Holomorphic invariants of 3–manifolds and q-gamma functions. [1]

Ruth Lawrence

Department of Mathematics
University of Michigan
Ann Arbor, Michigan

Abstract. We give a procedure which it is conjectured will construct an invariant, $Z^0(M)$, of rational homology 3-spheres, M, which is defined as a holomorphic function of a parameter q. It is defined in a way similar to the $\mathfrak{sl}_2$ Witten-Reshetikhin-Turaev invariant, $Z_K(M)$, an invariant defined only at roots of unity, q with $q^K\!=\!1$. It is believed that $Z^0(M)$ is related to the trivial connection contribution to the stationary phase expansion of the Feynman integral form for $Z_K(M)$ and that they share many other properties, which will be investigated further in a forthcoming paper.

1: Introduction

Suppose that M is a compact oriented 3–manifold without boundary. For any Lie algebra, $\mathfrak{g}$, and integral level, k, there is defined an invariant, $Z_{k+\check{c}_\mathfrak{g}}(M,L)$, of embeddings of links L in M, known as the Witten-Reshetikhin-Turaev invariant (see [W], [RT]). It is known that for links in S^3, $Z_K(S^3,L)$ is a polynomial in $q=\exp\frac{2\pi i}{K}$, namely the generalised Jones polynomial of the link L.

Now assume that M is a rational homology sphere, with $H=|H^1(M,\mathbf{Z})|$. In the normalisation for which the invariant for S^3 is 1, denote the invariant for the pair $(M,\emptyset)$, as an algebraic function of q at $K^{\rm th}$ roots of unity, by $Z_K(M)$. For a rational homology sphere M, the $SU(2)$-invariant $Z'_K(M)$ (see [KM2]) lies in $\mathbf{Z}[q]$ by [M1], so that for some $a_{m,K}(M)\in\mathbf{Z}$, one has

$$Z_K(M)=\sum_{m=0}^{\infty}a_{m,K}(M)h^m$$

where $q=1+h$. Although the $a_{m,K}$ are not uniquely determined, if K is prime then by [O1] and [M2], there exist rational numbers $\lambda_m(M)$ such that,

$$a_{m,K}(M)\equiv\lambda_m(M)$$

as elements of $\mathbf{Z}_K$ for all $K>2m+3$, while $\lambda_0(M)=H^{-1}$ and $\lambda_1(M)=6H^{-1}\lambda(M)$ where $\lambda(M)$ denotes the $\big(SU(2)$–$\big)$Casson-Walker invariant of M in Casson's normalisation. As a result, one may define a formal power series

$$Z_\infty(M)=\sum_{m=0}^{\infty}\lambda_m h^m\ ,$$

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with rational coefficients, which is an invariant of integral homology 3–spheres, M, and is expected to be the asymptotic expansion of the trivial connection contribution to $Z_K(M)$.

In previous work of the author with L. Rozansky [LR1] (see also [L1], [L2]), it has been seen that for Seifert fibred manifolds, $Z_K(M)$ can be written as a sum of terms, each of which define holomorphic functions of q. One of these terms has an asymptotic expansion which is $Z_\infty(M)$, and which converges K-adically to $Z_K(M)$ at $K^{\rm th}$ roots of unity. The present paper aims to define a function $Z^0(M)$ of q, for general rational homology spheres, which is an invariant and which it is hoped will satisfy similar properties relating it to $Z_K(M)$ as for the case of Seifert manifolds. This will be investigated further in [LR2].

An outline of the present paper is as follows. In §2.1, the basic notation of q-numbers, q-factorials, $q6j$-symbols and q-Clebsch Gordon consraints is given for q a root of unity. The theory of q-gamma functions, both at and away from roots of unity, is developed in §2.3 and §2.4. This leads in §2.5 to a function $\Gamma(z;K)$ of two variables, which is an extension of the q-gamma function defined for rational K, obtained for non-real K as a quotient of two q-gamma functions. This function provides appropriate generalisations of q-symbols for arbitrary q and non-integer arguments. In §3, a reformulation of the Reshetikhin-Turaev state model for $Z_K(M)$ is given in which all weights involve only products of q-factorials; a version of the Symmetry Principle of [KM2] is also easily derived. Finally in §4, the extended q-symbols are applied to give invariants defined away from roots of unity.

2: q-NUMBERS AND GAMMA FUNCTIONS

2.1 The q-symbols at roots of unity

Throughout this paper, $q=\mathbf{e}^{\frac{2\pi i}{K}}$. In the discussion of 'traditional' Witten-Reshetikhin-Turaev invariants of 3-manifolds of §3.1, K will be a positive integer so that q is a root of unity, while our aim is to produce a formulation valid away from roots of unity. This section contains notation used in §3.1, which is valid only for $K\in\mathbf{N}$, while in §2.5, much of it will be extended to the case of arbitrary K, using the properties of the q-gamma function. See also [KL2].

Set $I=\{0,1,\ldots,K-2\}$. Let $A=q^{1/4}$ be a $4K^{\rm th}$ root of unity. Define the q–numbers by

$$[n]_q=\frac{A^{2n}-A^{-2n}}{A^2-A^{-2}}\,.$$

The q–factorials are defined by $[n]!_q=\prod_{i=1}^n[i]_q$. The q–versions of multinomial coefficients are defined by $\begin{pmatrix} n \\ n_1\ n_2\ldots n_k\end{pmatrix}_q=\frac{[n]!_q}{[n_1]!_q[n_2]!_q\ldots[n_k]!_q}$ where $\sum\limits_{i=1}^k n_i=n$. A triple of non-negative integers (a,b,c) will be said to be q–*admissible* when $b+c-a$, $c+a-b$, $a+b-c$ and $2K-4-a-b-c$ are all positive and even.

If a is a non-negative integer, set $\Delta_a = (-1)^a[a+1]_q$. If (a,b,c) is a q–admissible triple, set

$$\lambda_c^{a,b} = (-1)^{(a+b-c)/2} A^{[a(a+2)+b(b+2)-c(c+2)]/2} ,$$
$$\theta(a,b,c) = (-1)^x \frac{[x+1]!_q[x-a]!_q[x-b]!_q[x-c]!_q}{[a]!_q[b]!_q[c]!_q} ,$$

where $2x = a+b+c$. The latter expression is known as a *θ–net*.

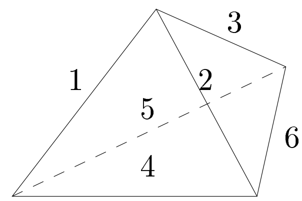

Figure 2: Tetrahedral net

Suppose that $\{a_i\}_{i=1}^6$ are non-negative integers such that (a_i,a_j,a_k) is a q–admissible triple for each $(i,j,k) \in S$ where $S = \{(1,2,3),(1,4,5),(2,4,6),(3,5,6)\}$. If the edges of a tetrahedron are numbered 1 to 6 as shown in Figure 2 then the elements of S are precisely those triples of numbers whose associated edges share a common vertex; that is, the elements of S index the vertices of the tetrahedron. Considering the integer a_i to be placed on the i^{th} edge, it is given that those triples of integers on edges emerging from any vertex form a q–admissible triple. Define the associated tetrahedral net (a variant of the quantum $6j$-symbol) to be

$$\begin{bmatrix} a_4 & a_5 & a_6 \\ a_3 & a_2 & a_1 \end{bmatrix} = \frac{\prod_{v,e}[y_e - x_v]!_q}{\prod_{i=1}^6 [a_i]!_q} \sum_{s=\max(x_v)}^{\min(y_e)} \frac{(-1)^s[s+1]!_q}{\prod_v [s-x_v]!_q \prod_e [y_e - s]!_q} . \qquad (2.1.1)$$

Here $2x_v = \sum_{i\in v} a_i$ for each $v \in S$ while if e denotes a pair of opposite edges, of which there are three, then $2y_e = \sum_{i\notin e} a_i$.

It is easily verified that when all indices are elements of the set I, the quantity Δ_a along with the values of θ-nets and tetrahedral nets are all non-zero. The θ-net depends in a totally symmetric way on the three indices, while the tetrahedral net exhibits the S_4 symmetry of the tetrahedron. Special values of these symbols are

$$\lambda_a^{a,0} = 1, \qquad \theta(a,a,0) = \Delta_a, \qquad \begin{bmatrix} 0 & a & a \\ i & a & a \end{bmatrix} = \theta(a,a,i) .$$

2.2 Gamma functions

We shall use $\mathbf{N}$ to denote the natural numbers while $\mathbf{Z}^+$ and $\mathbf{Z}^-$ denote non-negative and non-positive integers, respectively. The usual gamma function is characterised by the properties

(i) $\Gamma(z+1) = z\Gamma(z)$,

(ii) $\Gamma(z)$ is meromorphic with no zeros and simple poles at $\mathbf{Z}^-$,

(iii) $\Gamma(1) = 1$,

(iv) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$.

It has an infinite product expansion, due to Euler, with

$$\Gamma(z) = z^{-1}\,\mathbf{e}^{-\gamma z} \prod_{n=1}^{\infty} \left(\frac{\mathbf{e}^{\frac{z}{n}}}{1+\frac{z}{n}} \right) ,$$

where $\gamma = \lim\limits_{N\to\infty} \left(\sum\limits_{n=1}^{N} \frac{1}{n} - \ln N \right)$ is Euler's constant. Its logarithmic derivative is the function

$$F(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n} \right) .$$

The following properties will be used repeatedly in what follows.

(a) $F(-z) - F(z) - z^{-1} = \pi \cot \pi z = \frac{1}{z} + \sum\limits_{n\in\mathbf{N}} \frac{2z}{z^2-n^2}$,

(b) $F'(z)+F'(-z)-z^{-2} = \pi^2 \mathrm{cosec}^2 \pi z = \frac{1}{z^2} + \sum\limits_{n=1}^{\infty} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right) = \sum\limits_{n\in\mathbf{Z}} \frac{1}{(z+n)^2}$.

(c) $\sum\limits_{n\in\mathbf{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$,

(d) $-z^2\Gamma(z)\Gamma(-z) = \prod\limits_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)^{-1} = \frac{\pi z}{\sin \pi z}$.

A generalisation of $\Gamma(z)$ to a q-gamma function appeared in the work of Jackson [J]. Using a Jackson integral in place of a Riemann integral and a q-version of the exponential function, he defined a function $\Gamma_q(z)$ for $|q| < 1$, so as to satisfy a modified version of (i) above, with $\Gamma_q(z+1) = [z]\Gamma_q(z)$. Here $[z]$ is a q-number, which Jackson defined in the asymmetric way $\frac{q^z-1}{q-1}$.

We begin by constructing a function, $\widetilde{\Gamma}(z;K)$, which is slightly different from Jackson's, and is characterised by properties similar to (i)–(iv) above. This can only be done when $K \in \mathbf{C}\backslash(\mathbf{R}\backslash\mathbf{Q})$, while the cases in which $K \in \mathbf{Q}$ (that is q is a root of unity) and $K \in \mathbf{C}\backslash\mathbf{R}$ (so that $|q| \neq 1$) must be dealt with separately.

2.3 The q-gamma function for $K \notin \mathbf{R}$

Let ϵ denote the sign of $\Im(K)$, so that $K \in \mathbf{H}^\epsilon$, the upper or lower half plane. The characterising properties of $\widetilde{\Gamma}(z;K)$ will be

(i) $\widetilde{\Gamma}(z+1;K) = [z]_q \widetilde{\Gamma}(z;K)$,

(ii) $\widetilde{\Gamma}(z;K)$ is a meromorphic function of z, with no zeros and simple poles at $\mathbf{Z}^- + K\mathbf{Z}$,

(iii) $\widetilde{\Gamma}(1;K) = 1$.

Observe that since $[z]_q$ vanishes precisely when $z \in K\mathbf{Z}$, assuming that $\widetilde{\Gamma}(z;K)$ has no zeros, the positions of the poles are forced by the first condition. Note that $\widetilde{\Gamma}(z;K) = \widetilde{\Gamma}(z;-K)$. From (i) and (iii), it can be deduced that the residue of the pole at $z=0$ is $\frac{K}{\pi}\sin\frac{\pi}{K}$. Any function satisfying (ii) may be written as the exponential of an entire function times

$$G(z) = \frac{1}{z}\prod_{s=0}^{\infty}\prod_{\substack{n=-\infty \\ (n,s)\neq \mathbf{0}}}^{\infty}\left(1-\frac{z}{nK-s}\right)^{-1}\mathbf{e}^{-\frac{z}{nK-s}-\frac{z^2}{2(nK-s)^2}},$$

a product which is absolutely convergent. The function $\frac{G(z+1)}{G(z)}$ has zeros at $K\mathbf{Z}$ and no poles and thus, for some entire function $g(z)$,

$$\frac{G(z+1)}{G(z)} = \mathbf{e}^{g(z)}\sin\frac{\pi z}{K}.$$

Taking logarithmic derivatives of both sides yields

$$\begin{aligned} g'(z) &= -\frac{\pi}{K}\cot\frac{\pi z}{K}+\frac{1}{z}-\frac{1}{z+1}+\sum_{s=0}^{\infty}\sum_{\substack{n=-\infty \\ (n,s)\neq\mathbf{0}}}^{\infty}\left(\frac{1}{nK-s-z-1}-\frac{1}{nK-s-z}-\frac{1}{(nK-s)^2}\right) \\ &= -\frac{\pi^2}{3K^2}-\lim_{S\to\infty}\left(\frac{\pi}{K}\cot\frac{\pi(z+S+1)}{K}+\sum_{s=1}^{S}\frac{\pi^2}{K^2}\operatorname{cosec}^2\frac{\pi s}{K}\right) \end{aligned}$$

where the sum has been evaluated as a limit of partial sums $0 \le s \le S$ and properties (a)–(c) have been used. Observe that

$$\sum_{s=1}^{\infty}\frac{\pi^2}{K^2}\operatorname{cosec}^2\frac{\pi s}{K} = \sum_{s\in\mathbf{N}}\sum_{n\in\mathbf{Z}}\frac{1}{(s+nK)^2} = \frac{\pi^2}{6}E_2(K)+\frac{\pi\epsilon}{iK}-\frac{\pi^2}{6K^2}$$

so that $g'(z) = -\frac{\pi^2}{6}\left(E_2(K)+\frac{1}{K^2}\right)$. This determines $g(z)$ up to a constant. Observe next that in the limit $z\to 0$, $zG(z)\to 1$, so that $G(1) = \frac{\pi}{K}\mathbf{e}^{g(0)}$ and

$$\begin{aligned} g(0) &= \lim_{S\to\infty}\sum_{s=1}^{S}\left(\frac{\pi s}{K}\cot\frac{\pi s}{K}-\frac{\pi^2 s^2}{2K^2}\operatorname{cosec}^2\frac{\pi s}{K}\right)-\ln\sin\frac{\pi(S+1)}{K} \\ &= -\frac{\pi^2}{12K^2}-\frac{\pi^2}{12}E_2(K)+\frac{\pi i\epsilon}{2}(1-K^{-1})+\ln 2+\frac{2\pi i\epsilon}{K}\sum_{s=1}^{\infty}(q^{\epsilon s}-1)^{-1}. \end{aligned}$$

This enables the unique function $\widetilde{\Gamma}(z;K)$ satisfying all the characteristics (i)–(iii) to be written down,

$$\widetilde{\Gamma}(z;K)=\frac{K}{2\pi z}\left(2\sin\frac{\pi}{K}\right)^{1-z}\mathbf{e}^{a_K z^2-b_K z}\prod_{s=0}^{\infty}\prod_{\substack{n=-\infty\\(n,s)\neq\mathbf{0}}}^{\infty}\frac{\mathbf{e}^{-\frac{z}{nK-s}-\frac{z^2}{2(nK-s)^2}}}{1-\frac{z}{nK-s}}\,,$$

where

$$a_K=\frac{\pi^2}{12}\left(E_2(K)+\frac{1}{K^2}\right)\,,$$

$$b_K=\frac{\pi i\epsilon}{2}(1-K^{-1})+\frac{2\pi i\epsilon}{K}\sum_{s=1}^{\infty}\frac{1}{q^{\epsilon s}-1}\,.$$

Observe now that $\widetilde{\Gamma}(z;K)\widetilde{\Gamma}(-z;K)$ is a meromorphic function with poles at $\mathbf{Z}+K\mathbf{Z}$, all of which are simple, except for double poles at $K\mathbf{Z}$. Indeed,

$$\widetilde{\Gamma}(z;K)\widetilde{\Gamma}(-z;K)=-\frac{K\left(\sin\frac{\pi}{K}\right)^2}{\sin\pi z\sin\frac{\pi z}{K}}\prod_{a=1}^{\infty}\frac{\sin^2\pi aK}{\sin^2\pi aK-\sin^2\pi z}\,,$$

and hence using (i) that

(iv) $\widetilde{\Gamma}(z;K)\widetilde{\Gamma}(1-z;K)=\frac{K\sin\frac{\pi}{K}}{\sin\pi z}\prod_{a=1}^{\infty}\left(1-\frac{\sin^2\pi z}{\sin^2\pi aK}\right)^{-1}$.

Next observe, from property (ii), that $\frac{\widetilde{\Gamma}(z+K;K)}{\widetilde{\Gamma}(z;K)}$ has no zeros or poles and can therefore be written as $\mathbf{e}^{h(z)}$ for some entire function $h(z)$. Taking logarithmic derivatives yields,

$$h'(z)=\frac{1}{z}-\frac{1}{z+K}+2Ka_K+\sum_{s=0}^{\infty}\sum_{\substack{n\in\mathbf{Z}\\(n,s)\neq\mathbf{0}}}\left(\frac{1}{z-nK+s}-\frac{1}{z+K-nK+s}-\frac{K}{(nK-s)^2}\right)$$

$$=i\pi\epsilon\,.$$

To determine $h(z)$ completely, it is only necessary to evaluate it at $z=0$, where one finds that $\mathbf{e}^{h(0)}$ gives the ratio of the residues of $\widetilde{\Gamma}(z;K)$ at $z=K$ and at $z=0$. Thus

$$\mathbf{e}^{h(0)}=-\left(2\sin\frac{\pi}{K}\right)^{-K}\mathbf{e}^{a_K K^2-b_K K-\frac{3}{2}}\prod_{s=0}^{\infty}\sum_{\substack{n\in\mathbf{Z}\\(n,s)\neq\mathbf{0},(1,0)}}\frac{\mathbf{e}^{-\frac{K}{nK-s}-\frac{K^2}{2(nK-s)^2}}}{1-\frac{K}{nK-s}}$$

$$=-i\epsilon\left(2\sin\frac{\pi}{K}\right)^{-K}$$

and hence

(v) $\widetilde{\Gamma}(z+K;K)=-i\epsilon\left(2\sin\frac{\pi}{K}\right)^{-K}\mathbf{e}^{\pi i\epsilon z}\widetilde{\Gamma}(z;K)$.

2.4 The q-gamma function for $K \in \mathbf{N}$

Although it is possible to work with any rational value of K, for our purposes it will only be necessary to consider integer values and so we restict our attention to $K \in \mathbf{N}$. The characterising properties of $\Gamma(z; K)$ will be

(i) $\Gamma(z+1; K) = [z]_q \Gamma(z; K)$;

(ii) $\Gamma(z; K)$ is a meromorphic function of z, with all its zeroes and poles at integer points on the real line, the order of the zero at $n \in \mathbf{Z}$ being $\left[\frac{n-1}{K}\right]$, which when it is negative represents a pole;

(iii) $\Gamma(1; K) = 1$.

Note that property (ii) is significantly different from the corresponding property in §2.3, and indeed this is a direct consequence of (i) which forces the degree of vanishing of $\Gamma(z; K)$ at $z \in \mathbf{Z}$, to increase by one after each multiple of K. Any function satisfying (ii) may be written as the exponential of an entire function times

$$G(z) = \frac{1}{z} \prod_{\substack{t \in \mathbf{Z} \\ t \neq 0}} \left(\left(1 - \frac{z}{t}\right) \mathbf{e}^{\frac{z}{t} + \frac{z^2}{2t^2}} \right)^{\left[\frac{t-1}{K}\right]},$$

a product which is absolutely convergent. The function $\frac{G(z+1)}{G(z)}$ has zeroes at $K\mathbf{Z}$ and no poles. It can therefore be written as

$$\frac{G(z+1)}{G(z)} = \mathbf{e}^{g(z)} \sin \frac{\pi z}{K},$$

where $g(z)$ is an entire function, as in §2.3. By taking logarithmic derivatives along with an evaluation of $G(1)$, one obtains

$$g(z) = \ln 2 - \frac{1}{K} - \frac{\pi^2}{6}\left(1 + \frac{1}{K^2}\right)\left(z + \frac{1}{2}\right) - \frac{\pi}{K^2} \sum_{s=1}^{K-1} s \cot \frac{\pi s}{K}.$$

This allows an appropriate normalisation of $G(z)$ to be made, so as to make the resulting function satisfy properties (i) and (iii),

$$\Gamma(z; K) = \frac{K}{2\pi z}\left(2 \sin \frac{\pi}{K}\right)^{1-z} \mathbf{e}^{A_K z^2 - B_K z} \prod_{\substack{t \in \mathbf{Z} \\ t \neq 0}} \left(\left(1 - \frac{z}{t}\right) \mathbf{e}^{\frac{z}{t} + \frac{z^2}{2t^2}} \right)^{\left[\frac{t-1}{K}\right]},$$

where

$$A_K = \frac{\pi^2}{12}\left(1 + \frac{1}{K^2}\right),$$

$$B_K = -\frac{1}{K}\left(1 + \sum_{s=1}^{K-1} \frac{\pi s}{K} \cot \frac{\pi s}{K}\right).$$

The function $\Gamma(z;K)\Gamma(1-z;K)$ has poles precisely at $\mathbf{Z}$, all of which are simple and the precise form of the analogue of relation (iv) in §2.3 in this case is

(iv) $\Gamma(z;K)\Gamma(1-z;K)=\frac{K\sin\frac{\pi}{K}}{\sin\pi z}$.

Since K is here an integer, the relation between $\Gamma(z+K;K)$ and $\Gamma(z;K)$ may be derived in this case directly from (ii) giving

(v) $\Gamma(z+K;K)=\frac{2\sin\pi z}{(2\sin\frac{\pi}{K})^K}\Gamma(z;K)$.

2.5 Extensions for general q

In order to compare the two versions of the q-gamma function, $\widetilde{\Gamma}(z;K)$ defined in §2.3 away from the unit circle, and $\Gamma(z;K)$ defined in §2.4 at roots of unity, it is first necessary to realise that they have been normalised in completely different ways. Thus the former has only poles, all of which lie in the half-lattice $\mathbf{Z}^-+K\mathbf{Z}$, while the latter has both poles and zeroes distributed symmetrically in $\mathbf{Z}$. Suppose that $K\in\mathbf{C}\backslash\mathbf{R}$. Define

$$\Gamma(z;K)=\frac{\widetilde{\Gamma}(z;K)}{\widetilde{\Gamma}\left(1-\frac{z}{K};\frac{1}{K}\right)}(2\sin\pi K)^{\frac{z}{K}}\,\mathbf{e}^{-\frac{\pi i\epsilon}{2K}z(z+1-K)}\;,$$

where the last factor has been inserted so as to maintain the correct transformation properties under $z\to z+1$. From properties (i)–(v) in §2.3, we obtain

(i) $\Gamma(z+1;K)=\frac{\sin\frac{\pi z}{K}}{\sin\frac{\pi}{K}}\Gamma(z;K)$;

(ii) $\Gamma(z;K)$ has simple poles at $\mathbf{Z}^-+K\mathbf{Z}^-$ and simple zeroes at $\mathbf{N}+K\mathbf{N}$;

(iii) $\Gamma(1;K)=1$;

(iv) $\Gamma(z;K)\Gamma(1-z;K)=\frac{K\sin\frac{\pi}{K}}{\sin\pi z}$;

(v) $\Gamma(z+K;K)=\frac{2\sin\pi z}{\left(2\sin\frac{\pi}{K}\right)^K}\Gamma(z;K)$.

Despite the fact that the set of zeros and poles of $\Gamma(z;K)$, is antisymmetric under $z\leftrightarrow 1+K-z$, rather than under $z\leftrightarrow 1-z$, we have left (iv) in this form, since it may then be more readily compared with the case of $K\in\mathbf{N}$. This symmetry, along with that under $(z,K)\leftrightarrow\left(\frac{z}{K},\frac{1}{K}\right)$, are expressed algebraically by

(iv′) $\Gamma(z;K)\Gamma(1+K-z;K)=K\left(2\sin\frac{\pi}{K}\right)^{1-K}$;

(vi) $\Gamma\left(\frac{z}{K};\frac{1}{K}\right)=\frac{1}{K}\frac{(2\sin\pi K)^{1-\frac{z}{K}}}{\left(2\sin\frac{\pi}{K}\right)^{1-z}}\Gamma(z;K)$.

The last two properties can be neatly combined by multiplying them together, to give, where we have put $w=1-\frac{z}{K}$,

(vi′) $\Gamma\left(1-w;\frac{1}{K}\right)\Gamma(1+Kw;K)=\frac{(2\sin\pi K)^w}{\left(2\sin\frac{\pi}{K}\right)^{Kw}}$.

Using the infinite product expressions for q-gamma functions constructed in §2.3, it is possible to derive

$$\Gamma(z;K)=\frac{K}{2\pi z}\left(2\sin\frac{\pi}{K}\right)^{1-z}\mathbf{e}^{A_K z^2-B_K z}\prod_{\substack{t\in T_K\\ t\neq 0}}\left(\left(1-\frac{z}{t}\right)\mathbf{e}^{\frac{z}{t}+\frac{z^2}{2t^2}}\right)^{\epsilon(t)}, \qquad (2.5.1)$$

where t runs over all non-zero complex numbers of the form $nK+s$, in which n and s are either both positive integers, or both non-positive integers. The quantity $\epsilon(t)$ is $+1$ or -1, according as $n,s\in\mathbf{N}$ or $n,s\in\mathbf{Z}^-$. Note that since $K\notin\mathbf{R}$, this set T_K will consist of distinct elements; indeed this also holds for irrational real values of K. Here, A_K is as given in §2.4 while the constant B_K is given, in terms of the constants of §2.3, by

$$\begin{aligned}B_K&=b_K+\frac{1}{K}b_{\frac{1}{K}}+\frac{\pi i\epsilon}{2K}(1-K)\\ &=\frac{\pi i\epsilon}{2K}(K-1)+\frac{\pi}{K}\sum_{s=1}^{\infty}\left(\cot\frac{\pi s}{K}+K\cot\pi sK+i\epsilon(K-1)\right)\end{aligned} \qquad (2.5.2)$$

from which it is seen that $B_{\frac{1}{K}}=KB_K$. From property (i), another form for B_K may be derived,

$$B_K=\gamma\left(1+\frac{1}{K}\right)+\ln\frac{\Gamma\left(\frac{1}{K}\right)}{2\pi}+\sum_{t\in T_K^+}\left(\frac{2}{t}+\ln\frac{t-1}{t+1}\right), \qquad (2.5.3)$$

which has the advantage over (2.5.2), of being well defined for real K. Here T_K^+ denotes those elements $t\in T_K$ with $\epsilon(t)=+$, that is, $nK+s$ with $n,s\in\mathbf{N}$. Next observe that equations (2.5.1) and (2.5.3) may be used to define a meromorphic function $\Gamma(z;K)$ of z, satisfying properties (i)–(v) above, for any positive real value of K, as well as for non-real values. The only change that occurs for real K is when K is rational, there being repetitions in the set T_K and it is important that these repetitions be counted in the product in (2.5.1) and in the sum in (2.5.3). Note that when K is an integer, (2.5.3) agrees with the form given in §2.4. By direct comparison with the previous section, we arrive at the following proposition.

Proposition 2.5.4 *There is a unique function $\Gamma(z;K)$ of complex variables z and K, which is defined when $K\notin\mathbf{R}^-$ and $z\notin(\mathbf{N}+K\mathbf{N})\cup(\mathbf{Z}^-+K\mathbf{Z}^-)$, while satisfying properties (i)–(v) above. In addition, for fixed z, this function is a holomorphic function of K off the negative real axis and its behaviour under the transformation $K\to\frac{1}{K}$ is given by property (vi′) above. Furthermore, when $K\in\mathbf{N}$, $\Gamma(z;K)$ is identical with the function defined in §2.4.*

As a corollary one may deduce that $\overline{\Gamma(z;K)} = \Gamma(\overline{z},\overline{K})$, where the bar denotes complex conjugation. The asymmetry under $K \leftrightarrow -K$ comes from the initial choice of zeroes and poles to be limited to the first and third quadrants of the $(1,K)$ lattice. If the other pair of opposite quadrants had been used instead, the resulting function would have been $\Gamma(z,-K)$.

3: Witten-Reshetikhin-Turaev invariants

3.1 A state model

Suppose that M is a 3–manifold obtained by surgery around the framed link L in S^3. Represent L by a link diagram, $\mathcal{D}$, with the blackboard framing and place the checkerboard colouring on the regions into which $\mathcal{D}$ divides the plane where the exterior region is unshaded. The $\mathfrak{sl}_2$ Witten-Reshetikhin-Turaev invariant of the empty link in M, at the root of unity q, will be denoted $Z_K(M)$. A state-sum formulation of $Z_K(M)$ was given in the original work of Reshetikhin and Turaev; we here give a reformulation, using the normalisations and conventions given in Kauffman-Lins [KL2].

For a link diagram $\mathcal{D}$, let $R_{\mathcal{D}}$ denote a formal disjoint union of the set of components of the complement of $\mathcal{D}$ in the plane, with the set of components of the link. Let $S_{\mathcal{D}}$ denote the formal disjoint union of $R_{\mathcal{D}}$ with the set of crossings in $\mathcal{D}$. Thus $R_{\mathcal{D}}$ and $S_{\mathcal{D}}$ are finite sets. Define a state model in which by a state is meant an allowed assignment of an element of I to each of the components of L as well as to each of the regions into which $\mathcal{D}$ divides the plane, that is, it defines a map $R_{\mathcal{D}} \longrightarrow I$. Such an assignment is said to be *allowed* so long as the infinite region is labelled 0 and, for each edge of $\mathcal{D}$, the triple of integers assigned to the two adjacent regions and the component containing the edge form a q–admissible triple. For a fixed state, σ, define local weights on each vertex, edge, face and component of $\mathcal{D}$ as follows. If e is an edge of $\mathcal{D}$, set

$$w_e(\sigma) = \theta(a,b,c)^{-\chi}\ ,$$

where a,b,c are the assignments given by σ to the component of L containing e and the two regions adjacent to e and χ is the Euler characteristic of the edge (1 unless the edge contains no vertices, in which case it is 0). If f is a face or component of $\mathcal{D}$ set

$$w_f(\sigma) = \Delta^{\chi}_{\sigma(f)}\ ,$$

where $\chi = 1$, unless f is a face containing no vertices in which case $\chi = 0$. Finally, if v is a vertex of $\mathcal{D}$, set

$$w_v(\sigma) = (\lambda^{a,i}_b)^{-\epsilon}(\lambda^{d,i}_c)^{\epsilon}\begin{bmatrix} a & b & j \\ c & d & i \end{bmatrix}\ ,$$

where i,j and a,b,c,d are the labels assigned to the two components of L and the four regions meeting at v, respectively, while $\epsilon = \pm 1$ according to the orientation of the crossing (over/under) relative to the local shading of regions. The convention on local labels and the sign ϵ is determined by Figure 3 in which the sign is positive.

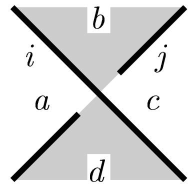

Figure 3: Local labels at a vertex in $\mathcal{D}$

To the state σ we now assign a global weight

$$W_{\mathcal{D}}(\sigma) = \prod_{\substack{\text{vertices}\\ v}} w_v(\sigma) \prod_{\substack{\text{edges}\\ e}} w_e(\sigma) \prod_{\substack{\text{regions}\\ f}} w_f(\sigma) \prod_{\substack{\text{components}\\ c}} w_c(\sigma)\,.$$

The invariant $Z_K(M)$ is now obtained from the partition function of this state model by renormalisation, so that

$$Z_K(M) = G_+^{-n_+} G_-^{-n_-} \sum_{\text{states } \sigma} W_{\mathcal{D}}(\sigma)\,,$$

where n_+ and n_- are the numbers of positive and negative eigenvalues, respectively, of the linking matrix defined by the framed link L. Also G_+ and G_- denote the partition function evaluations on an unknot with framings 1 and -1 respectively.

Finally, to simplify computations, it may noted that if a link is changed by altering the framing on one of its components, then the global weight associated with a state scales by the term

$$(-1)^{at} A^{a(a+2)t}\,,$$

where n denotes the number assigned by the state to the component and t denotes the number of positive twists added. Applying this fact to compute $G_\pm$ one obtains

$$G_\epsilon = \sum_{a\in I} (-1)^a A^{\epsilon a(a+2)} \Delta_a^2 = \frac{A^{-3\epsilon}}{\epsilon(A^{-2}-A^2)} \sum_{a=0}^{2K-1} (-1)^a A^{\epsilon a^2}\,, \tag{3.1.1}$$

which is a Gauss sum. For odd K, putting $q = A^4$, one has

$$G_\epsilon = \frac{A^{-3\epsilon}(1-A^{\epsilon K^2})}{\epsilon(A^{-2}-A^2)} \sum_{a=0}^{K-1} q^{\epsilon a^2}. \tag{3.1.2}$$

One important property of the invariant $Z_K(M)$ is that it transforms according to $q \longmapsto q^{-1}$ when the manifold M is replaced by its mirror image.

3.2 Reformulations

In this section we will reformulate the state model of §3.1 into a form which is easier to handle, and from which symmetries are more apparent. The only difference between the models is in the local weights; the states and the global weights associated with them are identical.

Pick square-roots for -1 and A and let $\lambda_a=(-1)^{a/2}A^{a(a+2)/2}$, by which we mean that $\lambda_a=(\sqrt{-1})^a(\sqrt{A})^{a(a+2)}$. Then $\lambda_c^{a,b}=\frac{\lambda_a\lambda_b}{\lambda_c}$. The factors λ appearing in the terms associated with each vertex can be written as $\lambda_a^{-\epsilon}\lambda_b^{\epsilon}\lambda_c^{-\epsilon}\lambda_d^{\epsilon}$. It may therefore be distributed amongst the four regions incident at a vertex by adding a factor $\lambda_a^{\pm}$ on a region labelled by a, for each vertex on its boundary, where the sign depends on the orientation of the crossing. The new weight associated with a face, f, containing vertices will be

$$w'_f(\sigma)=\Delta_{\sigma(f)}\left(\lambda_{\sigma(f)}\right)^{\epsilon(f)}$$

where $\epsilon(f)$ is an integer index defined as follows. On tracing the boundary of any region, f, in an anti-clockwise direction, one will pass a number of vertices and at each one the change in the segment of the link represented by the change in the associated segment of $\mathcal{D}$ will result in either an upward (that is out of the paper) or downward (that is into the paper) jump. We associate the local index -1 in the first case and $+1$ in the second case; the integer $\epsilon(f)$ is now defined as the sum of the local contributions at all the vertices on the boundary of f. The weight associated to a component, c, will be

$$w'_c(\sigma)=\Delta_{\sigma(c)}=(-1)^{\sigma(c)}[\sigma(c)+1]_q\,.$$

Observe also that each vertex in the link diagram $\mathcal{D}$ has four edges eminating from it and one may therefore distribute the first term in (2.1.1) amongst these four edges. Since there are two vertices which contribute such factors, the resulting weight on an edge containing vertices is

$$w'_e(\sigma)=\frac{1}{\theta(a,b,c)}\left(\frac{[x-a]!_q[x-b]!_q[x-c]!_q}{[a]!_q^{1/2}[b]!_q^{1/2}[c]!_q^{1/2}}\right)^2=(-1)^x\frac{[x-a]!_q[x-b]!_q[x-c]!_q}{[x+1]!_q}$$

where $x=1/2(a+b+c)$. The weight associated to a vertex will now be just the sum in (2.1.1).

We remove the summation in the vertex weight by increasing the state space to consist of maps $\sigma\colon S_{\mathcal{D}}\longrightarrow I$, so that labels are also assigned to crossings in $\mathcal{D}$. In addition to the K–Clebsch Gordon condition on edges, there is a constraint at each crossing x, namely that $\max(x_v) \leq s_x \leq \min(y_e)$. The local weights on components, regions, edges and crossings are now given by

$$\begin{aligned}
w'_c(\sigma) &= (-1)^{\sigma(c)}[\sigma(c)+1]_q;\\
w'_f(\sigma) &= (-1)^{\sigma(f)}[\sigma(f)+1]_q \left(\lambda_{\sigma(f)}\right)^{\epsilon(f)};\\
w'_e(\sigma) &= (-1)^x \begin{pmatrix} K-2 \\ K-2-x \; x-a \; x-b \; x-c \end{pmatrix}_q^{-1};\\
w'_x(\sigma) &= (-1)^s \begin{pmatrix} K-2 \\ \{s-x_v\} \; \{y_{e'}-s\} \end{pmatrix}_q,
\end{aligned}$$

where in the last expression the set $\{e'\}$ consists of four elements at each vertex, namely the three opposite-edge pairs of the associated tetrahedron, along with another (formal) element, for which $y_{e'} = K-2$. We have here used the fact that $[K-n]_q = [n]_q$ and hence that $[K-n-1]!_q[n]!_q = [K-1]!_q$. After all these operations the resulting reformulated theory has local weights w'; the new global weight $W_{\mathcal{D}}(\sigma)$ of a diagram obtained from w' in place of w, is identical to that in §3.1.

Example 3.2.1 Figure 4 shows a link diagram $\mathcal{D}$ whose associated blackboard framed link defines $M_{5,0}$. There are seven regions into which this link diagram divides the plane, namely the exterior region, innermost region and five other regions. The indices $\epsilon(f)$ for these regions are 5, 5 and five -2's. Observe that in all cases the sum of the local indices over all regions must be zero.

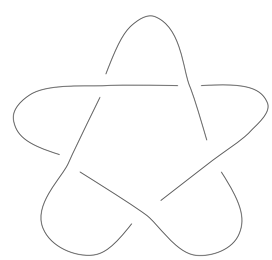

Figure 4: Link diagram for $M_{5,0}$

3.3 A Symmetry Principle

In this section we use the reformulation given in §3.2 to extract a factor from $Z_K(M)$, employing a Symmetry Principle similar to that obtained in [KM2].

There is an involution on I given by $\tau\colon n \longmapsto K-2-n$. Suppose that L' is a sublink of the link represented by $\mathcal{D}$, that is, it is determined by a subset of the set of components of $\mathcal{D}$, or equivalently by its characteristic function, a map θ from the set of components of $\mathcal{D}$ to $\{0,1\} = \mathbf{Z}/2\mathbf{Z}$ which takes value 1 on those components contained in L'. Extend θ to a map, $\alpha\colon R_{\mathcal{D}} \longrightarrow \mathbf{Z}/2\mathbf{Z}$ by

$$\alpha(f) = \begin{cases} 0, & \text{where } f_0 \text{ is the exterior region;} \\ \alpha(f') + \theta(c), & \text{for adjacent regions } f, f' \text{ separated by the component } c. \end{cases}$$

Such a sublink defines an involution, ι_θ, on the set of maps $\sigma\colon R_{\mathcal{D}} \longrightarrow I$, by

$$\iota_\theta(\sigma)(f) = \tau^{\alpha(f)}(\sigma(f))$$

for all $f \in R_{\mathcal{D}}$.

Lemma 3.3.1 *ι_θ defines an involution on the set of states in the state models of §3.1 and §3.2.*

Proof: It is only necessary to verify that admissibility is preserved by ι_θ. Fix a state, σ. Observe that $\alpha(f_0) = 0$ so that $\iota_\theta(\sigma)$ takes the same value as σ on the external region f_0. For any edge of $\mathcal{D}$, there is associated a triple $f_1, f_2, f_3 \in R_{\mathcal{D}}$, namely the component containing the edge and the two regions on either side of the edge. By the construction of α, $\alpha(f_1) + \alpha(f_2) + \alpha(f_3) = 0$. Thus when the values associated with f_1, f_2 and f_3 in the state $\iota_\theta(\sigma)$ are compared with those associated in the state σ, they are either identical or precisely two differ, via the application of τ. The lemma follows upon noting that if (a,b,c) is a q-admissible triple, then so is $(a, \tau(b), \tau(c))$. ∎

We now compare the local weights associated to σ and $\sigma' = \iota_\theta(\sigma)$ in the state model of §3.2. Since $[\tau(c)+1]_q = [c+1]_q$ and $(-1)^{\tau(c)} = (-1)^K(-1)^c$, thus $\Delta_{\tau(c)} = (-1)^K \Delta_c$. Also, $\lambda_{\tau(a)} = (-1)^{K/2} A^{K^2/2} A^{K(a+1)} \lambda_a$. Hence, for components, c,

$$\frac{w'_c(\sigma')}{w'_c(\sigma)} = (-1)^{K\theta(c)}$$

while for faces, f,

$$\frac{w'_f(\sigma')}{w'_f(\sigma)} = \left((\sqrt{-1})^{(\epsilon(f)-2)K} (\sqrt{A})^{\epsilon(f)K^2} A^{K\epsilon(f)(\sigma(f)+1)} \right)^{\alpha(f)} .$$

For edges, note that $\{2K-4-a-b-c, b+c-a, c+a-b, a+b-c\}$ is invariant under $(a,b,c) \longmapsto (a, \tau(b), \tau(c))$. The only local edge weights affected come from those edges for which precisely two of the three associated labels have been changed by the application of τ, say b and c, and then

$$\frac{w'_e(\sigma')}{w'_e(\sigma)} = (-1)^{(K+b+c)} .$$

For vertices, the labels will be affected by ι_θ in such a way that the transformation on the collections $\{x_v\}$ and $\{y_{e'}\}$ is given by an isometry of $\mathbf{R}$, that is either a translation or a reflection. Indeed, the effect on the six labels local to a vertex must be either trivial or the changed labels must correspond to a collection of edges of a tetrahedron which form a closed loop (either three or four edges). The transformation is now trivial in the first case and in the other cases is $x \longmapsto \delta_v \pm x$ with the sign being positive or negative according as the number of local changes is four or three, respectively and

$$\delta_v = \begin{cases} K - 2 - {}^1\!/_2(\text{sum of changed labels}) & \text{if four labels are changed locally} \\ K - 2 + {}^1\!/_2(\text{sum of unchanged labels}) & \text{if three labels are changed locally.} \end{cases}$$

In particular, for vertices affected by the transformation,

$$\frac{w'_v(\sigma')}{w'_v(\sigma)} = (-1)^{\delta_v} .$$

One may define a shading of the regions, by making a region f shaded precisely when $\alpha(f) = 1$. This defines a checkerboard shading of the subdiagram, $\mathcal{D}'$, of $\mathcal{D}$ defined by $L' \subseteq L$. Observing that $A^{2K} = -1$, so that A^K is a square-root of -1, one finds that the ratio of the global weights

$$\frac{W_{\mathcal{D}}(\sigma')}{W_{\mathcal{D}}(\sigma)} = (A^{K/2})^{\Delta(L')}$$

where $\Delta(L') \in \mathbf{Z}/8\mathbf{Z}$ is expressed as a sum of local terms,

$$\begin{cases} 4K & \text{for each component in } L', \\ -4K + (3K + 2 + 2\sigma(f))\epsilon(f) & \text{for each shaded region, } f, \\ 4(K + a + b) & \text{for each internal edge,} \\ 4(K + a + c) & \text{for each boundary edge,} \\ 4(K - 2) - 2(a + b + c + d) & \text{for each internal vertex,} \\ 4(K - 2) + 2(a + b + c) & \text{for each boundary vertex,} \\ 4(K - 2) - 2(a + b + c + d) & \text{for each vertex of } L'. \end{cases}$$

Here edges and vertices are referred to as internal and boundary with respect to the shaded region, while a, b, c, d are used to denote the labels on the shaded regions or components adjacent to the edge or vertex concerned.

Suppose now that the initial state, σ, is such that all the labels on the components are even; such a state will be said to be *even*. Since σ is an admissible state, all the labels on regions are also even. The above expression can now be simplified to one dependent only upon the subdiagram $\mathcal{D}'$, and not on the particular state σ,

$$\begin{aligned} \Delta(L') = 4K(\#\text{ components in } L') + 4K(\#\text{ shaded regions in } \mathcal{D}') \\ + 2(K - 2)(\#\text{ vertices of } \mathcal{D}' \text{ counted with sign}) \end{aligned}$$

where the sign associated to a vertex of $\mathcal{D}'$ is determined by the relative orientation of the crossing and the shading, Figure 3 showing the positive sign.

For the rest of this section, we suppose that K is odd. For any state, σ', let L' denote the sublink of L defined by those components for which $\sigma'(c)$ is odd. Then $\sigma' = \iota_\theta(\sigma)$, where θ is the characteristic function of L' and σ is an even state. Thus one may now write the WRT invariant as

$$Z_K(M) = G_+^{-n_+} G_-^{-n_-} \sum_{L' \subseteq L} A^{K\Delta(L')/2} \sum_{\text{even states } \sigma} W_{\mathcal{D}}(\sigma)\,, \tag{3.3.2}$$

where the sum is over all sublinks, L', of L. Note that $\Delta(L')$ is always even and so the first sum is a sum of fourth roots of unity.

Lemma 3.3.3 $Z_3(M) = (A^K - 1)^{-n} A^{-Kn_-} \sum_{L' \subseteq L} A^{K\Delta(L')/2}$ *where* $n = n_+ + n_-$.

Proof: Note that for $K = 3$, the states are maps $R_{\mathcal{D}} \longrightarrow \{0, 1\}$. There is precisely one even state, namely that for which $\sigma(f) = 0$ for all components and faces f. The weight associated with this one even state is 1. Observe that A^K is a fourth root of unity, independently of the value of K, so that the first sum in (3.3.2) is actually independent of K. The result now follows by applying (3.3.2) for $K = 3$ while noting that $G_+(K = 3) = A^K - 1$ and $G_-(K = 3) = A^{-K} - 1$. ■

As a corollary we deduce that

$$Z_K(M) = (A^K - 1)^n A^{Kn_-} G_+^{-n_+} G_-^{-n_-} Z_3(M) \sum_{\text{even states } \sigma} W_{\mathcal{D}}(\sigma)$$

and the term on the right hand side obtained by removing the factor $Z_3(M)$ is also a manifold invariant which we shall denote by $Z'_K(M)$; Z'_K is the $SU(2)$ invariant as opposed to the $SO(3)$ invariant Z_K, see [KM2].

The local weight system of §3.2 gives the following state model for $W_{ev}(\mathcal{D})$. The states of the model are maps $\sigma \colon S_{\mathcal{D}} \longrightarrow I$ for which the labels on link components and regions are all even, with the outer region mapping to zero and the appropriate form of the triangle inequality holding on edges and at crossings. The local weights are defined by

$$\begin{aligned}
w''_c(\sigma) &= [2\overline{\sigma}(c) + 1]_q;\\
w''_f(\sigma) &= [2\overline{\sigma}(f) + 1]_q q^{\epsilon(f)\overline{\sigma}(f)(\overline{\sigma}(f)+1)/2};\\
w''_e(\sigma) &= (-1)^x \begin{pmatrix} K-2 \\ K-2-x\ \ x-2a\ \ x-2b\ \ x-2c \end{pmatrix}_q^{-1};\\
w''_x(\sigma) &= (-1)^s \begin{pmatrix} K-2 \\ \{s - x_v\}\ \{y_{e'} - s\} \end{pmatrix}_q.
\end{aligned}$$

Here, $\overline{\sigma}$ is obtained by halving the labels from σ. For the local weight assigned to an edge, a, b and c denote the $\overline{\sigma}$-labels associated in a state to the adjacent regions and the component to which the edge belongs. For a vertex, x_v are sums of triples of $\overline{\sigma}$-labels and y_e are sums of quadruples of $\overline{\sigma}$-labels. Note that since $q^{1/2} = -q^{\frac{K+1}{2}}$ and $\lambda_{2a} = (-1)^a q^{a(a+1)/2}$, thus the sum of weights over even states may be written as a polynomial in q.

4: Holomorphic manifold invariants

4.1 Formal invariants for generic q

In §3.2, a state model was given in which the states consisted of assignments of elements of I to every component, region and vertex of $\mathcal{D}$, with the restriction that certain inequalities hold on each edge and at each vertex. It will be convenient to change variables, replacing σ by a map $\overline{\sigma}$ into $I' = I + 1 = \{1, \ldots, K-1\}$, with $\overline{\sigma}(i) = \sigma(i) + 1$ for all regions and components i. The Clebsch-Gordan constraints on an edge are now that, for each triple of labels defined by an edge and the two adjacent regions

$$\overline{CG}_K: \quad |\overline{a}_1 - \overline{a}_2| < \overline{a}_3 < \overline{a}_1 + \overline{a}_2\,, \quad \overline{a}_1 + \overline{a}_2 + \overline{a}_3 \text{ is odd and } < 2K$$

The state sum of this model is a handle-slide invariant, and when suitably normalised to be also invariant under the second Kirby move, gives the $\mathfrak{sl}_2$-WRT invariant $Z_K(M)$. The state sum may be decomposed as a sum

$$Z_K(M) = G_+^{-n_+} G_-^{-n_-} \sum_{\phi:\{c\}\longrightarrow I'} (-1)^{\phi(c)-1} [\sigma(c)]_q Z_K(L, \phi)\,,$$

where $Z_K(L,\phi)$ is the state sum obtained by summing only over those states σ which match a given assignment ϕ on the components. In other words, $Z(L,\phi)$ is the Jones polynomial for the link L, in which the component c is coloured with the representation of dimension $\phi(c)$, evaluated at the K^{th} root of unity. See [RT].

Let us fix the assignment $\phi\colon \{c\} \longrightarrow I'$ on the components. The set, $\mathcal{R}_K(\phi)$, of matching states σ, that is of allowed assignments of labels to the faces and vertices of $\mathcal{D}$ compatible with ϕ, will be an increasing function of K, which stabilises for large K to $\mathcal{R}(\phi)$ say, with respect to the inclusion order on sets. Then $\mathcal{R}(\phi)$ is the set of labels on faces and vertices which satisfy the Clebsch-Gordan condition on each edge, without K-constraints; this set is finite and the corresponding state sum $Z(L,\phi)$ is the ϕ-coloured Jones polynomial of L.

The argument used to prove invariance under handle slides of the state sum in which the bound K is present, formally applies also when this bound is removed;

see p.143 of [KL2]. It only relies on an interchange of summation signs, one of which remains finite. Formally, therefore

$$\sum_{\phi} Z(L,\phi)\prod_{c}(-1)^{\phi(c)-1}[\phi(c)]_q$$

is invariant under handle-slides. In line with §3.1, the normalisation required to give invariance under the second of Kirby's moves, would extend literally to give $G_+^{-n_+}G_-^{-n_-}$, where n_+ and n_- are the numbers of positive and negative eigenvalues of the linking matrix of L, and G_ϵ are infinite Gauss-type sums given by similar formulae to those in §3.1,

$$G_\epsilon = \frac{A^{-\epsilon}}{q^\epsilon - 1}\sum_{a=-\infty}^{\infty}(-1)^a A^{\epsilon a^2}\ .$$

Thus, up to convergence questions, this defines an invariant.

4.2 Holomorphic invariants using integrals

The main technique for turning a function defined only at integer values of K, by a sum over a set dependent on K, into a holomorphic function of K as employed in [LR1], is that of replacing sums by integrals, using Cauchy's residue theorem, once the summands have been transformed into functions defined away from integer arguments. This works for one-dimensional sums, and by iteration, for sums over any hypercuboid, that is, over a product of intervals. Our second step, then, is to convert the current sum into one over a set of states determined by *independent* parameters in I'.

By the fundamental properties of the function $\Gamma(z;K)$, it is known that at integer arguments $\Gamma(n;K) = [n-1]!_q$. This may be used to replace all the q-binomial coefficients appearing in the local weights by ratios of products of q-gamma functions, making all the local weights well-defined away from integer arguments, and for non-integer values of K. This completes the first step.

The second step is to replace the constraints on the labels specifying a state by ones which make the labels independent. For vertex labels, note that the only occurence of a vertex label is in the local weight associated with that vertex,

$$g(s) = (-1)^s\frac{\Gamma(s+1;K)}{\prod_v \Gamma(s-x_v;K)\prod_e \Gamma(y_e - s;K)}\ .$$

Observe that, if $K\in\mathbf{Z}$ and the labels on regions and components satisfy the Clebsch-Gordon constraint on each edge, then $\{x_v\}$ and $\{y_e\}$ are sets of integers, with $x_v \le y_e$ for all v and e. In this case, it follows from the properties of $\Gamma(z;K)$

that $g(s+K)=g(s)$ while $g(s)$ will be a holomorphic function of s will no poles, and zeros at those integer values outside $[c,d]+K\mathbf{Z}$ where $c=\max(\{x_v\})$ and $d=\min(K-2,\{y_e\})$. One may therefore equally well remove the constraints on vertex labels.

For the Clebsch-Gordan constraints, the Verlinde formula

$$\sum_{\beta=1}^{K-1}\frac{\sin\frac{\beta\alpha_1}{K}\sin\frac{\beta\alpha_2}{K}\sin\frac{\beta\alpha_3}{K}}{\sin\frac{\beta}{K}}=\frac{K}{2}\delta_{(\alpha_1,\alpha_2,\alpha_3)\in\mathbf{CG}_K}\ ,$$

removes each edge constraint while introducing an extra label on the edge. This completes the second step, since now

$$Z_K(M)=G_+^{-n_+}G_-^{-n_-}\sum_{\rho\in[1,K-1]^{M'}}f(\rho)$$

where M' denotes the sum of the numbers of regions, vertices, edges and components in $\mathcal{D}$, and $f(\rho)$ is product of evaluations of $\Gamma(z;K)$ at values of z which are linear functions of ρ.

Finally one may apply the residue theorem to replace the sums by a complex integral over an M'-dimensional torus. Unfortunately, the function obtained, while being a holomorphic extension of $Z_K(M)$, need not be independent of the link L used in its construction.

4.3 Holomorphic invariants using sums

Instead of considering q as a formal variable, as in the last section, or as a root of unity, as in §3.1, we suppose that $|q|>1$, that is $K\in\mathbf{H}^+$. Then G_- is a convergent series, while G_+ is not.

Each term in the state sum is a ratio of products of q-factorials and q-numbers. Each can therefore be written, in a unique way, in the form of an element of $q^t\mathbf{Z}[[q^{-1}]]$, for some half-integer t. All the series involved will be absolutely convergent. In particular, $\frac{n-1}{2}$ and $\frac{n(n-1)}{4}$ are the values of t for $[n]_q$ and $[n]!_q$, respectively. The sum of these terms, over all labels with a fixed assignment ϕ on the components, is finite, and may therefore also be written in this form. The degree $t(\sigma)$ associated with the state σ is a quadratic function of the state labels, while the leading coefficient is ±1. For fixed region labels, $t(\sigma)$ has a maximum value over all possible vertex labels, which is attained only on the one state where

$$s=\min(\{y_e\})\ ,$$

at each vertex, in the notation of (2.1.1). After summing over face labels, we obtain,

$$\deg Z(L,\phi)\le\frac{1}{8}\max\Big(\sum_f\big(\epsilon(f)f^2+4f\big)-\sum_x\big((\alpha-\beta)(\alpha-\gamma)+2(\alpha-2)\big)\Big)\ ,$$

where the maximum is taken over all possible face labels compatible with ϕ. At each crossing x, the symbols α, β and γ here denote the three sums of pairs of labels of opposite edges of the tetrahedral symbol associated with that vertex, while α is chosen to be the largest of the three. Equality holds if the maximum is attained only one set of face labels. The maximum is evaluated over a subset of $\mathbf{N}^M$ defined by the triangle inequality and parity constraint on each edge,

$$\overline{CG}: \quad \left| \, \overline{a}_1 - \phi(c) \, \right| < \overline{a}_2 < \overline{a}_1 + \phi(c) \; , \quad \overline{a}_1 + \overline{a}_2 + \phi(c) \text{ is odd}$$

for adjacent regions with labels $\overline{a}_1$ and $\overline{a}_2$ separated by part of a component with label c, and M denotes the number of interior regions. Now restrict to only odd labels, in accordance with the symmetry principle of §3.3, so that the conditions on labels are simple inequalities. When the labels $\phi(c)$ on components are all sufficiently large, the degree $t(\phi)$ of $Z(L,\phi)\prod_c[\phi(c)]_q$ as a polynomial in q, is a quadratic function $T(\phi)$, of the individual labels $\phi(c)$. This degree is an invariant of the link L.

A single unknot with framing α has $t(\phi) = \frac{\alpha}{4}a^2 + a - 1 - \frac{\alpha}{4}$ where a is the label assigned by ϕ to that component.

We may now define

$$Z^0(M) = \lim_{a\longrightarrow\infty} \frac{\sum Z(L,\phi)\prod_c(-1)^{\phi(c)-1}[\phi(c)]_q}{\sum \pm q^{T(\phi)}}$$

where both sums are over maps $\phi\colon \{c\} \longrightarrow \{1,\ldots,a\}$. The signs are generated by the leading coefficients of $Z(L,\phi)$. We conjecture that this is well defined and invariant under Kirby moves, and moreover that it defines a holomorphic function of q outside the unit circle, with limiting values related to $Z_K(M)$. Indeed, one expects that the homogeneous part of $4T(\phi)$ is the quadratic form in $\phi(c)$ given by the linking matrix of the components, so that the denominator reduces, for rational homology spheres at a root of unity, to

$$G_+^{n_+} G_-^{n_-} \left(\frac{H}{K}\right)$$

by the properties of Gauss sums, since $|\det(lk\,L)| = |H_1(M,\mathbf{Z})| = H$. Thus we expect the function so generated to be related to $\left(\frac{H}{K}\right) Z'_K(M)$, the $SO(3)$-invariant in precisely the normalisation used by Ohtsuki.

This leaves many open questions!

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References

[J] F.H. Jackson, 'On q-definite integrals', *Quart. J. Pure Appl. Math.* **41** (1910) p.193–203.

[J] L.C. Jeffrey, 'Chern-Simons-Witten Invariants of Lens Spaces and Torus Bundles, and the Semiclassical Approximation', *Commun. Math. Phys.* **147** (1992) p.563–604.

[KL2] L.H. Kauffman, S. Lins, 'Temperley–Lieb recoupling theory and invariants of 3–manifolds', Princeton University Press (1994).

[KM2] R. Kirby, P. Melvin, 'The 3–manifold invariants of Witten and Reshetikhin-Turaev for $sl(2,\mathbf{C})$', *Invent. Math.* **105** (1991) p.473–545.

[L1] R. Lawrence, 'Asymptotic expansions of Witten-Reshetikhin-Turaev invariants for some simple 3–manifolds', *J. Math. Phys.* **36** (1995) p.6106–6129.

[L2] R. Lawrence, 'Witten-Reshetikhin-Turaev invariants of 3-manifolds as holomorphic functions', *Lecture Notes in Pure and Applied Mathematics* **184** (1996) p.363–377.

[LR1] R. Lawrence, L. Rozansky, 'Witten-Reshetikhin-Turaev invariants for Seifert manifolds', *preprint* (1997).

[LR2] R. Lawrence, L. Rozansky, 'Asymptotic expansions of Witten-Reshetikhin-Turaev invariants', *in preparation.*

[Mo] L.J. Mordell, 'The definite integral $\int_{-\infty}^{\infty}\frac{\mathbf{e}^{ax^2+bx}}{\mathbf{e}^{cx}+d}dx$ and the analytic theory of numbers', *Acta Mathematica* **61** (1933) p.323–360.

[M1] H. Murakami, 'Quantum $SU(2)$–invariants dominate Casson's $SU(2)$–invariant', *Math. Proc. Camb. Phil. Soc.* **115** (1993) p.253–281.

[M2] H. Murakami, 'Quantum $SO(3)$–invariants dominate the $SU(2)$–invariant of Casson and Walker', *preprint* (1993).

[O1] T. Ohtsuki, 'A polynomial invariant of integral homology 3–spheres', *Math. Proc. Camb. Phil. Soc.* (to appear).

[O2] T. Ohtsuki, 'Finite type invariants of integral homology 3–spheres', *preprint* (1994).

[O3] T. Ohtsuki, 'A polynomial invariant of rational homology 3–spheres', *Invent. Math.* **123** (1996) p.241–257.

[RT] N.Yu. Reshetikhin, V.G. Turaev, 'Invariants of 3-manifolds via link polynomials and quantum groups', *Invent. Math.* **103** (1991) p.547–597.

[R1] L. Rozansky, 'Witten's invariant of 3–dimensional manifolds: Loop expansion and surgery calculus', *preprint* (1993).

[R2] L. Rozansky, 'A contribution of the trivial connection to the Jones polynomial and Witten's invariant of 3d manifolds I, II', *preprint* (1994).

[TV] V.G. Turaev, O.Ya. Viro, 'State Sum Invariants of 3-manifolds and quantum $6j$–symbols', *Topology* **31** (1992) p.865–902.

[Wa] K. Walker, 'An Extension of Casson's Invariant', *Annals of Mathematical Studies* **126** Princeton University Press (1992).

[W] E. Witten, 'Quantum Field Theory and the Jones Polynomial', *Commun. Math. Phys.* **121** (1989) p.351–399.

E-mail: lawrence@math.lsa.umich.edu

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