Algebras and triangle relations [1]

R.J. Lawrence[2]

Department of Mathematics
University of Michigan
Ann Arbor MI 48109

Abstract. In this paper the new concept of an n-algebra is introduced, which embodies the combinatorial properties of an n-tensor, in an analogous manner to the way ordinary algebras embody the properties of compositions of maps. The work of Turaev and Viro on 3-manifold invariants is seen to fit naturally into the context of 3-algebras. A new higher dimensional version of Yang-Baxter's equation, distinct from Zamolodchikov's equation, which resides naturally in these structures, is proposed. A higher dimensional analogue of the relationship betweeen the Yang-Baxter equation and braid groups is then seen to exhibit a similar relationship with Manin and Schechtman's higher braid groups.

1: Introduction

The central theme of this paper is the interplay existing between algebras and decompositions of polyhedra; see for example [S]. Central, to the concept of classical algebras and groups, is their abstract embodiment of the properties of actions and their compositions and hence, it is their representation theories which play a prominent role in the understanding of their natures.

Let G be a group and $\{h_j\}$ be a set of generators. Recall that the Cayley graph of G is defined to have vertices indexed by the elements of G, while an oriented edge joins the vertex, x, to the vertex, y, with label h_j if, and only if, $y = h_j x$. For $1 \leq i \leq n$, choose $g_i \in \{h_j\}$. The product $g_1 \ldots g_n$ defines a path of length n in the Cayley graph, starting from any given point $x \in G$, namely, it joins x to $g_n x$ to $g_{n-1} g_n x \ldots$ to $g_1 \ldots g_n x$. It is, therefore, natural to represent a product $g_1 \ldots g_n$ in an algebra, A, more generally, by a labelled sub-division of an interval into n subintervals. The labelling defines,

(i) an orientation on each sub-interval, compatible with neighbouring orientations;

(ii) an element of A associated to each sub-interval.

An evaluation of such a product is given by defining a bracketting, or equivalently, by a binary tree with one root and n leaves. Such a tree specifies how the $(n-1)$ multiplications should be performed so as to effect an evaluation.

For present purposes, it is convenient to replace such a tree (evaluation) by its dual, namely a triangulation of an $(n+1)$-gon with one marked edge, and no internal vertices in the triangulation. Thus the n unmarked boundary edges are labelled with $g_1, \ldots, g_n$, while the number of triangles, $(n-1)$, gives the number of multiplications in an evaluation. The result of the product can be considered as appearing on the marked edge.

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The Stasheff polyhedron [S] associated with such a product has vertices labelled by the triangulations of the $(n+1)$-gon, while two such triangulations, T_1, T_2 are connected by an edge if they differ only in one line, which divides a quadrilateral into two triangles in the two possible ways, one in each of T_1 and T_2. It follows from the connectivity of this polyhedron, that associativity (independence of the result of an evaluation of a product upon the chosen evaluation) is ensured by this property holding just for quadrilaterals. The latter is the usual associativity relation,

$$(ab)c = a(bc) ,$$

which is given in Fig. 1.

A representation of a group, in the usual sense, may also be viewed in the above framework. Suppose ρ defines a representation on a vector space V. Then one may associate a copy of V to every vertex of the Cayley graph, and an element of $\mathrm{End}(V)$ to every oriented edge. The property that ρ is a representation is now expressed as the commutativity of closed loops. The geometric nature of the discussion of the preceeding paragraphs leads to an immediate generalisation to higher dimensions, which is the subject of this paper. Indeed, one may replace intervals by triangles, and thus embody the properties of combinations of 3-tensors. In §2 the formal definitions of the notion of a 3-algebra and its representation are given, while the associated notions of product, evaluation and associativity are discussed. It is seen in §3 that a 3-algebra may be constructed using $6j$-symbols, starting from a quantum group; in this case associativity is guaranteed by the Elliot-Biedenharn relation.

In defining the notion of a quasi-triangular Hopf algebra, the triangle relation (quantum Yang-Baxter equation) played a prominent role; see [D 1]. This relation may be considered as the commutativity of a hexagon, which should not be confused with the pentagon and hexagon relations of conformal field theory which are central to the notion of a quasi-Hopf algebra in [D 2]. The duality of the triangular and honeycomb lattices leads to a procedure for generating the QYBE directly from a triangular lattice, which may be naturally generalised to higher dimensions. In three-dimensions, it leads to the consideration of commutativity of combinations of 3-algebra elements associated with closed polyhedra (as opposed to the hexagon in two-dimensions). This is discussed in §4, along with relations with Manin and Schechtman's higher braid group representations.

The n-dimensional generalisations of such concepts are discussed in §5, and some comments on relations with other work are made in §6.

2: 3-ALGEBRAS

2.1 Definition of a 3-algebra

As noted in §1, a 3-algebra embodies the properties of combinations of 3-tensors.

Definition 2.1 *A 3-algebra, A, over a field K, is a vector space over K, endowed with K-linear maps,*

$$\begin{aligned} P\colon &\quad A \longrightarrow A \qquad (\text{of order 3, } P^3 = \mathrm{id}) \\ m\colon &\quad A\otimes A\otimes A \longrightarrow A \\ \overline{b}\colon &\quad A\otimes A \longrightarrow A\otimes A \end{aligned}$$

which satisfy the following conditions,

(i) $m(m\otimes 1\otimes 1) = m(1\otimes 1\otimes m)P_{34}(1\otimes \overline{b}\otimes 1\otimes 1)P_{34}$;

(ii) $(1\otimes m)P_{23}(\overline{b}\otimes 1\otimes 1) = \overline{b}(1\otimes m)P_{12}(P^{-1}\otimes 1\otimes 1\otimes 1)(\overline{b}\otimes 1\otimes 1)(P\otimes P\otimes 1\otimes 1)P_{23}$;

(iii) $\overline{b}(m\otimes 1) = (1\otimes m)P_{12}(P^2\otimes \overline{b}\otimes 1)(1\otimes 1\otimes \overline{b})P_{12}P_{23}$;

(iv) $(1\otimes \overline{b})P_{12}(1\otimes \overline{b}) = (\overline{b}\otimes 1)(1\otimes \overline{b})(P\otimes P\otimes 1)(\overline{b}\otimes 1)(1\otimes P^{-1}\otimes 1)$;

(v) $(1\otimes m)P_{23}(\overline{b}\otimes P^2\otimes 1) = (m\otimes 1)(1\otimes 1\otimes \overline{b})$;

(vi) $Pm = m(P\otimes P\otimes P)P_{23}P_{12}$;

(vii) $\overline{b}$ *commutes with* $(P^2\otimes P)P_{12}$.

In this notation, P_{ij} denotes the action on $A^{\otimes n}$ of transposing the i^{th} and j^{th} factors. Each of the conditions (i)–(v) represents the coincidence of two maps $A^{\otimes n} \longrightarrow A^{\otimes(6-n)}$, in which $n = 5, 4, 4, 3, 4$, respectively. Meanwhile, (vi) and (vii) express the coincidence of the two actions on $A^{\otimes n}$ with $n = 3$ and 2, respectively. A is said to be **an orthogonal 3-algebra** if, in addition to (i)–(vii), the following axiom also holds,

(viii) $(1\otimes P^2)P_{12}\overline{b}(P\otimes P)\overline{b} = Q$ is a projection $A\otimes A \longrightarrow A\otimes A$, while m vanishes on $(\ker Q)\otimes A$.

Definition 2.2 *If A is a 3-algebra over a field K, then by a representation of A is meant a K-linear map $\rho\colon A \longrightarrow V\otimes V\otimes V$ for some K-vector space V with positive definite inner product, such that, for all $a, b, c \in A$,*

$$\rho(Pa) = P_{23}P_{12}\big(\rho(a)\big)\ ,$$
$$\rho\big(m(a\otimes b\otimes c)\big) = \sum_{\substack{l,m,n\\ i,j,k}} a_{inm}b_{njl}c_{mlk}e_{ijk}\ ;$$

in which a_{ijk} denotes the coefficient of $e_i\otimes e_j\otimes e_k = e_{ijk}$ in $\rho(a)$, where $\{e_i\}$ is an orthonormal basis in V. It is also required that if,

$$\rho\big(m'(a\otimes b)\big) = \sum A_{ijk}^{lmn} e_{ijk}\otimes e_{lmn}$$

then,

$$\sum_k A_{ijk}^{lkn} = \sum_\lambda a_{i\lambda l}b_{\lambda jn}\,.$$

2.2 The meaning of a 3-algebra

Recall the summary given in §1, of how 2-algebras may be viewed in terms of subdivisions of intervals, and triangulations. To move from 2–algebras to 3–algebras, replace 'interval' and 'triangles' by 'triangle' and 'tetrahedra', respectively. By a **product** in a 3-algebra, A, is meant a labelled triangulation of a triangle; that is, a triangulated surface endowed with a labelling, whose boundary is a triangle in which further,

(i) each triangle has labels 1,2,3 placed on its sides, in such a way that the orientations of adjacent triangles so induced, match;

(ii) each internal triangle is labelled by an element of A.

In particular a product, Π, provides a set of vertices and a set of 3-sets, namely, the vertices of the triangles appearing in the triangulation. There is also a distinguished 3-set, given by the triangle bounding the surface associated with Π, called the **base triangle** and which will be denoted by Π_0. It is required that the surface represented by the product is topologically just a disc. Let T_Π refer to the triangulation of Π_0 given by the product Π, without the additional data on the faces and edges.

By an **ordered evaluation**, $\mathcal{T}$, of such a product Π, is meant a sequence of triangulations, starting at T_Π and ending at the trivial triangulation of Π_0 by a single triangle, each triangulation being able to be obtained from the previous one via a local operation which is of one of the types depicted in Fig. 2. This may be expressed alternatively by a set of tetrahedra with no internal vertices (that is, 4-sets of vertices of T_Π), in which each 3-set appears at most twice, the triangles associated with those appearing once being precisely the base triangle and those belonging to T_Π. In other words, it is a decomposition into tetrahedra of a 3-region which is topologically a 3-ball, whose boundary is the triangulated sphere, $\overline{T_\Pi}$, associated with T_Π with an additional triangle glued onto the boundary of this surface. The term **evaluation** will be used to refer to such an (unlabelled) subdivision into tetrahedra, while by an abuse of notation, it will also be denoted by $\mathcal{T}$. Note that many different ordered evaluations may give rise to the same evaluation (change of order), while not all subdivisions into tetrahedra will arise from ordered evaluations, it being required that a type of shellablility condition be satisfied.

The first move is analogous to ordinary multiplication in (2-)algebras. Using this first move alone, most products cannot be evaluated. For example, no product based on the surface of an octahedron, with one triangle removed, can be evaluated with this move. The second move, which has no analogue in the theory of ordinary algebras, must therefore also be included.

The basic structures in a 3–algebra, A, may be interpreted geometrically as actions on the labels of local pieces of triangulations, in which P diagramatically represents rotations, as shown in Fig 3, while m and $\overline{b}$ are shown in Fig 4. Thus $\big(P(a)\big)_{ijk} = a_{kij}$. Any ordered evaluation $\mathcal{T}$, of a product Π, now produces a result as follows. Each intermediate triangulation, T_i, in $\mathcal{T}$, once supplied with a labelling of edges may be endowed additionally with an element of $A^{\otimes|\Delta_i|}$, as a result of recursive applications to the initial vector in $A^{\otimes|T_\Pi|}$ given by Π, of the operations P, m and $\overline{b}$ in A, as appropriate, according to the types of the moves in $\mathcal{T}$. The product on Π_0 resulting from the last stage of this sequence will be called the **result** of the evaluation; it will be determined, up to the action of P, by an element of A.

Recall that a representation of a 3-algebra associates to any $a \in A$ an element $\rho(a) \in V \otimes V \otimes V$. Place a copy of V on each edge of the triangulation; the labelling of a triangulation required to specify a product now has a natural meaning, specifying the order in which the copies of V enter into the description of the element associated with the interior of a triangle. Let $\{e_i\}_{i \in I}$ be an orthonormal basis for V, and suppose that Π is a product. Then set,

$$\rho(\Pi) = \sum_{i,j,k \in I} \Big(\sum_{\sigma} \prod_{\text{triangles } \Delta} \rho_\sigma(a_\Delta) \Big) e_{ijk} \in V \otimes V \otimes V \,. \tag{2.1}$$

Here the product is over all triangles Δ in the triangulation describing T, while a_Δ is the element of A associated with Δ. The sum is taken over all assignments, σ, of elements of I, to edges of the triangulation, coinciding with i, j, k on the boundary, Π_0. Finally, for given σ and Δ, $\rho_\sigma(a_\Delta)$ describes the element of K,

$$\big[\rho(a_\Delta)\big]_{\sigma(\delta_1)\sigma(\delta_2)\sigma(\delta_3)} \,,$$

where $\delta_1, \delta_2, \delta_3$ are the edges of Δ with labels 1,2,3.

The definition of 3-algebra and its representations axiomatises the principle that the choice of evaluation of a product should not affect the final result, be it an element of A, or of $V \otimes V \otimes V$. In the latter case, it should be an expression of the form (2.1), a generalisation of the formula for matrix products appearing in 2-algebras.

Any (shellable) decomposition of a 3-ball with boundary $\overline{T_\Pi}$, into tetrahedra with no internal vertices, leads to an evaluation of the product Π. One may pass between any two such decompositions by a sequence of moves, each one having the form of a local 'variation' or its inverse. The local variations replace the join of two adjacent tetrahedra by a union of three,

$$1234 \,\cup\, 1235 \longrightarrow 1245 \,\cup\, 1345 \,\cup\, 2345 \,.$$

In this situation, decompositions refer to those of geometric polyhedra in $\mathbf{R}^3$. A (combinatorial) triangulation, $\overline{T}$, of S^2 defines a collection of polyhedra in $\mathbf{R}^3$, namely those whose boundary is equivalent to $\overline{T}$. In particular, one may consider the subset $\mathcal{C}_{\overline{T}}$ of this collection consisting of polyhedra in general position, that is, such that no four vertices are coplanar. Clearly $\mathcal{C}_{\overline{T}}$ may be identified with an open subset of $\mathbf{R}^{3n}$ where n is the number of vertices in $\overline{T}$. Elements of $\mathcal{C}_{\overline{T}}$ lying in a common connected component behave identically in all the constructions that arise; such a component is called a **geometric realisation** of $\overline{T}$. A decomposition, $\mathcal{T}$, of the interior of $\overline{T}$ into tetrahedra, refers to a subdivision of the interior of some geometric realisation of $\overline{T}$ into a number of tetrahedra with disjoint interiors. It is only such decompositions which are allowed as evaluations.

Axioms (i)–(iv) generate all the relations given by such associativity conditions. In Figs. 5–8, the starting and ending configurations, with appropriate labels, are supplied for each of the axioms. In the figures, the 3-region whose boundary is the union of the triangulation shown on the left with that shown on the right (which has opposite orientation), may be subdivided into tetrahedra with no internal vertices in precisely two ways, each of them describing a local evaluation. Indeed, each diagram in a figure should be viewed as a local piece of the triangulation associated with a product.

Fix a 3-algebra, A. Note that for any oriented (not necessarily closed) two-dimensional manifold Σ, and a set of 'allowed' triangulations $\mathcal{C}$ of Σ, closed under the moves of Fig. 2, one may define the product space $A_{\mathcal{C}}(\Sigma)$,

$$A_{\mathcal{C}}(\Sigma) \equiv \{\text{labelled triangulations of } \Sigma\}/\sim,$$

where $\sim$ is an equivalence relation generated by the moves m, $\overline{b}$ and P. Indeed, $A_{\mathcal{C}}(\text{triangle}) = A$, while $A_{\mathcal{C}}(\text{square}) \cong \ker(Q-I)$. Here, $Q = (1\otimes P^2)P_{12}\overline{b}(P\otimes P)\overline{b}$ denotes the k-linear map $A\otimes A \longrightarrow A\otimes A$ associated with a double application of $\overline{b}$. If Σ has genus 0, with $\partial\Sigma \sim S^1$, one may pick n points on the boundary, and a base triangulation T_Σ with $n-2$ triangles. Using m, $\overline{b}$ and P, any product on Σ may be reduced to a labelling of T_Σ, and thus $A(\Sigma)$ may be identified with a quotient of $A^{\otimes n-2}$. Here, the allowed set of triangulations, $\mathcal{C}$, consists simply of those whose vertices on the boundary are precisely the chosen n vertices.

In this spirit, each of the axioms (i)–(iv) may also be viewed as specifying the equivalence of two procedures for re-expressing a product of $(8-k)$ elements based on a k-gon, in terms of a standard triangulation of the k-gon. Here $k=3$, 4, 4 and 5, respectively. See Figs. 5–8.

Axiom (v) ensures that products are independent of the position of internal vertices; see Fig 9.

Finally, axioms (vi) and (vii) specify the symmetries one expects in m and $\overline{b}$; namely those of Figs. 10 and 11. Axiom (vi) ensures that m commutes with rotation through $2\pi/3$. Axiom (vii) specifies that $\overline{b}$ commutes with rotation of the base quadrilateral through π, given by $a\otimes b \longmapsto P^2 b\otimes Pa$.

An orthogonal 3-algebra is such that the square of the operation $\overline{b}$ is a projection. The extra factors of P and P_{12} are required to match the positions of the indices 1,2,3 on the edges. Any product involving adjacent triangles with labels in $\ker Q$ vanishes; and thus Q specifies a compatability condition in the sense of [TV].

The dual of the generic diagram describing (i)–(iv) is shown in Fig. 12.

Theorem *Suppose A is an orthogonal 3-algebra and Π is a labelled triangulation of the triangle. The composition of m, $\overline{b}$ and P maps, specified by an evaluation $\mathcal{T}$ of Π, has an image in A which is independent of the choice of $\mathcal{T}$.*

For a fixed geometric realisation of the triangulation $\overline{T_\Pi}$ of S^2, independence of the result of the product upon the evaluation is ensured by A being a 3-algebra. Invariance under variations of the realisation within $\mathcal{C}_{\overline{T}}$ requires A to be an orthogonal 3-algebra. Note that unlike the situation in 2-algebras, the number of tetrahedra in an evaluation $\mathcal{T}$ is not determined by the product Π, indeed the generating moves change pairs of tetrahedra into triples, and conversely.

Full 3-algebras

The notion of 3-algebra introduced above may be extended to what will be called a **full 3-algebra**, in order to deal with (arbitrary) triangulations of manifolds, rather than just shellable triangulations of a 3-ball. Thus, in a full 3-algebra A, there are maps $m_j: A^{\otimes j} \longmapsto A^{\otimes(4-j)}$ for each j with $0 \leq j \leq 4$, in addition to an element, P, of $\mathrm{End}(A)$ of order 3. The relations to be satisfied by these maps are best motivated by the associated geometric interpretation. Thus, the maps m_j are visualised as lying on a tetrahedron in which, on each face, labels 1,2 and 3 have been placed on the edges and a copy of A or its dual is placed on each face, according as the orientation on the face defined by the labels, matches or otherwise, the orientation induced from that of the tetrahedron. The maps m_j are required to be invariant under the subgroup of the direct symmetry group, A_4, of the tetrahedron which preserves the face orientations.

By a **product** in a full 3-algebra is meant a labelled combinatorial triangulation, the labelling supplying for each triangular face, numbers 1,2 and 3 on its edges, a sign $\pm$ and an element of A or A^* according as the sign is $-$ or $+$, in such a way that adjacent faces have compatible orientations. The surface, Σ, which enters here is only defined combinatorially, and not topologically; it is just a collection of triples of points in which each pair of points appears at most twice, those appearing once forming the edges of $\partial\Sigma$. Let $\overline{\Sigma}$ denote the closed surface obtained from Σ by adjoining n discs, where n denotes the number of components in $\partial\Sigma$.

By an **evaluation** of a product, Π, is meant a combinatorial triangulation of a 3-manifold whose boundary is $\overline{\Sigma}$ along with, for each tetrahedron, a sign on each face. This data must be such that its restriction to Σ is that determined by Π, while the signs associated with an interior face are opposite in the two bounding 3-simplices.

Given an evaluation $\mathcal{T}$ of a product Π, we now define the result of this evaluation as follows. On each tetrahedron in $\mathcal{T}$, one may place a copy of the vector space A or its dual, according as the sign placed on that face in $\mathcal{T}$ is $+$ or $-$, respectively. Each face of Σ may also be adorned with a copy of A^* or A according as the sign in Π is $+$ or $-$. Thus, each internal face of $\mathcal{T}$ and each face of Σ will be adorned with dual vector spaces from the two tetrahedra incident on this face and hence there is defined a contraction operation from the tensor product of all the vector spaces to a tensor product involving only those factors associated with faces in $\overline{\Sigma}\backslash\Sigma$. If j is the number of negative signs on the bounding faces of a tetrahedron in $\mathcal{T}$, then one may place a copy of m_j on that tetrahedron. The result of the contraction of the tensor product of these vectors defines the result of the evaluation, it is an element of a tensor product of n factors, each either A or A^*. (Note that there may be extra operations P entering in order to match up the two sets of labels placed on the edges of an internal face.) This procedure may perhaps be more familiar as contraction of tensors on the 4-valent oriented graph dual to $\mathcal{T}$. The orientation on this graph is defined by requiring that an edge is oriented away from a particular vertex precisely when the sign on the associated triangle, as a face of the tetrahedron represented by that vertex, is +.

To ensure that the result of an evaluation of a product is independent of the choice of evaluation, it is necessary that the result be unchanged under local moves on triangulations. That is, for any combination of arrows on the external legs of Fig. 12, identical on the two sides of the equivalence, and any choice of arrows on the internal edges, the results of the contraction of the combinations of m_j defined by Fig. 12, are identical. In addition, the similar statement, associated with a move replacing a single tetrahedron by a union of four, a new (internal) vertex having been introduced, must hold. This describes the conditions to be satisfied by the m_j's in a full 3-algebra. Those conditions which involve only m_2 and m_3 are identical to the constraints on the structures $m = m_3$ and $\overline{b} = m_2$ in a 3-algebra.

Theorem *Suppose A is an orthogonal full 3-algebra and Π is a product based on a triangulated closed surface Σ. The composition of m_j maps, specified by an evaluation $\mathcal{T}$ of Π, has an image in $A(\Sigma)$ which is independent of the choice of $\mathcal{T}$. Thus A determines an invariant of 3-manifolds with boundary Σ.*

3: An example

In this section we shall show how a 3-algebra may be constructed starting from a quantum group. Suppose, more generally, that a set I is given, together with maps $w: I \longrightarrow K$ and,

$$f: I^6 \longrightarrow K\ .$$

The latter map will be denoted, for later convenience by,

$$(a,b,c,i,j,k) \longmapsto \begin{pmatrix} a & b & c \\ i & j & k \end{pmatrix}. \tag{3.1}$$

Any such set of six elements of I may be placed on the edges of a tetrahedron whose vertices are labelled 1,2,3,4, as shown in Fig. 13. The direct symmetry group, A_4, of the tetrahedron acts on these six labels in a clear way, and we assume that f is invariant under this action.

It is now possible to use this data to attempt to construct a 3-algebra, A, whose structure as a K-vector space is simply that generated by a basis indexed by I^3. Let e_{ijk} denote the basis element associated with $(i,j,k) \in I^3$.

Lemma 3.1 *The K-linear space, A, generated by $\{e_{ijk} \mid i,j,k \in I\}$ together with maps,*

$$\left.\begin{aligned} P(e_{ijk}) &= e_{jki} \\ m(e_{akj} \otimes e_{k'bi} \otimes e_{j'i'c}) &= \delta_{ii'}\delta_{jj'}\delta_{kk'} \begin{pmatrix} a & b & c \\ i & j & k \end{pmatrix} e_{abc} \\ \overline{b}(e_{j_2bc} \otimes e_{b'aj_1}) &= \delta_{bb'} \sum_j w_j^2 \begin{pmatrix} j_2 & a & j \\ j_1 & c & b \end{pmatrix} e_{j_2aj} \otimes e_{cjj_1} \end{aligned}\right\} \tag{3.2}$$

define a 3-algebra, if, and only if, for all $a, b, c, e, f, j_1, j_2, j_3, j_{23} \in I$,

$$\sum_j w_j^2 \begin{pmatrix} e & j_3 & j \\ j_2 & a & j_{23} \end{pmatrix} \begin{pmatrix} j & c & j_1 \\ b & a & j_2 \end{pmatrix} \begin{pmatrix} j_3 & c & f \\ j_1 & e & j \end{pmatrix} = \begin{pmatrix} j_3 & c & f \\ b & j_{23} & j_2 \end{pmatrix} \begin{pmatrix} e & f & j_1 \\ b & a & j_{23} \end{pmatrix} . \quad (3.3)$$

The axioms (i)–(iv) are each seen to be equivalent to (3.3) when the symmetries of (3.1) are considered. This lemma allows the construction of a number of examples of 3-algebras. Suppose B is a simple Lie group, and $U_q\mathfrak{b}$ is a quantum group obtained as the quantisation of the universal enveloping algebra of the Lie algebra, $\mathfrak{b}$ of B, evaluated at a root of unity, q. Let I be a suitable set, labelling the irreducible representations of $U_q\mathfrak{b}$, and let V_i denote the representation labelled by $i \in I$.

For any three irreducible representations V_{i_1}, V_{i_2}, V_{i_3}, one may construct a basis for $V_{i_1} \otimes V_{i_2} \otimes V_{i_3}$ in two distinct ways. Firstly, as a join of bases for $V_i \otimes V_{i_3}$ as i ranges over elements of I, for which V_i occurs in the decomposition of $V_{i_1} \otimes V_{i_2}$ into irreducible representations. Secondly, a basis may be constructed as a join of bases for $V_{i_1} \otimes V_j$ as j ranges over elements of I, for which V_j occurs in the decomposition of $V_{i_2} \otimes V_{i_3}$ into irreducible representations. Thus, if V_k appears in the decomposition of $V_{i_1} \otimes V_{i_2} \otimes V_{i_3}$, then two distinct bases are known for the component transforming according to V_k. The change of basis transformation has coefficients known as the quantum $6j$-symbols and is traditionally denoted,

$$\begin{pmatrix} i & i_1 & i_2 \\ j & i_3 & k \end{pmatrix} . \quad (3.4)$$

An element $\{i, j, k\}$ of I^3 is said to be **admissible** if $V_i \otimes V_j \otimes V_k$ contains a copy of the trivial representation (the assumption of simplicity then implies that it occurs with multiplicity one). As defined above, (3.4) is only meaningful when $\{i_1, j, k\}$, $\{i, i_1, i_2\}$, $\{i_3, j, i_2\}$ and $\{k, i, i_3\}$ are all admissible triples. We extend its definition to all of I^6 by defining it to be zero in all cases where it would not otherwise be meaningful.

The symmetries of such quantum $6j$-symbols, together with the Elliot-Biedenharn relation ensure, by Lemma 3.1, that a 3-algebra results. See [KR] for precise definitions of the quantum 6j-symbols. Note also, that the orthogonality relation amongst $6j$-symbols guarantees the orthogonality of the resulting 3-algebra, while the associated projection Q of axiom (viii) has,

$$A \otimes A /_{\ker Q} \cong \langle e_{ijk} \otimes e_{jlm} \mid (i, j, k) \text{ and } (j, l, m) \text{ are admissible} \rangle .$$

The purpose of using a quantum group at a root of unity is that the relevant representation theory is then finite, that is, the resulting 3-algebra is finite-dimensional as a vector space, so that problems of convergence of results of evaluations of products do not arise.

The (Racah-Wigner) quantum $6j$-symbols for $U_q\mathfrak{sl}_2$ were found in [KR]. The map $w \colon I \longrightarrow \mathbf{C}$ is simply given by $w \colon n \mapsto w_n = i^{2n}[2n+1]^{1/2}$, where the indexing set, I, is

$^1/_2\mathbf{Z}^+$. Here, $[n]$ denotes the q-number, $q^{n-1} + q^{n-3} + \cdots + q^{3-n} + q^{1-n}$. At the sixth root of unity $q = \mathbf{e}^{i\pi/3}$, the resulting 3-algebra may easily be written down. The allowable representations have spins in $I = \{0, {}^1/_2\}$, while the 'admissible' triples are,

$$(0,0,0),\ ({}^1/_2, 0, {}^1/_2),\ ({}^1/_2, {}^1/_2, 0),\ (0, {}^1/_2, {}^1/_2)\ .$$

The relevant quantum $6j$-symbols are,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1\,, \quad \begin{pmatrix} {}^1/_2 & {}^1/_2 & 0 \\ {}^1/_2 & {}^1/_2 & 0 \end{pmatrix} = -1\,, \quad \begin{pmatrix} 0 & {}^1/_2 & {}^1/_2 \\ {}^1/_2 & 0 & 0 \end{pmatrix} = -i\,;$$

while $w_0 = 1$, $w_{{}^1/_2} = i$. The associated 3-algebra A is thus four dimensional as a vector space over $\mathbf{C}$, with basis elements as in Fig. 14.

A complete description of the 3-algebra, A, is given by,

$$P(a) = c, \quad P(b) = a, \quad P(c) = b, \quad P(d) = d\,; \tag{3.5}$$

together with the maps m and $\overline{b}$, below.

$$\left.\begin{array}{ll} m(a \otimes b \otimes c) = -id & \overline{b}(a \otimes b) = -id \otimes b \\ m(a \otimes c \otimes b) = -a & \overline{b}(a \otimes c) = a \otimes c \\ m(b \otimes a \otimes c) = -c & \overline{b}(b \otimes a) = -c \otimes b \\ m(b \otimes d \otimes b) = -ib & \overline{b}(b \otimes d) = ib \otimes c \\ m(c \otimes b \otimes a) = -b & \overline{b}(c \otimes b) = b \otimes a \\ m(c \otimes c \otimes d) = -ic & \overline{b}(c \otimes c) = -ic \otimes d \\ m(d \otimes a \otimes a) = -ia & \overline{b}(d \otimes a) = ia \otimes a \\ m(d \otimes d \otimes d) = d & \overline{b}(d \otimes d) = d \otimes d\,. \end{array}\right\} \tag{3.6}$$

Note that in the above list the images of basis elements have been given, with those not mentioned having zero images.

The dual vector space has a dual 3-algebra structure, with maps P, Δ and $\overline{\Delta}$ replacing p, m and $\overline{b}$,

$$\begin{array}{rl} \Delta: & A^* \longrightarrow A^* \otimes A^* \otimes A^*\,; \\ \overline{\Delta}: & A^* \otimes A^* \longrightarrow A^* \otimes A^*\,. \end{array}$$

If A, B, C, D denote the associated dual basis for A^*, then,

$$P(A) = B, \quad P(B) = C, \quad P(C) = A, \quad P(D) = D\,;$$

while Δ is given by,

$$\begin{aligned} \Delta(A) &= -A \otimes C \otimes B - iD \otimes A \otimes A \\ \Delta(B) &= -iB \otimes D \otimes B - C \otimes B \otimes A \\ \Delta(C) &= -B \otimes A \otimes C - iC \otimes C \otimes D \\ \Delta(D) &= -iA \otimes B \otimes C + D \otimes D \otimes D\,. \end{aligned}$$

The dual of $\overline{b}$, namely $\overline{\Delta}$, is specified by the images of basis elements of $A^* \otimes A^*$; those which are non-zero are listed below.

$$\begin{aligned} &\overline{\Delta}(A\otimes A) = iD\otimes A && \overline{\Delta}(C\otimes B) = -B\otimes A \\ &\overline{\Delta}(A\otimes C) = A\otimes C && \overline{\Delta}(C\otimes D) = -iC\otimes C \\ &\overline{\Delta}(B\otimes A) = C\otimes B && \overline{\Delta}(D\otimes B) = -iA\otimes B \\ &\overline{\Delta}(B\otimes C) = iB\otimes D && \overline{\Delta}(D\otimes D) = D\otimes D\,. \end{aligned}$$

These maps satisfy dual axioms to (i)–(vii) above. For example, the dual to axiom (i) is,

$$(\Delta\otimes 1\otimes 1)\Delta = P_{34}(1\otimes\overline{\Delta}\otimes 1\otimes 1)P_{34}(1\otimes 1\otimes\Delta)\Delta\,.$$

It is easily seen that all the axioms are satisfied, whilst Q is the projection onto

$$\langle a\otimes b,\ a\otimes c,\ b\otimes a,\ b\otimes d,\ c\otimes b,\ c\otimes c,\ d\otimes a,\ d\otimes d\rangle\,.$$

A non-equivalent 3-algebra over $\mathbf{C}$, also of dimension four as a vector space is given by $\langle a,b,c,d\rangle$ with similar relations to (3.5) and (3.6), except that in (3.6) all coefficients ± 1 and $\pm i$ are replaced by $+1$. The resulting 3-algebra has a representation of dimension two, with,

$$\begin{aligned} &a\longmapsto e_1\otimes e_2\otimes e_2\,; && b\longmapsto e_2\otimes e_1\otimes e_2\,; \\ &c\longmapsto e_2\otimes e_2\otimes e_1\,; && d\longmapsto e_1\otimes e_1\otimes e_1\,. \end{aligned}$$

The invariants of 3-manifolds discussed in [TV] are, in the case of no internal vertices in $\mathcal{T}$, just given by the multiplication map $A(\Sigma_1)\to A(\Sigma_2)$. The particular 3-algebra described by (3.5), (3.6) above gives rise to the invariants discussed in §8 of [TV]. Here A is the 3-algebra constructed from a quantum group at a root of unity, as above, and $\Sigma_1\cup\overline{\Sigma}_2$ is a subdivision of the boundary of the 3-manifold. To deal with arbitrary 3-manifolds, general triangulations must be considered. The moves required to pass between two topologically equivalent triangulations are more complex, and were dealt with in [TV] and [M]. An extra 'bubble' move introduces a contribution from internal vertices into the state model description of invariants in [TV]; see also [P].

4: Triangle relations

4.1 Lattices and triangle relations

In the theory of the quantum groups, the star-triangle relation, or quantum Yang-Baxter equation,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,, \tag{4.1}$$

for an element $R\in A\otimes A$, plays a central role. In statistical mechanics, the solution of the honeycomb Ising model directly involves the duality of the honeycomb and triangular lattices, as well as the star-triangle transformation of Fig. 15. See [B].

Both these transformations produce a honeycomb lattice from a triangular lattice, and conversely; and their combination enables the solution of the model. In this respect, a triangular lattice refers to a tesselation by a rhombus divided into two triangles. The resulting dual honeycomb lattice has two types of vertex, corresponding to the two orientations of triangle in the initial lattice. The three orientations of lines in the initial triangular lattice give rise to three edge types in the honeycomb lattice. Let us label the honeycomb lattice as follows. Place an element of $A \otimes A \otimes A$ on each edge, it being R_{12}, R_{13}, or R_{23} according to the type of edge; see Fig. 16. These elements may be regarded as maps $A \otimes A \otimes A \longrightarrow A \otimes A \otimes A$, namely left multiplications. The commutativity of all closed loops of such maps on the honeycomb lattice is ensured by that for the generating loops, that is the commutativity of the hexagons themselves. In other words, (4.1) may be interpreted as expressing the commutativity of a suitably labelled honeycomb lattice.

The natural generalisation is obtained by considering a tetrahedral lattice, that is, the tesselation by a cuboid divided into six tetrahedra. Its dual lattice has six different vertex types and may be generated by a polyhedron which we call Γ_3. It is a truncated cube/octahedron, with 24 vertices and 14 faces (6 squares and 8 hexagons).

In general consider an n-dimensional lattice obtained by the tesselation of a hypercube divided into $n!$ n-simplices. That is, subdivide $\mathbf{R}^n$ by $1/2 n(n+1)$ hyperplanes,

$$\sum_{s=t}^{u} x_s \in \mathbf{Z}\,, \tag{4.2}$$

where $n \geq u \geq t \geq 1$. For $u = t$, (4.2) reduces to the hyperplanes, $x_t \in \mathbf{Z}$, defining the standard n-dimensional cuboidal lattice. For $n = 2$, this gives x_1, x_2 and $x_1 + x_2 \in \mathbf{Z}$, the standard triangular lattice. Let Γ_n denote the polyhedron generating the dual graph of the lattice of (4.2), the n–dimensional permutahedron. Thus Γ_2 is a hexagon while Γ_3 is a tetrakaidekahedron.

Let $\mathcal{P}_{n,k}$ consist of partitions into r terms of total order $(k + r)$ chosen from $\{0, 1, \ldots, n\}$; that is, an element of $\mathcal{P}_{n,k}$ consists of an integer r together with r pairwise disjoint subsets $S_1, \ldots, S_r$ of $\{0, 1, \ldots, n\}$ with $\sum |S_i| = k + r$ and $|S_i| > 1$ for all i. Then $\mathcal{P}_{n,k}$ labels the k-dimensional faces of Γ_n. Indeed, such a face is defined by an arrangement of k hyperplanes in general position, each of the type specified in (4.2). A hyperplane of type (4.2) is given by a subset $\{t-1, u\}$ of $\{0, 1, \ldots, n\}$ of order 2. Fix a collection of $(n+1)$ linearly independent vectors $\mathbf{v}_0, \ldots, \mathbf{v}_n$ in some vector space. Then to each set $\{t-1, u\}$ associate the vector $\mathbf{v}_{t-1} - \mathbf{v}_u$. A collection of k, 2-sets corresponds to an arrangement of k hyperplanes in general position precisely when the associated k vectors are linearly independent, while the span of these vectors determines a face of Γ_n (up to translation). It is apparent that $\mathcal{P}_{n,k}$ labels those k-dimensional vector spaces spanned by vectors of the type $\mathbf{v}_{t-1} - \mathbf{v}_u$, and hence to each $\underline{\alpha} \in \mathcal{P}_{n,k}$, there corresponds a face ${\Gamma_n}^{\alpha}$ of Γ_n.

The geometrical type of ${\Gamma_n}^{\alpha}$ is determined solely by r and the orders $|S_j|$, $j = 1, 2, \ldots, r$. The set $\{|S_j| - 1\}_{j=1}^{r}$ defines a partition of k into r positive integers, and therefore the geometrical types of k-dimensional faces of Γ_n are indexed by the partitions of k.

Suppose α is a partition of k. The geometrical types of the facets of type-α faces of Γ_n, are labelled by sub-partitions of α. That is, a partition of $k' < k$ obtained from α by reducing or removing some, or all, terms in the partition. Indeed, if $\alpha \in \mathcal{P}_{n,k}$, the k'-dimensional faces of ${\Gamma_n}^\alpha$ are given by ${\Gamma_n}^\beta$ with $\beta \in \mathcal{P}_{n,k'}$ of form $\{T_1, \ldots, T_{r'}\}$ in which,

(i) $T_i \subset \{0, 1, \ldots, n\}$ are disjoint subsets for $i = 1, \ldots, r'$;

(ii) $\sum_i |T_i| = k' + r'$;

(iii) for each i, $1 \leq i \leq r'$, $\exists j$ with $1 \leq j \leq r$ and $T_i \subset S_j$.

Here $\{S_1, \ldots, S_r\}$ specifies $\alpha \in \mathcal{P}_{n,k}$. Define an equivalence relation $\sim_\beta$ on $\{1, 2, \ldots, n\}$ in which $a \sim_\beta b$ for distinct a and b, if, and only if, $a, b \in T_j$ for some j. Under $\sim_\beta$, $\{S_j\}$ is reduced to another partition, γ. The number of faces of ${\Gamma_n}^\alpha$ of type ${\Gamma_n}^\beta$ is $\prod_{j=1}^r \gamma_j!$ where γ_j is the size of the j^{th} segment in γ.

Example 1 $k = 1$. The set $\mathcal{P}_{n,1}$ consists of pairs of integers in $\{0, 1, \ldots, n\}$. It has order $\frac{1}{2}n(n+1)$, and each element of $\mathcal{P}_{n,1}$ specifies an edge of Γ_n up to orientation.

Example 2 $k = 2$. An element $\mathcal{P}_{n,2}$ consists of $r \in \mathbf{N}$ and r pairwise disjoint subsets, each of order at least two, and whose union has order $r + 2$. Clearly, either $r = 1$ with one set of order 3, or $r = 2$ with two sets of order 2. The elements of $\mathcal{P}_{n,2}$ may therefore be labelled,

$$ijk \qquad \text{or} \qquad ij; kl\,. \tag{4.3}$$

These two types of elements are specified by the two partitions of k, namely, 2 and 1+1, and correspond to the two possible geometrical types of 2-faces, namely, hexagonal and square, respectively. Following the discussion of the relation (4.1), one may associate $R_{ij} \in A^{\otimes n}$ to the edge of Γ_n labelled by $\{i, j\} \in \mathcal{P}_{n,1}$. The commutativity of all 1-cycles in Γ_n is equivalent to that of all 2-faces of Γ_n. The commutativity of faces labelled by (4.3) gives relations,

$$\begin{aligned} R_{ij} R_{ik} R_{jk} &= R_{jk} R_{ik} R_{ij} \\ [R_{ij},\ R_{kl}] &= 0\,, \end{aligned}$$

respectively. This is just the Yang-Baxter equation in the form natural in the context of quantum groups.

Example 3 Take $\alpha = \{\{0, 1, 2, 3, 4\}\} \in \mathcal{P}_{n,4}$. It gives rise to ${\Gamma_n}^\alpha$, a 4-dimensional face of Γ_n. The 3-dimensional faces of ${\Gamma_n}^\alpha$ are labelled by partitions, β, of $\{0, 1, 2, 3, 4\}$ into r sets, of total size $r + 3$. Those with $r = 1$ are five in number, while those with $r = 2$ are ten in number. Examples of each are $\big\{\{0, 1, 2, 3\}\big\}$ $(r = 1)$ and $\big\{\{0, 1, 2\},\ \{3, 4\}\big\}$ $(r = 2)$. The total number of three-dimensional faces of ${\Gamma_n}^\alpha$ is 30, each of the above occurring twice in antipodal faces.

Example 4 $k = 3$. The 3-dimensional faces of Γ_n come in three types, corresponding to the three possible partitions, 3, $2+1$ and $1+1+1$, of 3. Such faces are labelled by elements of $\mathcal{P}_{n,3}$, with parameters $ijkl$, $ijk;lm$ and $ij;kl;mp$, respectively. Here i,j,k,l,m and p are distinct elements of $\{0,1,\ldots,n\}$. The polyhedra so defined have the shapes of, a tetrakaidekahedron, a hexagonal prismoid and a cube, respectively. Following example 2, it is now natural to take the commutation of all 2-cycles of Γ_n, as the 3-dimensional analogue of the quantum Yang-Baxter equation; that is, the condition is equivalent to the commutation of the three types of 3-dimensional face discussed above. In this context, commutation is amongst 3-algebra elements, $S_{ijk} \in B(\Sigma_6)$ and $T_{ij;kl} \in B(\Sigma_4)$, where Σ_n denotes an n-gon. These elements are placed on each face and combined in a similar manner to the way R_{ij} was placed on each edge in example 2. See Figs. 17, 18 and 19. Note that here, B, is a 3-algebra and it is a suitable analogue of $A^{\otimes n}$ appearing in the case of 2-algebras.

The form of the 3-dimensional triangle relation descibed above should be compared with Zamolodchikov's relation (see [Z 1], [Z 2]). The tetrahedral equations appeared in Zamolodchikov's work while considering random arrays of planes in the context of 3-dimensional statistical mechanics; this clearly has relations with the geometrical structures above.

It is proposed, more generally, that the correct k-dimensional analogue of the triangle relation is a relation between elements which live naturally in the context of k-algebras. These relations merely express the commutivity of all k-dimensional faces of Γ_n, and are between elements $R^{(\gamma)} \in B(X^{(\gamma)})$, defined for each partition γ of $k-1$, of the k-algebra object associated with $X^{(\gamma)}$, a $(k-1)$-dimensional face of Γ_n with type γ. We refer to §5 for such generalisations.

In [FM], a solution of the classical version of Zamolodchikov's tetrahedral equation is quantised, but it is shown that a similar procedure breaks down in general for the simplex equation in dimensions higher than three. It is hoped that the geometrical nature of the proposed alternative higher dimensional equations will lead to a better fate for their solutions, when they are constructed. Note also that in [KV], a formulation of Zamolodchikov's equations, differing somewhat from the original simplex equation, is obtained in three dimensions, and it appears that polyhedra of the types of those depicted in Figs. 17–19 enter. Their formulation involves 2-categories which, although giving rise to similar geometrical pictures to those of our approach, take account of fewer of the inherent symmetries.

A few more comments about the relations represented in Figs. 17–19 are in order. Consider the situation from the standpoint of a supplied representation of B, on a vector space W, say. Then S_{ijk}, $T_{ij;kl}$ are now elements of $W^{\otimes 6}$ and $W^{\otimes 4}$ respectively, or equivalently, maps $W^{\otimes 3} \to W^{\otimes 3}$ and $W^{\otimes 2} \to W^{\otimes 2}$. Fig. 20 depicts these maps, in which the edges have been labelled by pairs of elements of $\{0,1,\ldots,n\}$ as in Example 1.

A trivial solution of the relations may be obtained by supposing the actions of S_{ijk} and $T_{ij;kl}$ to be independent of their indices, say s and t, with $s \in \mathrm{End}(W^{\otimes 3})$ and $t \in$

$\mathrm{End}(W^{\otimes 2})$. The relations then become,

$$\left.\begin{aligned} t_{12}t_{23}t_{12} &= t_{23}t_{12}t_{23} \\ t_{12}t_{23}t_{34}s_{123} &= s_{234}t_{12}t_{23}t_{34} \\ t_{34}s_{123}s_{345}t_{56}t_{23}s_{345}s_{123} &= s_{456}s_{234}t_{45}t_{12}s_{234}s_{456}t_{34}\ , \end{aligned}\right\} \tag{4.4}$$

in which s_{ijk} and t_{ij} denote elements of $\mathrm{End}(W^{\otimes n})$, trivial on all factors except those specified by the indices, on which the action is given by s and t, respectively. It is at once apparent that if P_{ij} denotes the action on $W^{\otimes n}$ of transposing the i^{th} and j^{th} factors, $Pt = R$ is a solution of the quantum Yang-Baxter equation. Put $\Phi = P_{13}s$. Then (4.5) reduces to the following relations.

$$\left.\begin{aligned} R_{34}R_{24}R_{14}\Phi_{123} &= \Phi_{123}R_{14}R_{24}R_{34} \\ R_{34}\Phi_{356}\Phi_{246}R_{16}R_{25}\Phi_{145}\Phi_{123} &= \Phi_{123}\Phi_{145}R_{25}R_{16}\Phi_{246}\Phi_{356}R_{34}\ . \end{aligned}\right\} \tag{4.6}$$

A solution of these equations is $\Phi = R_{12}R_{13}R_{23}$. This solution should be regarded in much the same way as the transposition matrix P_{12} with respect to the standard 2-dimensional triangle relation. It is striking to note the similarity between the first relation in (4.6) and the fundamental relation of [FRT]. See [CS1], [CS2] for other solutions, also derived from quantum groups.

4.2 Dual triangle relations and higher braid groups

The commutativity relations discussed in §4.1 for the 3-dimensional faces of Γ_n have the dual representations shown in Figs. 21–23. Here the actions of S and T are represented by 6- and 4-point vertices.

The specification of a continuous map $f\colon \mathbf{R} \to \mathbf{R}^{1/2 n(n-1)}$ is equivalent to that of $1/2 n(n-1)$ continuous maps $\mathbf{R} \to \mathbf{R}$ which may conveniently be labelled by f_{ij}, for $1 \le i < j \le n$. Consider the space $\mathcal{C}$ of such continuous maps f which are such that,

(i) $f(x) = \mathbf{f}_0$ for all $|x| > N$ and suitably large N, for some constant vector $\mathbf{f}_0 \in \mathbf{R}^{1/2 n(n-1)}$;

(ii) $f_{ij}(x) - f_{kl}(x)$ has a finite number of zeroes, in x, for all distinct pairs $i < j$ and $k < l$;

(iii) whenever x and $1 \le i < j < k \le n$, are such that at least two of $f_{ij}(x)$, $f_{ik}(x)$ and $f_{jk}(x)$ are equal, then all three are equal.

An element $f \in \mathcal{C}$ may be depicted by $1/2 n(n-1)$ curves in $\mathbf{R}^2$, each being a vertical line outside a sufficiently large region. These curves C_{ij}, are labelled by pairs of elements of $\{1, 2, \ldots, n\}$ and are such that the only intersections allowed between such curves, are combinations of crossings of C_{ij} and C_{kl} (for i, j, k, l distinct) or of C_{ij}, C_{ik} and C_{jk} (for i, j, k distinct). Let $\mathcal{C}'$ denote the subset of $\mathcal{C}$, consisting of elements for which all crossings are either double or triple points, of the elemental types just mentioned. Figs. 21–23 define generators of an equivalence relation $\sim$ on $\mathcal{C}'$. Any S and T satisfying our 3-dimensional triangle relations will provide an invariant for $\mathcal{C}'$ under $\sim$.

Viewed in this way, Figs 21–23 may be considered as a natural generalisations of the Yang-Baxter equation in another context. For, Fig. 21 is the classical depiction of the QYBE, in which each line is labelled with a single element of $\{1, 2, \ldots, n\}$, while a crossing of lines labelled i and j represents the action of R_{ij}. Suppose now that the labels are changed to be pairs of elements of $\{1, 2, \ldots, n\}$ with the extra assumption that lines labelled ij, ik and jk intersect at the triple points only. Figs. 22 and 23 naturally arise as the closest approximations to Fig. 21 in which three lines with labels ij, kl and mn intersect with i, j, k, l, m and n, not necessarily all distinct. Thus a double point, at which lines with labels ij and kl intersect, represents the action of $T_{ij;kl}$, while a triple point given by lines with labels ij, ik and jk represents the action of S_{ijk}. This viewpoint 'explains' the solution $\Phi = R_{12}R_{13}R_{23}$ of (4.6). Just as the braid group B_n and permutation group S_n are associated with QYBE, it is possible to construct analogous objects for our three dimensional generalisation. Let $B_n^{(3)}$ denote the groupoid whose objects are orderings of the set of pairs of elements of $\{1, 2, \ldots, n\}$, and whose morphisms are generated by β_{ijk}, $\alpha_{ij;kl}$ and their inverses, with relations specified by Figs. 21–23. Here β_{ijk} and $\alpha_{ij;kl}$ are the morphisms depicted in Fig. 24, which are defined between suitable pairs of objects. Thus β_{ijk} is only defined between objects in which ij, ik and jk are adjacent in an increasing order on the tail of the arrow, and in decreasing order on the head, the two orderings being identical, except from these three elements. A similar definition holds for $\alpha_{ij;kl}$.

More precisely, suppose P_1 and P_2 are two objects in $B_n^{(3)}$, that is, they are total orderings of the set of subsets of $\{1, 2, \ldots, n\}$ of order 2. By a morphism between P_1 and P_2 is meant a sequence of α's, β's and their inverses, compatible with P_1 and P_2 at their ends, defined up to an equivalence generated by the relations of Figs. 21–24.

Let $X(n, 2)$ consist of all arrangements of n lines in $\mathbf{C}^2$ parallel to n given real lines $l_1^0, \ldots, l_n^0$ in $\mathbf{C}^2$, which are in general position. Its fundamental group is Manin and Schechtman's higher braid group, $B(n, 2)$, see [MS 1] and [MS 2]. One may construct a natural map from the fundamental groupoid of $X(n, 2)$, whose objects are real arrangements, to a quotient of $B_n^{(3)}$, in which an arrangement of n lines is mapped to the image of the $1/2\, n(n-1)$ pairwise intersection points, under a projection onto a line l_0. Here $l_0, l_1^0, \ldots, l_n^0$ are $(n+1)$ real lines in general position in $\mathbf{C}^2$, with say, increasing positive slopes. The generators of $B(n, 2)$ are given by one line l_i, passing through the intersection point of two others, l_j and l_k and this is mapped to β_{ijk} with $i < j < k$. The quotient is generated by local moves represented by α's.

A solution of the relations of Figs. 21–23, provides a representation of $B(n, 2)$, so long as the α's are mapped to 1, in which case the relations reduce to Zamolodchikov's relations.

Note: The above discussion indicates an analogue of the braid group which is a '3-group' generated by S_{ijk} and $T_{ij;kl}$ with relations as represented by Figs. 17–19. Note that the corresponding construction with 2-groups gives the group,

$$G_n = \langle R_{ij} \mid R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij} \text{ and } R_{ij}R_{kl} = R_{kl}R_{ij} \rangle \,,$$

in which i, j, k, l run over all distinct elements of $\{1, 2, \ldots, n\}$. The twisted product $G_n \rtimes S_n$, in which S_n acts on G_n by permuting the indices,

$$\sigma(R_{ij}) = R_{\sigma(i)\sigma(j)} = \sigma R_{ij} \sigma^{-1}, \qquad \sigma \in S_n$$

contains a normal subgroup H_n of index $n!$, generated by $\tilde{R}_{ij} = (i\,j)R_{ij}$. This group has a presentation,

$$H_n = \langle \tilde{R}_{ij} \mid \tilde{R}_{ij}\tilde{R}_{jk}\tilde{R}_{ij} = \tilde{R}_{jk}\tilde{R}_{ij}\tilde{R}_{jk} \text{ and } \tilde{R}_{ij}\tilde{R}_{kl} = \tilde{R}_{kl}\tilde{R}_{ij} \rangle$$

and contains many copies of the braid group B_n. A representation of G_n (and thus also of H_n and B_n) is provided from any quasi-triangular Hopf algebra (quantum group) [D 1].

5: Generalisations

Some of the structures constructed in this paper may be extended to the d-dimensional case in a natural way. Thus, a d-algebra is obtained in a similar way to a 3-algebra, with $(d-1)$-simplices replacing triangles and d-simplices replacing tetrahedra. It will describe the behaviour of d-tensors under composition (via contractions).

Formally, a product in a d-algebra, A, is a labelled decomposition of a $(d-1)$-simplex into other such simplices, with no internal vertices. The labelling defines an element of A for each $(d-1)$-simplex, and a labelling $1, 2, \ldots, d$ on the faces of each $(d-1)$-simplex in the decomposition. These labelled simplices must have compatible orientations. This allows a freedom for the labelling on each simplex, described by the alternating group, A_d.

An evaluation of a product is a set of d-simplices based on the vertices as defined by the products, such that each $(d-1)$-simplex appears as a face at most twice, those occurring once, being precisely the simplices defining the product, and the base simplex. This is a combinatorial description for a d-dimensional manifold whose boundary is the $(d-1)$-dimensional manifold formed by adjoining the base simplex to the surface defined by the product.

The structure of a d-algebra is so arranged that, given any product and an associated evaluation, a result may be obtained in this algebra. Moreover, this result depends only upon the topology of the d-dimensional manifold bounded by the surface defined by the product. The operations present in a d-algebra, like those of a 3-algebra, enable the necessary symmetries and d-simplices to be described. There are, however, many difficulties in the higher dimensional case, the foremost of which being that the moves required to translate between any two topologically equivalent subdivisions of a d-dimensional manifold with triangulated boundary, become complex. See the review in [TV] for details on how such moves may be defined. Another problem is that some subdivisions into d-simplices may not be capable of being viewed as compositions of multiplication maps. This is because it is possible for no d-simplex to contain at least $\frac{1}{2}(d+1)$ faces in the boundary.

It is possible to remove some of these problems by restricting allowed evaluations to those with trivial topology and introducing geometric realisations as in §2. An evaluation is then a decomposition of the d-dimensional interior of a geometric realisation of the product into d-simplices, whose interiors are pairwise disjoint, and which contain no interior vertices.

By a d-algebra is meant a vector space over some field k, together with k-linear maps,

$$m_i \colon A^{\otimes(d+1-i)} \longrightarrow A^{\otimes i}\ ,$$

for $i = 1, 2, \ldots, [1/2(d+1)]$, and an action P of A_d upon A, satisfying suitable axioms. The axioms come in three types.

Type I Associativity axioms.

For any $j \in \{2, 3, \ldots, [1/2(d+1)]\}$, it is possible to subdivide a suitable union of j d-simplices into $(d+2-j)$ d-simplices. This corresponds to all possible projections of a $(d+1)$-simplex onto a d-dimensional plane without internal vertices. Let Σ_j denote the boundary surface of the union of j d-simplices, of the form just noted. Then Σ_j is a closed union of $j(d+2-j)$, $(d-1)$-simplices. The surface Σ_j may be projected onto a $(d-1)$-plane, providing two decompositions into k and $j(d+2-j)-k$, d-simplices, say $\Sigma_j^{(1)}$ and $\Sigma_j^{(2)}$. This is equivalent to decomposing Σ_j into two (topological) discs. One may now attempt to interpret the different decompositions of the interior of Σ_j into d-simplices, as evaluations from $A(\Sigma_j^{(1)})$ to $A(\Sigma_j^{(2)})$. When this is possible, there is exhibited an axiom giving the equivalence of the corresponding compositions of the m-maps; j on one side of the equation and $(d+2-j)$ on the other side.

Type II Symmetry axioms.

For each $j = 2, 3, \ldots, [1/2(d+1)]$, there is an axiom requiring of m_j the appropriate symmetry of the union of i, d-simplices. The full symmetry group of such a union is $S_j \times S_{d+1-j}$, and the direct symmetry group $S_{d,j}$ is an index-2 subgroup. The associate axiom specifies that m_j commutes with the action of $S_{d,j} = S_{d,d+1-j}$, upon $A^{\otimes j}$ on one side and $A^{\otimes(d+1-j)}$ on the other. The action of an element of $S_{d,j}$ upon $A^{\otimes j}$ is the composition of an appropriate permutation of the factors, with actions of the form $P(\sigma)$, for suitable $\sigma \in A_d$, on individual factors, A, of $A^{\otimes j}$.

Type III Invariance under change of internal vertices.

These may be considered as type I equivalences. Consider a segment of a product, described by a collection of $(d-1)$-dimensional simplices, whose union is topologically a $(d-1)$-dimensional disc. Introducing a notion of the geometrical realisation of such a triangulation, it is seen that the allowed evaluations depend on the choice of a geometric realisation of the product. Type III axioms ensure the equality of results obtained from evaluations associated with different geometric realisations of the product.

For $d = 3$, (i)–(iv) of definition 2.1 have type I and (vi)–(vii) have type II, while (v) is of type III. One may also define a full d-algebra, in which there are maps $m_i\colon A^{\otimes(d+1-i)} \longrightarrow A^{\otimes i}$, for all $i = 1, \ldots, (d+1)$.

By a representation of a d-algebra A we mean a map $A \longrightarrow V^{\otimes d}$ where V is a k-vector space with non-degenerate inner product, such that the multiplication maps are compatible with contraction, in the same sense as in §2.

Example Let us consider the case $d = 5$. We are considering products given by combinations of 4-simplices. Start with six points $P_1, \ldots, P_6$ in $\mathbf{R}^4$ whose convex hull, C, does not contain any of the initial points in its interior. Then C may be expressed as a union of 4-simplices, the vertices being drawn entirely from the given points. Depending upon the configuration of $\{P_i\}$, it may be necessary to use two or three 4-simplices.

In the case when C may be decomposed into two 4-simplices, $P_1P_2P_3P_4P_5$ and $P_1P_2P_3P_4P_6$, say, the boundary of C is a union of eight tetrahedra, $P_1P_2P_3P_5$, $P_1P_2P_4P_5$, $P_1P_3P_4P_5$, $P_2P_3P_4P_5$, $P_1P_2P_3P_6$, $P_1P_2P_4P_6$, $P_1P_3P_4P_6$ and $P_2P_3P_4P_6$. Choose a seventh point, P_7, in the interior of C. A decomposition of C into eight 4-simplices is achieved, using cones with vertex P_7 and whose bases are the eight tetrahedral faces of C. This defines a product in a 5-algebra, when endowed with a suitable labelling. To compute the product one must combine the three types of multiplication maps appropriately.

$$\begin{aligned} m_1 &: A^{\otimes 5} \longrightarrow A \\ m_2 &: A^{\otimes 4} \longrightarrow A^{\otimes 2} \\ m_3 &: A^{\otimes 3} \longrightarrow A^{\otimes 3} \,. \end{aligned}$$

As a function of the elements of A associated with the eight 4-simplices involved in the product, the result of the product is a map,

$$A^{\otimes 8} \longrightarrow A^{\otimes 2} \,,$$

where A is the 5-algebra. Recall that an evaluation may be denoted by a subdivision of the 5-dimensional region bounded by the eight 4-simplices subdividing C, together with $P_1P_2P_3P_4P_5$ and $P_1P_2P_3P_4P_6$ (the latter two with opposite orientations to the rest). It is apparent that this 5-dimensional region may be expressed as the union of two 5-simplices $P_1P_2P_3P_4P_6P_7$ and $P_1P_2P_3P_4P_5P_7$ (or as the union of five 5-simplices $P_1P_2P_3P_5P_6P_7$, $P_1P_2P_3P_4P_5P_6$, $P_1P_2P_4P_5P_6P_7$, $P_1P_3P_4P_5P_6P_7$ and $P_2P_3P_4P_5P_6P_7$). The result of the product may now be expressed in terms of the composition of maps m_i. Below are written the 4-simplices as they are combined, starting with the conical subdivision of C, and ending with two 4-simplices. At each step, only the six 4-simplices involved in the m_i map are noted.

$$\begin{aligned} &P_1P_2P_3P_5P_7, P_1P_2P_4P_5P_7, P_1P_3P_4P_5P_7, P_2P_3P_4P_5P_7 \xrightarrow{m_2} P_1P_2P_3P_4P_7, P_1P_2P_3P_4P_5 \\ &P_1P_2P_3P_4P_7, P_1P_2P_3P_6P_7, P_1P_2P_4P_6P_7, P_1P_3P_4P_6P_7, P_2P_3P_4P_6P_7 \xrightarrow{m_1} P_1P_2P_3P_4P_6 \,. \end{aligned}$$

Another evaluation is obtained by interchanging P_5 and P_6. The equivalence of the resulting two maps $A^{\otimes 8} \longrightarrow A^{\otimes 2}$, both of the form $m_1 \circ m_2$ (with additional actions of A_5 and permutations of factors) provides an axiom of type III.

Another axiom of type III appears from the case when C can be decomposed into three 4-simplices $P_1P_2P_3P_4P_5$, $P_1P_2P_3P_4P_6$ and $P_1P_2P_3P_5P_6$, say. The boundary of C is a union of nine tetrahedra, and adjoining a point P_7 inside C, subdivides C into nine cones. A product $A^{\otimes 9} \longrightarrow A^{\otimes 3}$ is now defined, and evaluations may be obtained from decompositions of a five-dimensional object with twelve faces, $P_1P_2P_4P_5P_7$, $P_1P_3P_4P_5P_7$, $P_2P_3P_4P_5P_7$, $P_1P_2P_4P_6P_7$, $P_1P_3P_4P_6P_7$, $P_2P_3P_4P_6P_7$, $P_1P_2P_5P_6P_7$, $P_1P_3P_5P_6P_7$, and $P_2P_3P_5P_6P_7$ and the three initial 4-simplices, $P_1P_2P_3P_4P_5$, $P_1P_2P_3P_4P_6$, $P_1P_2P_3P_5P_6$. An evaluation is shown below.

$$\begin{aligned} &P_1P_2P_4P_5P_7, P_1P_3P_4P_5P_7, P_2P_3P_4P_5P_7 \xrightarrow{m_3} P_1P_2P_3P_5P_7, P_1P_2P_3P_4P_7, P_1P_2P_3P_4P_5 \\ &P_1P_2P_3P_4P_7, P_1P_2P_3P_6P_7, P_1P_3P_4P_6P_7, P_2P_3P_4P_6P_7 \xrightarrow{m_2} P_1P_2P_3P_6P_7, P_1P_2P_3P_4P_6 \\ &P_1P_2P_3P_6P_7, P_1P_2P_5P_6P_7, P_1P_3P_5P_6P_7, P_2P_3P_5P_6P_7, P_1P_2P_3P_5P_7 \xrightarrow{m_1} P_1P_2P_3P_5P_6 \,. \end{aligned}$$

There are six different such evaluations, all of the form $m_1 \circ m_2 \circ m_3$. Geometrically the above evaluation involves three 5-simplices which all define meaningful maps when P_7 lies inside $P_1P_2P_3P_5P_6$. The six possible evaluations come from varying the position of P_7 inside C, so as to obtain differential geometric realisations of the product. The equivalence of these six evaluations gives another axiom of type III.

The d-dimensional Yang-Baxter equations are given by the possible d-dimensional faces of Γ_n (arbitrary n). These are relations amongst objects in a d-algebra, residing on the $(d-1)$-dimensional faces of Γ_n. Recall that the number of such objects is the number of partitions of $(d-1)$, while the number of relations is the number of partitions of d.

Define a groupoid $B_n^{(d)}$ whose objects are total orderings of $\mathcal{P}_{n,d-2}$ and whose morphisms are generated by operations corresponding to the $(d-1)$-dimensional faces of Γ_n. The operation associated with a face, Δ, permutes the terms of the ordering associated with elements of $\mathcal{P}_{n,d-2}$ labelling the faces of Δ. Let $X(n,d-1)$ denote the space of arrangements, in general position, of n hyperplanes in $\mathbf{C}^{\alpha-1}$, parallel to a fixed set of n hyperplanes, also in general position. A map may be defined from the fundamental groupoid of the space, $X(n,d-1)$, to a quotient of $B_n^{(d)}$. The fundamental group of $X(n,d-1)$ is Manin and Schechtman's higher braid group $B(n,d-1)$. This establishes a relation between higher triangle relations and higher braid groups. Note, however, that the remarks at the end of §4 apply again here, so that a more natural object of study in this context, is not a group, but rather what may be called a 'd-group'.

6: Further remarks

The axioms for a 3-algebra given in §2.1 lead to many interesting new equations. As was seen in §3, quantum groups give rise to solutions of these equations via $6j$-symbols, in which a basis for the 3-algebra is labelled by triples of irreducible representations. Recall that P acts by cyclically permuting the labels. One may attempt to construct a 3-algebra based on a vector space $\langle e_n \mid n \in \mathbf{Z}\rangle$ in which,

$$\left.\begin{aligned} P(e_n) &= e_n \; ; \\ m(e_k \otimes e_l \otimes e_n) &= \Delta_{k,l,n} e_{k+l+n} \; ; \\ \bar{b}(e_k \otimes e_l) &= \sum_a w_{l,k,a} e_a \otimes e_{k+l-a} \; . \end{aligned}\right\} \tag{6.1}$$

Such data provides a 3-algebra analogue of the ring of Laurent polynomials in one indeterminate, and m and $\bar{b}$ preserve the sum of indices. The 3-algebra axioms give relations satisfied by Δ and w. Define $f(i,j,k,l) = \Delta_{i,j,k}$ when $i+j+k+l=0$ and $i,j,k,l \in \mathbf{Z}$.

Lemma *Set $w_{l,k,a} = \Delta_{k,l,a-k-l}$ and suppose that f is A_4-symmetric. Then (6.1) defines a 3-algebra on $\langle e_n \mid n \in \mathbf{Z}\rangle$ if, and only if, for all λ, μ, ν, ρ, $\sigma \in \mathbf{Z}$,*

$$\sum_{a \in \mathbf{Z}} \Delta_{\rho,\mu,-a} \Delta_{\nu,\mu+\rho-a,\sigma} \Delta_{\lambda,a,\mu+\nu+\rho+\sigma-a} = \Delta_{\lambda,\mu,\nu} \Delta_{\lambda+\mu+\nu,\rho,\sigma} \; . \tag{6.2}$$

Denote $\Delta_{ijk} = f(i,j,k,-i-j-k)$ by a tetrahedral vertex, in which each edge is oriented and labelled by a flow as given in Fig. 25. The total flow into such a vertex is zero.

In terms of this vertex notation for Δ the relation (6.2) is depicted in Fig. 26. Notice the similarity with Fig. 12. Fig. 26 should be considered as expressing the equivalence of the two networks of tetrahedral vertices shown, each with six inputs. For given inputs, with net flow zero, the flow between the two vertices on the left hand side is determined, while on the right hand side there is a one parameter freedom possible in the flows between the three vertices.

The relations (6.2) are similar in form to the pentagon relation arising in the context of quasi-Hopf algebras [D 2]. However, it is not clear for which fields, k, a non-trivial solution of (6.2) exists over k. This question may be generalised by replacing $\mathbf{Z}$ by an arbitrary abelian group. Note that putting $\lambda = \mu = \nu = 0$ in (6.2) gives,

$$\sum_a \sigma_{\rho,a-\rho}\sigma_{\rho-a,\sigma}\sigma_{a,\rho+\sigma-a} = \sigma_{\rho,\sigma}\sigma_{0,0}\,, \tag{6.3}$$

where $\sigma_{a,b} = \Delta_{0,a,b}$. Solutions to (6.3) come from the usual fusion algebras of conformal field theory. For example, $g_{ijk} \equiv \sigma_{ij}$ for $i+j+k=0$ provides a solution with g cyclically symmetric, and

$$g_{1\,0\,-1} = g_{0\,0\,0} = -g_{1\,-1\,0} = 1\,,$$

defining the non-zero terms. This solution of (6.3) cannot, however, be extended to a solution of (6.2).

The above example serves to illustrate the complexity of the structures arising out of 3-algebras. It is apparent that a minimal 3-algebra on one generator is spanned, as a vector space, by a set labelled by possible (genus zero) triangulations of a triangle. Any triangulation into n triangles, may be transformed via $\overline{b}$ into a standard such triangulation. It is thereby seen that there need be only one independent (vector space) generator, e_n, for each integer n. Hence, (6.1) defines a 3-algebra structure with one (algebra) generator, whenever solutions of (6.2) are given. The construction of more general universal enveloping algebras involves equations which are more complex but of a type similar to (6.2), whose solutions are fundamental in the construction of a 'higher' quantisation process. Just as quantum groups appear as deformations of universal enveloping algebras of Lie algebras, one may attempt to quantise 3-algebras. Since quantum groups may be used to provide examples of 3-algebras (see §3), their quantisations can be expected to be 'doubly quantised objects'.

In further work, we intend to develop deeper the notion of a 3-algebra and its relationship to solutions of higher dimensional statistical mechanical systems, as well as to the essentially geometric nature of topological field theories. The author believes that because the notion of a 3-algebra contains within it inherent 3-dimensional symmetries, it provides a natural setting for generalisations of many concepts in algebra and topology.

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