BRAID GROUP REPRESENTATIONS ASSOCIATED WITH $\mathfrak{sl}_m$

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Abstract. It has been seen elsewhere how elementary topology may be used to construct representations of the Iwahori-Hecke algebra associated with two-row Young diagrams, and how these constructions are related to the production of the same representations from the monodromy of n-point correlation functions in the work of Tsuchiya & Kanie and to the construction of the one-variable Jones polynomial. This paper investigates the extension of these results to representations associated with arbitrary multi-row Young diagrams and a functorial description of the two-variable Jones polynomial of links in S^3.

Keywords: Braid representations, homological constructions, Knizhnik-Zamolodchikov equation, local coefficient systems, configuration spaces, Jones polynomial.

1. Introduction

In [L 1], a construction for special types of Hecke algebra representations was given, using elementary topology. Let $H_n(q)$ be the **Iwahori-Hecke algebra** with generators $\sigma_1 \ldots, \sigma_{n-1}$ and relations,

$$\left.\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\ ; \qquad \text{for} \left|i-j\right| > 1\ ; \\ (\sigma_i - 1)(\sigma_i + q) &= 0\ . \end{aligned}\right\} \tag{1.1}$$

For generic q, the representation theory of $H_n(q)$ has the same structure as that of the symmetric group S_n (see [We]). In other words, to each Young diagram Λ with n squares, there is associated an irreducible representation π_Λ of $H_n(q)$. When Λ is a two-row Young diagram, π_Λ was constructed in [L 1] from the monodromy representation of a vector bundle over the configuration space X_n of n points in $\mathbf{C}$, on which a natural flat connection is defined. This construction will be briefly outlined in §2. It was also seen in [L 1], how these methods are related to those of [TK], where the same representations were arrived at from a discussion of the monodromy of correlation functions of a conformal field theory on $\mathbf{P}^1$. Indeed, it was shown in [TK], that the n-point correlation functions considered satisfy a system of differential equations known as the Knizhnik-Zamolodchikov equations, and the structure of the associated monodromy representations was established by investigating this system of equations.

The discussion of [TK] was based on the Lie algebra $\mathfrak{sl}_2$ and the spin-$1/2$ (vector) representation. However, similar constructions can be carried out for arbitrary Lie algebras and representations, see [SV]. Indeed, it was shown in [K], that using $\mathfrak{sl}_m$ and the m-dimensional vector representation, the monodromy representation of the associated system of differential equations again provides a representation of $H_n(q)$. The homological analogue of the constructions for the case of $\mathfrak{sl}_2$ and the spin-j representation was discussed in [L 4]. In this paper, the constructions for the case of $\mathfrak{sl}_m$ with the m-dimensional vector representation will be investigated. The general case of $\mathfrak{sl}_m$ with an arbitrary representation does not lead to representations of $H_n(q)$ but only to representations of the braid group $B_n = \pi_1(\widetilde{X}_n)$ where $\widetilde{X}_n$ is the configuration space of n distinct unordered points in $\mathbf{C}$. These braid groups have a presentation consisting of generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations given by the first two in (1.1); that is, $H_n(q)$ can be viewed as a quotient of the group algebra $\mathbf{C}B_n$ of B_n.

Any link, L, in S^3 can be represented by a braid in at least two distinct ways. Any element of the braid group B_n on n strings can be depicted by a diagram of n strings joining two sets of n points with 'over-crossings' and 'under-crossings' marked, as in Fig 1. Such a braid $\beta \in B_n$ may be closed by joining corresponding pairs of points in the two sets, so as to produce a link, denoted by $\widehat{\beta}$ (see Fig 2(a)). When n is even, another method for producing a link from β, is to join up adjacent pairs of points; that is, each set of n points is divided into $n/2$ adjacent pairs which are joined to produce the link denoted by $\widetilde{\beta}$ (see Fig 2(b)). Indeed, any systematic procedure for joining up the $2n$ points bounding a braid provides a closure. However, the closures $\widehat{\beta}$ and $\widetilde{\beta}$, whose particular significance is mainly historical, are known as the *braid* and *plait* closures, respectively, of the braid β. The two-variable Jones polynomial $X_L(q, \lambda)$ of the link L, expressed in the form of a braid closure $\widehat{\beta}$, may be evaluated in terms of a linear combination of characters of B_n associated with representations which factor through $H_n(q)$ (see [J] and §§2 and 5). The one-variable specialisation of X_L given by $\lambda = q$, may be expressed as a linear combination of characters of $H_n(q)$ evaluated on the image of $\beta \in B_n$ in $H_n(q)$, and associated with two-row Young diagrams. Using this fact, a functorial representation of $V_L(q) = X_L(q, q)$ was given in [L 2], which used only elementary topological constructions. This construction will be outlined in §2.

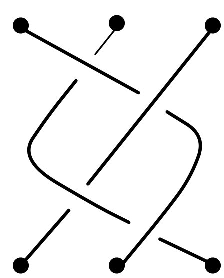
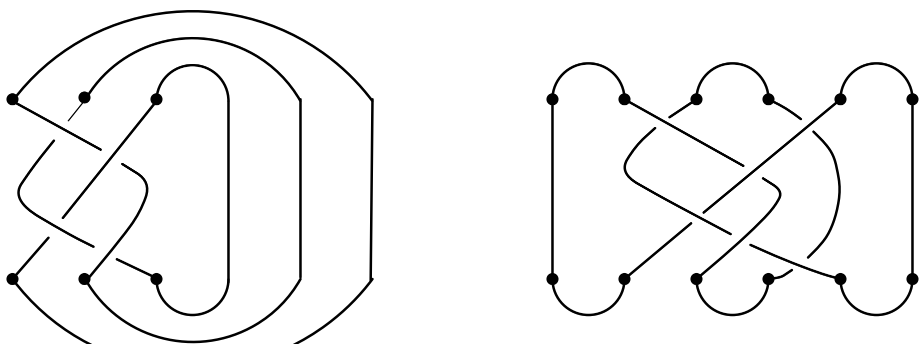


Figure 1

In §3, the basic representation theory of $\mathfrak{sl}_m$ will be recalled. The $\mathfrak{sl}_2$ construction of §2 will be generalised in §4 to provide the necessary geometric formulation



(a) (b)

Figure 2

of general Iwahori-Hecke algebra representations, which can be compared with the generalised Tsuchiya-Kanie theory. In §5, the homological approach of [L 1] and [L 2] is generalised in line with §4, to produce a topological interpretation of the two-variable Jones polynomial.

2. The $\mathfrak{sl}_2$ theory

Let V_j denote the spin-j representation of the Lie algebra $\mathfrak{sl}_2$. If H, E and F denote the standard generators for $\mathfrak{sl}_2$, with commutators,

$$[H,E]=2E, \qquad [H,F]=-2F, \qquad [E,F]=H,$$

then let Ω denote the polarised Casimir operator,

$$\tfrac{1}{2}(H\otimes H+2E\otimes F+2F\otimes E)\,. \tag{2.1}$$

The methods of [TK] produce representations of B_n as the monodromy of the system of differential equations,

$$\kappa\frac{\partial f}{\partial z_i}=\sum_{j\neq i}\frac{\mathbf{\Omega}_{ij}}{z_i-z_j}f\,, \tag{2.2}$$

for the vector-valued function $f\colon X_n\to W_t$. In their approach, Tsuchiya & Kanie construct f as the n-point correlation function,

$$\big\langle v \mid \Phi_1(u_1;z_1)\cdots\Phi_n(u_n;z_n) \mid \mathrm{vac}\big\rangle\,, \tag{2.3}$$

where $\Phi_1,\ldots,\Phi_n$ are spin-$\tfrac{1}{2}$ vertex operators, $(z_1,\ldots,z_n)\in X_n$, $u_1,\ldots,u_n\in V_{1/2}$, $\big|\mathrm{vac}\big\rangle\in V_0$ and $\big\langle v\big|\in\check V_t$. Such a function can be viewed as a map,

$$X_n\longrightarrow \check V_{1/2}^{\otimes n}\otimes V_t\,,$$

where $(z_1,\ldots,z_n)$ is mapped to the functional (2.3) on $V_{1/2}^{\otimes n}\otimes\check{V}_t$. Here W_t is defined to be the $\mathfrak{sl}_2$-invariant part of $\check{V}_{1/2}^{\otimes n}\otimes V_t$ (where t is a half-integer), and $\mathbf{\Omega}_{ij}$ acts on W_t by acting as $\mathbf{\Omega}$ on the i^{th} and j^{th} factors in $\check{V}_{1/2}^{\otimes n}$, and as the identity on the rest. Since the Casimir operator commutes with the action of $\mathfrak{sl}_2$, the action of $\mathbf{\Omega}_{ij}$ is well-defined on W_t. The dimension of W_t is the multiplicity of the representation V_t in $V_{1/2}^{\otimes n}$, and is given by,

$$n_t=\binom{n}{m}-\binom{n}{m-1},$$

where $m={}^{n}\!/_{2}-t$. That is, there is a decomposition,

$$V_{1/2}^{\otimes n}=\bigoplus(n_tV_t),$$

given by the $\mathfrak{sl}_2$-module structure, in which the sum is over all half-integers t, $0\le t\le {}^{n}\!/_{2}$ with ${}^{n}\!/_{2}-t\in\mathbf{Z}$. The Young diagram Λ_m is defined to have two rows with $n-m$ and m squares in them. It gives rise to a representation, π_{Λ_m}, of $H_n(q)$ of dimension n_t.

Theorem 2.1 [TK] *The monodromy representation obtained from (2.2) where f is considered as a W_t-valued function on X_n is a scaled version of the single irreducible representation π_{Λ_m} of $H_n(q)$, where m and t are related by ${}^{n}\!/_{2}-t=m$, while $q=\exp\left({}^{2\pi i}\!/_{\kappa}\right)$ is not a root of unity.*

The monodromy representation given by the **Knizhnik-Zamolodchikov equations** (2.2), where f is considered as a $V_{1/2}^{\otimes n}$-valued function on X_n, consists of a direct sum over m, of $(2t+1)$ copies of a scaled version of the representation π_{Λ_m} of $H_n(q)$.

The homological approach of [L 1] to these same irreducible representations π_{Λ_m} proceeds as follows. There is a natural map $X_{m+n}\to X_n$ given by taking the first n points, and this map has fibre $Y_{\mathbf{w},m}$ over $\mathbf{w}\in X_n$, which is the configuration space of m points in the punctured complex plane $\mathbf{C}\backslash\mathbf{w}$. The fundamental group $\pi_1(Y_{\mathbf{w},m})$ is a generalised version of the pure braid group on m strands, $P_m=\pi_1(X_m)$. It has generators, denoted $\beta_{\lambda\mu}$, in which all z_i are fixed except for one, say $z_j=\lambda$, which follows a loop with winding number $\delta_{\mu\nu}$ around each $\nu\in\{z_1,\ldots,z_m,w_1,\ldots,w_n\}\backslash\{\lambda\}$. The relations satisfied by these generators are quite complex, but their abelianisations are trivial; i.e. the abelianisation $H_1(Y_{\mathbf{w},m})$ is a free Abelian group. The generators $\beta_{\lambda\mu}$ are defined for all $\lambda\in\{z_1,\ldots,z_m\}$ and $\mu\in\{z_{j+1},\ldots,z_m,w_1,\ldots,w_n\}$, where $\lambda=z_j$. Thus, a local coefficient system χ on $Y_{\mathbf{w},m}$ may be defined by,

$$\chi\colon\pi_1(Y_{\mathbf{w},m})\longrightarrow\mathbf{C}^*,$$

where the images of the generators $\beta_{\lambda\mu}$ of $\pi_1(Y_{\mathbf{w},m})$ are specified as α if $\mu\in\{z_{j+1},\ldots,z_m\}$ and q if $\mu\in\{w_1,\ldots,w_n\}$. The middle-dimensional cohomology $H^m(Y_{\mathbf{w},m},\chi)$ may now be evaluated, and defines a vector bundle E^m over X_n. It is clear from the definitions that the natural actions of S_m and S_n on X_{m+n} given by permuting the points $z_1,\ldots,z_m$ and $w_1,\ldots,w_n$ also act on E^m, and preserve the fibres. Thus there is a related vector bundle $\widetilde{E}^m$ over $\widetilde{X}_n$ with identical fibres to E^m on which a natural action of S_m is defined. Since homology is homotopy invariant, there is a natural flat connection defined on $\widetilde{E}^m$, and the associated monodromy representation of $\pi_1(\widetilde{X}_n)=B_n$ is given by the following Theorem.

Theorem 2.2 [L 1] *The monodromy representation of B_n obtained on the S_m-antisymmetric part of $H^m(Y_{\mathbf{w},m},\chi)$ contains, as a subrepresentation on a subspace, $W_{n,m}$, the representation π_{Λ_m} of $H_n(q)$, whenever $\alpha=q^{-2}$ and q is not a root of unity.*

In [L 1], the subspace $W_{n,m}$ was defined explicitly. Let $\mathcal{S}_m$ be the set of m-tuples $\alpha_1,\ldots,\alpha_m$ with $\alpha_i\in\{z_{i+1},\ldots,z_m,w_1,\ldots,w_n\}$. For each $\underline{\alpha}\in\mathcal{S}_m$, define an embedding $\gamma_{\underline{\alpha}}\colon[0,1]^m\to Y_{\mathbf{w},m}$ of the hypercube, in which the r^{th} generator is mapped into a loop in which z_r goes around α_r with winding number 1. Let $\mathcal{T}_m$ be the subset of $\mathcal{S}_m$ consisting of m-tuples for which $\{\alpha_i\}$ are distinct elements of $\{w_1,\ldots,w_n\}$. Then the embedded m-cubes $\gamma_{\underline{\alpha}}$ (for $\underline{\alpha}\in\mathcal{S}_m$) may be lifted to embedded m-cubes in any covering of $Y_{\mathbf{w},m}$, and one may consider holomorphic functions, f, on $Y_{\mathbf{w},m}$ which twist according to the local coefficient system χ, while,

$$\int_{\gamma_{\underline{\alpha}}} f=0\,,\qquad \forall\underline{\alpha}\in\mathcal{S}_m\backslash\mathcal{T}_m\,.$$

Such functions define elements $f\,dz_1\ldots dz_m$ of the Dolbeault cohomology space $H^{m,0}(Y_{\mathbf{w},m},\chi)$, and the space spanned by such elements defines $W_{n,m}$. An explicit basis for $W_{n,m}$, with indexing set $\mathcal{T}_m$, was constructed in [L 1] and it was also shown in [L 1] and [L 3] that an isomorphism could be constructed between the approaches of Theorems 2.1 and 2.2.

In [L 4], this result was generalised to the case $\alpha=q^{-1/j}$. The associated homology construction gave rise to a representation of B_n which no longer factored through the quadratic algebra $H_n(q)$. Indeed, the generators σ_i of B_n were mapped to matrices which possessed $(2j+1)$ eigenvalues, and had dimension,

$$\binom{n}{m}_j-\binom{n}{m-1}_j\,,$$

where $\binom{n}{m}_j$ is the generalisation of the binomial coefficients $\binom{n}{m}$ to multinomial coefficients; $\binom{n}{m}_j$ is the coefficient of x^m in $(1+x+\cdots+x^{2j})^n$. These representations give eigenvalues with multiplicities for the generators σ_i, which coincide with those obtained from the approach of [TK] when generalised to $\mathfrak{sl}_2$ with higher representations; see [L 4].

Suppose L is a link, expressed as the plait closure of a braid $\beta\in B_{2n}$. Associate to the plane $\mathbf{C}$ with $2n$ marked points $w_1,\ldots,w_{2n}$, the vector space,

$$W_{2n,n}\subseteq H^n(Y_{\mathbf{w},m},\chi)\,.$$

The braid β induces an action $\pi(\beta)$ on $W_{2n,n}$.

Theorem 2.3 [L 2] *The one-variable Jones polynomial of the link $L = \widetilde{\beta}$ where $\beta \in B_{2n}$ may be expressed in the form,*

$$\big\langle v_n \bigm| \pi(\beta) \bigm| v_n \big\rangle \,,$$

where $\big|v_n\big\rangle$ and $\big\langle v_n\big|$ are elements of $W_{2n,n}$ and $W^*_{2n,n}$ independent of β. Moreover $\big|v_n\big\rangle$ is an element of the one-dimensional subspace of $W_{2n,n}$ which transforms totally antisymmetrically under the action of $S_2 \times \cdots \times S_2$ (n copies) in which each S_2 acts by interchanging adjacent pairs of points.

[L 2] went on to describe V_L in terms of the link L in a functorial way; that is, it associated to any collection of curves joining a set of $2m$ points on one plane to $2n$ points on a parallel plane, a map $W_{2n,n} \to W_{2m,m}$. The vectors $\big|v_n\big\rangle$, $\big\langle v_n\big|$ and the map $\pi(\beta)$ are all special cases of this map (see Figure 3). However, in this paper, it will only be necessary to note the result in the form of Theorem 2.3. In §5, we will see how Theorem 2.3 can be generalised to give a similar result for the full Jones polynomial, X_L.

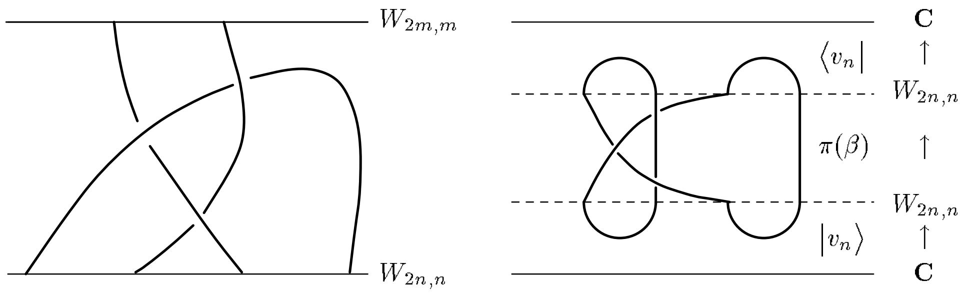


Figure 3

3. Representation theory for $\mathfrak{sl}_m$

The Lie group G, of all $m \times m$ matrices of determinant $+1$ has associated Lie algebra $\mathfrak{g} = \mathfrak{sl}_m$ generated by A_k and E_{ij} for $1 \le i, j \le m$, $i \ne j$ with commutation relations,

$$\begin{aligned}
[A_i, A_j] &= 0\,; \\
[A_i, E_{jk}] &= \alpha_{ijk} E_{jk}\,; \\
[E_{ij}, E_{kl}] &= \delta_{jk} E_{il}\,, && \text{for } i \ne j\,; \\
[E_{ij}, E_{ji}] &= A_i + A_{i+1} + \cdots + A_{j-1}\,, && \text{for } i < j\,;
\end{aligned}$$

where $\alpha_{ijk} = \delta_{ij} - \delta_{ik} - \delta_{i+1\,j} + \delta_{i+1\,k}$. The generator E_{ij} in the m-dimensional vector representation of $\mathfrak{sl}_m$ is given by the $m \times m$ elementary matrix with entries 0 everywhere except 1 in position (i, j). The generator A_i in this representation is given by the diagonal matrix with the only non-zero entries being 1 and -1 in the i^{th} and $i{+}1^{\text{th}}$ positions, respectively.

The Lie algebra $\mathfrak{g}$ consists of all trace-free $m \times m$ matrices. The maximal torus, $T \subseteq G$, consists of all diagonal matrices of determinant 1, while the associated Lie algebra $\mathfrak{t} = \operatorname{Lie} T$ consists of trace-free diagonal matrices, that is, it is generated by $A_1, \ldots, A_{m-1}$. The Dynkin diagram for G is shown in Fig 4.

$(m-1)$ nodes

Figure 4

Let

$$\Omega = \sum_{i=1}^{m-1} i(m-i)A_i^2 + \sum_{i<j} 2i(m-j)A_iA_j + \sum_{i \neq j} E_{ij}E_{ji} \,. \tag{3.1}$$

It may be easily checked by using the above commutation relations, that Ω commutes with all E_{jk} $(j \neq k)$ and all A_i. Indeed Ω is the Casimir operator for $\mathfrak{sl}_m$, up to a factor. In the special case of $m = 2$, the expression above for Ω reduces to,

$$A_1^2 + 2E_{12}E_{21} + 2E_{21}E_{12} \,,$$

while A_1, E_{12} and E_{21} correspond to H, E and F in the notation of §2.

Let V denote the standard vector representation of $\mathfrak{sl}_m$. Then $\Omega = (2m-1).I$ in this representation. The action of the polarisation of Ω,

$$\sum_{i=1}^{m-1} i(m-i)A_i \otimes A_i + \sum_{i<j} i(m-j)(A_i \otimes A_j + A_j \otimes A_i) + m \sum_{i \neq j} E_{ij} \otimes E_{ji} \,, \tag{3.2}$$

on $V \otimes V$ is given by,

$$e_i \otimes e_j \longmapsto m e_j \otimes e_i - e_i \otimes e_j \,,$$

whenever $1 \leq i, j \leq m$. Hence this action has eigenvalues $m{-}1$ and $-m{-}1$ with multiplicities ${}^1\!/_2 m(m{+}1)$ and ${}^1\!/_2 m(m{-}1)$, respectively.

Consider the system of differential equations of Knizhnik-Zamolodchikov type:

$$\kappa \frac{\partial f}{\partial z_i} = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} f \,, \tag{3.3}$$

where $f \colon X_n \to V^{\otimes n}$ is a vector-valued holomorphic function, and Ω_{ij} acts on $V^{\otimes n}$ by Ω, as given in (3.2), on the i^{th} and j^{th} factors, and the identity on the remaining factors. In [K], it was shown that this system gives rise to a monodromy representation of B_n which factors through the Hecke algebra $H_n(q)$, up to scaling. Here, $q = \mathrm{e}^{2\pi i\, m/\kappa}$. Indeed, the representation of B_n produced gives an action of σ_i on $V^{\otimes n}$ with eigenvalues,

$$\exp\left({}^{\pi i(m-1)}\!/_{\kappa}\right), \quad -\exp\left(-\,{}^{\pi i(m+1)}\!/_{\kappa}\right) .$$

That is after scaling by $-q^{(m+1)/2m}$, one obtains eigenvalues 1 and $-q$.

The representation of $\mathfrak{sl}_m$ on $V^{\otimes n}$ decomposes into irreducible representations,

$$V^{\otimes n} = \bigoplus_r n_r V_r \ , \tag{3.4}$$

where n_r is the multiplicity with which V_r occurs. The representation of $H_n(q)$ on $V^{\otimes n}$, defined by the monodromy of (3.3), decomposes into irreducible representations,

$$V^{\otimes n} = \bigoplus_r (\dim V_r)\pi_r \ , \tag{3.5}$$

where π_r is an irreducible representation of $H_n(q)$ and q is assumed not to be a root of unity. We may therefore index the decomposition (3.5) by Young diagrams with n squares; it follows, from the work of Kohno [K], that the only Young diagrams appearing in (3.5) are those with at most m rows. In the next section it will be seen how such Young diagrams can be used in a natural way to label the decomposition in (3.4), and in particular, to index irreducible representations of $\mathfrak{sl}_m$. The two relations (3.4) and (3.5) are dual: the multiplicities in (3.4) are the dimensions of the Hecke algebra representations π_r of (3.5).

4. Decompositions of representations of $\mathfrak{sl}_m$

Let V denote the (vector) m-dimensional representation of $\mathfrak{sl}_m$. Then $G = SL_m$ acts on $V^{\otimes nm}$ by,

$$\mathbf{A}(\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_{nm}) = \mathbf{A}\mathbf{v}_1 \otimes \cdots \otimes \mathbf{A}\mathbf{v}_{nm} \qquad \text{for } \mathbf{A} \in SL_m \ ,$$

whenever $\mathbf{v}_1, \ldots, \mathbf{v}_{nm} \in V \simeq \mathbf{C}^m$. The action of the Lie group $\mathfrak{g} = \mathfrak{sl}_m$ on $V^{\otimes nm}$ is similarly given.

Lemma 4.1 *The $\mathfrak{sl}_m$-invariant part of $V^{\otimes nm}$ has dimension,*

$$C_{m,n} \equiv \frac{(mn)!(m-1)!!(n-1)!!}{(m+n-1)!!} \ ,$$

and has a basis which may be indexed by reduced tableax of shape $\Lambda_{n,m}$, the rectangular Young diagram with m rows of n squares each. Here $m!!$ denotes the product $m!(m-1)!\cdots 2!1!$.

Proof: Suppose $\mathbf{v} \in V^{\otimes nm}$ is invariant under the action of $\mathfrak{sl}_m$. Denote by $v(\alpha_1, \ldots .\alpha_{nm})$ be the $m \times \cdots \times m$ (with nm m's) dimensional tensor specifying the components of $\mathbf{v}$. Under the action of any $A \in \mathfrak{sl}_m$, it is necessary that,

$$\sum_{r=1}^{nm} \sum_{\alpha_i\text{'s}} v(\alpha_1, \ldots, \alpha_{nm}) e_{\alpha_1} \otimes \cdots \otimes A e_{\alpha_r} \otimes \cdots \otimes e_{\alpha_{nm}} \ , \tag{4.1}$$

In particular applying this with $A = A_i$ leads to the fact that $v(\alpha_1, \ldots, \alpha_{nm})$ can only be non-zero if the numbers of occurrences of i and $i{+}1$ in the sequence $\alpha_1, \ldots, \alpha_{nm}$ are equal for all i. That is, the only non-zero components of v come from those sequences $\underline{\alpha}$ in which each element of $\{1, 2, \ldots, m\}$ appears exactly n times. There are,

$$\binom{nm}{n}\binom{n(m-1)}{n}\cdots\binom{2n}{n} = \frac{(mn)!}{(n!)^m}$$

such sequences $\underline{\alpha}$, and thus this provides an upper bound on $\dim\big(V^{\otimes nm}\big)_0$.

Next, apply (4.1) with $A = E_{ij}$. This gives,

$$\sum_{\beta_r = i} v(\beta_1, \ldots, \beta_{r-1}, j, \beta_{r+1}, \ldots \beta_{nm}) = 0 \;,$$

for all i, j and $\underline{\beta} = (\beta_1, \ldots, \beta_{nm})$ sequences of integers in $\{1, 2, \ldots, m\}$ containing $n + \delta_{ik} - \delta_{jk}$ occurrences of the integer k. The sum is over the $n{+}1$ positions r in the sequence $\underline{\beta}$ corresponding to the integer i; and the statement is that the sum of the v's associated with the $n{+}1$ sequences obtained from $\underline{\beta}$ by replacing one of the occurrences of i with j, vanishes. It may be shown that these relations enable any $v(\underline{\alpha})$ to be uniquely expressed as a linear combination of those associated with $\underline{\alpha} \in \mathcal{S}$, where $\mathcal{S}$ consists of sequences in which the r^{th} occurrences of $1, 2, \ldots, m$ occur in this order for all $r = 1, 2, \ldots, n$. Any $\underline{\alpha}$ defines a Young tableau of shape $\Lambda_{n,m}$ in which the number placed in the j^{th} square in row i is the position of the j^{th} occurrence of i in the sequence $\underline{\alpha}$. It is seen that $\underline{\alpha} \in \mathcal{S}$ if, and only if, the associated Young diagram tableau is **reduced**; that is, the entries increase when scanning along any row from left to right, and along any column from top to bottom. The dimension of the $\mathfrak{sl}_m$-invariant part of $V^{\otimes nm}$ is thus the order of $\mathcal{S}$, the number of reduced tableaux of shape $\Lambda_{n,m}$.

However, the representation theory of the symmetric group S_{nm} leads to a construction of the representation of S_{nm} associated with Young diagram $\Lambda_{n,m}$ on a vector space indexed by reduced tableaux of shape $\Lambda_{n,m}$. Thus the order of $\mathcal{S}$ is exactly the dimension of the representation of S_{nm} associated with $\Lambda_{n,m}$. The hook length formula may be used to evaluate this. ■

In this lemma, the numbers $C_{m,n}$ are generalised Catalan numbers, while $C_{2,n}$ gives the usual Catalan number $\frac{1}{n+1}\binom{2n}{n}$ and $C_{m,n} = C_{n,m}$. Let Λ be a Young diagram with n squares and at most m rows. Let $l_1, \ldots, l_m$ be the lengths of the rows of Λ, with $l_m \geq l_{m-1} \geq \ldots \geq l_1 \geq 0$, and $l_1 + \cdots + l_m = n$. Set $j_r = {}^n\!/\!_m - l_r$ for $r = 1, 2, \ldots, m$. Then $\{j_r\}$ are multiples of ${}^1\!/\!_m$ congruent to ${}^n\!/\!_m \pmod 1$, with sum 0. Since $\{l_r\}$ is a non-decreasing sequence, $\{j_r\}$ is a non-increasing sequence. Such a Young diagram Λ gives rise to a term in the decomposition (3.5). In [K], it was shown that the multiplicity of π_Λ in (3.5) is,

$$\frac{\prod_{i>j}(l_i - l_j + i - j)}{(m-1)!!} = \frac{\prod_{1\leq j<i\leq m-1}(j_j - j_i + i - j)}{(m-1)!!} \cdot \prod_{i=1}^{m-1}(j_1 + \cdots + 2j_i + \cdots + j_{m-1} + m - i). \tag{4.2}$$

We thus expect, from the Weyl duality mentioned at the end of §3, that the irreducible representations of $\mathfrak{sl}_m$ exist with the above dimensions.

Lemma 4.2 *The action of $\mathfrak{sl}_m$ on $V^{\otimes n}$ may be decomposed as,*

$$\bigoplus_{j_i} N_n(j_1,\ldots,j_{m-1})V(j_1,\ldots,j_{m-1})\ ,$$

where $V(j_1,\ldots,j_{m-1})$ is an irreducible representation of $\mathfrak{sl}_m$, defined for any non-increasing sequence of $(m-1)$ elements of ${}^1\!/\!{}_m.\mathbf{Z}$, congruent to each other $\bmod 1$, *and with $j_1+\cdots+j_{m-2}+2j_{m-1}\geq 0$. The decomposition is indexed by those j's for which $j_i\equiv {}^n\!/\!{}_m (\bmod 1)$, $j\leq {}^n\!/\!{}_m$. Moreover, the multiplicity $N_n(j_1,\ldots,j_{m-1})$ is the dimension of the representation of S_n associated with the Young diagram Λ containing ${}^n\!/\!{}_m+j_1+\cdots+j_{m-1}$, ${}^n\!/\!{}_m-j_{m-1},\ldots,{}^n\!/\!{}_m-j_1$ squares in its m rows.*

We will now make a few remarks on how this Lemma fits into the classical representation theory of the Lie algebras. The maximal torus T may be parametrised as,

$$\begin{pmatrix} \mathrm{e}^{2\pi i\theta_1} & & & \\ & \ddots & & \\ & & \mathrm{e}^{2\pi i\theta_{m-1}} & \\ & & & \mathrm{e}^{-2\pi i(\theta_1+\cdots+\theta_{m-1})} \end{pmatrix},$$

and then the system of roots obtained is generated by $\theta_1-\theta_2,\ldots,\theta_{m-2}-\theta_{m-1}$ and $\theta_1+\cdots+\theta_{m-2}+2\theta_{m-1}$. The Weyl group in this case is just S_m, and the generators $(i\,i{+}1)$ act on the roots by transposing θ_i and θ_{i+1}, the result being a reflection in the hyperplane on which one of the above roots vanishes. A suitable fundamental Weyl chamber is bounded by such hyperplanes; for example,

$$\left.\begin{array}{l} \theta_1\geq\theta_2\geq\cdots\geq\theta_{m-1}\ ; \\ \theta_1+\cdots+\theta_{m-2}+2\theta_{m-1}\geq 0\ . \end{array}\right\} \tag{4.3}$$

Irreducible representations of $\mathfrak{sl}_m$ are now indexed by points in this region for which,

$$\begin{array}{l} \theta_i-\theta_{i+1}\in\mathbf{Z}\ ,\qquad (i=1,2,\ldots,m-2) \\ \theta_1+\cdots+\theta_{m-2}+2\theta_{m-1}\in\mathbf{Z}\ . \end{array}$$

Such points have $m\theta_i\in\mathbf{Z}\quad\forall i$, while having $\theta_1,\ldots,\theta_{m-1}$ all congruent to each other modulo 1.

The point $(0,\ldots,0)$ gives rise to the trivial (1-dimensional) representation, while the vector representation V, is associated with the point $({}^1\!/\!{}_m,\ldots,{}^1\!/\!{}_m)$. The tensor product of two representations may be decomposed into irreducible representations, and one obtains a relation analogous to the Clebsch-Gordon relation for $\mathfrak{sl}_2$. For tensor products with V, this relation is,

$$({}^1\!/\!{}_m,\ldots,{}^1\!/\!{}_m)\otimes(j_1,\ldots,j_{m-1})=(j_1+{}^1\!/\!{}_m,\ldots,j_{m-1}+{}^1\!/\!{}_m)\oplus\bigoplus_{r=1}^{m-1}(j_s-\delta_{rs}+{}^1\!/\!{}_m)_{s=1}^{m-1}. \tag{4.4}$$

Thus one obtains a decomposition into m terms. This relationship should be interpreted as meaning that the tensor product representation on l.h.s. can be decomposed into irreducible representations as given on the r.h.s., when these lie within the region defined by (4.3). Any term which does not satisfy (4.3) should be omitted. By repeated application of (4.4), Lemma 4.2 may be verified. The dimension of the representation $V(j_1,\dots,j_{m-1})$ is given by (4.2). For example, it is easily seen that when $j_i = {}^1\!/\!_m\ \forall 1\le i\le m-1$, (4.2) reduces to,

$$\big((m-2)!!.m!\big)\big((m-1)!!\big)^{-1} = m\,.$$

Example Consider the case $m=3$. The irreducible representations of $\mathfrak{sl}_3$ are indexed by those j_1 and j_2 multiples of $1/3$ which are congruent mod 1, with $j_1 \ge j_2$ and $j_1+2j_2 \ge 0$. The dimension of $V(j_1,j_2)$ is $1/2(j_1-j_2+1)(2j_1+j_2+2)(j_1+2j_2+1)$ and the multiplicity with which it occurs in the decomposition of the 3^n-dimensional representation of $\mathfrak{sl}_3$ on $V^{\otimes n}$ is,

$$\frac{n!(j_1-j_2+1)(2j_1+j_2+2)(j_1+2j_2+1)}{(n/3-j_1)!(n/3-j_2+1)!(n/3+j_1+j_2+2)!}\,,$$

so long as $j_1 \le n/3$ (see (4.2)).

In Fig. 5, the fundamental Weyl chamber described above is drawn, and the points associated with irreducible representations of $\mathfrak{sl}_3$ are pictured for $j_1 \le 5/3$. The numbers written next to each such point give the dimensions of these representations.

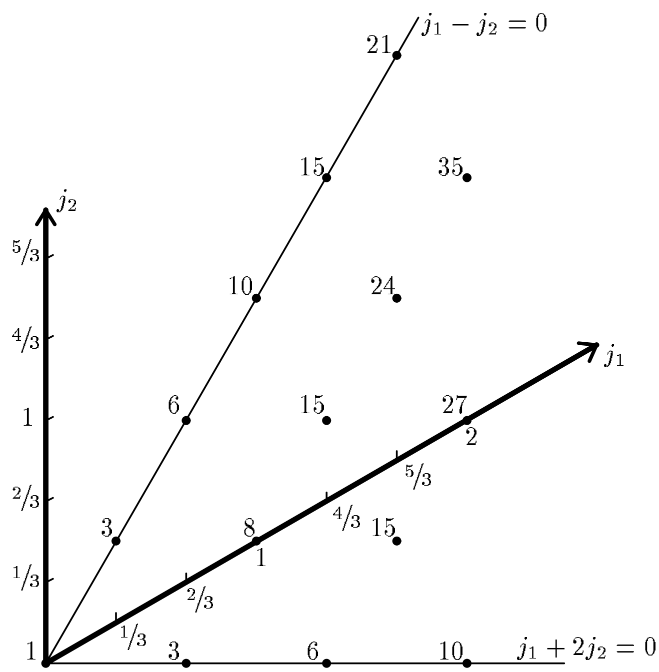


Figure 5

Thus for $n = 5$, it is seen that $V^{\otimes 5}$ may be decomposed as,

$$6V({}^2/_3,{}^2/_3) \oplus V({}^5/_3,{}^5/_3) \oplus 5V({}^2/_3,-{}^1/_3) \oplus 4V({}^5/_3,{}^2/_3) \oplus 5V({}^5/_3,-{}^1/_3)\ ,$$

while the representations $V(j_1, j_2)$ mentioned here have dimensions 6,21,3,24,15, respectively, and $6.6 + 1.21 + 5.3 + 4.24 + 5.15 = 243 = 3^5$.

5. The general topological construction and the Jones polynomial

Let Λ be any Young diagram with m rows of lengths $l_1, l_2, \ldots, l_{m-1}$ and $n - l_1 - l_2 - \cdots - l_{m-1}$ from the bottom row upwards. Define $k_1, \ldots, k_m$ by,

$$\begin{aligned} k_r &= l_1 + \cdots + l_r\ , \qquad \text{for } 1 \leq r \leq m{-}1 \\ k_m &= n\ . \end{aligned}$$

Let $K = k_1 + \cdots + k_{m-1} = \sum_{r=1}^{m-1}(m-r)l_r$. Then we may consider $Y_{\mathbf{w},K}$ with local coordinates $\{z_s^{(r)}\}_{s=1}^{k_r}{}_{r=1}^{m-1}$, where $\mathbf{w} = (w_1, \ldots, w_n)$. A local coefficient system χ may be placed on $Y_{\mathbf{w},K}$ in such a way that when $z_s^{(r)}$ follows a loop of winding number 1 around $z_{s'}^{(r')}$, the system twists by,

$$\begin{aligned} &q\ , && \text{if } |r - r'| = 1\ ; \\ &q^{-2}\ , && \text{if } r = r'\ ; \\ &1\ , && \text{if } |r - r'| > 1\ . \end{aligned}$$

It is also necessary to specify that when $z_s^{(r)}$ follows a loop of winding number 1 around w_i, the system twists by,

$$\begin{aligned} &1\ , && \text{if } r < m{-}1\ ; \\ &q\ , && \text{if } r = m{-}1\ . \end{aligned}$$

On the space $Y_{\mathbf{w},K}$, there is a natural action of S_K. There is a natural subgroup of S_K of the form $S_{k_1} \times S_{k_2} \times \cdots \times S_{k_{m-1}}$ given by permuting $\{z_s^{(r)}\}$ while leaving r invariant. The analogue of Theorem 2.2 involves the part of $H^K(Y_{\mathbf{w},K}, \chi)$ which transforms totally antisymmetrically under the action of $S_{k_1} \times \cdots \times S_{k_{m-1}}$. This is meaningful since the local system χ is invariant under the action of this subgroup of S_K.

Theorem 5.1 *The monodromy action of B_n on the part of $H^K(Y_{\mathbf{w},K}, \chi)$ transforming totally antisymmetrically under $(S_{k_1} \times \cdots \times S_{k_{m-1}})$ contains a large irreducible part which factors through the Hecke algebra $H_n(q)$, and is associated with the Young diagram Λ.*

Sketch of Proof: This result follows in a similar way to that of Theorem 2.2 in [L 1]. Define an order on the formal symbols $\{z_q^{(p)} \mid 1 \le p \le m, 1 \le q \le k_p\}$ by $z_q^{(p)} < z_s^{(r)}$ if, and only if, $p < r$ or $p = r$ and $q < s$, where for convenience, $z_s^{(m)} \equiv w_s$ with $k_m \equiv n$. Just as in §2 where the set $\mathcal{S}_m$ indexed a basis for a set of chains, so one may define a set of maps on formal symbols,

$$\mathcal{S}_{\mathbf{k}} = \left\{ \underline{\alpha}\colon \{z_s^{(r)}\} \longrightarrow \{z_s^{(r)}\} \cup \{w_i\} \;\middle|\; \underline{\alpha}(z_s^{(r)}) > z_s^{(r)} \quad \forall r, s \right\}.$$

For any $\underline{\alpha} \in \mathcal{S}_{\mathbf{k}}$, there is an associated embedded K-dimensional hypercube $\gamma_{\underline{\alpha}}$ in $Y_{\mathbf{w},m}$ which may be lifted to the cover of $Y_{\mathbf{w},m}$ associated with the local system χ. The actions of B_n and $S_{k_1} \times \cdots \times S_{k_{m-1}} < S_K$ upon these chains, and thereby also on homology, as the kernel of a suitable boundary map on chains, may be deduced using the general recurrence relations of [L 1] which relate such actions to one another, as they progress along a tower,

$$Y_{\mathbf{w},K} \longrightarrow Y_{\mathbf{w},K-1} \longrightarrow \cdots \longrightarrow Y_{\mathbf{w},1} = \mathbf{C} \backslash \mathbf{w}\ .$$

The dual space is also indexed by the sets $\mathcal{S}_{\mathbf{k}}$. Let $(\underline{\alpha})$ denote a cochain dual to $\gamma_{\underline{\alpha}}$. It is found that there is a subspace of the space of cochains invariant under the action of B_n, and indexed by the subset $\mathcal{T}_{\mathbf{k}}$ of $\mathcal{S}_{\mathbf{k}}$, defined by,

$$\mathcal{T}_{\mathbf{k}} = \Big\{ \text{injective } \underline{\alpha} \in \mathcal{S}_{\mathbf{k}} \;\Big|\; \underline{\alpha}(z_s^{(r)}) \in \{ z_p^{(r+1)} \mid 1 \le p \le k_{r+1} \} \quad \text{whenever } 1 \le s \le k_r \text{ and } 1 \le r \le m-1 \Big\}.$$

We now give explicit formulae for the $S_{k_1} \times \cdots \times S_{k_{m-1}} < S_K$ totally antisymmetric part, denoted $W'_{n,k_{m-1},\ldots,k_1}$, of the space of cochains just described, along with the space of boundaries and the action of B_n.

It will first be necessary to define some notation. For any set $\lambda \subset \mathbf{N}$, $\bar{\lambda}$ will denote the ordered list of elements of λ, in an increasing sequence. Let l and h denote the maps,

$$\begin{aligned} l\colon S_n &\longrightarrow \mathbf{N} \cup \{0\}, \\ h\colon S_n &\longrightarrow H_n(q), \end{aligned}$$

defined on the symmetric group S_n, by

$$\begin{aligned} h(\sigma) &= \sigma_{i_1} \ldots \sigma_{i_k}\ , \\ l(\sigma) &= k\ , \end{aligned}$$

where $\sigma \in S_n$ can be written as a reduced word $(i_1\, i_1 + 1) \ldots (i_k\, i_k + 1)$ in the generators $\{(i\, i{+}1) \mid 1 \le i \le n{-}1 \quad \forall p\}$. It should be noted that h is well-defined, but is not a homomorphism, while $l(\sigma)$ is known as the **length** of σ.

It is convenient to describe the formulae in a recursive manner. Suppose that $\lambda_i \subset \{1, 2, \ldots, k_{i+1}\}$ are sets of order k_i, for $1 \le i \le m{-}1$. We define $[\lambda_1, \ldots, \lambda_r]$

for $1 \le r \le m-1$, and the representations, π_r, of $H_{k_{r+1}}(q)$ $(0 \le r \le m-1)$ on the spaces of such symbols explicitly by,

$$[\lambda_1, \ldots, \lambda_r] = \sum_{\sigma \in S_{k_r}} q^{-l(\sigma)} \Big(\pi_{r-1}\big(h(\sigma)\big)[\lambda_1, \ldots, \lambda_{r-1}], \sigma(\overline{\lambda}_r) \Big) , \tag{5.1}$$

and,

$$\pi_r(\sigma_i)[\lambda_1, \ldots, \lambda_r] = \begin{cases} [\lambda_1, \ldots, \lambda_r] , & \text{if } i, i+1 \notin \lambda_r; \\ q[\lambda_1, \ldots, \lambda_{r-1}, (i\, i+1)\lambda_r] , & \text{if } i \notin \lambda_r, i+1 \in \lambda_r; \\ (1-q)[\lambda_1, \ldots, \lambda_r] + [\lambda_1, \ldots, (i\, i+1)\lambda_r] , & \text{if } i \in \lambda_r,\ i+1 \notin \lambda_r; \\ \big[\pi_{r-1}(\sigma_t)[\lambda_1, \ldots, \lambda_{r-1}], \lambda_r\big] , & \text{if } i, i+1 \in \lambda_r; \end{cases} \tag{5.2}$$

where, in the last line, t denotes the position at which i occurs in $\overline{\lambda}_r$. These recursive formulae are initialised with $\pi_0 = \text{id}$. It can be checked that the recursive formulae are well-defined, so that the above definitions of actions of the σ_i for $1 \le i \le k_{r-1}$ under π_r extend to a Hecke algebra representation. Then,

(1) the symbol $[\lambda_1, \ldots, \lambda_{m-1}]$ is a formal combination of symbols $(\mu_1, \ldots, \mu_{m-1})$, where μ_i is an ordered list of length k_i, of elements of $\{1, 2, \ldots, k_{i+1}\}$. It thus represents a cochain via the identifications of $(\mu_1, \ldots, \mu_{m-1})$ with $\underline{\alpha} \in \mathcal{T}_{\mathbf{k}}$ in which $\underline{\alpha}(z_s^{(r)})$ is $z_p^{(r+1)}$, where p is the s^{th} element of the list μ_r;

(2) the space $\langle \{[\lambda_1, \ldots, \lambda_{m-1}]\} \rangle$ defines a subspace $W'_{n,k_{n-1},\ldots,k_1}$ of the space of cochains indexed by $\mathcal{S}_{\mathbf{k}}$, which is antisymmetric under the action of $S_{k_1} \times \cdots \times S_{k_{m-1}}$;

(3) the monodromy action of B_n upon $W'_{n,k_{m-1},\ldots,k_1}$ defined by the Gauss-Manin connection on the vector bundle over $\widetilde{X}_n$ with fibres $H^K(Y_{\mathbf{w},K}, \chi)$ is given by a quotient of the representation π_{m-1}.

In order to compute the monodromy representation in (3) it is still necessary to give the form of the cocycles, that is, the dual of the boundaries map, by whose image we should divide out. This invariant subspace of $W'_{n,k_{n-1},\ldots,k_1}$ spanned by the boundaries, can also be simply written down in terms of π_r defined in (5.2). It is spanned by,

$$\left\{ \sum_{\substack{j=1 \\ j \notin \lambda_r}}^{k_{r+1}} q^{-x} \Big[\pi_{r-1}(\sigma_x \ldots \sigma_1)[\lambda_1, \ldots, \lambda_{r-1}], \lambda_r \cup \{j\}, \lambda_{r+1}, \ldots, \lambda_{m-1} \Big] \right\} , \tag{5.3}$$

for $1 \le r \le m-1$ where $\lambda_i \subset \{1, 2, \ldots, k_{i+1}\}$ for $1 \le i \le m-1$ while $|\lambda_i| = k_i - \delta_{ir}$. In this formula, x denotes the number of elements of λ_r less than j. The quotient of $W'_{n,k_{m-1},\ldots,k_1}$ by the space spanned by vectors of the form (5.3) defines a subspace of cohomology, denoted by $W_{n,k_{m-1},\ldots,k_1}$.

The proof of the above formulae relies very heavily on the particular form of the twisting parameters in χ. Note for $m = 2$, π_1 is seen to be the two-row Hecke algebra representation $\pi_{\Lambda_{k_1}}$ of Theorem 2.2. It can be seen recursively from the above explicit form for π_r that it is the irreducible representation of $H_{k_{r+1}}(q)$ associated with the $r+1$-row Young diagram with rows of lengths $k_{r+1}-k_r,\ k_r-k_{r-1},\ldots,k_2-k_1$ and k_1. The $(m-1)^{\text{th}}$ stage gives that π_{m-1} is the irreducible representation, π_Λ, of $H_n(q)$ associated with Λ. ■

The recursive nature of the formulae (5.1) and (5.2) is not essential, and was only used to ease difficult notation. It is interesting to note that (5.2) can be used to define a repesentation π' of $H_n(q)$ on $V^{\otimes\binom{n}{k}}$ whenever a representation π of $H_n(q)$ is given, for any $n = k$. This jacking up procedure is an iterated version of that in [L 1], using k iterations, and doesn't seem to come simply from any of the usual operations used in representation theory. It should also be noted that the local coefficient system χ defined here is identical to that defined in [SV] for the case of $\mathfrak{sl}_m$ and the vector representation. Thus, to each node in the Dynkin diagram of $\mathfrak{sl}_m$, [SV] associates a collection of variables, $\{z_s^{(r)} \mid 1 \leq s \leq k_r\}$ to the r^{th} node. The twistings in χ are then defined to be $q^{-a_{rr'}}$ where $(a_{rr'})$ is the Cartan matrix of $\mathfrak{sl}_m$.

We shall now develop the analogue of the connection with the Jones polynomial of §2, in this more general case. Let L be a link, expressed as the braid closure of a braid $x \in B_n$. In [J] it was seen that the two-variable Jones polynomial of L is,

$$X_L(q,\lambda) = \left(\frac{\lambda q - 1}{\sqrt{\lambda}(q-1)}\right)^{n-1} (\sqrt{\lambda})^e \operatorname{tr}\bigl(\pi(x)\bigr)\,, \tag{5.4}$$

where e is the exponent sum of x as a word in the braid group generators σ_i; and $\pi\colon B_n \to H_n(q)$ is the natural quotient map. The map tr is the **Ocneanu trace** defined on $\coprod_n H_n(q)$ by,

$$\operatorname{tr}(1) = 1,$$
$$\operatorname{tr}(x\sigma_n) = z\operatorname{tr}(x),$$

whenever x is a word in $\sigma_1,\ldots,\sigma_{n-1}$. Here z denotes $\frac{1-q}{\lambda q-1}$. It was also seen in [J], that the restriction of tr to $H_n(q)$ can be expressed as a linear combination of characters χ_Λ on $H_n(q)$ associated with Young diagrams Λ with n squares. This explicit relation is,

$$\operatorname{tr}(x) = \sum_\Lambda W_\Lambda(q,z)\chi_\Lambda(x)\,, \tag{5.5}$$

where the sum is over all Young diagrams with n squares, and the coeficients $W_\Lambda(q,z)$ are defined by,

$$\prod_{(i,j)} \left(\frac{q^i z - q^j w}{1 - q^{l(i,j)}}\right) = W_\Lambda(q,z)\,. \tag{5.6}$$

The product is over all squares Λ, indexed by row i and column j, while $w = 1-q+z$ and $l(i,j)$ is the hook length associated with this square.

Now $w/qz = \lambda$. The exact form of the expression of $W_\Lambda(q,z)$ is unimportant here; it is only necessary to note that when $\lambda = q^{m-1}$, $W_\Lambda(q,z) = 0$ for all Young diagrams Λ with more than m rows. Therefore the Jones polynomial $X_L(q,q^{m-1})$

may be expressed as a combination of characters associated with Young diagrams of n squares and at most m rows. For example,

$$\begin{aligned} X_L(q,1) &= 1, && (m=1) \\ X_L(q,q) &= V_L(q), && (m=2) \\ X_L(q,q^{-1}) &= \Delta_L(q), && (m=0) \end{aligned}$$

where V_L is the one-variable Jones polynomial, and Δ_L is the Alexander polynomial. Note that this last result is obtained in the limit $\lambda \to q^{-1}$, and that $X_L(q,\lambda)$ has a (non-essential) singularity at $\lambda = q^{-1}$ when defined by (5.4).

The one-variable Jones polynomial may be expressed as in Theorem 2.3, where the space $W_{2n,n}$ is associated with a slice of L involving $2n$ points. The space $W_{2n,n}$ is associated with the two-row Young diagram with n squares in each row. It is natural to suppose that in the generalisation to arbitrary m, this is replaced by the space on which the representation of $H_{mn}(q)$ associated with the m-row Young diagram $\Lambda_{m,n}$. By Theorem 5.1, this space is $W_{nm,n(m-1),\ldots,n}$. It has dimension $C_{m,n}$ and may thus be identified with the $\mathfrak{sl}_m$-invariant part of $V^{\otimes mn}$ (see Lemma 4.1). However, such a space can only be defined when there are nm points $\{w_i\}$, and yet the slices of L obtained only give $2n$ points. This problem is resolved by the following construction.

Suppose L is a link, expressed as the plait closure of a braid $\beta \in B_{2n}$. Put an orientation on L. Then the orientations induced on the strands of β are such that any slice of β contains $2n$ points with n oriented one way and n oriented the other way. Construct a diagram L^* from L as follows. Replace each strand in β oriented downwards by $m-1$ parallel strands, while leaving all strands in β which are oriented upwards. One now has a braid β^* with mn strands. The link L was obtained from β by plait closure, and in a similar way L^* is defined by performing a 'generalised plait closure' to β^*, in which nodes are produced joining m consecutive strands, as shown in Fig. 6. Then there is a natural action of β^* on $W = W_{nm,n(m-1),\ldots,n}$. The generalised plait closure operation gives rise to special elements of W and W^*.

The analogue of Theorem 2.3 can now be stated.

Theorem 5.2 *Suppose L is the plait closure of a braid $\beta \in B_{2n}$. Then,*

$$X_L(q,q^{m-1}) = \left\langle v_{n,m} \mid \pi_{n,m}(\beta^*) \mid v_{n,m} \right\rangle ,$$

where $\pi_{n,m}$ is the tensor product of the one-dimensional representation $\sigma_i \longmapsto q^{1/2(m-1)}$ with the representation of B_{mn} on $W = W_{nm,n(m-1),\ldots,n}$, and $\langle v_{n,m}|$ and $|v_{n,m}\rangle$ are elements of W^* and W respectively, dependent only on m and n (independent of β^*). Moreover $|v_{n,m}\rangle$ transforms totally antisymmetrically under the natural action of $H_m(q) \times \cdots \times H_m(q)$ upon W.

Proof: In this Theorem the action of $B_m \times \cdots \times B_m$ on W is given by that of the subgroup of B_{mn} in which only those points within the same cluster of m points may be permuted amongst each other. These clusters are determined by the $2n$ nodes introduced in L^*. The part of W transforming totally antisymmetrically under the action of $H_m(q) \times \cdots \times H_m(q)$ is one-dimensional, just as for the case $m = 2$. Indeed, $W'_{mn,(m-1)n,\ldots,n}$ has dimension $(mn)!/(n!)^m$ and is spanned by $[\lambda_1, \ldots, \lambda_{m-1}]$ with $|\lambda_i| = in$, $\lambda_i \subset \{1, 2, \ldots, (i+1)n\}$ for $1 \leq i \leq m-1$. Represent such a term by an array of marks built up, row by row, on top of each other, as follows. The 'ground' row consists of mn positions, labelled $1, \ldots, mn$, in which a mark is placed in a position whenever the position's label lies in λ_{m-1}. The second row is placed vertically above the first, marks only being placed in those $(m-1)n$ columns above marks already present, with a mark appearing in the j^{th} allowed position whenever $j \in \lambda_{m-2}$. Continuing in this way results in $\frac{1}{2}mn(m-1)$ marks placed in $m-1$ rows. There are mn columns, precisely n having each of the heights $0, 1, \ldots, m-1$. We will refer to this diagram as the *picture* of $[\lambda_1, \ldots, \lambda_{m-1}]$. For $n = 1$, the permutation defined by the heights establishes a correspondence between $W'_{m,m-1,\ldots,1}$ and $\mathbf{C}S_m$ considered as a vector space. In this way,

$$\sum_{\sigma \in S_m} (-q)^{l(\sigma)} \cdot \sigma ,$$

defines an element, $\mathbf{v}_{1,m}$, of $W'_{m,m-1,\ldots,1}$, and it is this element that we shall associate with a single node $|v_{1,m}\rangle$. Then $|v_{n,m}\rangle$ is just a tensor product of such terms, applied to the n sets of m w's, each with $m-1$ $z^{(m-1)}$'s,..., two $z^{(2)}$'s and one $z^{(1)}$'s attached; that is, the image of $\mathbf{v}_{1,m}^{\otimes n}$ under the natural map ${W'_{m,m-1,\ldots,1}}^{\otimes n} \longrightarrow W'_{mn,(m-1)n,\ldots,n}$ in which the new picture is obtained by placing n pictures in a line one after the other. Similarly, the associated 'bra vector', $\langle v_{n,m}|$ is, up to scaling, specified by the element of $W'^*_{m,m-1,\ldots,1}$ given by,

$$\sigma \longmapsto \operatorname{sgn}(\sigma) = (-1)^{l(\sigma)} .$$

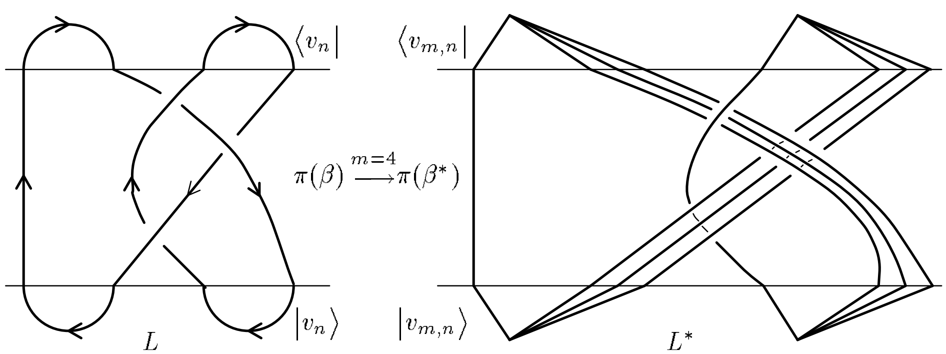


Figure 6

The proof of the result parallels that of Theorem 2.3, as given in [L2]. Firstly, the construction of L^* from L given above, is extended from that for links expressed as plait closures to those given by arbitrary link diagrams. Thus, suppose that $\mathcal{D}$ is an oriented link diagram in the plane, on which a height function is given, such that critical points of the height function on $\mathcal{D}$ do not occur at double points. Define $\mathcal{D}^*$ to be the unoriented diagram obtained from $\mathcal{D}$ by replacing each of the sections of $\mathcal{D}$ between consecutive critical points, on which the height function is decreasing, by $m-1$ parallel strands. Thus $\mathcal{D}^*$ may be subdivided by slices defined by level sets of the height function, into elemental sections, namely vertices in which m strands either emerge or enter, and braids, each deriving from the elemental sections, cup, cap and braid in a subdivision of $\mathcal{D}$.

Let $\mathcal{T}$ denote the subcategory of the category of oriented tangles defined by restricting to objects with equal numbers of upwardly and downwardly oriented strands. The objects in $\mathcal{T}$ consist of pairs (n,σ), where $n \in \mathbf{N} \cup \{0\}$ and σ is a sequence of $2n$ signs, precisely n of them being '+'. Next define a functor $\theta\colon \mathcal{T} \longrightarrow \mathcal{V}$ as folows, where $\mathcal{V}$ denotes the category of vector spaces. For any object $(n,\sigma) \in \mathcal{T}$ there is associated a subdivision of mn objects into $2n$ clumps of sizes 1 and $m-1$ as determined by σ. One may now define,

$$\theta(n,\sigma) = W(n,m) \subseteq W_{nm,n(m-1),\dots,n} \ ,$$

to be the subspace consisting of those elements which transform totally antisymmetrically under the action of the subgroup $B_{m-1} \times \cdots \times B_{m-1}$ (n times) of B_{nm} generated by those σ_i's permuting pairs of strands within the same clump. Morphisms, β, of braid type transform according to $\theta(\beta) = \pi(\beta^*)$, where $\beta \in \operatorname{Morph}\Big((n,\sigma),(n,\tau)\Big)$ becomes $\beta^* \in B_{nm}$ under the replicating procedure and π is the representation of B_{nm} on $W(n,m)$ of Theorem 5.1, scaled by $q^{1/2(m-1)} \equiv \alpha$, so that $\pi(g)\mathbf{v} = \alpha^{e(g)}\pi_{\Lambda_{n,m}}(g)\mathbf{v}$ for $g \in B_{nm}$ of exponent sum $e(g)$. Thus π is the restriction of $\pi_{n,m}$ to $W(n,m)$.

For any consecutive set of m strands, there is associated a subalgebra, $H_m(q)$, of $H_{m(n+1)}(q)$ generated by σ_i's associated with the strands in the given cluster. The part of $W(m,n+1)$ transforming totally antisymmetrically under this action of $H_m(q)$ gives a representation of $H_{mn}(q)$ isomorphic to that naturally defined on $W(m,n)$. The image of a cup morphism in $\mathcal{T}$, corresponding to the diagram $\mathcal{D}^*$ in which m strands emerge from a vertex is defined, up to scaling, to be this inclusion map. Explicitly, if there are t strands to the left of the insertion, the map is,

$$\begin{aligned} W(m,n) &\longrightarrow W(m,n+1) \ , \\ [\lambda] = [\lambda_1,\dots,\lambda_{m-1}] &\longmapsto q^{r_t(\lambda)}(-\sqrt{q})^{-(m-1)t} \sum_{\sigma \in S_m} (-q)^{l(\sigma)}[\sigma_t(\lambda)] \ , \end{aligned}$$

where $\sigma_t(\lambda) \in \mathcal{T}_{(n+1)m,\dots,n+1}$ is such that its picture is obtained from that of $\lambda \in \mathcal{T}_{(nm,\dots,n)}$ by inserting m new columns immediately after the t^{th} column, with marks placed in these columns according to σ, as in the description of $\mathbf{v}_{1,m}$ above. Here $r_t(\lambda)$ denotes the number of marks to the left of column t, in the picture of λ.

The image of a cap morphism with t strands to the left of the m strands defining the vertex, is similarly defined to be the map $W(m,n) \longrightarrow W(m,n-1)$ in which

$[\lambda]$ maps to $\mathbf{0}$ when the heights of the columns $t+1,\ldots,t+m$ in the picture of λ are not all distinct, and otherwise to,

$$A_m{}^{-1}(-\sqrt{q})^{(m-1)t}q^{-r_t(\lambda)}(-1)^{l(\sigma)}[\mu]\,.$$

Here $\sigma\in S_m$ is determined by the permutation of $\{0,1,\ldots,m-1\}$ giving the heights of columns $t+1,\ldots,t+m$, and μ has picture obtained from that of λ by removing these columns. Also,

$$A_m=(\sqrt{q})^{m-1}\prod_{r=1}^{m-1}(1+q+\cdots+q^{r-1}),$$

this particular value arising from the relations,

$$\alpha^{-1}\sum_{\substack{\sigma\in S_m\\ \sigma(m)=s}}q^{l(\sigma)+s-1}=A_m=\alpha\sum_{\substack{\sigma\in S_m\\ \sigma(1)=s}}q^{l(\sigma)-s+1}\,,$$

valid for arbitrary $s=1,\ldots,m$.

The category $\mathcal{T}$ is generated by braid, cup and cap morphisms and therefore the above definitions fix the images under θ of arbitrary elements of $\mathcal{T}$. The relations satisfied by these generators have simple local forms (see [Y] and [T]). The scalings involved above in the images of elementary morphisms in $\mathcal{T}$, have been chosen in such a way that θ defines a functor; this is directly verified by checking that the generating relations are preserved, in a way similar to that for the special case of $m=2$ to be found in [L2].

The image of the empty object in $\mathcal{T}$ under θ is the one-dimensional vector space $\mathbf{C}$. Thus for any link L, $\theta(L)\colon\mathbf{C}\longrightarrow\mathbf{C}$ is given by multiplication by a quantity which is an invariant of L. To identify this invariant of links, observe that since π is a multiple of a restriction of the Hecke algebra $\pi_{\Lambda_{n,m}}$, $\theta(L)$ satisfies a skein relation. Indeed,

$$(\pi(\sigma_i)-\alpha)(\pi(\sigma_i)+q\alpha)=0\,,$$

for $1\le i\le mn-1$ where $\alpha=q^{1/2(m-1)}$. Suppose that L_+,L_0,L_- are three links, identical away from a single crossing, that crossing appearing in L_+, L_- with positive and negative orientations, respectively, while being split in L_0. Then,

$$\theta(L_+)+(q-1)\alpha\theta(L_0)-q^m\theta(L_-)=0\,.$$

This identifies the invariant as $B.X_L(q,q^{m-1})$ where $B=\theta(\text{unknot})$. Since an unknot may be obtained as the composition of cup and cap morphisms,

$$B=A_m{}^{-1}\sum_{\sigma\in S_m}q^{l(\sigma)}=(\sqrt{q})^{1-m}(1+q+\cdots+q^{m-1})=\frac{\sqrt{q}^m-\sqrt{q}^{-m}}{\sqrt{q}-\sqrt{q}^{-1}}\,,$$

which is the value of $X_.(q,q^{m-1})$ on the two component unlink.

Finally observe that if L and β are as in the theorem, then $\theta(L) = \langle v_{n,m} \mid \pi(\beta^*) \mid v'_{n,m} \rangle$ where $\langle v_{n,m} \mid$ and $\mid v'_{n,m} \rangle$ are the images under θ of the closure sections of L^*, as illustrated in Figure 6. ■

The result of Theorem 5.2 gives $X_L(q, q^{m-1})$ as a matrix element of $\pi_{n,m}(\beta^*)$. The action of β^* on W may be given alternatively as an action of β on a subspace of $H^K(Y_{\mathbf{w}',K}, \chi')$ where $\mathbf{w}' \in X_{2n}$, and the local coefficient system χ' is defined to have twists of $z_s^{(r)}$ around w'_i given by,

q ,	if $r = m-1$, and w'_i is a strand with upwards orientation;
q^{m-1} ,	if $r = m-1$, and w'_i is a strand with downwards orientation;
1 ,	otherwise.

All the other twistings are defined in the same way as for χ. This interpretation of X_L puts q and λ (the two variables in this Jones polynomial) on an almost equal footing, as twistings around parts of L with upward and downward orientations. Note that this explains why in the special case $\lambda = q^{-1}$ it is especially easy to deal with; for, in this case the local coefficient system is given by a twisting of q around upward strands and q^{-1} around downward strands. This may be specified as a local coefficent system on $S^3 \backslash L$, namely, precisely the system used in the topological description of the Alexander polynomial $\Delta_L(q) = X_L(q, q^{-1})$ (see [L 1],§4.1 for more details).

Suppose that L is the braid closure of the braid $\gamma \in B_n$. Then L^* will be obtained fron $\gamma^* = \gamma \otimes \mathrm{id} \in B_{nm}$ by joining up the nm strands at the two ends of this braid, into n clumps, the r^{th} clump joining the r^{th} strand together with the $m-1$ consecutive strands numbered from $nm-r(m-1)+1$ to $nm-(r-1)(m-1)$ (see Fig. 7). We then obtain, by the proof of Theorem 5.2, an expression for $X_L(q, q^{m-1})$ as a matrix element $\langle w_{n,m} \mid \pi(\gamma^*) \mid w_{n,m} \rangle$ where $w_{n,m} \in W$ is associated with the closing method employed. Thus π is the same as in Theorem 5.2, while $w_{n,m}$ differs from $v_{n,m}$, although it still depends only upon n and m. However, the representation $\pi_{\Lambda_{n,m}}$ of $H_{nm}(q)$ in Theorem 5.2 is associated with the Young diagram $\Lambda_{n,m}$. It induces a representation of $H_n(q) \times \cdots \times H_n(q)$ (m copies) which is no longer irreducible. Indeed, it has a direct sum decomposition into terms of the form,

$$\pi_{\Lambda^{(1)}} \otimes \cdots \otimes \pi_{\Lambda^{(m)}} ,$$

where the Young diagrams $\Lambda^{(i)}$ each have n squares and at most m rows, and form a partition of $\Lambda_{n,m}$. Hence $\pi(\gamma^*)$ may be decomposed as a direct sum of terms $\pi_{\Lambda^{(1)}}(\gamma) \otimes \mathrm{id}$, and its matrix element leads to a linear combination of the traces of $\pi_\Lambda(\gamma)$ as Λ ranges over all Young diagrams with n squares and at most m rows, namely, that given by (5.4), (5.5) and (5.6).

The monodromy representation of B_n obtained from the system (3.3) factors through $H_n(q)$ [K] and decomposes into a direct sum over j's of $\dim V(j_1, \ldots, j_{m-1})$ copies of a Hecke algebra representation of dimension $N_n(j_1, \ldots, j_{m-1})$; for details see Lemma 4.2. This gives the generalised version of the Pimsner-Popa-Temperley-Lieb representation obtained for $m = 2$. Irreducible representations of $H_n(q)$ are

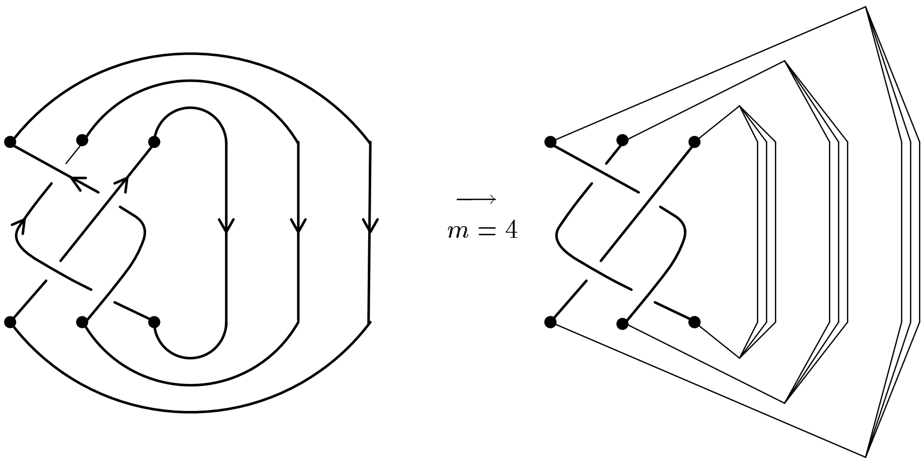


Figure 7

obtained when we consider that part of the monodromy representation coming from (3.3), where f is thought of as a vector-valued function $f\colon X_n \to (V^{\otimes n} \otimes \check{V}_{\mathbf{j}})_0$ with values in that part of $V^{\otimes n}$ transforming according to $V_{\mathbf{j}}$ under the action of $\mathfrak{sl}_m$. This is analogous to Theorem 2.1. The coefficients of π_Λ in $X_L(q, q^{m-1})$ (namely, W_Λ in (5.6)) may thus be viewed as a sum of $N_n(j_1, \ldots, j_{m-1})$ factors, giving the weights of the different copies of $V_{\mathbf{j}}$ appearing in the weighted trace of the full representation of $H_n(q)$ on $V^{\otimes n}$.

We conclude this section with the general construction of the vector spaces W to be associated with a slice of the link, in the topological field theory associated with a general Lie algebra, $\mathfrak{g}$. It is well known that given $\mathfrak{g}$, an invariant can be computed for any oriented link whose components are endowed with representations of $\mathfrak{g}$. This is embodied in [RT] in its "quantised" form, in terms of modular Hopf algebras and their representations, and in [Wi], [A] and [S] in the topological field theory context.

Let d be the rank of $\mathfrak{g}$, and $\alpha_1, \ldots, \alpha_d$ be simple roots. Suppose that L is an oriented link in S^3 whose components are labelled with representations of $\mathfrak{g}$. A slice of L gives an even number of points, $w_i \in \mathbf{C}$, for $i = 1, 2, \ldots, 2n$, each suitably labelled with representations, ρ_i, of $\mathfrak{g}$, and orientations $\beta_i \in \{\pm 1\}$. Let λ_i be the weight of the representations ρ_i if $\beta_i = +1$, and of $\check{\rho_i}$ if $\beta_i = -1$. Define $\{k_r\}$ by,

$$\sum_{r=1}^{d} k_r(\alpha_r, \alpha_s) = \sum_{i=1}^{2n} \alpha_s(\lambda_i) ,$$

for $1 \le s \le d$. Set $K = \sum_{r=1}^{d} k_r$, and consider the fibration of X_{K+2n} over X_{2n} with fibre $Y_{\mathbf{w},K}$ over $\mathbf{w} \in X_{2n}$. Subdivide the K local coordinates in $Y_{\mathbf{w},K}$ into d sets of k_r points, for $1 \le r \le d$. Define an Abelian local system χ on $Y_{\mathbf{w},K}$ for which the loop with $z_p^{(r)}$ going around $z_q^{(s)}$ has twisting $q^{-(\alpha_r, \alpha_s)}$, while a loop in which $z_p^{(r)}$ goes around w_i has twisting $q^{\alpha_r(\lambda_i)}$ (see also [SV]). Then just as in Theorem 5.1, there is a subspace W of the $(S_{k_1} \times \cdots \times S_{k_d})$ totally antisymmetric part of the middle-dimensional cohomology $H^K(Y_{\mathbf{w},K}, \chi)$.

If q is a root of unity of sufficiently high order, it is claimed that there is a correspondence between constructions of [A] and [S] of a functor, and the above construction. The vector space $Z(\Sigma)$ associated with a slice of L, corresponds to the above vector space, W, while the image of a braid-like tangle is a morphism corresponding to the map between the W vector spaces given by the parallel transport associated with the Gauss-Manin connection.

6. Further remarks

In this paper we have seen how $X_L(q,\lambda)$ may be re-interpreted using methods similar to those in [L 2]. It is possible to produce a *completely functorial* description of $X_L(q,\lambda)$ along the lines of [L 2], where one associates the space $W_{nm,n(m-1),\ldots,n}$ to the slice of L^* with nm points, and defines maps between such spaces in which n increases or decreases by 1. Here L^* is defined generally along similar lines to §5, by first choosing some unit vector and then transforming all 'downward' sections of L into $m-1$ parallel copies, and all turning points (with respect to the chosen unit vector) into nodes with degree m (see Fig. 8 below). This should be compared with the functorial approach of E. Witten in [Wi], where a topological quantum field theory with finite dimensional Hilbert space (the analogue of $W_{nm,n(m-1),\ldots,n}$) was employed. See also [S]. The relationship between this form of parallel construction and X_L, is seen from [R], [KR] to be due to the fact that the $\mathfrak{sl}_m$-invariant part of $V^{\otimes m}$ is one-dimensional i.e. V^{m-1} contains precisely one copy of $\check{V}$.

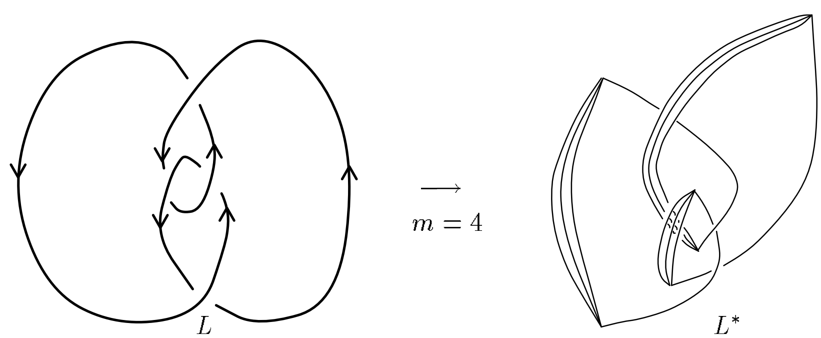


Figure 8

Note that the parallel construction involved here, for any $m > 2$ requires the the introduction of a framing (in this case, a vertical framing was used); cf. [Wi], where framings also enter.

One may construct the homology $H^K(Y_{\mathbf{w},K},\chi)$ in terms of rational holomorphic forms with appropriate twisting. The subrepresentation of Theorem 5.1 can now be realised on a set of differential forms $f\cdot\omega_0$, where ω_0 is a suitable holomorphic differential on $Y_{\mathbf{w},K}$, of degree K, twisting with local system χ, and f is a holomorphic single-valued function on $Y_{\mathbf{w},K}$. It can be shown that the functions f are spanned by the symmetrisations of a product,

$$f_0\cdot\prod_{r,s}\left(z_s^{(r)}-\alpha(z_s^{(r)})\right)^{-1},$$

in which $\underline{\alpha} \in \mathcal{S}_{\mathbf{k}}$. Note that the $\underline{\alpha}$'s employed here are precisely those used in the last section to characterise chains. The cohomology may be calculated as a quotient of a vector space whose basis is indexed by the above $\underline{\alpha}$'s, and the relations existing between such elements are fairly complex, having a strong combinatorial structure. Finally, note that an isomorphism exists between the constructions in [K] and [TK], and the topological construction of Theorem 5.1. According to this, π in Theorem 5.2 is given by the monodromy of a system of equations of the form (3.3), in which f is a vector-valued function on X_{nm}, with values in the $\mathfrak{sl}_m$-invariant part of $V^{\otimes nm}$. Lemma 4.1 provides the bridge between this interpretation and the topological approach described above.

When q is a root of unity of the form $\mathbf{e}^{2\pi i/\kappa}$ some integer $\kappa \leq n$, the topological constructions described in §5 give rise to a space, W, of dimension larger than that in the case of generic q, on which the associated representations are thus not irreducible. However, in this case Wenzl [We] has shown that semi-simple irreducible representations π_Λ of $H_n(q)$ can still be defined, although their dimensions are now less than those of the associated symmetric group representations, and they are only defined for special types of Young diagram, Λ. In [TK], a set of algebraic relations were obtained, for the case of $\mathfrak{sl}_2$, dependent on $l = \kappa - 2 \in \mathbf{N}$, which together with the differential equations (2.2) give rise to those irreducible representations constructed by Wenzl associated with two-row Young diagrams. In a similar way, one should replace the vector space W, of Theorem 5.2 by a subspace obtained by imposing a set of extra conditions, see [FSV], analogous to the algebraic relations of [TK]. The vector spaces so obtained should be isomorphic to the finite dimensional Hilbert spaces of [Wi], in the case of arbitrary Lie algebras.

Note that the Hilbert spaces in [Wi] are *only* defined when q is a primitive root of unity, $\mathbf{e}^{2\pi i r/\kappa}$. However, when κ is sufficiently large (in particular, $> n$), its dimension is independent of κ, and it is isomorphic to W in Theorem 5.2. It is only for smaller values of κ that truncation of the Hilbert spaces occurs (c.f. [V]).

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