Witten-Reshetikhin-Turaev Invariants of 3-manifolds as Holomorphic Functions

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Abstract. For any Lie algebra, $\mathfrak{g}$, and integral level, k, there is defined an invariant, $Z_k^*(M,L)$, of embeddings of links L in 3-manifolds M, known as the Witten-Reshetikhin-Turaev invariant. When $\mathfrak{g}=\mathfrak{sl}_2$, $Z_k^*(S^3,L)$ is a polynomial in $q=\exp\frac{2\pi i}{(k+2)}$, namely the generalised Jones polynomial of the link L. This paper discusses the invariant $Z^*_{r-2}(M,\varnothing)$ when $\mathfrak{g}=\mathfrak{sl}_2$ for a simple family of rational homology 3-spheres, $M_{n,t}$, obtained by integer surgery around $(2,n)$ type torus knots. In earlier work of the author it was shown that there is an associated holomorphic function $Z_\infty(M_{n,t})$ of $\ln q\in\mathbb{C}\backslash i\mathbb{R}$, related to Ohtsuki's invariants, from which $Z^*_{r-2}(M_{n,t},\varnothing)$ may be derived for all sufficiently large primes r. The current paper extends the results to prime powers and odd composite numbers coprime to $|H_1(M_{n,t})|$, showing how the invariant $Z^*_{r-2}(M_{n,t},\varnothing)$ may be extracted from $Z_\infty(M_{n,t})$.

1. Introduction and Main Results

Suppose that M is a compact oriented 3-manifold without boundary. In [38], Witten formally defined a topological invariant $Z_{k+2}(M)$, dependent on some additional data, namely a choice of a Lie algebra $\mathfrak{g}$ and of a level $k\in\mathbb{Z}$, in the form of a functional integral,

$$Z_{k+2}(M)=\int_{\mathcal{A}/\mathcal{G}}\mathrm{e}^{\frac{ik}{4\pi}\int_M\langle A,dA+\frac{1}{3}[A,A]\rangle\,d\tau}\mathcal{D}A,\tag{1.1}$$

over a quotient of the space of G-connections on M by an appropriate gauge group, $\mathcal{G}$. For the integrand to be well-defined, that is invariant under $\mathcal{G}$, one needs k to be

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an integer. Although many attempts have been made to give a direct and calculable meaning to this Feynman integral, it remains only a formal expression from which valid results can be derived when the functional integral is manipulated according to certain rules; see for example [2, 3, 4, 5, 6, 32, 33, 34]. The approaches which are closest in spirit to that of (1.1) employ the notion of a topological field theory (see [1]) whose definition is based on Segal's mathematical definition of conformal field theory.

The situation is much like that which existed for divergent series in the last century. For example, at integer values of $m \geqslant 2$, one has

$$\zeta(m) = \sum_{n \neq 0} n^{-m} = -\frac{B_m}{m!}(2\pi i)^m,$$

where B_m is the mth Bernoulli number. However, Euler was happy to consider the same sum for negative integral m, writing that

$$\sum_{n=1}^{\infty} n^m = -\frac{B_{m+1}}{m+1}, \qquad \text{for } m \geqslant 1.$$

One may ask in what sense it is meaningful to claim (using $m = 3$) that the sum of the positive integer cubes is $-B_4/4 = -\frac{1}{168}$. As will be seen in this paper there are some rather close connections between this story and that of the Feynman integral (1.1), including the occurrence of Bernoulli numbers, and so there is no reason to think that in the future such Feynman integrals will not be able to be put on a completely rigorous foundation.

Many alternative and completely rigorous formulations of $Z_{k+2}(M)$ have been obtained, primarily using the description of a compact, connected, orientable 3-manifold M, without boundary, as obtained by Dehn surgery around a suitable link L_M, in S^3. Reshetikhin and Turaev [31] found $Z_{k+2}(M)$ as a combination of the quantum invariants of L obtained from all possible choices of irreducible representations attached to the components of L. This sum will only be finite when q is a root of unity. However, it is still something of a mystery that while quantum invariants of links in S^3 are defined for all values of q, being polynomials in q, this happy state of affairs is not true of any of the definitions so far known for $Z_k^*(M, L)$ when $M \neq S^3$.

From the formulation of [RT], it is seen that $Z_k^*(M, \varnothing)$ can be defined for all roots of unity q, rather than just those of the form $\mathbf{e}^{\frac{2\pi i}{r}}$. Very few concrete computations of $Z_k^*(M, \varnothing)$, as a function of $r = k + 2$ (the order of the root of unity q), have been carried out—see [9, 14, 15, 17, 18, 27] for some such computations.

It follows quickly from its definition that, for fixed order r of the root of unity q, $Z_k^*(M, \varnothing)$ can be written as an algebraic function of q, with rational coefficients. In the normalisation for which the invariant for S^3 is 1, denote the invariant, as an algebraic function of q at rth roots of unity, for the pair $(M, \varnothing)$ by $Z_r(M)$. Kirby and Melvin [18] derived a symmetry principle for terms in the sum arising in $Z_r(M)$ and thereby obtained a slightly finer invariant, which is just the associated $SO(3)$-invariant, $Z'_r(M)$. We now describe some of the results of Murakami and Ohtsuki on the forms of these functions of $h = q - 1$ when r is an odd prime. The reader is referred to [25, 26, 28, 29, 30] for details.

THEOREM 1.1 (MURAKAMI/OHTSUKI). *Suppose that r is an odd prime and M is an oriented $\mathbb{Z}/r\mathbb{Z}$-homology sphere. Let $N = |H_1(M, \mathbb{Z})|$, so that $r \nmid N$.*

(a) *As a function of q, $Z'_r(M) \in \mathbb{Z}[h]$, so that for some $a_{m,r}(M) \in \mathbb{Z}$, one has $Z'_r(M) = \sum_m a_{m,r}(M)h^m$. For $0 \leqslant m \leqslant r-2$, $a_{m,r}(M)$ is uniquely determined by this condition as an element of $\mathbb{Z}/r\mathbb{Z}$.*

(b) *There exist rational numbers $a_{m,\infty}(M) \in \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2m+1}, \frac{1}{N}]$ such that, for any prime $r \geqslant 2m+3$, $a_{m,r}(M) \equiv \left(\frac{N}{r}\right) a_{m,\infty}(M)$ as elements of $\mathbb{Z}/r\mathbb{Z}$, where $\left(\frac{N}{r}\right)$ denotes the quadratic residue while*

$$a_{0,\infty} = N^{-1}, \qquad a_{1,\infty}(M) = 6N^{-1}\lambda(M),$$

where $\lambda(M)$ denotes the Casson-Walker [37] *invariant of M in Casson's normalisation.*

As a result of part (b) of this Theorem, Ohtsuki defines a formal power series

$$Z_\infty(M) = \sum_{m=0}^{\infty} a_{m,\infty} h^m,$$

with rational coefficients, which is an invariant of rational homology 3-spheres, M. It is expected that a stationary phase expansion of (1.1) will lead to precisely this series and indeed, according to Rozansky [34, 35] this has been verified for three-fibred Seifert manifolds.

In this paper we restrict our attention to a particular two-parameter family of rational homology 3-spheres, $\{M_{n,t}\}$, given by integer t-surgery around a $(2,n)$-torus knot. Here $N \equiv |H_1(M_{n,t}, \mathbb{Z})| = |n+t|$ and so this family contains a sub-family of integral homology spheres, namely those for which $|n+t| = 1$ while the Poincaré homology sphere is included as $M_{-3,2}$. The Poincaré homology sphere is realised as $M_{3,-2}$ and the framed knot in S^3 giving rise to this manifold is shown in Figure 1, where the knot is given the blackboard framing. This diagram also serves to identify positive twists, the two extra curls being negative twists.

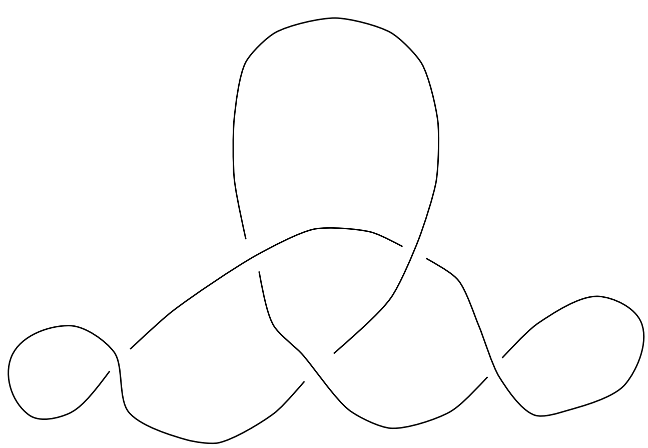

FIGURE 1. Knot for Poincaré homology sphere

THEOREM 1.2. [20]. *Suppose that n is an odd integer and $t \neq -n$ is an integer. Put $h = q-1$ and $N = |t+n|$.*

(i) *There is a formal power series, $Z_\infty(M_{n,t}) \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{N}\right][[h]]$ which is such that it converges r-adically to $\left(\frac{N}{r}\right) Z'_r(M_{n,t})$, when $1+h=q$ is an rth root of unity for some odd prime r not dividing $t+n$.*

(ii) *There is a holomorphic function of* $\ln q \in \mathbb{C}\backslash i\mathbb{R}$ *whose asymptotic expansion around* $q=1$ *gives the formal power series* $Z_\infty(M_{n,t})$. *This function may be defined by*

$$Z_\infty(M_{n,t}) = \begin{cases} 2\frac{(-1)^t q^{\frac{n}{2}}}{1-q^{-\delta}} \int\limits_{-\infty}^{\infty} \frac{q^{\Delta_1(iz)} - q^{\Delta_2(iz)}}{e^{2\pi z}+e^{-2\pi z}}\, dz, \\ \frac{(-1)^t q^{\frac{n}{2}}}{1-q^{-\delta}} \sum\limits_{m=-\infty}^{\infty} (-1)^m \left(q^{\Delta_1(\frac{m}{2}-\frac{1}{4})} - q^{\Delta_2(\frac{m}{2}-\frac{1}{4})}\right) \end{cases} \tag{1.2}$$

according as the integral or sum converges, where $\delta = \operatorname{sgn} t + n$ *and* $\Delta_1(x)$, $\Delta_2(x)$ *are the quadratic functions given by*

$$\begin{aligned} 2(t+n)\Delta_1(x) &= 4n(t-n)x^2 + 4tx - \frac{1}{4}n(t+n) - \frac{1}{2}, \\ 2(t+n)\Delta_2(x) &= 4n(t-n)x^2 + 4(t-2n)x - \frac{1}{4}n(t+n) - \frac{9}{2}. \end{aligned} \tag{1.3}$$

For the particular family of manifolds under consideration, the first part of Theorem 1.2 is much stronger than Theorem 1.1(a), since it allows the reconstruction of $Z_r(M)$ from $Z_\infty(M)$ for all but finitely many primes r. Indeed, the r-adic convergence of Theorem 1.2(i) may be expressed in more elementary terms as stating that the difference between $Z_\infty(M_{n,t})$, a formal power series in h, and $\left(\frac{N}{r}\right) Z_r'(M_{n,t})$, a polynomial in h, is divisible by

$$\frac{(1+h)^r - 1}{h}, \tag{1.4}$$

within the ring of formal power series in h with rational coefficients whose denominators are coprime to r. Since for prime r, the coefficient of h^m in (1.4) is divisible by r whenever $0 \leqslant m < r-1$, one may deduce that

$$a_{m,r} \equiv \left(\frac{N}{r}\right) a_{m,\infty} \qquad \text{whenever } m \leqslant r-2,$$

and not merely when $m \leqslant \frac{r-3}{2}$ as in Theorem 1.1(a).

For the manifolds $M_{n,t}$, the result on the denominators involved in $a_{m,\infty}$ obtained from Theorem 1.2 is also somewhat stronger than that in Theorem 1.1, namely they involve only powers of 2 and divisors of N. From numerical evidence in the computation of $a_{m,\infty}$ for integer homology spheres of our family, the following conjecture was made.

CONJECTURE 1.3. [20]. *If* M *is a* $\mathbb{Z}$*-homology sphere then* $Z_\infty(M) \in \mathbb{Z}[[h]]$.

Currently this cannot be proved even for all integer homology spheres within the family $\{M_{n,t}\}$. The restriction to the particular family of manifolds discussed in this paper is necessitated by the fact that it is only for these manifolds that the associated state-sum expressions for $Z_r(M)$ involve only "trivial" quantum $6j$-symbols which cancel, leaving a relatively simple sum.

In the current paper we discuss the main steps in the proof of Theorem 1.2 and examine the extent to which they may be generalised to values of r which are prime powers and, more generally, arbitrary odd integers. Specifically, it is seen in §4.1 that Theorem 1.2 holds equally well when r is an odd prime power coprime to N. When r is composite it is impossible to speak of r-adic convergence of a formal power series in $h = q-1$, since then $\zeta_r - 1$ is a unit in $\mathbb{Z}[\zeta_r]$, where $\zeta_r = \exp \frac{2\pi i}{r}$. However, it will be seen in §4.2 that it is still possible to reconstruct $Z'_r(M)$ from $Z_\infty(M)$, so long as r is assumed odd and coprime to N.

2. Witten-Reshetikhin-Turaev Invariants

Throughout this paper, $r \in \mathbb{N}$ will denote the order of a root of unity q. Set $I = \{0, 1, \ldots, r-2\}$. Let $A = q^{1/4}$ and define the q-numbers by

$$[n]_q = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}.$$

The q-factorials are defined by $[n]!_q = \prod_{i=1}^{n} [i]_q$. A triple of non-negative integers (a, b, c) will be said to be *q-admissible* when $b+c-a$, $c+a-b$, $a+b-c$ and $2r-4-a-b-c$ are all positive and even. If a is a non-negative integer, set $\Delta_a = (-1)^a [a+1]_q$.

Suppose that M is a 3-manifold obtained by surgery around the framed link L in S^3. Represent L by a link diagram, $\mathcal{D}$, with the blackboard framing. The $\mathfrak{sl}_2$ Witten-Reshetikhin-Turaev invariant of the empty link in M, at the root of unity q, will be denoted $Z_r(M)$. It can be computed generally as a suitably normalised version of the partition function of a certain state model, whose states are allowed assignments of an element of I to each of the components of L, as well as to each of the regions into which $\mathcal{D}$ divides the plane. Such an assignment is said to be *allowed* if the infinite region is labelled 0 and, for each edge of $\mathcal{D}$, the triple of integers assigned to the two adjacent regions and the component containing the edge, form a q-admissible triple. The weight, $W_{\mathcal{D}}$, assigned to a state, σ, is defined as the product of local weights associated with each vertex, edge, face and component of $\mathcal{D}$, each of which is a certain ratio of products of q-factorials, namely quantum dimensions, theta nets and quantum $6j$ symbols.

The invariant $Z_r(M)$ is now obtained from the partition function of this state model by renormalisation, so that

$$Z_r(M) = G_+^{-n_+} G_-^{-n_-} \sum_{\text{states } \sigma} W_{\mathcal{D}}(\sigma),$$

where n_+ and n_- are the numbers of positive and negative eigenvalues, respectively, of the linking matrix defined by the framed link L. Also G_+ and G_- denote the partition function evaluations on an unknot with framings 1 and -1 respectively; they take the form of Gauss sums.

It turns out that for the manifolds $M_{n,t}$, the quantum $6j$ symbols in the local weights are particularly simple and the weight of a state can be reduced to a product of quantum dimensions. Let $\mathcal{D}_n$ be the link diagram of the $(2, n)$ torus knot containing n vertices with $n+2$ regions. Figure 2 shows $\mathcal{D}_5$; the associated blackboard framed link defines $M_{5,0}$. The states are indexed by a pair $a, i \in I$, where a is the label assigned to the single component of L (and thus also to the n shaded regions) and i is the label assigned to the interior unshaded region, with the constraint that (a, a, i) be a q-admissible triple. The invariant for the manifold $M_{n,t}$ is therefore

$$Z_r(M_{n,t}) = \frac{1}{G_\delta} \sum_{a=0}^{r-2} \sum_{j=0}^{\min a, r-2-a} (-1)^{at} A^{a(a+2)t}.\Delta_a \Delta_{2j} \left((-1)^{a-j} A^{a(a+2)-2j(j+1)} \right)^{-n},$$

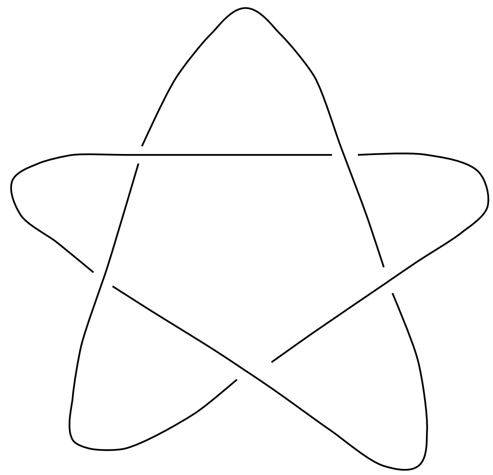

FIGURE 2. Link diagram for $M_{5,0}$

where $\delta = \operatorname{sgn} n + t$ and we have put $i = 2j$. See [20] for details of the calculation. By some simple algebraic manipulations, this may be rewritten as

$$Z_r(M_{n,t}) = \frac{c_r(n+t)}{c_r(\delta)} \overline{Z}_r(M_{n,t}),$$

where $c_r(m) = A^{-m}(1 + A^{-mr^2})$ and

$$\overline{Z}_r(M_{n,t}) = (-1)^{t+\alpha} A^{2n} \frac{\sum_{(x,y)\in X} (-1)^y q^{Q(x,y)}}{(q^{-\delta} - 1) \sum_{s=0}^{r-1} q^{\delta s^2}}. \tag{2.1}$$

Here

$$Q(x,y) = \frac{n}{8}(x+y)^2 + \frac{t-n}{16}(x-y)^2 - \frac{n}{8} + \frac{1}{4}(3x+y)$$

is a $\mathbb{Z}/r\mathbb{Z}$-valued quadratic form on $\mathbb{Z} \times \mathbb{Z}$, while $\alpha = \frac{r-1}{2}$ and X denotes the set of integral points in $[-\alpha, \alpha] \times [-\alpha, \alpha]$ whose coordinates have opposite parity. When $M_{n,t}$ is an integral homology sphere, $n + t = \delta$ and so $\overline{Z}_r(M_{n,t}) = Z_r(M_{n,t})$.

3. Sketch of Proof of Theorem 1.2

In this section we give the main steps in the proof of Theorem 1.2, starting from the formulation of $\overline{Z}_r(M_{n,t})$ in (2.1). We will be assuming that r is an odd prime which does not divide $N = |t + n|$ and all congruences are modulo r unless explicitly stated to the contrary. Let $\mathbb{Z}_r \subset \mathbb{Q}$ denote the set of those rationals whose denominators are not divisible by r. If $a, b \in \mathbb{Z}_r$ then we will write $a \equiv b$ to mean that $\frac{b-a}{r} \in \mathbb{Z}_r$. There is a natural map $\mathbb{Z}_r \longrightarrow \mathbb{Z}/r\mathbb{Z}$ defined by $x \longmapsto \overline{x}$ with $x \equiv \overline{x}$.

3.1. Reformulation of Sum. Since q is an rth root of unity,

$$\left(\sum_{s=0}^{r-1} q^{\delta s^2}\right) \cdot \left(\sum_{s=0}^{r-1} q^{-\delta s^2}\right) = r$$

and therefore the ratio of sums in (2.1) may be rewritten as

$$\frac{1}{r}\sum_{\substack{(x,y)\in X\\ s\in\mathbb{Z}/r\mathbb{Z}}}(-1)^y q^{Q(x,y)-\delta s^2}=\frac{1}{r}\sum_{\substack{x=-\alpha\\ x\ \text{even}}}^{\alpha}\sum_{y,s\in\mathbb{Z}/r\mathbb{Z}}\left(q^{Q(y,x)-\delta s^2}-q^{Q(x,y)-\delta s^2}\right). \qquad (3.1)$$

In the last step, the region X was broken into a union of sets of pairs of points of the form (x,y) and (y,x) where x is even and y is odd; the antisymmetry of the summand on the right hand side now enables the restriction on the parity of y to be removed without any effect on the resulting sum. The coefficient of q^P in (3.1) is therefore

$$\frac{1}{r}\sum_{\substack{x=-\alpha\\ x\ \text{even}}}^{\alpha}\left(\#\left\{(y,s)\in\mathbb{Z}/r\mathbb{Z}\times\mathbb{Z}/r\mathbb{Z}\;\middle|\;\delta(Q(y,x)-P)\equiv s^2\right\}\right.$$
$$\left.-\#\left\{(y,s)\in\mathbb{Z}/r\mathbb{Z}\times\mathbb{Z}/r\mathbb{Z}\;\middle|\;\delta(Q(x,y)-P)\equiv s^2\right\}\right) \qquad (3.2)$$

for any $P\in\mathbb{Z}/r\mathbb{Z}$.

LEMMA 3.1. *For any $a,\Delta\in\mathbb{Z}/r\mathbb{Z}$, with $a\not\equiv 0$,*

$$\#\left\{(y,s)\in\mathbb{Z}/r\mathbb{Z}\times\mathbb{Z}/r\mathbb{Z}\;\middle|\;ay^2+\Delta\equiv s^2\right\}=(r\delta_{\Delta\equiv 0}-1)\left(\frac{a}{r}\right)+r,$$

where δ_T denotes the Dirac delta function δ_T which is 1 if T is true and 0 otherwise.

As a corollary, one obtains that for any quadratic $Q(y)$ whose coefficient of y^2 is a, not divisible by r,

$$\#\left\{(y,s)\in\mathbb{Z}/r\mathbb{Z}\times\mathbb{Z}/r\mathbb{Z}\;\middle|\;Q(y)\equiv s^2\right\}=(r\delta_{\Delta\equiv 0}-1)\left(\frac{a}{r}\right)+r,$$

where $\Delta\in\mathbb{Z}_r$ denotes the discriminant $c-\frac{b^2}{4a}$. Applying this result to the two quadratics $Q(x,y)$ and $Q(y,x)$, considered as functions of y with x fixed, and noting that they have a common leading coefficient, namely $\frac{t+n}{16}$, one finds that (3.2) may be simplified to

$$\sum_{X=-[\frac{\alpha}{2}]}^{[\frac{\alpha}{2}]}\left(\delta_{\Delta_2(X)\equiv P}-\delta_{\Delta_1(X)\equiv P}\right)\left(\frac{|t+n|}{r}\right),$$

where $\Delta_1(x)$ and $\Delta_2(x)$ are the discriminants of $Q(2x,2y)$ and $Q(2y,2x)$ as quadratics in y. It may be verified that $\Delta_1(x)$ and $\Delta_2(x)$ are given by (1.3). Hence (3.1) reduces to

$$\sum_{X=-[\frac{\alpha}{2}]}^{[\frac{\alpha}{2}]}\left(q^{\overline{\Delta_2(X)}}-q^{\overline{\Delta_1(X)}}\right)\left(\frac{|t+n|}{r}\right),$$

and (2.1) now gives

$$\overline{Z}_r(M_{n,t})=\frac{(-1)^tA^{2n}}{1-q^{-\delta}}\left(\frac{N}{r}\right)(-1)^{\alpha}\sum_{X=-[\frac{\alpha}{2}]}^{[\frac{\alpha}{2}]}\left(q^{\overline{\Delta_1(X)}}-q^{\overline{\Delta_2(X)}}\right). \qquad (3.3)$$

3.2. Bernoulli numbers and holomorphic extensions. Let B_m denote the mth Bernoulli number, as defined by the generating function

$$\sum_{m=0}^{\infty}\frac{B_m z^m}{m!}=\frac{z}{\mathrm{e}^z-1}.$$

Following [8], we will use a symbolic notation employing the symbol "B" so that B^m refers to B_m.

LEMMA 3.2. *For any polynomial g, $\sum\limits_{i=a}^{b-1} g(i)=\int\limits_{B+a}^{B+b} g(x)\,dx$ whenever $a,b\in\mathbb{Z}$.*

When the integrand is not a polynomial, it is not in general possible to make sense of $\int_{a+B}^{b+B} g(x)\,dx$. However, if $g(x)$ has the form $q^{f(x)}$ where f is a polynomial, then the integrand may be considered as a formal power series in $h=q-1$ whose coefficients are polynomials and this allows the integral to be evaluated giving an element in $\mathbb{Q}[[h]]$ as the result. The following lemma allows the limits of such an integral to be shifted by multiples of r in $\mathbb{Z}_r$.

LEMMA 3.3. *Suppose that $f(y)$ is an $\mathbb{Z}_r$-valued polynomial function and that $a,b\in\mathbb{Z}_r\subset\mathbb{Q}$ with $b-a$ divisible by r in $\mathbb{Z}_r$. Then,*

$$\int_{B+a}^{B+b} q^{f(y)}\,dy-\frac{b-a}{r}\left(q^{f(0)}+\cdots+q^{f(r-1)}\right)$$

is divisible by q^r-1 in $\mathbb{Z}_r[[h]]$.

LEMMA 3.4. *If $x\in\mathbb{Z}_r$ then q^x, as a formal power series in $h=q-1$, defines an element of $\mathbb{Z}_r[[h]]$. Moreover, whenever $x\equiv y$ in $\mathbb{Z}_r$, then q^x-q^y is divisible by $(1+h)^r-1$ in $\mathbb{Z}_r[[h]]$.*

Set

$$\overline{B}_m=2\int_{B+\frac14}^{B+\frac34} x^m\,dx.$$

Then $2^{2m}\overline{B}_m$ is the mth Euler number and other expressions for $\overline{B}_m$ are

$$\overline{B}_m=4i.m!\sum_{s=-\infty}^{\infty}(-1)^s\big(2\pi i(2s+1)\big)^{-m-1}=4\int_{-\infty}^{\infty}\frac{(iz)^m\,dz}{\mathrm{e}^{2\pi z}+\mathrm{e}^{-2\pi z}}.$$

A symbolic notation may be employed in which $\overline{B}^m$ means $\overline{B}_m$ and then one has the following lemma.

LEMMA 3.5. *Whenever g is a polynomial, $g(\overline{B})=2\int_{B+\frac14}^{B+\frac34} g(x)\,dx$.*

Suppose that g is any analytic function. Then ascribe a meaning to $g(\overline{B})$ via the following formulae,

$$g(\overline{B})=\begin{cases}4\int\limits_{-\infty}^{\infty}\frac{g(iz)\,dz}{\mathrm{e}^{2\pi z}+\mathrm{e}^{-2\pi z}},\\ 2\sum\limits_{m\in\mathbb{Z}+\frac12}\mathrm{e}^{\pi i(m+\frac12)}g(\tfrac{m}{/}2),\end{cases}\tag{3.4}$$

for those functions g for which at least one of these expressions converges. Note that use of Cauchy's theorem allows one to pass formally between the two expressions, so that if it happens that both the sum and integral converge then they will be equal. From Lemma 3.5, it may be seen that whenever $g(x)$ is a polynomial function of x, the integral presentation for $g(\overline{B})$ given above matches the symbolic formulation.

LEMMA 3.5′. *When $g(x) = q^{f(x)}$ where f is a real quadratic function,*

$$g(\overline{B}) = 2\int_{B+\frac{1}{4}}^{B+\frac{3}{4}} g(x)\,dx,$$

where the right hand side is a formal power series in $h = q-1$ and the left hand side is a holomorphic function of $\ln q \in \mathbb{C}\backslash i\mathbb{R}$ and the meaning of the equality sign is that the formal power series is an asymptotic expansion of the holomorphic function around $q = 1$.

Applying Lemmas 3.4, 3.2, 3.3 and 3.5′ in turn, we obtain

$$\begin{aligned}
(-1)^{\alpha}\sum_{X=-\left[\frac{\alpha}{2}\right]}^{\left[\frac{\alpha}{2}\right]}\left(q^{\overline{\Delta_1(X)}} - q^{\overline{\Delta_2(X)}}\right) &= (-1)^{\alpha}\int_{B-\left[\frac{\alpha}{2}\right]}^{B+\left[\frac{\alpha}{2}\right]+1}\left(q^{\Delta_1(X)} - q^{\Delta_2(X)}\right)dX \\
&= \int_{B+\frac{1}{4}}^{B+\frac{3}{4}}\left(q^{\Delta_1(X)} - q^{\Delta_2(X)}\right)dX \\
&= \frac{1}{2}\left(q^{\Delta_1(\overline{B})} - q^{\Delta_2(\overline{B})}\right),
\end{aligned}$$

where in the second step we have used the fact that $-\left[\frac{\alpha}{2}\right]$ and $\left[\frac{\alpha}{2}\right]+1$ are congruent to $\frac{1}{4}$ and $\frac{3}{4}$ in either this or the reverse order according as α is even or odd. Throughout we are operating in the quotient of the ring of formal power series $\mathbb{Z}_r[[h]]$ in $h = q-1$, by the ideal generated by $\frac{q^r-1}{q-1}$. This is precisely what is meant by r-adic convergence and gives part (i) of Theorem 1.2.

Finally, in the last step, we know that by Lemma 3.5′, the formal power series may be viewed as the asymptotic expansion around $q = 1$ of a certain holomorphic function of $\ln q$ as defined by (3.4), namely (1.2) and this gives part (ii) of Theorem 1.2.

4. Other Roots of Unity

In this section we investigate the extent to which Theorem 1.2(i) can be generalised to deal with evaluations at roots of unity whose order is not prime, by observing how the steps in the proof given in §3 can be extended to such cases.

4.1. The Case of Prime Powers. Suppose $r = p^m$ $(m \geqslant 1)$, where p is an odd prime not dividing $N = |t+n|$. In the reformulation of the ratio of sums in §3.1, the only change is in Lemma 3.1, whose appropriate extension is given below. Again all congruences will be modulo r, unless stated otherwise, while for any $x \in \mathbb{Z}/r\mathbb{Z}$,

$$[x]_p \text{ denotes the integer } \begin{cases} \beta, & \text{if } p^\beta | x,\ p^{\beta+1} \nmid x,\ 0 \leqslant \beta < m, \\ m, & \text{if } x \equiv 0, \end{cases}$$

that is, the largest power of p in $\{0, 1, \ldots, m\}$ dividing x.

Lemma 3.1′. *For any $a, \Delta \in \mathbb{Z}/r\mathbb{Z}$, with a coprime to $r = p^m$,*

$$\#\left\{(y,s) \in \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z} \;\middle|\; ay^2 + \Delta \equiv s^2\right\}$$
$$= \begin{cases} r(1-p^{-1})([\Delta]_p + 1) + rp^{-1}\delta_{\Delta\equiv 0}, & \text{if } \left(\frac{a}{p}\right) = 1; \\ 0, & \text{if } \left(\frac{a}{p}\right) = -1,\ [\Delta]_p < m \text{ is odd;} \\ r(1+p^{-1}), & \text{if } \left(\frac{a}{p}\right) = -1,\ [\Delta]_p < m \text{ is even;} \\ p^{2[\frac{m}{2}]}, & \text{if } \left(\frac{a}{p}\right) = -1 \text{ and } \Delta \equiv 0. \end{cases}$$

As before, the same result holds when $ay^2 + \Delta$ is replaced by an arbitrary quadratic function $Q(y)$ whose leading coefficient is a and whose discriminant is Δ. It allows (3.3) to be derived in this case also, where now $\left(\frac{N}{r}\right)$ is replaced by the factor

$$\begin{cases} p^{-1} + (1-p^{-1})C_+, & \text{if } \left(\frac{N}{p}\right) = 1; \\ p^{-1}(-1)^{m+1} + (1+p^{-1})C_-, & \text{if } \left(\frac{N}{p}\right) = -1, \end{cases}$$

where $C_+ = \sum_{P \in \mathbb{Z}/r\mathbb{Z}} q^P [P]_p$ and $C_- = \sum_{P \in \mathbb{Z}/r\mathbb{Z}} q^P \delta_{[P]_p \text{ even}}$. The fact that q has order exactly p^m enables these sums to be evaluated to give $C_+ = 1$ and $C_- = (-1)^m$ and hence the total factor becomes $\left(\frac{N}{p}\right)^m$, which is nothing more than $\left(\frac{N}{r}\right)$.

In §3.2, r only enters in Lemmas 3.3 and 3.4. When r is only a prime power, these lemmas still hold if $\mathbb{Z}_r$ is everywhere replaced by $\mathbb{Z}_p$ while $(1+h)^r - 1$ is replaced by $h\phi_r(1+h)$, where ϕ_r is the cyclotomic polynomial. Lemmas 3.4, 3.2, 3.3 and 3.5′ may now be applied in just the same way as at the end of §3, except that now the terms should be viewed as elements of the quotient of $\mathbb{Z}_p[[h]]$ by the ideal generated by $\phi_r(1+h)$.

Theorem 1.2(i)′. *If $r = p^m$ is an odd prime power coprime to N then the formal power series $Z_\infty(M_{n,t}) \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{N}\right][[h]]$ evaluated at $h = q - 1$ where q is an rth root of unity, converges r-adically to $\left(\frac{N}{r}\right)\overline{Z}_r(q)$. That is, they are equal as elements of the quotient of $\mathbb{Z}_p[[h]]$ by the ideal generated by the cyclotomic polynomial $\phi_r(1+h)$.*

Since $\left[\binom{ip^{m-1}}{j}\right]_p = m - 1 - [j]_p$ and $\binom{ip^{m-1}}{jp^{m-1}} \equiv \binom{i}{j}$ for $1 \leqslant j \leqslant i < p$, thus the cyclotomic polynomial $\phi_r(1+h)$, which is

$$\frac{(1+h)^{p^m} - 1}{(1+h)^{p^{m-1}} - 1} = \sum_j h^j \sum_{i=0}^{p-1} \binom{ip^{m-1}}{j},$$

is a monic polynomial in h, all of whose coefficients, except the leading one, are divisible by p. Hence even when r is any prime power, similar statements to Ohtsuki's may be made, at least for manifolds $M = M_{n,t}$, namely

$$a_{l,r}(M) \equiv \left(\frac{N}{r}\right) a_{l,\infty}(M) \pmod p,$$

for $r = p^m$ and $0 \leqslant l < r(1-p^{-1})$.

4.2. Roots of Unity of Composite Order. When r is composite, say $r = \prod_{i=1}^k p_i^{m_i}$, the analogue of Lemma 3.1 is obtained by observing that for any $a, \Delta \in \mathbb{Z}/r\mathbb{Z}$,

$$\#\{(y,s) \mid ay^2 + \Delta \equiv s^2\} = \prod_{i=1}^k \#\left\{(y,s) \in (\mathbb{Z}/p_i^{m_i}\mathbb{Z})^2 \;\middle|\; ay^2 + \Delta \equiv s^2 \pmod{p_i^{m_i}}\right\},$$

where on the left hand side all expressions are considered to be in $\mathbb{Z}/r\mathbb{Z}$. Thus, whenever a and r are coprime, this number may be calculated with the help of Lemma 3.1$'$. Once again it will be found that (3.3) holds, where now $\left(\frac{N}{r}\right) = \prod_{i=1}^k \left(\frac{N}{p_i}\right)^{m_i}$ denotes the Legendre symbol.

Unfortunately though, the equivalent of Theorem 1.2(i) is now meaningless, since whenever $k > 1$, so that r is composite, $\phi_r(1+h)$ is invertible in $\mathbb{Z}_r[[h]]$. One can, however, in general make a direct link between the holomorphic function of (1.2) and $\overline{Z}_r(q) \in \mathbb{Z}[q]$. The main observation is that since $\Delta_1(x)$ and $\Delta_2(x)$ are quadratics with the same leading term and the same discriminant, they are therefore just translations of each other.

Lemma 4.1. *Suppose that r is an odd integer and that $\Delta(x) = ax^2+bx+c$ is a quadratic with rational coefficients whose denominators all divide some integer S coprime to r, while $\epsilon \in \mathbb{Z}$. Then the asymptotic expansion of the function*

$$2\int_{-\infty}^{\infty} \frac{\mathbf{e}^{k\Delta(iz)} - \mathbf{e}^{k\Delta(iz+\epsilon)}}{\mathbf{e}^{2\pi z} + \mathbf{e}^{-2\pi z}}\, dz$$

in k, around $k = \frac{2\pi i s}{r}$, has leading term

$$(-1)^\alpha \sum_{X=-[\frac{\alpha}{2}]}^{[\frac{\alpha}{2}]} \left(\mathbf{e}^{\frac{2\pi i s}{r}\overline{\Delta(X)}} - \mathbf{e}^{\frac{2\pi i s}{r}\overline{\Delta(X+\epsilon)}}\right),$$

where $r = 2\alpha+1$ and, for $X \in \mathbb{Z}$, $\overline{\Delta(X)}$ denotes the element of $\mathbb{Z}/r\mathbb{Z}$ defined by $\Delta(X) \in \mathbb{Q}$ and s is any integer divisible by $16S$. Indeed, when $|bs| < r$, the leading term in the expansion of the first term of the integral is precisely the value of that integral at $k = \frac{2\pi i}{r}$.

This lemma has a simple proof using the residue theorem for ϵ integral. The lemma can be extended to other ϵ for which $S\Delta(x+\epsilon) \in \mathbb{Z}[x]$ and thus the following theorem may therefore be deduced from (3.3).

Theorem 4.2. *For any root of unity, ζ, of odd order coprime to $N = |n+t|$, an appropriate sheet of the function $Z_\infty(M_{n,t})$ has an asymptotic expansion around $q = \zeta$ whose constant term is $\left(\frac{N}{r}\right)\overline{Z}_r(M_{n,t})$ evaluated at $q = \zeta$.*

5. Some Examples

The manifold $M_{n,t}$ is a $\mathbb{Z}$-homology sphere for $|t+n|=1$. In this case, $t+n=\delta$ and so $Z_r(M_{n,t}) = \overline{Z}_r(M_{n,t})$. Since $M_{n,-n-1}$ is the mirror image of $M_{-n,n+1}$, thus

$$Z_\infty(M_{-n,n+1})(q) = Z_\infty(M_{n,-n-1})(q^{-1}).$$

Indeed, as a power series in $h=q-1$ one has

$$Z_\infty(M_{n,-n-1}) = \lambda_0 + \lambda_1 h + \lambda_2 h^2 + \cdots,$$

where $\lambda_0 = 1$ and λ_m is a polynomial in n of degree $2m$. By geometric arguments it follows that $Z_\infty = 1$ when $n=\pm 1$ and so one may write $\lambda_m = \frac{(-1)^m}{8}(n^2-1)\beta_m$, for $m \in \mathbb{N}$, where β_m is a polynomial in n of degree $2m-2$. Indeed, it may be computed that $\beta_1 = 6$, $\beta_2 = \frac{1}{4}(25n^2-16n+3)$ and $\beta_3 = \frac{1}{48}(427n^4 - 528n^3 + 230n^2 - 48n + 15)$. It can now be seen explicitly that these first few coefficients are integral. As follows from Theorem 1.1, $\frac{1}{6}\lambda_1$ is always an integer, namely the Casson invariant of the 3-manifold. From the above explicit calculation it can be seen that λ_2 is divisible by 3 and is an odd multiple of $\frac{1}{6}\lambda_1$; the first statement has been shown to hold for general integer homology 3-spheres by Lin & Wang [23].

For the Poincaré homology sphere, $M_{-3,2}$, the first 14 terms of the expansion are

$$\begin{aligned} Z_\infty(M_{-3,2}) = 1 &- 6h + 45h^2 - 464h^3 + 6224h^4 - 102816h^5 + 2015237h^6 \\ &- 45679349h^7 + 1175123730h^8 - 33819053477h^9 + 1076447743008h^{10} \\ &- 37544249290614h^{11} + 1423851232935885h^{12} - 58335380481272491h^{13} + \cdots. \end{aligned}$$

The coefficient of h^m may be computed using the following line of Mathematica [39].

```
Zinf[m_]:=Sum[Coefficient[Binomial[30x^2-16x+9/8,m+1]-
    Binomial[30x^2-4x-7/8,m+1],x,2*i]*2^(-4*i)*EulerE[2i],{i,0,m}]/2
```

The ratio between mth and $(m-1)$th coefficients in the series for the $\mathbb{Z}$ homology sphere $M_{n,-n-1}$ in powers of h is asymptotically $\frac{2}{\pi^2}n(1-2n)m$. Further computations of Z_∞ for the family $\{M_{n,t}\}$ may be found in [20].

According to [24], the manifolds in this family are all three-fibred Seifert manifolds with multiplicities 2, n and $n-t$. Rozansky [34] has explicitly computed the asymptotic expansion of $Z_r(M)$ from the Witten-Chern-Simons path integral for Seifert manifolds and, according to Rozansky [35], the results obtained agree with the numerical values for the first few terms given above.

By work of Murakami and Ohtsuki, λ_m may be defined for all rational homology spheres. When M is a $\mathbb{Z}$-homology sphere, $\lambda_0(M) = 1$ and we may let $z_\infty(M)$ denote the formal power series $\ln Z_\infty(M)$, in $h\mathbb{Q}[[h]]$. Then z_∞ behaves additively under the operation of connected sum, while the coefficient of h^m in $z_\infty(M_{n,-n-1})$ is a polynomial in n (an odd integer) of degree $2m$. It follows that for each m, there exist integers A_n (for $1 \leqslant n \leqslant m$) for which

$$\sum_{n=1}^{m} A_n z_\infty(M_{2n+1,-2n-2})$$

has non-zero coefficient of h^m but zero coefficients of $h^1, \ldots, h^{m-1}$. For $\epsilon = +,-$, let $I_\epsilon = \{n \mid \epsilon A_n > 0, 1 \leqslant n \leqslant m\}$. Define X_ϵ to be the manifold obtained as the direct sum of ϵA_n copies of $M_{2n+1,-2n-2}$, over all $n \in I_\epsilon$. By construction the power series

$z_\infty(X_+)$ and $z_\infty(X_-)$ have their first discrepancy in the coefficient of h^m. Thus X_+ and X_- cannot be distinguished by $\lambda_1, \dots, \lambda_{m-1}$, but can be distinguished by λ_m.

COROLLARY 5.1. *For each $m > 1$, the manifold invariant $\lambda_m(M)$ is independent of the previous invariants in the sequence $\lambda_1, \dots, \lambda_{m-1}$.*

6. Conclusions

Since (3.3) was seen in §3 to hold for all odd r coprime to $N = |H_1(M, \mathbb{Z})|$, it follows that for this particular family of manifolds, $Z'_r(M) \in \mathbb{Z}[q]$ for *all* such r. It has been shown by Murakami [26] that $Z'_r(M) \in \mathbb{Z}[q]$ whenever r is an odd prime and M is a $\mathbb{Z}/r\mathbb{Z}$-homology 3-sphere.

CONJECTURE 6.1. *For all rational homology 3-spheres M and integers r coprime to $|H_1(M, \mathbb{Z})|$, the invariant $Z'_r(M)$ lies in $\mathbb{Z}[q]$.*

Indeed it is natural to suppose that once the correct normalisation has been found, this will also hold for all 3-manifolds, not just for $\mathbb{Q}$ homology spheres. Conjecture 1.3 on the integrality of the coefficients in the formal power series Z_∞ for integer homology spheres would lead one to believe that all these coefficients should have a neat combinatorial interpretation similar to that for the first of these, namely Casson's invariant; see [23] for the first step in this direction, namely the second coefficient. It should also be possible to directly compare the coefficients in the formal power series $Z_\infty(M)$, for an arbitrary manifold M, with those obtained from an expansion of (1.1).

If it is possible to generalise the approach in [20] and the current paper, to deal with general $\mathbb{Q}$-homology spheres, then one would hope for a generalisation of Theorem 4.2.

CONJECTURE 6.2. *There exists an invariant $Z_\infty(M)$ of rational homology spheres M, taking values in holomorphic functions of $\ln q \in \mathbb{C}\backslash i\mathbb{R}$ such that, for any odd integer r coprime to $N = |H_1(M, \mathbb{Z})|$ and any primitive rth root of unity ζ, an appropriate sheet of $Z_\infty(M)(q)$ has an asymptotic expansion about $q = \zeta$ whose constant term is $\left(\frac{N}{r}\right)$ times the value of the $SO(3)$ Witten-Reshetikhin-Turaev invariant $Z'_r(M)$ at $q = \zeta$.*

This conjecture would imply that almost all the information on $\mathfrak{sl}_2$-WRT invariants at roots of unity is contained in a new invariant, $Z_\infty(M)$, which is a holomorphic function of $\ln q$. It would be very interesting to understand the properties of this function when q is away from the unit circle.

For general Lie algebras, similar results are expected to hold. Paralleling the world of Vassiliev invariants for knots, there should be a universal 'quantum' invariant of 3-manifolds whose values are holomorphic fuctions of $\ln q$ in the graph cohomology of M; see [19]. It is already known how many of the algebraic structures existing in the theory of Vassiliev invariants (see [6], [7]) can be carried over to the theory of finite-type invariants of 3-manifolds; the reader is referred to [10], [29], [11], [13] and [12] for details. See [22] for a construction of a power series 3-manifold invariant, $\Omega(M)$, based on Kontsevich's universal Vassiliev invariant for links. According to [21], $\Omega(M)$ is a universal invariant, containing all finite type invariants of rational homology 3-spheres.

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