\int_c g_i(\beta)q^{-\frac{\beta}{2P_i}}\,d\beta$ where $g_i(\beta)$ is obtained from $f(\beta)$ by removing the i^{th} term. Thus

$$\int_C f(\beta)\,d\beta=-2\sum_{m=2}^{\infty}c'_m\int_C q^{-\frac{H\beta^2}{4P}+\frac{\beta}{2}\left(N-2-\sum_{j\neq i}P_j^{-1}\right)}(q^\beta-1)^m q^{-\frac{\beta}{2P_i}}\,d\beta\,,$$

which may be integrated as a sum of Gaussians and leads to an expression similar to (4.13),

$$\widetilde{Z}_\infty(M) = -\frac{H}{h}\sum_{m=2}^{\infty} c'_m \sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s} q^{\frac{P}{H}\left(s-1+\frac{1}{2}\sum(1-P_j^{-1})\right)^2-\frac{\phi}{4}-6\lambda(M)+\frac{1}{2}}\,.$$

The factor P_i does not appear in the denominators of c'_m and can only appear in $Z_\infty(M)$ from the expansions of power of q appearing in this last equation. However, by (4.1), $-\frac{\phi}{4}+\frac{P}{H}\frac{1}{4P_i^2}$ does not contain powers of P_i in its denominator. Thus $\widetilde{Z}_\infty(M)\in \mathbf{Z}[\frac{1}{2H},\frac{P_i}{P}]$. Furthermore, using (4.1) it can be seen that for H odd there will be no factors of 2 in the denominator of the exponent of q in $\widetilde{Z}_\infty(M)$. Since $v^{2a}\binom{u/v}{a}\in\mathbf{Z}$ whenever $u\in\mathbf{Z}$ and $v,a\in\mathbf{N}$, thus we deduce that $\widetilde{Z}_\infty(M)\in \mathbf{Z}[\frac{1}{H},\frac{P_i}{P}][[h]]$. Since this holds for any choice of i, we deduce the following result.

Theorem 5 *If M is a Seifert fibred manifold then $\widetilde{Z}_\infty(M)\in\mathbf{Z}[\frac{1}{H}][[h]]$.*

4.7 K-adic convergence

We now assume that $K\in\mathbf{Z}$ is an odd prime power, say $K=p^r$, coprime to H. Since the P_j are coprime, K must be coprime to all but at most one of the P_j's, which we denote P_i. In a similar way to the previous section, perform an expansion

$$f_i(h)=\frac{\prod_{j=1}^{N}\left(q^{\frac{1}{2P_j}}-q^{-\frac{1}{2P_j}}\right)}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{N-2}}=\sum_{m=2}^{\infty} d_m h^m\,, \tag{4.14}$$

so that the coefficients d_m are rationals with denominators coprime to K, that is, $d_m\in\mathbf{Z}_K$.

Observing that $G_{-1,4PK}=2(1-i)\sqrt{PK}$, the $SO(3)$–invariant $Z'_K(M)$ is now given, from (4.2), by

$$\frac{G_{-H,4PK}}{G_{-1,4PK}}Z'_K(M)=Z_K(M)=\frac{q^{\frac{1}{2}-\frac{\phi}{4}}}{hG_{-1,4PK}}\sum_{\substack{\beta=-PK\\ K\not|\beta}}^{PK} q^{-\frac{H\beta^2}{4P}}\left(q^{\frac{\beta}{2P_i}}-q^{-\frac{\beta}{2P_i}}\right)f_i(q^\beta-1)\,.$$

By [O3], $Z'_K(M)\in\mathbf{Z}[h]$. Meanwhile, from (4.10),

$$\sqrt{H}Z_K^0(M)=\frac{q^{\frac{1}{2}-\frac{\phi}{4}}}{2h}\sqrt{\frac{iH}{2KP}}\int_C q^{-\frac{H\beta^2}{4P}}\left(q^{\frac{\beta}{2P_i}}-q^{-\frac{\beta}{2P_i}}\right)f_i(q^\beta-1)d\beta$$

This last expression has a unique asymptotic expansion around $q = 1$, which may be obtained by substituting the expansion (4.14) for $f_i(q^\beta - 1)$, and the result will be an element of $\mathbf{Z}_K[[h]]$, by Theorem 5.

As complex numbers, rather than algebraic numbers,

$$2q^{-\frac{P}{HP_i^2}} \sum_{\beta=-PK}^{PK-1} q^{-\frac{H\beta^2}{4P}} q^{\frac{\epsilon\beta}{2P_i}+s\beta} d\beta = q^{\frac{P}{4P_i^2}(H^*-H^{-1})} q^{PH^*s^2+\frac{\epsilon PsH^*}{P_j}} G_{-H,4PK}\ ,$$

$$q^{-\frac{P}{HP_i^2}} \sqrt{\frac{iH}{2KP}} \int_C q^{-\frac{H\beta^2}{4P}} q^{\frac{\epsilon\beta}{2P_i}+s\beta} d\beta = q^{\frac{Ps^2}{H}+\frac{\epsilon Ps}{HP_j}}\ ,$$

for $\epsilon = \pm 1$, where H^* denotes the inverse of H mod K. Let A_s denote the term on the right hand side of either of the last two expressions, where in the first case the factor $G_{-H,4PK}$ is omitted. Then A_s defines an element of $\mathbf{Z}_K[[h]]$, and the two possible choices of A_s will give identical elements in $R = \mathbf{Z}_K[[h]]/(\phi_K(1+h))$, using $q^a = q^b$ in R whenever $b - a \in K\mathbf{Z}_K$. Next observe that the combination

$$\frac{1}{2h} q^{\frac{1}{2}+\frac{P}{4P_i^2}-\frac{\phi}{4}} \sum_{m=2}^{\infty} \sum_{\epsilon=\pm} \epsilon d_m \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} A_s$$

will give $Z'_K(M)$ or $\sqrt{H} Z^0_K(M)$, respectively, as elements of R, according to the choice of A_s. For the first case, we can ignore contributions from $\beta \in K\mathbf{Z}$ since then $q^\beta - 1 = 0$ in R. The innermost sum is an element of $h^m R$, while all the coefficients in this combination lie in $\mathbf{Z}_K$ since $\frac{\phi}{4} \in \frac{P}{4HP_i^2} + \mathbf{Z}_K$, by (4.1).

Theorem 6 (See also [L1], [L2], [R6].) *For Seifert manifolds M and odd prime powers K coprime to P, the asymptotic expansion of $\sqrt{H} Z^0_K(M)$ in powers of $h = q - 1$ converges K–adically to $Z'_K(q) \in \mathbf{Z}[q]$, in the sense that they agree when considered as elements of $\mathbf{Z}[\frac{1}{2H}][[h]]/(\phi_K(1+h))$, where $\phi_K(q)$ is the K^{th} cyclotomic polynomial.*

Note that the factor of $\sqrt{H}$ which appears in the comparison of the trivial connection contribution to $Z_K(M)$ and the K–adic series for $Z'_K(M)$ appears in the above calculation as the result of the ratio of a Gaussian sum and the associated Gaussian integral. It should also be pointed out that our version of the $SO(3)$-invariant, $Z'_K(M)$, differs by a sign from the Kirby-Melvin normalisation [KM2].

4.8 Comparison of integrals

In §4.3, an integral expression was found for Z_∞ for Seifert manifolds. However, in [L1], another such expression was found for the case of the manifold $M_{n,t}$, obtained by integer surgery around a $(2,n)$ torus knot with t additional twists. That is, for a two parameter family of manifolds indexed by the odd integer n and the integer t, with $\left|H^1(M_{n,t},\mathbf{Z})\right| = |n+t|$. Another description of $M_{n,t}$ is as a three-fibred Seifert manifold $X\left(\frac{2}{-1},\frac{n}{(n+1)/2},\frac{t-n}{1}\right)$; see [Mos]. The formula of [L1] is

$$Z_\infty = 2\frac{(-1)^t q^{\frac{n}{2}}}{1-q^{-\delta}} \int_{-\infty}^{\infty} \frac{q^{\Delta_1(iz)} - q^{\Delta_2(iz)}}{\mathrm{e}^{2\pi z} + \mathrm{e}^{-2\pi z}}\, dz,$$

where

$$2(t+n)\Delta_1(x) = 4n(t-n)x^2 + 4tx - \frac{1}{4}n(t+n) - \frac{1}{2}\,,$$
$$2(t+n)\Delta_2(x) = 4n(t-n)x^2 + 4(t-2n)x - \frac{1}{4}n(t+n) - \frac{9}{2}\,.$$

In this case, $P = 2n(t-n)$, $H = t+n$ and $N = 3$. The integral coming from Theorem 2 is

$$Z_K(M) = \frac{BG_0}{2\pi i} q^{-\frac{\phi}{4}} \sum_{r=0}^{H-1} \int \mathrm{e}^{\frac{(t+n)Kiy^2}{16\pi n(t-n)} - rKy} \frac{\left(\mathrm{e}^{\frac{y}{2n}} - \mathrm{e}^{-\frac{y}{2n}}\right)\left(\mathrm{e}^{\frac{y}{2(t-n)}} - \mathrm{e}^{-\frac{y}{2(t-n)}}\right)}{\mathrm{e}^{\frac{y}{4}} + \mathrm{e}^{-\frac{y}{4}}}\, dy\,.$$

Both integrals share the same asymptotic expansion, and when the contours are suitably turned and shifted so as to follow paths of steepest descent, both give the same numerical results. However, it should be noted that they are distinct!

4.9 Example

For the Poincaré homology sphere (opposite orientation), $\mathbf{p} = (2,3,5)$, at $K = 11$ we have $P = 30$, $H = 1$, $\lambda = -1$ and $\phi = \frac{181}{30}$, giving values

$$Z_{11}(M) = 2.49611 + 1.29639i,$$
$$Z_{11}^{\mathrm{res}}(M) = 2.54609 + 1.75882i,$$
$$Z_{11}^{\mathrm{int}}(M) = -0.04998 - 0.46243i,$$

where the integral was evaluated on the standard contour C which is a line in the direction $1-i$. The contributions of flat connections are thus,

$$Z_{11}^{0}(M) = -0.04998 - 0.46243i,$$
$$Z_{11}^{A_1}(M) = -2.45952 + 1.87797i,$$
$$Z_{11}^{A_2}(M) = 5.00562 - 0.11916i,$$

there being two irreducible flat connections providing non-trivial contributions, aside from the trivial connection, on the Poincaré homology sphere. In terms of the sum over m running from 1 to $2P-1=59$ which makes up Z_{11}^{res}, these two connections come from the terms with m-values in $\{1,11,19,29,31,41,49,59\}$ and $\{7,13,17,23,37,43,47,53\}$, respectively. In terms of the labelling of connections by l and m_j, these connections both have $l=1$ while $\mathbf{m}=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2},\frac{3}{2})$, all the other connections providing trivial contributions. Their contributions are

$$Z_K^{A_j}(M) = -\sqrt{\frac{K}{5}}\,\frac{\mathbf{e}^{\frac{\pi i}{4}}}{\sin\frac{\pi}{K}}\,\mathbf{e}^{-\frac{181\pi i}{60K}}\,\sin\frac{j\pi}{5}\,\mathbf{e}^{-\frac{m^2\pi i K}{60}}\,,$$

as also computed in [FG]. Note however that we here have an *exact* accounting of all of $Z_{11}(M)$, even for a relatively small value of K, whereas in [FG], only the irreducible (non-trivial) connection contributions were used, and they thus only gave a good approximation for numerically large K.

To work out the value of the invariant at the second 11^{th} root of unity, we use $s=13$, seeing that it is congruent to 1 mod 4, and coprime to $P=30$. The sign appearing in the expression for $Z_{\frac{13}{11}}(M)$ in the evaluation (4.7) for rational K, is

$$(-1)^{\frac{s-1}{4}}\left(\frac{s}{2r|P|}\right) = (-1)^3\left(\frac{13}{11.3.5}\right) = -1\,.$$

The value of the WRT invariant of the Poincaré sphere at $q=\mathbf{e}^{\frac{4\pi i}{11}}$ is thus

$$-Z_{\frac{13}{11}}(M) = 1.55773 - 0.08019i\,.$$

This value may be obtained from the polynomial for the 11^{th} roots of unity, namely

$$Z_{11}(M) = 1 + q - q^7\,,$$

by substituting the particular value of q. From the alternative description as a holomorphic function involving a sum of integrals and residues, the individual terms contributing are

$$\begin{aligned} Z_{\frac{13}{11}}^{\mathrm{int}}(M) &= 0.49086 + 0.10697i\,,\\ Z_{\frac{13}{11}}^{\mathrm{stat}}(M) &= -2.03617 - 2.63598i\,,\\ Z_{\frac{13}{11}}^{\mathrm{res}}(M) &= -0.01242 + 2.60918i\,,\end{aligned}$$

whose sum can be seen to agree with the value of $Z_{\frac{13}{11}}(M)$ given above. Here, Z^{int} is the sum of the integrals with contours shifted so as to pass through stationary points, Z^{stat} is the sum of residues acquired in the process of this shifting of contours and Z^{res} is the sum of residues acquired in the process of converting sums into integrals in §4.4. Meanwhile the trivial connection contribution, given by the term with $t=0$ alone, is $Z_{\frac{13}{11}}^0(M) = -0.136139 - 0.0296678i$

5. Torus knots

For coprime integers P and Q, let $K_{P,Q}$ denote a torus knot of type (P,Q). Choose integers R and S so that $PS-QR=1$. Consider a four component link $L_0 \amalg L_1 \amalg L_\alpha \amalg L_2$ in which the last three components form an unlink, the first component having linking number 1 with them. Observe that rational surgery on S^3 around the three components L_0, L_1 and L_2 with framings 0, $-\frac{P}{R}$ and $\frac{Q}{S}$ reproduces the 3-sphere while transforming the unknot labelled α to $K_{P,Q} \subseteq S^3$. The Jones polynomial of $K_{P,Q}$ may therefore be described as a combination of invariants on the 4-link in S^3, or equivalently, by doing the $\mathbf{S}$-surgery around L_0, as a combination of invariants on the three-link in $S^1 \times S^2$. The sum is over labels β, α_1 and α_2 on the three components of the link on which surgery is carried out, leaving

$$Z(S^3, K^\alpha_{P,Q}) = \mathrm{e}^{\frac{\pi i}{4}\frac{K-2}{K}\Phi(L,\mathbf{\Lambda})} \sum_{\beta=1}^{K-1} \frac{d^{\mathbf{S}}_\beta}{\sin\frac{\pi}{K}\sin\frac{\pi\beta}{K}} \sum_{\alpha_1,\alpha_2=1}^{K-1} d^{\Lambda_1}_{\alpha_1} d^{\Lambda_2}_{\alpha_2} \sin\frac{\pi\beta\alpha_1}{K}\sin\frac{\pi\beta\alpha_2}{K}\sin\frac{\pi\beta\alpha}{K}.$$

Here $\Lambda_1 = \begin{pmatrix} P & -Q \\ -R & S \end{pmatrix}$ and $\Lambda_2 = \begin{pmatrix} Q & -P \\ S & -R \end{pmatrix}$ describe the two non-zero surgeries. As in §4.1, we apply (3.3) to evaluate the sums over α_1 and α_2 and include the appropriate framing correction, giving (see [R5]),

$$\begin{aligned} J_\alpha(K_{P,Q}) &= G_0\, \mathrm{e}^{-\frac{\pi i}{2K}PQ(\alpha^2-1)} \sum_{\beta=1}^{K-1} \frac{\sin\frac{\pi\alpha\beta}{K}}{\sin\frac{\pi\beta}{K}} d^{\mathbf{S}\Lambda_1}_\beta d^{\mathbf{S}\Lambda_2}_\beta \\ &= \frac{\mathrm{e}^{\frac{\pi i}{2K}(PQ(1-\alpha^2)-\frac{P}{Q}-\frac{Q}{P})}}{2i\sin\frac{\pi}{K}\sqrt{2K|PQ|}}\, \mathrm{e}^{\frac{\pi i}{4}\mathrm{sign}\,PQ} . \Sigma \end{aligned}$$

by a substitution of the formula for d^Λ_β from §2.2, where

$$\Sigma = \sum \frac{\sin\frac{\pi\alpha\beta}{K}}{\sin\frac{\pi\beta}{K}} \boldsymbol{\mu}\, \mathrm{e}^{-\frac{\pi i\beta^2}{2KPQ} - \frac{\pi i\beta}{K}\left[2K(\frac{n_1}{P}+\frac{n_2}{Q})\right] + \frac{\mu_1}{P}+\frac{\mu_2}{Q}\right] - 2\pi i\left(\frac{n_1\mu_1 Q}{P}+\frac{n_2\mu_2 P}{Q}+\frac{KQn_1^2}{P}+\frac{KPn_2^2}{Q}\right)}.$$

The sum is over $1 \le \beta \le K-1$, $0 \le n_1 \le P-1$, $0 \le n_2 \le Q-1$, $\mu_j = \pm 1$ and $\boldsymbol{\mu} = \mu_1\mu_2$. Observe that the summand is invariant under the change of variables,

$$(n_1, n_2, \mu_1, \mu_2, \beta) \longrightarrow \big(n_1+a, n_2+b, \mu_1, \mu_2, \beta - (2aQ+2bP)K\big)\,,$$

for any integers a and b, while it is periodic in n_1 and n_2 of periods P and Q, respectively. Thus one may carry out the sum equivalently by keeping $n_1 = n_2 = 0$ and letting β range over

$$\big\{c + 2Kd \,\big|\, 1 \le c \le K-1, \quad 0 \le d < PQ\big\}\,.$$

Performing the sum over $\boldsymbol{\mu}$ and noting that the resulting expression is symmetric under $\beta \longrightarrow -\beta$, one now obtains,

$$\Sigma = -2 \sum_{\substack{\beta=1 \\ K \not| \beta}}^{2PQK-1} f_\alpha(\beta)$$

where

$$f_\alpha(\beta) = \frac{\sin \frac{\pi\alpha\beta}{K}}{\sin \frac{\pi\beta}{K}} \sin \frac{\pi\beta}{KP} \sin \frac{\pi\beta}{KQ} \, \mathrm{e}^{-\frac{\pi i \beta^2}{2KPQ}} \, .$$

This is a sum of exactly the same form as (4.2) and therefore can be transformed into the sum of an integral and a sum of residues, as was done in (4.5) in §4.3. In this case, one obtains as a function of $q = \mathrm{e}^{\frac{2\pi i}{K}}$,

$$J_\alpha(K_{P,Q}) = \frac{q^{\frac{PQ}{4} - \frac{P}{4Q} - \frac{Q}{4P}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}} \frac{\mathrm{e}^{\frac{\pi i}{4} \operatorname{sign} PQ}}{\sqrt{|PQ|}} \sqrt{\frac{2}{K}} q^{-\frac{PQ\alpha^2}{4}} \left(\int_C f_\alpha(\beta) d\beta + I^{\mathrm{res}} \right)$$

using the fact that $f_\alpha(\beta + 2PQK) = f_\alpha(\beta) \, \mathrm{e}^{-2\pi i\beta}$. The contour C here is a line through the origin in the direction $1 - i$, while

$$\begin{aligned} I^{\mathrm{res}} &= -2\pi i \sum_{m=1}^{2PQ-1} \operatorname{Res}\left(\frac{f_\alpha(\beta)}{1 - \mathrm{e}^{-2\pi i\beta}}, \beta = mK \right) \\ &= -\frac{\pi\alpha}{K} \sum_{m=1}^{2PQ-1} (-1)^{m(\alpha+1)} \sin \frac{\pi m}{P} \sin \frac{\pi m}{Q} \, \mathrm{e}^{-\frac{\pi i K m^2}{2PQ}} \, . \end{aligned}$$

Asymptotic behaviour

Considering α to be of the same order as K, by putting $a = \frac{\alpha}{K}$ and doing a stationary phase expansion for large K with a staying finite, one finds that the integrand $f_\alpha(\beta)$ splits into two parts, with distinct (symmetric) stationary phase points at $\beta = \pm PQKa$. Shifting the contour so as to pass through the stationary phase points one obtains

$$\int_C f_\alpha(\beta) d\beta = -i \, \mathrm{e}^{\pi i PQKa^2/2} \int_C \mathrm{e}^{-\frac{\pi i \beta^2}{2PQK}} \frac{\sin\left(\frac{\pi\beta}{KP} + \pi Qa\right) \sin\left(\frac{\pi\beta}{KQ} + \pi Pa\right)}{\sin\left(\frac{\pi\beta}{K} + \pi PQa\right)} d\beta + I^{\mathrm{stat}}$$

where the residues acquired are

$$I^{\mathrm{stat}} = 2K \sum_{m=1}^{[PQa]} (-1)^m \, \mathrm{e}^{\pi i \alpha m} \, \mathrm{e}^{-\frac{\pi i K m^2}{2PQ}} \sin \frac{\pi m}{P} \sin \frac{\pi m}{Q} \, .$$

When the parameter a is sufficiently small, $a < \frac{1}{PQ}$, the term I^{stat} completely disappears. This gives the entire asymptotic behaviour of $J_\alpha(K_{P,Q})$ and in particular

$$J_\alpha(K_{P,Q}) = J^0(a,K) + J^{\mathrm{res}}(a,K)\,,$$

where J^0 is the trivial connection contribution and is an analytic function of the two variables $a = \frac{\alpha}{K}$ and K. The term J^{res} is a sum of two sets of residues, coming from a combination of I^{res} and I^{stat}.

6. Generalisations

In this paper we have carried out a precise analysis of the invariants of Witten and Reshetikhin-Turaev associated with $\mathfrak{sl}_2$, in the case of Seifert fibred manifolds and of coloured torus knots in S^3. In both cases, it has been seen how, by transforming a sum into an integral with additional residue contributions, the various terms can be identified with contributions which arise from a stationary phase expansion of the corresponding Chern-Simons-Witten path integral. This technique also allows these invariants to be naturally extended to holomorphic functions. In the case of the invariants of 3-manifolds, the new function is a holomorphic function of the parameter K which usually defines the order of the root of unity, and in the case of the coloured Jones polynomial for torus knots, it is the colour variable which has been extended to non-integer values. In both cases, the stationary phase expansion was exact, for finite K. The trivial connection contribution has been seen to play a major role, providing the complete invariant at a K-adic level.

It is conjectured that similar results hold in general, for all WRT invariants of pairs of links contained in 3-manifolds.

Conjecture *Let M be a rational homology sphere with $H = |H^1(M,\mathbf{Z})|$.*

(a) *The stationary phase expansion, $\sum_A Z_K^A(M)$, of the Chern-Simons-Witten path integral (1.1) for $Z_K(M)$, decomposed into terms labelled by conjugacy classes, A, of flat connections, is exact.*

(b) *There is an integral expression for the trivial connection contribution $Z_K^0(M)$ which is also valid (up to signs) for rational K.*

(c) *The normalised trivial connection contribution $\widetilde{Z}_\infty(M) = q^{-6\lambda(M)}H^{\frac{3}{2}}Z_K^0(M)$ has an asymptotic expansion in $h = q-1$ which lies in $\mathbf{Z}[\frac{1}{H}][[h]]$.*

(d) *For any odd prime power K coprime to H, $\sqrt{H}Z_K^0(M)$ has an asymptotic expansion in powers of $h = q-1$ which converges K–adically to $Z'_K(M)$.*

(e) *The coefficient $L_n(M)$ in the expansion of (c) is a finite type invariant (see* [G], [Le] *and* [LMO]*) of order n, with $L_0(M) = 1$, $L_1(M) = 0$ (by* [M1]*). Furthermore, $L_2(M) \in 6\mathbf{Z}$ and $L_2(M) + L_3(M) \in 48\mathbf{Z}$, if M is an integer homology sphere.*

The reader is referred to [L3] for a construction of a holomorphic invariant of 3-manifolds for general rational homology spheres. It is hoped that just as there is a perfect and calculable correspondence (see [BN2] and [K]) between Feynman diagrams coming from a perturbative expansion of (1.1) and contributions to the quantum invariants of links labelled by chord diagrams, the same will come to hold for invariants in general 3–manifolds. The better understanding of the decomposition of the invariants into terms $Z_K^A(M)$ and of their behaviour away from integer values of K is a step in this programme.

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