On Ohtsuki's invariants of homology 3-spheres. [1]

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Abstract. By analysing Ohtsuki's original work in which he produced a formal power series invariant of rational homology 3-spheres, we obtain a simplified explicit formula for them, which may also be compared with Rozansky's integral expression. We further show their relation to the exact $SO(3)$ Witten-Reshetikhin-Turaev invariants at roots of unity in a stronger form than that given in Ohtsuki's original work.

1: Introduction

Suppose that M is a rational homology 3–sphere. For any integer r and root of unity q of order r, Reshetikhin and Turaev [RT] constructed an invariant $\tau_r(M)$. It is convenient to use the refined form, $\tau'_r(M)$, which was extracted by Kirby and Melvin [KM] by the use of a symmetry principle; it is nothing other than the $SO(3)$ Witten-Reshetikhin-Turaev invariant of M, in which are included only terms of $\tau_r(M)$ corresponding to 'half' of the representations. (See also [W].) It is known that $\tau'_r(M)\in\mathbf{Z}[q]$ from [M1], so that one may write

$$\tau'_r(M)=\sum_{j=0}^{\infty}\lambda_{j,r}(M)h^j\ ,$$

where $h=q-1$ and $\lambda_{j,r}\in\mathbf{Z}$, all but finitely many vanishing. The coefficients $\lambda_{j,r}$ are not uniquely determined since the relation $\phi_r(1+h)\equiv\frac{(1+h)^r-1}{h}=0$ holds.

In [O2], Ohtsuki constructed a power series invariant, $\tau(M)\in\mathbf{Q}[[h]]$, of rational homology spheres, multiplicative under connected sums, with the defining property that for any odd prime, r, not dividing $H=|H_1(M;\mathbf{Z})|$,

$$\left(\frac{H}{r}\right)\lambda_{j,r}\equiv\lambda_j\ (\mathrm{mod}\ r),\qquad \text{for } 0\le j\le\frac{r-3}{2}\ ,$$

where $\tau(M)=\sum_{j=0}^{\infty}\lambda_j h^j$ and $\lambda_j\in\mathbf{Q}$. Here $\left(\frac{\cdot}{r}\right)$ denotes the Jacobi symbol describing quadratic residues modulo r.

For various special cases, much stronger results connecting τ and τ'_r are known. For all Seifert fibred 3-spheres, [LR] shows that

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(a) $\tau(M) \in \mathbf{Z}[\frac{1}{H}][[h]]$;

(b) $\tau(M)$ converges r-adically to $\left(\frac{H}{r}\right)\tau'_r(M)$ for all prime r not dividing H, that is $\tau(M) - \left(\frac{H}{r}\right)\tau'_r(M) \in \phi_r(1+h)\mathbf{Z}[\frac{1}{H}][[h]]$;

(c) there is an explicitly presentable collection of holomorphic functions $Z^A(r)$ of r, indexed by classes of flat connections whose sum, evaluated at integer r, gives $\tau'_r(M)$ at $q = \mathbf{e}^{\frac{2\pi i}{r}}$;

(d) $\tau(M)$ can be viewed as an asymptotic expansion of $Z^0(M)$.

In this note we will refine the computations of [O2] so as to produce an explicit and useable formula for $\tau(M)$, in Theorem 3.1.4, which is also directly comparable with values of $\tau'_r(M)$; see §4. Specific results are given in §3.4 for the first few coefficients in the power series expansion. See also [LW], [L1], [L2], [R2] and [R4] for related results. The method also gives rise to a formal reformulation of $\tau(M)$ in a shape which is very similar to the Reshetikhin–Turaev construction of $\tau_r(M)$; see §3.3.

2: Reformulation of $\tau'_r(M)$

2.1 Notation

We will deliberately use notation compatible with the papers [O1], [M2] and [O2], rather than more usual and simpler notation, so as to make comparisons easy. Thus $\overline{a}$ refers to the multiplicative inverse of a in $\mathbf{Z}/r\mathbf{Z}$, while expressions such as binomial coefficients and powers with vector entries are defined multiplicatively based on corresponding entries, for example $\mathbf{x}^{\mathbf{j}} = \prod_c x_c^{j_c}$. The q-numbers are defined by $[k]_q = \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$. Also $G(q)$ denotes the Gauss sum $\sum\limits_{k=0}^{r-1} q^{k^2}$ where $q = 1+h$ is a root of unity of order r. The notation $[.]_{t^i}$ will be used to denote the coefficient of t^i in the expression inside the square brackets, when expanded in increasing and positive powers of t; sometimes we will have occasion to use expansions in more than one variable. Also $\mathbf{Z}_r$ will denote the intersection of the r-adic integers with the rationals, that is, it consists of all rationals whose denominators are not divisible by r.

When L is a (framed) link, $|L|$ will denote the number of its components and $L^{\mathbf{j}}$ will denote the cabling in which there are j_c parallels in place of the c^{th} component, obtained using the framing on L. The Jones polynomial of L normalised to be muliplicative under disjoint union is denoted $<L>$. Ohtsuki defines another invariant

$$\Phi(L) = \sum_{L' \subseteq L} (-1)^{|L|-|L'|} [2]_q^{-|L'|} <L'> \,,$$

which has the following properties (see [O2])

(i) $<L> = [2]_q^{|L|} \sum_{L' \subseteq L} \Phi(L')$;

(ii) $\Phi(L^{\mathbf{j}})$ is an element of $h^{\mathbf{j}+\max(\mathbf{j})}\mathbf{Z}[q]$.

2.2 Computation of $\tau_r'(M)$

Suppose that M is a rational homology 3-sphere. It is known (see [O2]) that M, or possibly a connected sum of M with some Lens spaces $L(n,1)$, can be expressed as integer surgery of S^3 around an algebraically split link. Since all the invariants considered are multiplicative under connected sums, and our results can be easily checked for Lens spaces, we shall assume henceforth that $M = S^3_L$, where L is an algebraically split link. Thus the linking matrix of L is diagonal, say with entry f_c corresponding to the framing on the c^{th} component of the link. Also the value of $H = |H_1(M;\mathbf{Z})|$ is just $|\prod_c f_c|$.

As in [O2], we start with Murakami's form for $\tau_r'(M)$ (see [M2]), as

$$(-1)^{\overline{\sigma}} q^{3\dot{\overline{4}}\sigma-\overline{2}|L|}\left(\frac{q-1}{G(q)}\right)^{|L|}\sum_{\mathbf{k}=1}^{\frac{r-1}{2}} q^{\overline{4}\mathbf{f}(\mathbf{k}^2-1)}[\mathbf{k}]_q \sum_{\mathbf{j}=\mathbf{0}}^{\frac{\mathbf{k}-1}{2}}(-1)^{\mathbf{j}}\binom{\mathbf{k}-\mathbf{j}-\mathbf{1}}{\mathbf{j}} < L^{\mathbf{k}-2\mathbf{j}-\mathbf{1}} > ,$$

Here σ is the signature of the link, $\sigma_+ - \sigma_-$, and $\overline{\sigma}$ is σ_+ or σ according as $r\equiv 1$ or 3 modulo 4, where $\sigma_\pm$ count the number of positive/negative elements in $\mathbf{f}$. Thus $(-1)^{\overline{\sigma}} = \left(\frac{\operatorname{sgn}(\mathbf{f})}{r}\right)\prod\limits_c(-\operatorname{sgn}(f_c))$.

Converting appearances of $\Phi(L^{\mathbf{j}})$ into those of $<L^{\mathbf{k}}>$ for $\mathbf{k}\le\mathbf{j}$ yields

$$\left(\frac{H}{r}\right)\tau_r'(M) = \sum_{\mathbf{i}=0}^{\frac{r-3}{2}}\frac{\Phi(L^{\mathbf{i}})}{(q-1)^{\mathbf{i}}}\prod_{c=1}^{|L|} Q^{(r)}_{i_c,f_c}\ , \tag{2.2.1}$$

where

$$Q^{(r)}_{i,f} = C_f'\left(\frac{f}{r}\right)\frac{(q-1)^{i+1}}{qG(q)}\sum_{k=1}^{\frac{r-1}{2}} q^{\overline{4}fk^2}[k]_q\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k-j-1}{j}\binom{k-2j-1}{i}[2]_q^{k-2j-1} \tag{2.2.2}$$

are functions of q dependent on r, non-zero integer f (mod r) and non-negative integer i. The only non-zero terms in the sum for $Q^{(r)}_{i,f}$ are those with $k\ge i+1$ and $j\le\frac{k-i-1}{2}$; however, we choose to write the expression in the above form for later convenience. In particular, observe that $Q^{(r)}_{i,f}=0$ whenever $i\ge\frac{r-1}{2}$. Also, we have set

$$C_f' = -\operatorname{sgn}(f)q^{\overline{2}-\overline{4}f+3\dot{\overline{4}}\operatorname{sgn}(f)}\ .$$

We first perform the sum over j,

$$\begin{aligned}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k-j-1}{j}\binom{k-2j-1}{i}[2]_q^{k-2j-1} &= \sum_{j=0}^{\left[\frac{k-1}{2}\right]}\left[(p(1+t)[2]-p^2)^{k-j-1}\right]_{t^ip^{k-1}}\\ &= [2]_q^i[(1+p^2-p[2]_q)^{-i-1}]_{p^{k-i-1}}\\ &= \frac{[2]_q^i}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{i+1}}[(\alpha-\beta)^{i+1}]_{p^k}\end{aligned}$$

where a decomposition into partial fractions was used in the last line, and $\alpha = (1-pq^{\frac{1}{2}})^{-1}$, $\beta = (1-pq^{-\frac{1}{2}})^{-1}$. Observe that $\alpha\beta = \frac{\beta - q\alpha}{1-q}$, from which functions $A_{i,n}$ and $B_{i,n}$ of q can be found for which

$$(\alpha-\beta)^{i+1} = \sum_{n=1}^{i+1}(A_{i,n}\alpha^n + B_{i,n}\beta^n)\,.$$

Substituting into the above and (2.2.2) as well as evaluating the coefficient of p^k gives

$$Q_{i,f}^{(r)} = C'_f\left(\frac{f}{r}\right)\frac{(q+1)^i}{q^{\frac{1}{2}}G(q)}\sum_{k=1}^{\frac{r-1}{2}} q^{\bar{4}fk^2}\sum_{n=1}^{i+1}[k]_q(A_{i,n}q^{\frac{k}{2}} + B_{i,n}q^{-\frac{k}{2}})(-1)^k\binom{-n}{k}\,.$$

Observe that the product of the last two terms is just $f_n(k) = \binom{n+k-1}{n-1}$, a polynomial in k. Recalling the definition of $[k]_q$, it is seen that the inner sum is a linear combination of q^k, q^{-k} and 1, whose coefficients are all polynomials in k of degree at most i, and as such, may be extended to a well-defined function for negative k. This function is even and its value at $k=0$ is zero, so that by introducing a factor of $\frac{1}{2}$, the sum may be extended over $\frac{1-r}{2} \le k \le \frac{r-1}{2}$, a complete residue class giving

$$Q_{i,f}^{(r)} = C'_f\frac{(q+1)^i}{2h}\sum_{n=1}^{i+1}\left\langle\big(A_{i,n}(q^k-1) - B_{i,n}(q^{-k}-1)\big)f_n(k)\right\rangle_r\,, \tag{2.2.3}$$

where the angle brackets denote the taking of an expectation value with respect to the variable k thought of as an integer-valued random variable taking values between $\frac{1-r}{2}$ and $\frac{r-1}{2}$ with distribution $q^{\bar{4}fk^2}$ (normalised). That is,

$$<f(k)>_r \equiv \frac{\sum\limits_{k=\frac{1-r}{2}}^{\frac{r-1}{2}} f(k)q^{\bar{4}fk^2}}{\sum\limits_{k=\frac{1-r}{2}}^{\frac{r-1}{2}} q^{\bar{4}fk^2}}\,.$$

In (2.2.3), we have also used the fact that the denominator here is none other than $\left(\frac{f}{r}\right)G(q)$.

Next we perform the sum over n in (2.2.3). From the definitions of $A_{i,n}$ and $B_{i,n}$, one can find that $A_{i,i+1} = 1$ while for $n \le i$,

$$A_{i,n} = (1-q)^{n-i-1}\sum_{p=1}^{i+1-n}\binom{i+1}{p}\binom{i-n}{p-1}q^p \tag{2.2.4}$$

from which $\sum_{n=1}^{i+1} A_{i,n} f_n(k)$ can be computed to be

$$\binom{i+k}{i} + \sum_{p=1}^{i} \binom{i+1}{p} (-q)^p h^{-i} \sum_{n=1}^{i+1-p} \binom{-k-1}{n-1} \binom{-p}{i-n-p+1} (-h)^{n-1} ,$$

where we have interchanged the order of summation of n and p. The sum over n may be expressed as the coefficient of x^{i-p} in $(1+x)^{-p}(1-hx)^{-k-1}$, and then the sum over p may be performed, giving

$$\sum_{n=1}^{i+1} A_{i,n} f_n(k) = h^{-i} \left[\frac{(1-hx)^{i-k}}{(1+x)^{i+1}} \right]_{x^i} .$$

Note that interchanging q and q^{-1} will interchange $A_{i,n}$ and $(-1)^{i+1} B_{i,n}$, so that

$$\sum_{n=1}^{i+1} B_{i,n} f_n(k) = -(1-q^{-1})^{-i} \left[\frac{(1+(1-q^{-1})x)^{i-k}}{(1+x)^{i+1}} \right]_{x^i} .$$

Substituting these results into the formula for $Q_{i,f}^{(r)}$ gives

$$Q_{i,f}^{(r)} = C_f' \frac{(q+1)^i}{2h^{i+1}} < \left[\frac{(1-hx)^{i-k} - (q+hx)^{i-k}}{(1+x)^{i+1}} (q^k - 1) \right]_{x^i} >_r .$$

Suppose now that x and y are related by $x - qy - hxy = 0$. This provides a transformation from x to y which fixes both 0 and -1. A power series in x thus transforms to one in y, and it can be shown that whenever the series f and g are related by $f(x) = g(y)$ then

$$\left[\frac{f(x)}{1+x} \left(\frac{q+hx}{1+x} \right)^i \right]_{x^i} = \left[\frac{g(y)}{1+y} \left(\frac{1-hy}{1+y} \right)^i \right]_{y^i} .$$

This transformation may be applied to the second term in the expression for $Q_{i,f}^{(r)}$ above, and will reduce it to the first term with y replacing x, since $1+(1-q^{-1})x = (1-hy)^{-1}$. Thus the two terms are equal and

$$Q_{i,f}^{(r)} = C_f' \frac{(q+1)^i}{h^{i+1}} \left[\frac{<(1-hx)^{i-k}(q^k-1)>_r}{(1+x)^{i+1}} \right]_{x^i} . \tag{2.2.5}$$

Note that none of the reasoning up to this point depends on i lying in the range $0 \le i \le \frac{r-3}{2}$, but would equally well hold for any non-negative integer i. On the other hand, from the initial definition of $Q_{i,f}^{(r)}$ we know that it vanishes for $i \ge \frac{r-1}{2}$.

2.3 Integrality of $Q_{i,f}^{(r)}$

In this section we show that $Q_{i,f}^{(r)}$, which is naturally an element of $\mathbf{Q}[q]$, actually lies in $\mathbf{Z}[q]$. This is trivial when $i \geq \frac{r-1}{2}$, as then $Q_{i,f}^{(r)} = 0$, so we assume that $i \leq \frac{r-3}{2}$. Note that by (2.2.2), the inner sum over j is an element of $\mathbf{Z}[q]$ for each integer value of k. However, $h^{\frac{1-r}{2}}G(q)$ is an invertible element of $\mathbf{Z}[q]$, as follows from the fact that $G(q) = \prod_{k=1}^{\frac{r-1}{2}}(q^{2k-1} - q^{1-2k})$. As a result, we know that $h^{\frac{r-3}{2}-i}Q_{i,f}^{(r)} \in \mathbf{Z}[q]$. This is enough to guarantee that $Q_{i,f}^{(r)} \in \mathbf{Z}[q]$, once it is known that $Q_{i,f}^{(r)} \in \mathbf{Z}_r[q]$.

From (2.2.4), observe that $A_{i,n} \in h^{n-i-1}\mathbf{Z}[h]$, with a similar statement for $B_{i,n}$. In the n^{th} term of the sum for $Q_{i,f}^{(r)}$ given by (2.2.3), the q^k term may be transformed by completing the square in the exponent of q and shifting k, so as to obtain the expectation value of a polynomial in k of degree $n-1$ with coefficients in $\mathbf{Z}_r$. (Note that this shift does affect the result, but not the integrality properties discussed; see also §3.1.) By Lemma 8.2 in [O1], $h^m{<}k^{2m}{>}_r \in \mathbf{Z}_r[h]$ and thus the n^{th} term in the right hand side of (2.2.3) contributes to $Q_{i,f}^{(r)}$ terms lying in $h^{-i+[\frac{n}{2}]}\mathbf{Z}_r[h]$. We conclude that $h^iQ_{i,f}^{(r)} \in \mathbf{Z}_r[h]$ and thus that $h^iQ_{i,f}^{(r)} \in \mathbf{Z}[q]$.

The stronger result that $Q_{i,f}^{(r)} \in \mathbf{Z}[q]$ is an immediate consequence of (7.3) in [O1].

Thus we have

$$\left(\frac{H}{r}\right)\tau_r'(M) = \sum_{\mathbf{i}=0}^{\infty} \frac{\Phi(L^{\mathbf{i}})}{(q-1)^{\mathbf{i}}} \prod_{c=1}^{|L|} Q_{i_c,f_c}^{(r)}\,, \tag{2.3.1}$$

where the first term of the product, $h^{-\mathbf{i}}\Phi(L^{\mathbf{i}})$, lies in $h^{\max(\mathbf{i})}\mathbf{Z}[h]$. Meanwhile, $Q_{i_c,f_c}^{(r)} \in \mathbf{Z}[h]$, from which Murakami's result (see [M2]) that $\tau_r'(M)$ lies in $\mathbf{Z}[q]$ is clear. An infinite sum has been used to stress the natural independence on r of this part of the computation of $\tau_r'(M)$, but in reality this is a finite sum since the $Q_{i,f}^{(r)}$ terms vanish for $i \geq \frac{r-1}{2}$.

3: A formula for $\tau(M)$

3.1 Computation of $\tau(M)$

The only dependence upon r in (2.3.1) is in the terms $Q_{i,f}^{(r)}$, and from (2.2.5) this dependence is limited to the process of taking the expectation value ${<}\cdot{>}_r$ and the form for C_f'.

In the previous section, the discussion was of elements of the cyclotomic ring $\mathbf{Z}[q]$ generated by an r^{th} root of unity, q, for a specific value of r. In this section we shift to power series in $h = q - 1$. When comparing these two, the strongest form of equivalence in $\mathbf{Z}_r[[h]]$ is up to the ideal generated by $\phi_r(1+h)$.

Lemma 3.1.1 *If $a, b \in \mathbf{Z}_r$ then $q^a - q^b \in \phi_r(1+h)\mathbf{Z}_r[[h]]$ whenever $a - b \in r\mathbf{Z}_r$.*

Thus in this sense, $C'_f \in \mathbf{Z}[q]$ is equivalent to

$$C_f = -\operatorname{sgn}(f) q^{\frac{1}{2} - \frac{f}{4} + \frac{3}{4}\operatorname{sgn}(f)},$$

an invertible element of $\mathbf{Z}[\frac{1}{2f}][[h]]$, independent of r. We are left with removing the r-dependence from the evaluation of the expectation value in (2.2.5).

Next let $<\cdot>_\infty$ denote the expectation value taken where k is considered as a continuous real-valued random variable whose distribution is proportional to $q^{fk^2/4}$, so that

$$<f(k)>_\infty \equiv \frac{\int_{-\infty}^{\infty} f(k) q^{\frac{fk^2}{4}}\, dk}{\int_{-\infty}^{\infty} q^{\frac{fk^2}{4}}\, dk}.$$

As written, this only makes sense (giving convergent integrals) if $|q| < 1$ and $f > 0$ or $|q| > 1$ and $f < 0$. However, in the general case, we interpret the integral to be a complex contour integral taken along a line through the origin in the direction of steepest descent.

Lemma 3.1.2 *For any odd prime r, $<(1-hx)^k>_r \equiv <(1-hx)^k>_\infty$ modulo $h^{\frac{r-1}{2}} x^{\frac{r-1}{2}}$. In this statement, the left hand side is considered as a formal power series in x whose coefficients lie in $\mathbf{Q}[h]/(\phi_r(1+h))$, while the right hand side is considered as a formal power series in x whose coefficients lie in $\mathbf{Q}[[h]]$. The statement is thus that for any non-negative integer $j \leq \frac{r-1}{2}$, these coefficients, which lie in different rings, agree as elements of the common quotient by the ideal generated by $h^{\frac{r-1}{2}}$.*

This follows immediately from the fact that $h^m<k^{2m}>_r \equiv h^m<k^{2m}>_\infty$ modulo $h^{\frac{r-1}{2}}$ for $m \leq \frac{r-1}{2}$, which is Lemma 8.2 of [O1].

Define

$$Q_{i,f} = C_f \frac{(q+1)^i}{h^{i+1}} \left[\frac{<(1-hx)^{i-k}(q^k-1)>_\infty}{(1+x)^{i+1}} \right]_{x^i}, \tag{3.1.3}$$

for any non-zero integer f and non-negative integer i. As a corollary of the previous lemma, we have

$$h^i Q_{i,f} \equiv h^i Q_{i,f}^{(r)} \qquad \text{modulo } h^{\frac{r-1}{2}}$$

for $i < \frac{r-1}{2}$, where the left hand side is in $\mathbf{Q}[[h]]$ and the right hand side in $\mathbf{Z}_r[h]/(\phi_r(1+h))$. To see this, first note that the q^k term in the expectation value can be transformed by completing the square in the exponent of q and then shifting k accordingly. This procedure is valid with respect to $<\cdot>_\infty$, while it introduces an extra term which is a multiple of $h^{-i}rG(q)^{-1}$ in the case of $<\cdot>_r$ due to the fact that the functions of k involved in the expectation value are not invariant under $k \longrightarrow k+r$. This process results in identical expressions from $Q_{i,f}$ and $Q^{(r)}_{i,f}$, apart from the term $<(1-hx)^{-k}>$ which must still be interpreted in the two different ways. The last lemma then completes the proof when it is noted that the only relevant coefficients of powers of x are those up to x^i.

From the definition (3.1.3), it is seen that $h^iQ_{i,f} \in \mathbf{Q}[[h]]$. However, we know that $Q^{(r)}_{i,f} \in \mathbf{Z}_r[q]$ and from above $h^iQ_{i,f} \equiv h^iQ^{(r)}_{i,f}$ modulo $h^{\frac{r-1}{2}}$; together they imply that $Q_{i,f} \in \mathbf{Q}[[h]]$ with the first $\frac{r-1}{2} - i$ coefficients lying in $\mathbf{Z}_r$. It now follows immediately from (2.3.1) that $\left(\frac{H}{r}\right)\tau'_r(M)$ is congruent modulo $h^{\frac{r-1}{2}}$ to an expression obtained by replacing $Q^{(r)}_{i,f}$ by $Q_{i,f}$ everywhere.

Theorem 3.1.4 *Ohtsuki's invariant of the 3-manifold, M, obtained by integer surgery around the algebraically split link L whose components, L_c, have framing numbers f_c is,*

$$\tau(M) = \sum_{\mathbf{i}=0}^{\infty} \frac{\Phi(L^{\mathbf{i}})}{(q-1)^{\mathbf{i}}} \prod_{c=1}^{|L|} Q_{i_c,f_c}\ , \tag{3.1.5}$$

as an element of $\mathbf{Q}[[h]]$, where $Q_{i,f} \in \mathbf{Q}[[h]]$ is given by (3.1.3). Note that since $h^{-\mathbf{i}}\Phi(L^{\mathbf{i}}) \in h^{\max(\mathbf{i})}\mathbf{Z}[h]$, the coefficients of powers of h in the sum for $\tau(M)$ above are finite combinations.

3.2 Properties of $Q_{i,f}$

By completing the square and shifting k, one obtains that $<q^{ak}>_\infty \equiv q^{-a^2/f}$. Applying this separately to the two expontials in k appearing in (3.1.3), it follows that

$$Q_{i,f} = C_f \frac{(q+1)^i}{h^{i+1}} \left[\frac{(1-hx)^i}{(1+x)^{i+1}} \left(q^{-\frac{1}{f}}(1-hx)^{\frac{2}{f}} - 1 \right) \mathrm{e}^{-\frac{\ln^2(1-hx)}{f\ln q}} \right]_{x^i}, \tag{3.2.1}$$

where all exponentials are evaluated in terms of power series. It follows from this form that $Q_{i,f}$ is a polynomial in $\frac{h}{\ln q}$ of degree $\left[\frac{i}{2}\right]$, whose coefficients are rational functions of $q^{\frac{1}{f}}$. Indeed the coefficients in $h^iQ_{i,f}$ are of the form $P(h) + Q(h)q^{-\frac{1}{f}}$ where $P, Q \in \mathbf{Q}[h]$.

Putting $Q_{i,f} = C_f q^{-\frac{1}{f}} \frac{(q+1)^i}{h^{i+1}} \overline{Q}_{i,f}$, explicit computations yield,

$$\overline{Q}_{0,f} = 1 - q^{\frac{1}{f}} \;;$$
$$\overline{Q}_{1,f} = (2+h)(q^{\frac{1}{f}} - 1) - \frac{2h}{f} \;;$$
$$\overline{Q}_{2,f} = \left(\frac{h^2}{f \ln q} - h^2 - 6h - 6\right)(q^{\frac{1}{f}} - 1) + \left(\frac{2}{f^2} + \frac{3}{f}\right)h^2 + \frac{6h}{f} \;;$$
$$\overline{Q}_{3,f} = \frac{2h^2}{f \ln q}\left(\frac{h}{f} - (2+h)(q^{\frac{1}{f}} - 1)\right) + (h^3 + 12h^2 + 30h + 20)(q^{\frac{1}{f}} - 1)$$
$$- \left(\frac{4}{3f^3} + \frac{4}{f^2} + \frac{11}{3f}\right)h^3 - \left(\frac{8}{f^2} + \frac{20}{f}\right)h^2 - \frac{20}{f}h \,.$$

It is always true that $C_{-f}(q^{-1}) = -q^{-1}C_f(q)$ and $Q_{i,-f}(q^{-1}) = (-q)^{-i}Q_{i,f}(q)$. The same transformation laws hold for C'_f and $Q^{(r)}_{i,f}$, while $\overline{Q}_{i,-f}(q^{-1}) = q^{-i}\overline{Q}_{i,f}(q)$.

When $f = \pm 1$, as occurs when we are considering only integer homology spheres, these formulae simplify considerably. In particular, $C_1 = -q$ and $C_{-1} = 1$ giving

$$Q_{0,f} = 1 \;;$$
$$Q_{1,f} = -f(q+1) = -f(2+h) \;;$$
$$Q_{2,1} = (q+1)^2\left(1 + \frac{1}{h} - \frac{1}{\ln q}\right) = 2 + \frac{7}{3}h + \frac{2}{3}h^2 + \frac{1}{45}h^3 + \cdots \;;$$
$$Q_{3,1} = q(q+1)^3\left(\frac{2}{h \ln q} - \frac{1}{h} - \frac{2}{h^2}\right) = -\frac{4}{3} + O(h) \,.$$

These results will be used in the next section. Note that the coefficients in the expression for $Q_{i,1}$ in terms of $\ln q$ and h are not necessarily integers, as can be checked by computing $Q_{4,1}$.

Re-writing (3.2.1) by introducing the parameter $y = -\log_q(1 - hx)$, which agrees with x to first order in h, gives the simpler-looking form,

$$Q_{i,f} = C_f(1+q)^i \left[q^y \frac{q^{-\frac{(y+1)^2}{f}} - q^{-\frac{y^2}{f}}}{(q^{y+1} - 1)^{i+1}} \right]_{x^i} . \qquad (3.2.2)$$

Proposition 3.2.3 *For an arbitrary non-zero integer f and non-negative integer i, we have $Q_{i,f} \in \mathbf{Q}[(\ln q)^{-1}, h, h^{-1}]$. Furthermore, $Q_{i,f}$ is an entire function of $\ln q$, so that it may be considered as an element of $\mathbf{Q}[[h]]$.*

Since the forms for $Q^{(r)}_{i,f}$ and $Q_{i,f}$ are identical except for the occurrence of a different expectation value, we may reverse the logic that led from (2.2.2) to (2.2.5), and then starting from (3.1.3) this will lead to

$$Q_{i,f} = C_f \frac{(q+1)^i}{2q^{\frac{1}{2}}} \frac{\int_{-\infty}^{\infty} [k]_q q^{\frac{fk^2}{4}} G_i(k,q)\, dk}{\int_{-\infty}^{\infty} q^{\frac{fk^2}{4}}\, dk} ,$$

where $G_i(k,q) = \sum_{n=1}^{i+1} (A_{i,n} q^{\frac{k}{2}} + B_{i,n} q^{-\frac{k}{2}}) \binom{n+k-1}{n-1}$ is a function of k which is a sum of polynomials times exponentials. Considering $G_i(k,q)$ now as a function of q, it may be expanded in powers of $[2]_q = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ (around $q=-1$) and this enables $Q_{i,f}$ to be written as

$$Q_{i,f} = \frac{hC_f}{2q} \frac{\int\limits_{-\infty}^{\infty} [k]_q q^{\frac{fk^2}{4}} \sum\limits_{j=i}^{\infty} \binom{j}{i} [2]_q^j H_j(k)\, dk}{\int_{-\infty}^{\infty} q^{\frac{fk^2}{4}}\, dk}$$

where $H_j(k) = \binom{\frac{k+j-1}{2}}{j} \sin\left(\frac{\pi}{2}(k-j)\right)$ are the coefficients of $[k]_q$ when expanded in powers of $[2]_q$.

3.3 Formal interpretation of $\tau(M)$ as an expectation value

Putting the form just found for $Q_{i,f}$ into the expression for $\tau(M)$, (3.1.5) gives

$$\tau(M) = \left(\prod_c \frac{hC_{f_c}}{2q}\right) \frac{\int [\mathbf{k}]_q q^{\frac{\mathbf{f}\mathbf{k}^2}{4}} \sum_{\mathbf{j}=0}^{\infty} H_{\mathbf{j}}(\mathbf{k}) {<} L^{\mathbf{j}} {>}\, d\mathbf{k}}{\int q^{\frac{\mathbf{f}\mathbf{k}^2}{4}}\, d\mathbf{k}}$$

where the integrals here are over variables k_c, one for each component of the link and the usual multi-index notation is being used. We have here interchanged the summations over $\mathbf{i}$ and $\mathbf{j}$, there being for each $\mathbf{j}$ only a finite number of relevant $\mathbf{i}$'s and the associated contributions from $\Phi(L^{\mathbf{i}})$ collapsing to give $<L^{\mathbf{j}}>$. This interchange thus results in a purely formal expression, whose form is very close to that for τ'_r, in which a sum, over colours $\mathbf{k}$ on components, is replaced by an integral. Note that for integral $\mathbf{k}$, the sum over $\mathbf{j}$ appearing in the last expression for $\tau(M)$ is finite and gives nothing but the evaluation of the bracket on L with colour $\mathbf{k}$, where we use the dimension of the representation as the colour label (see [KL] for combinatorial realisations of this bracket). When $\mathbf{k}$ is not integer, it should be compared with J_∞ appearing in [R1], [R2] and [LR]. The meaning of such expressions as analytic functions of q will be investigated further in [L3].

3.4 Computation of some coefficients in $\tau(M)$

In this section we compute the coefficients, $\tau_i(M)$, of h^i in $\tau(M)$ for $i = 0, 1, 2$ and 3 and verify some divisibility conjectures about them in the case of integer homology spheres, where $f_c = \pm 1$.

For general f,

$$Q_{0,f} = \frac{1}{|f|} + \left(\frac{3}{4f} - \frac{\operatorname{sgn}(f)}{4} - \frac{\operatorname{sgn}(f)}{2f^2}\right) h + O(h^2)$$
$$Q_{1,f} = -\frac{2\operatorname{sgn}(f)}{f^2} + O(h)$$

From (3.1.5), the only term contributing to $\tau_0(M)$ is that with $\mathbf{i} = \mathbf{0}$, which gives

$$\tau_0(M) = \prod_c \frac{1}{|f_c|} = \frac{1}{H}\,.$$

Contributions to $\tau_1(M)$ come from all $\mathbf{i} \le 1$, giving

$$H\tau_1(M) = \sum_c \left(\frac{3\operatorname{sgn}(f)}{4} - \frac{f}{4} - \frac{1}{2f}\right) + \sum_{l\subseteq L} [\Phi(l)]_{h^{|l|+1}} \prod_{c\in l}\left(-\frac{2}{f_c}\right)\,,$$

as obtained in [M2] and identified there as $6\lambda(M)$ where $\lambda(M)$ is the Casson-Walker invariant of M (see [Wa]) in the normalisation of Casson.

In order to simplify the formulae, we now restrict to integral homology spheres, although it should be noted that the computations are no harder in the general case. Then we get, using the computations of $Q_{i,f}$ from §3.2,

$$\tau_0(M) = 1$$
$$\tau_1(M) = \sum_{l_1} (-2f)^{l_1}\phi(l_1)$$
$$\tau_2(M) = \sum_{l_1,l_2} (-2f)^{l_1}2^{l_2}\phi(l_1,l_2) + \sum_{l_1}\frac{|l_1|}{2}(-2f)^{l_1}\phi(l_1)$$
$$\tau_3(M) = \sum_{l_1,l_2,l_3} (-2f)^{l_1}2^{l_2}\left(-\frac{4f}{3}\right)^{l_3}\phi(l_1,l_2,l_3)$$
$$+ \sum_{l_1,l_2}(-2f)^{l_1}2^{l_2}\left(\frac{|l_1|}{2} + \sum_{l_2}(1+\frac{f}{6})\right)\phi(l_1,l_2)$$

Here all the sums are over disjoint sublinks l_j, while $\phi(l_1,\ldots,l_m)$ denotes the coefficient of $h^{m+\sum j|l_j|}$ in $\Phi[l_1 \cup \cdots \cup l_m^m]$, the Φ polynomial evaluated on the link obtained from L by selectively cabling components. An expression of the form x^l denotes the product of evaluations of x over each component of the link l. As noted in [M1] (Proposition 3.1), $\phi(L)$ is just $6(-2)^{-|L|}$ times the coefficient of $z^{|L|+1}$ in the Conway polynomial of L. The formula for $\tau_2(M)$ is also obtained in [LW].

In the case of surgery around a knot, the coefficients in $\tau(M)$ reduce to combinations of $a_{i,j} = [\Phi(L^i)]_{h^j}$. When L is replaced by its mirror image, $\Phi(L^i)$ transforms according to $q \longrightarrow q^{-1}$. To obtain coefficients $\alpha_{i,j}$ which multiply by $(-1)^j$ under the taking of the mirror image, it is necessary to add to the $a_{i,j}$ certain combinations of lower order terms,

$$\alpha_{1,2} = a_{1,2}, \qquad \alpha_{1,3} = a_{1,3} + a_{1,2}, \qquad \alpha_{1,4} = a_{1,4} + {}^3\!/\!{}_2 a_{1,3},$$
$$\alpha_{2,4} = a_{2,4}, \qquad \alpha_{2,5} = a_{2,5} + 2a_{2,4}, \qquad \alpha_{3,6} = a_{3,6}.$$

As in [LR], we consider the normalised version of $\tau(M)$ in which the coefficient of h vanishes, namely $q^{-6\lambda(M)}\tau(M)$. This has coefficients L_i, with $L_0(M) = 1$, $L_1(M) = 0$ and

$$\begin{aligned} L_2(M) &= -2f\alpha_{1,3} + 2\alpha_{2,4} - 2\alpha_{1,2}^2 , \\ L_2(M) + L_3(M) &= -{}^4\!/\!{}_3\alpha_{1,2} f(2\alpha_{1,2}^2 + 1) - 4\alpha_{1,2}(\alpha_{1,3} + f\alpha_{2,4}) - 2f\alpha_{1,4} \\ &\qquad + {}^1\!/\!{}_3 f(\alpha_{2,4} - 4\alpha_{3,6}) + 2\alpha_{2,5} . \end{aligned}$$

with the property that L_2 is invariant while $L_2 + L_3$ changes sign, under reversal of orientation on M.

The coefficient $a_{i,j}$ is an integer-valued Vassiliev invariant of order j, considered as an invariant of the knot L. In order to understand the divisibility properties better, it is convenient to shift from $\alpha_{i,j}$ to $d_{m,n}$, the coefficients in the Melvin-Morton-Rozansky expansion of the coloured Jones polynomial,

$$<L,k> = [k]_q \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} d_{m,n}(L)(q^{\frac{k}{2}} - q^{-\frac{k}{2}})^{2m} h^n ,$$

where $<L,k>$ denotes the evaluation of the coloured Jones polynomial of the knot L in which the colour label k is the dimension of the representation. It is conjectured [R3] that $d_{m,n}(L)$, which are always Vassiliev invariants of L of order $2m + n$, are actually always integer-valued. In terms of these coefficients, the $\alpha_{i,j}$ may be computed,

$$\begin{aligned} \alpha_{1,2} &= 3d_{1,0}, & \alpha_{2,4} &= 3d_{1,0} + 30d_{2,0}, \\ \alpha_{1,3} &= 3d_{1,1}, & \alpha_{2,5} &= 3d_{1,1} + 30d_{2,1}, \\ \alpha_{1,4} &= -{}^1\!/\!{}_2 d_{1,0} + {}^3\!/\!{}_2 d_{1,1} + 3d_{1,2} + 15d_{2,0}, & \alpha_{3,6} &= 3d_{1,0} + 1230d_{2,0} + 630d_{3,0}. \end{aligned}$$

In terms of these coefficients, $\tau_1(M) = -6fd_{1,0}$, that is $-d_{1,0}(L)$ is the Casson invariant of the homology sphere obtained by surgery around the knot L. Furthermore,

$$\begin{aligned} L_2 &= -6fd_{1,1} + 6d_{1,0} + 60d_{2,0} - 18d_{1,0}^2, \\ L_2 + L_3 &= -6fd_{1,0}(12d_{1,0}^2 + 6d_{1,0} + 1 + 6fd_{1,1} + 60d_{2,0}) \\ &\quad - 180fd_{2,0} - 840fd_{3,0} + 60d_{2,1} - 6fd'_{1,2} + 6d_{1,1}\,, \end{aligned}$$

where we have here introduced $d'_{1,2} = d_{1,2} + \frac{1}{2}d_{1,1}$, as a more natural invariant than $d_{1,2}$, in the sense that under the operation of taking the mirror image, $d_{1,1}$ and $d_{2,1}$ change sign, while $d_{1,0}$, $d_{2,0}$, $d_{3,0}$ and $d'_{1,2}$ all remain invariant.

It follows that $L_2 \in 6\mathbf{Z}$; see also [LW], where it was shown that $\tau_2(M) \in 3\mathbf{Z}$. Furthermore, assuming $d'_{1,2}$ is integer and of the same parity as $d_{1,0}$, $L_2 + L_3$ is divisible by 12. One way of verifying the conjecture of [LR] that $L_2 + L_3 \in 48\mathbf{Z}$, at least for the case of surgery around a knot, it to explicitly compute the weight system for this Vassiliev invariant of order 6.

4: Relations between $\tau'_r(M)$ and $\tau(M)$

For Seifert manifolds [LR] and manifolds obtained by surgery around a knot [R2], it is known that $\tau(M) \in \mathbf{Z}\left[\frac{1}{H}\right][[h]]$ and that its value is $\left(\frac{H}{r}\right)\tau'_r(M) \in \mathbf{Z}[q]$, in the sense of r-adic convergence. Since we have not shown in general that $\tau(M) \in \mathbf{Z}\left[\frac{1}{H}\right][[h]]$, it is impossible to talk of such r-adic convergence. However, it is still possible to give a very tight relation between $\tau(M)$ and $\tau'_r(M)$.

Fix a prime r (odd and not dividing f in our case). Consider the $A = \mathbf{Q}[[h]]$ as a module over the ring $Z = \mathbf{Z}_r[[h]]$. Let A'_r denote the submodule of the Laurant series in h with rational coefficients generated by $\left\{(h \ln q)^{-n} \;\middle|\; n \geq 0\right\}$. Put $A_r = A'_r \cap A$; this is the submodule of A generated by $Q_{i,f}$ for all i. Also let B_r be the quotient of A by the ideal generated by $\phi_r(1+h)$; this contains the cyclotomic ring as a subring. Then there is a unique linear map defined by

$$\begin{aligned} \theta_r \colon A_r &\longrightarrow B_r \\ Q_{i,f} &\longmapsto Q_{i,f}^{(r)}\,. \end{aligned}$$

This map may be thought of as a restriction of a map $A'_r \longrightarrow B'_r$ defined by mapping $<k^{2l}>_\infty \longmapsto <k^{2l}>_r$, where B'_r is the Laurant series analogue of B_r. By the constructionof θ_r, we deduce the following.

Theorem 4.0.1 *If M is obtained as surgery around an algebraically split link L in S^3, and r is an odd prime not dividing $H = |H^1(M,\mathbf{Z})|$, then*

$$\left(\frac{H}{r}\right)\tau'_r(M) = \theta_r(\tau(M))\,.$$

Since H, τ and τ'_r are multiplicative invariants under connected sum, and any rational homology sphere, when taken in a connected sum with a suitable collection of Lens spaces $L(n,1)$, can be expressed as surgery around an ASL, the theorem also holds for all $\mathbf{Q}$HS. Here we use the fact that, although θ_r is not a homomorphism,

$$\theta_r(xy)=\theta_r(x)\theta_r(y)$$

whenever $x\in\mathbf{Z}_r[[h]]$ as is true for the value of $\tau(L(n,1))$; see [J]. Whenever integrality of $\tau(M)$ is known, meaning that $\tau(M)\in Z$, this gives r-adic convergence.

5: Summary

In this note, we have seen that the $SU(2)$ Witten–Reshetikhin–Turaev invariant, τ'_r, at roots of unity, and the Ohtsuki power series invariant, τ, may be written in comparable forms,

$$\left(\frac{H}{r}\right)\tau'_r(M)=\sum_{\mathbf{i}=0}^{\frac{r-3}{2}}\frac{\Phi(L^{\mathbf{i}})}{(q-1)^{\mathbf{i}}}\prod_{c=1}^{|L|}Q^{(r)}_{i_c,f_c}\ ,$$

and

$$\tau(M)=\sum_{\mathbf{i}=0}^{\infty}\frac{\Phi(L^{\mathbf{i}})}{(q-1)^{\mathbf{i}}}\prod_{c=1}^{|L|}Q_{i_c,f_c}\ .$$

The quantities $Q_{i,f}$ and $Q^{(r)}_{i,f}$ are functions of q independent of the manifold under consideration, and may themselves be written in identical forms, as the expection value of a certain function of a parameter k, which is viewed in one case as being an integer-valued random variable and in the other case as a continuous random variable, but in both cases with a Gaussian distribution. This observation allows the comparison of $\tau_r(M)$ and $\tau(M)$, as well as concrete computations of $\tau(M)$ to be carried out.

Another consequence of the closeness of the shapes of expressions for $\tau'_r(M)$ and $\tau(M)$ is that interpretations of the WRT invariants should have analogues for $\tau(M)$. In particular, $\tau'_r(M)$ has a formulation as the partition function of a state model (see for example [KL]), or as a weighted sum of bracket polynomials of a link with varying colours (representations of $SU(2)$) on the components. The analogue for $\tau(M)$ would seem to be a weighted *integral* with the colour variable now being allowed to vary continuously, and the integrand being an analogue of the coloured bracket polynomial extended to non-integer colours; see §3.3. Such extensions to non-integral colours have been observed previously in special cases (see [R1], [R2] and [LR]), and would seem to indicate the underlying presence of a representation theory other than that of $SU(2)$, namely that of $SL(2;\mathbf{R})$, see [L3].

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