Triangulations, Categories and Extended Topological Field Theories [1] [2]

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Abstract. The concept of a topological field theory is extended to encompass structures associated with manifolds of codimension >1. When all the manifolds involved are considered triangulated, it is seen that such structures may be constructed from a finite quantity of data, most conveniently viewed as associated with polyhedra and their decompositions. The special cases of 2 and 3 dimensions are briefly considered, the relations with structures of higher categories, algebras and vector spaces, becoming clear. A more detailed account is currently in preparation.

2:: Introduction

A d-dimensional topological field theory (TFT), associates a vector space $Z(\Sigma)$ to a closed $(d-1)$-dimensional manifold Σ, and a vector $Z(M)\in Z(\partial M)$ to a d-dimensional manifold M. In many cases the allowed manifolds M and Σ may be restricted in some way, or may be supplied with extra data, for example, a framing or triangulation. We shall assume that all manifolds considered are orientable, and supplied with an orientation, unless otherwise stated. The structure, Z, is constrained by the requirement that it satisfy certain axioms. Their precise form varies amongst authors, but the following general set are by now fairly standard [A].

NATURALITY An isomorphism of manifolds $\alpha\colon \Sigma_1^{d-1}\xrightarrow{\sim}\Sigma_2^{d-1}$ induces an isomorphism $\Sigma(\alpha)\colon Z(\Sigma_1)\xrightarrow{\sim} Z(\Sigma_2)$ of the corresponding vector spaces, with $Z(\beta\circ\alpha)=Z(\beta)\circ Z(\alpha)$ for any suitable $\beta\colon \Sigma_2^{d-1}\xrightarrow{\sim}\Sigma_3^{d-1}$.
MULTIPLICATIVITY $Z(\Sigma_1\coprod\Sigma_2)=Z(\Sigma_1)\otimes Z(\Sigma_2)$, where $\coprod$ denotes disjoint union.
VACUUM $Z(\varnothing)=K$, the base field.
DUALITY If Σ^{d-1} is a closed manifold, there is a natural isomorphism $Z(\Sigma^*)=Z(\Sigma)^*$ where Σ^* denotes the manifold Σ endowed with the opposite orientation.
NATURALITY An isomorphism $\alpha\colon M_1^d\xrightarrow{\sim} M_2^d$ induces an equality between the vectors $Z(M_1)$ and $Z(M_2)$ in the vector spaces $Z(\partial M_1)$ and $Z(\partial M_2)$, by the isomorphism $Z(\alpha|_{\partial M_1})$ of .
ASSOCIATIVITY If Σ_1, Σ_2 and Σ_3 are $(d-1)$-dimensional manifolds and

[1] This paper was presented at the AMS Meeting #876, held in Dayton, Ohio, on Oct. 31 1992.
[2] This paper is supported in part by NSF Grant No. 9013738.
[3] The author is a Junior Fellow of the Society of Fellows.

Y and Y' are cobordisms between Σ_1 and Σ_2, Σ_2 and Σ_3, respectively, then

$$Z(Y \cup Y') = Z(Y) \circ Z(Y') \in Z(\Sigma_1) \otimes Z(\Sigma_3)^*$$

where $Z(Y)$ and $Z(Y')$ are viewed as elements of $Z(\Sigma_1) \otimes Z(\Sigma_2)^*$ and $Z(\Sigma_2) \otimes Z(\Sigma_3)^*$, respectively, and here $\circ$ denotes the natural contraction map.

There are two other axioms that are sometimes included, viz., that of *completeness* and that variously called *duality* or *conjugation* at the level of d-dimensional manifolds. These, however, do not alter the main structure involved, the latter only being meaningful when $Z(\Sigma)$ is endowed with a $*$-structure, e.g. if it is a Hilbert space.

Suppose that M^d is a closed manifold. Then, by , $Z(M) \in K$ is the associated invariant. It may be computed from a splitting of M into M_1 and M_2 with common boundary Σ^{d-1} as follows. Here $\partial M_1 = \Sigma$ and $\partial M_2 = \Sigma^*$, say. Then M_1 and M_2 determine vectors, $Z(M_1) \in Z(\Sigma)$ and $Z(M_2) \in Z(\Sigma)^*$. The natural pairing on these two vectors gives $Z(M)$.

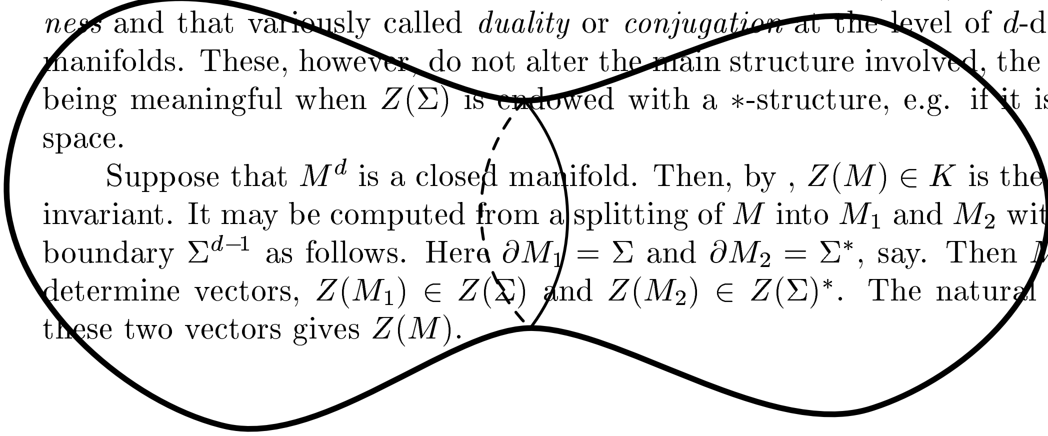

Figure 1: 3

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Suppose now that Σ, a closed $(d-1)$-dimensional manifold, is split into two parts, Σ_1 and Σ_2 with common boundary C^{d-2}. We would like to associate to Σ_1 and Σ_2 elements of an appropriate structure and its dual, dependent on C, in such a way that their natural pairing gives back $Z(\Sigma)$. Since $Z(\Sigma)$ is a vector space, the appropriate structure to be associated with C is a higher generalisation — a 2-vector space. In a similar way, closed $(d-r)$-dimensional manifolds will give rise, in an extended topological field theory (ETFT), to r-vector spaces.

In §2, the notion of an r-vector space, extending Kapranov and Voevodsky's notion of a 2-vector space, will be briefly introduced. Appropriate axioms for an ETFT are given in §3, and they are seen to embody the natural rules of composition (gluing laws) of manifolds and, therefore, an ETFT can be viewed as a functor from a universal object embodying such rules, to an object based on r-vector spaces. The precise formulation of these concepts are to appear in [L 1].

In this paper the basic ideas will be presented and illustrated in the case of dimension two. It will be seen that the formulation of invariants obtained is that of a generalised state model construction on a 'blow-up' of a triangulation of the manifold. The data involved in the state model is associated with a finite collection of suitably labelled elemental polyhedra, and it must satisfy relations geometrically expressed as the equivalences of distinct decompositions of a polyhedron into elemental forms. This is discussed in detail in §4 where $d = 2$. In §5 some examples are given; in particular the Euler characteristic and the invariant of Turaev–Viro. Further comments on the extensibility of the constructions are made in §6.

3:. Higher vector spaces

Suppose K is a field. A finite dimensional vector space V over K is specified, up to isomorphism, by its dimension, a non-negative integer, n. Given two vector spaces, V and W, over K, it makes no sense to ask whether V and W are *equal*, only whether they are *isomorphic*. In particular, a linear transformation, $V \to W$, is given by an $m \times n$ matrix over K, where $m = \dim W$. From such considerations it becomes natural to consider the category of coordinatised vector spaces, $\mathcal{C}_1$, whose objects are such vector spaces and whose morphisms are linear transformations.

The structure of a higher vector space is meant to be like that of a vector space, but where K has been replaced by a category, or even a higher category. For 2-vector spaces, this notion was introduced in [KV], there being also introduced three different 2-category structures of such 2-vector spaces, each with a different degree of coordinatisation. A fully coordinatised 2-vector space is specified, up to isomorphism, by its dimension. If V is a 2-vector space of dimension n, and $\langle e_i \rangle$ is a basis for V, the *elements* of V should be thought of as formal linear combinations,

$$\sum_{i=1}^{n} \lambda_i e_i \ ,$$

where $\lambda_i \in \mathcal{C}_1$ are vector spaces. That is, an element of V is equivalently an n-tuple of vector spaces. If W is a 2-vector space of dimension m, then a map $T\colon V \to W$ is given by an $m \times n$ matrix T_{ij} of elements of $\mathcal{C}_1$. The action of T is given by,

$$T(\sum \lambda_j e_j) = \sum \mu_i f_i \ ,$$

where,

$$\mu_i = \bigoplus (T_{ij} \otimes \lambda_j)$$

in which $\oplus$ and $\otimes$ have their usual meanings as on vector spaces. Two such maps S and $T\colon V \to W$ cannot be directly compared, just as vector spaces cannot be equated. Rather there may exist a natural transformation U from S to T, and it is specified by a family,

$$U_{ij}\colon S_{ij} \longrightarrow T_{ij} \ ,$$

of linear transformations, i.e., an array of matrices. Note that the dimensions of any two entries in the array need not be in any way related. This whole structure forms a 2-category in the sense of [MoSe], objects being 2-vector spaces, 1-morphisms being maps between them and 2-morphisms being natural transformations.

In general, an $(r+1)$-vector space should be thought of as a linear space over $\mathcal{C}_r$, the r-category of r-vector spaces. Up to equivalence, an r-vector space is specified by its dimension. There is a special object in $\mathcal{C}_r$, namely the r-vector space, $\mathbb{1}_r$, of dimension 1. Whenever V is an r-vector space, the space of linear maps $V \to \mathbb{1}_r$, thought of as maps between linear spaces over $\mathcal{V}_{r-1}$, form another r-vector space, denoted by V^*. There are clearly also notions of direct sum, and tensor product, defined in similar ways to those for the usual vector spaces. Finally, as can be seen from the case of 2-vector spaces above, the higher morphism structures involve multiple nested indices.

4:. Extended topological field theories

The axioms for a topological field theory, as given in §1, may be embodied in a statement of the form, Z is a functor from the category $\mathcal{M}_1$ to the category $\mathcal{V}_1$ of vector spaces, preserving suitable additional structures. Here and throughout this section the top dimension, d, of the theory will be assumed fixed and therefore all dependencies upon d will be omitted. The category $\mathcal{M}_1$ has objects and morphisms given by,

$$\mathrm{Obj}_{\mathcal{M}_1} = \{\text{closed } (d-1)\text{-dim. oriented manifolds up to isomorphism}\}$$

$$\mathrm{Morph}_{\mathcal{M}_1}(\Sigma_1, \Sigma_2) = \left\{ \begin{array}{l} d\text{-dimensional oriented manifolds } M \\ \text{with } \partial M \simeq \Sigma_1^* \coprod \Sigma_2, \text{ up to isomorphism} \end{array} \right\}$$

with additional special structures,

(i) $\emptyset \in \mathrm{Obj}_{\mathcal{M}_1}$, the empty object;
(ii) $\coprod$, disjoint union, providing a monoidal structure on $\mathcal{M}_1$;
(iii) $*$, the operation of reversing orientation, which gives a contravariant functor $\mathcal{M}_1 \longrightarrow \mathcal{M}_1$.

The category $\mathcal{V}_1$ consists of vector spaces, that is,

$$\mathrm{Obj}_{\mathcal{V}_1} = \mathbf{Z}^+$$
$$\mathrm{Morph}_{\mathcal{V}_1}(m, n) = \{\alpha\colon [m] \times [n] \longrightarrow K\}$$

where K is the chosen base field, and $[m] = \{1, 2, \ldots, m\}$ as in §2. The additional extra structures corresponding to those for $\mathcal{M}_1$ are,

(i) $1 \in \mathrm{Obj}_{\mathcal{V}_1}$, the vector space K;
(ii) $\otimes$, the operation of tensor product, which acts as multiplication on objects;
(iii) $*$, the duality operation, which on objects takes $n \longmapsto n$, and on morphisms takes $\alpha\colon [m] \times [n] \to K$ to $\alpha \circ P\colon [n] \times [m] \to K$, where P is the map permuting the first two factors.

With the above definitions of tensor category structures, the axioms – of §1 are now precisely embodied in the functorial nature of Z. The category $\mathcal{M}_1$, with the above extra structure, embodies the gluing rules of manifolds of dimension d, and of closed $(d-1)$-dimensional manifolds.

An extended TFT (ETFT) is a structure similar to a TFT in which gluing rules of manifolds of all codimensions up to d are embodied. An s-ETFT contains only gluing rules of manifolds with codimensions $\leq s$, so that a 1-ETFT is just a TFT, while a d-ETFT is just an ETFT. Thus an s-ETFT is a functor,

$$\mathcal{M}_s \longrightarrow \mathcal{V}_s \ ,$$

where $\mathcal{M}_s$ and $\mathcal{V}_s$ are suitable structures, both not unlike that of an s-category; $\mathcal{M}_s$ is a structure in which the r-morphisms label $(d-s+r)$-dimensional manifolds, $r \in [s]$, and the objects label closed $(d-s)$-dimensional manifolds. In $\mathcal{V}_s$, the r-morphisms are elements of $(s-r+1)$-vector spaces, as defined in §2, while the objects are s-vector spaces.

A d-dimensional s-ETFT associates to each closed $(d-r)$-dimensional manifold, M with $0 \leq r \leq s$, an r-vector space ${}_r(M)$; and to each $(d-r)$-dimensional manifold, M with $0 \leq r < s$, an element $Z_r(M)$ of the $(r+1)$-vector space ${}_{r+1}(\partial M)$; and does this in such a way that certain axioms are satisfied. As for the case of a TFT, the manifolds involved may be restricted in some way, or they may be endowed with extra structures. The properties which Z and must satisfy include the following.

NATURALITY An isomorphism, α, of closed manifolds, Σ_1^{d-r} and Σ_2^{d-r}, induces an isomorphism of the corresponding r-vector spaces, under which the elements corresponding to isomorphic manifolds M_1 and M_2 of dimension $d-r+1$ whose boundaries are Σ_i, are identified, whenever α is the restriction of the isomorphism to the boundary.

VACUUM ${}_r(\emptyset) = \mathbb{1}_r$, while if M is a closed $(d-r)$-dimensional manifold, ${}_r(M) \in \mathbb{1}_{r+1}$ may be identified with $Z_r(M) \in \mathcal{V}_r$.

DUALITY If Σ^{d-r} is a closed manifold $(1 \leq r \leq s)$, there is a pairing ${}_r(\Sigma) \otimes_r {}(\Sigma^*) \longrightarrow \mathbb{1}_r$, where Σ^* denotes Σ endowed with the opposite orientation.

MULTIPLICATIVITY ${}_r(\Sigma_1 \coprod \Sigma_2) =_r (\Sigma_1) \otimes_r (\Sigma_2)$.

ASSOCIATIVITY

The most important property is , the analogue of the associativity axiom (axiom of §1) for TFT's. This is a generalised gluing law, by which whenever a manifold M is decomposed into a union of other manifolds, M_i, with gluing taking place along boundaries (possibly only along part-boundaries), there is a procedure by which $Z(M)$ may be obtained naturally from $\{Z(M_i)\}$ by composing only structures already associated in the theory to the M_i, their boundaries, common part-boundaries,..., up to codimension s incidence properties. However, the exact statement of this property, in the general setting, is complex. The case of 2-ETFT with $d = 2$ is illustrated in the next section.

5:. 2-ETFT

In this section the structure of a 2-ETFT is investigated. The basic objects are d-, $(d-1)$-, and $(d-2)$-dimensional manifolds, the $(d-2)$-dimensional manifolds involved all being closed. There are gluing operations of the same type as those arising in 1-ETFT. Thus, if $\partial M_1 = \Sigma$ and $\partial M_2 = \Sigma^*$ where M_1 and M_2 are $(r-1)$-dimensional manifolds while Σ is a closed r-dimensional manifold, then there is a manifold M, obtained by gluing M_1 and M_2 along Σ, where $r = d$ or $d-1$. There is also the slightly more general gluing along a (still closed) part-boundary, as in of §1. However, in 2-ETFT, there is an additional type of gluing of d-dimensional manifolds M_i $(1 \leq i \leq n-1)$, for which $\partial M_i = \Sigma_{i+1} \cup \Sigma_i^*$, where Σ_i are $(d-1)$-dimensional manifolds with common boundary C, $(1 \leq i \leq n)$. Here C is a closed $(d-2)$-dimensional manifold. As in the case of the first type of gluing operation, there is a slightly more general form in which the codimension 2 manifold, C, along which the gluing takes place, is only a (still closed) part boundary of Σ_i. It may be observed that the types of gluing laws appearing in s-ETFT are independent of d, so that to study 2-ETFT, all the structures needed appear when $d = 2$.

We shall assume that all manifolds considered are endowed with a triangulation and an orientation. In this context, the gluing laws are best observed by transforming a triangulated k-dimensional manifold, $(M, \mathcal{T})$, to its 'blow-up' $(M, \mathcal{T}^*)$, in which each top-dimensional cell in $\mathcal{T}^*$ corresponds to a simplex or sub-simplex in $\mathcal{T}$ (see Figure 2).

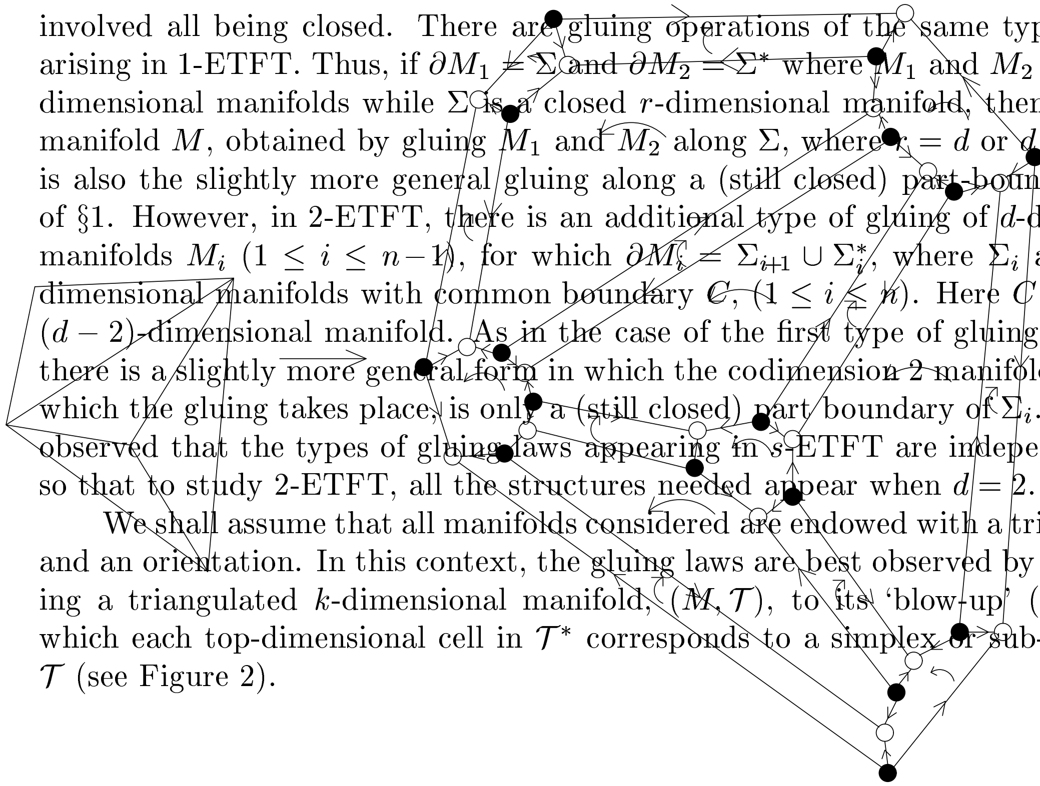

Figure 2: 4

0,20 When M is closed the vertices of $\mathcal{T}^*$ are labelled by flags of simplices in $\mathcal{T}$, $X_0 \subset \cdots \subset X_k$, while faces in $\mathcal{T}^*$ are labelled by incomplete flags in $\mathcal{T}$. When M is not closed, $\mathcal{T}^*$ is defined as for the closed case, except that $\mathcal{T}$ is altered by adjoining a 'virtual' top-dimensional simplex with the incidence property that it contains all simplices in ∂M. In this case, there is also another decomposition for which the vertices are in 1–1 correspondence with actual (complete) flags of simplices in $\mathcal{T}$. As geometric decompositions, $\mathcal{T}^*$ can be obtained from $\mathcal{T}_*$ by adjoining a cylinder on $(\partial\mathcal{T})^*$, where $\partial\mathcal{T}$ is the triangulation of ∂M induced from $\mathcal{T}$ by restriction. By definition, every cell, C, of $\mathcal{T}^*$ may be labelled by a subset, I, of $\{0, 1, \ldots, k\}$ specifying the dimensions of the simplices in the corresponding partial flag. In such a case, the dimension of C is $k+1-|I|$, and I is called the type of C. Let $i = \max(I)$. Given I, the geometric form of C is completely determined by the link of X_i in $\mathcal{T}$.

From an orientation on M it is possible to canonically define orientations on all cells and sub-cells in $\mathcal{T}_*$. For $k = 2$ the result is illustrated in Figure 2, in which the orientations on r-dimensional cells are depicted by arrows, for $r = 1$ and 2, and filled and open circles, for $r = 0$. The general definitions of the orientations on the cells and subcells of $\mathcal{T}^*$ and $\mathcal{T}_*$ may be found in [L 1].

For $k = 2$, there are 7 possible non-empty subsets, I, of $\{0, 1, 2\}$, the dimension of an associated cell in $\mathcal{T}^*$ being $3 - |I|$. For ease of notation, the type, I, of a cell will be denoted by the string of its elements, in ascending order, and without separators. There are two kinds of type-012 cell, namely points with positive or negative orientation. There is one kind of cell of each of the types 01, 02 and 12, each of these being intervals bounded by two type-012 cells, with opposite orientations. The cells in $\mathcal{T}^*$ of type 0,1 and 2 are 2-dimensional, there being one kind of each of the last two types. For any $n \geq 2$, there is a $2n$-gon of type 0 in $\mathcal{T}^*$, associated to a point in $\mathcal{T}$ on which n edges are incident, and we say such a cell is of type 0^n. See Figure 3.

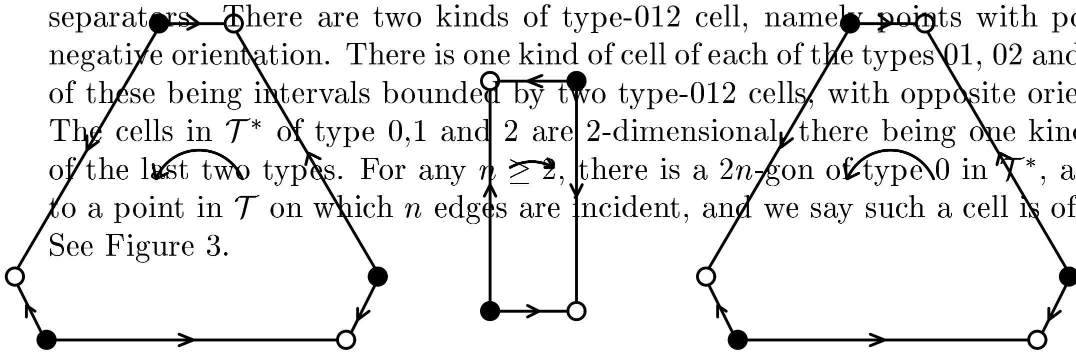

Figure 3: 3

4,30

Suppose now that M is a 2-dimensional manifold whose boundary C contains a point P, and that $\mathcal{T}$ is a triangulation of M for which P is a vertex. In $\mathcal{T}^*$ there will be a 2-dimensional, type-0^m cell associated with P, where m is the number of 1-simplices in M containing P. In $\mathcal{T}_*$ there will be similarly associated a truncated type-0 cell with $2(m-1)$ edges, illustrated for $m = 3$ in Figure 4(i); such a cell will be said to have type 0_m. Any cell in $\mathcal{T}_*$ corresponding to a point in $\mathcal{T}$ will be of type 0^m or of type 0_m, for some m, and can be decomposed into the two types, 0_3 and 0^2 of Figure 4, to be called $0a$ and $0b$, respectively. We now investigate the effect of a gluing operation upon the associated blow-ups.

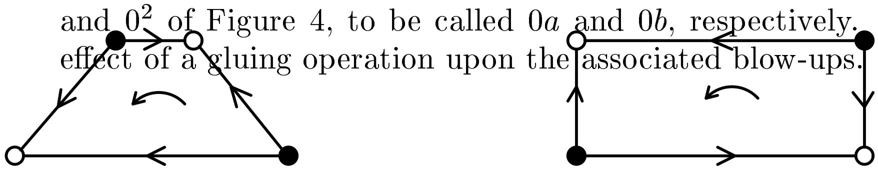

Figure 4: 0

,46

Suppose that N is a 2-dimensional manifold with boundary C^*, and triangulation $\mathcal{U}$ compatible with $\mathcal{T}$ on C. Let n be the number of 1-simplices in N containing P. Then $\mathcal{T}_*$ and $\mathcal{U}_*$ each contain a cell associated with P and of types 0_m and 0_n, respectively. The cell associated with P in the blow-up $(\mathcal{T} \cup \mathcal{U})_*$ will have type 0^{m+n-2}. This cell may be obtained by gluing together the two associated cells in $\mathcal{T}_*$ and $\mathcal{U}_*$, with a cell of type $0b$ between these two cells.

Suppose instead that N is such that its boundary C' contains P and shares a partial boundary C_1^* with C^* while $P \in \partial C_1$. Then the cell in $(\mathcal{T} \cup \mathcal{U})_*$ associated with P will be of type 0_{m+n-1}. This new cell may be obtained by simply gluing together the two cells associated with P in $\mathcal{T}_*$ and $\mathcal{U}_*$, with a single cell of type $0a$, between these two cells.

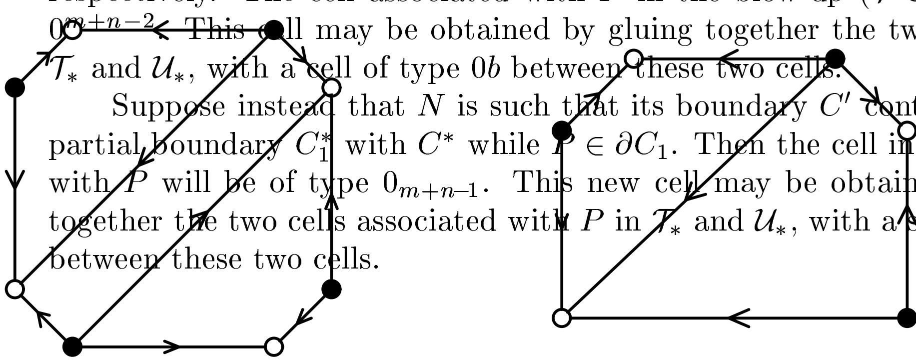

Figure 5: 6

5,30

For the case $m = n = 3$, the two local gluing rules just discussed are illustrated in Figure 5. The gluing rules in a 2-ETFT with $d = 2$, enable the invariant associated with M in the theory, to be obtained from those associated with the elemental triangles in $\mathcal{T}$, using gluing operations along the edges and vertices in $\mathcal{T}$. By changing our viewpoint from the consideration of $\mathcal{T}$ to that of $\mathcal{T}^*$, the structures associated with elemental triangles and the gluing operations are seen to be on an equal footing, both being geometrically described by top-dimensional cells of $\mathcal{T}^*$, the gluing operation coming from those cells of types $0a$, $0b$ and 1. To specify the weights for a 2-ETFT with $k = 2$, it thus suffices to give those on the cells of Figures 4(i) and 4(ii), along with the two right hand cells of Figure 3. What we will call a *polyhedral data* for a 2-ETFT consists of to below.

Two sets I_+ and I_-, to be considered as the allowed sets of labels on type-012 cells with positive and negative orientations, respectively.

Sets $I_{ij}(a, b)$ for each $(a, b) \in I_+ \times I_-$ and $\{i, j\} \subset \{0, 1, 2\}$, the allowed labels on type-ij cells whose boundary has already been labelled by a and b.

Weights $w_\lambda(\{a_v\}, \{\alpha_e\}) \in K$ for $\lambda \in \{0a, 0b, 1, 2\}$ whenever $\{a_v\}$ is an allowed vertex labelling and $\{\alpha_e\}$ is a compatible allowed edge labelling of a type-λ cell.

Thus, in , $a_v \in I_+ \cup I_-$ for each vertex v of a type-λ cell, while, for each edge e, α_e is an element of a suitable set $I_{ij}(a_{v_1}, a_{v_2})$. In order for the w_{0a} and w_{0b} data to determine, via contraction on internal edges, a well-defined weight on an arbitrary cell of type 0, it is necessary for,

w_{0b} to remain unchanged when the vertex and edge labels are changed by performing a rotation through π;

the combination in Figure 6(i) to be invariant under rotations through $\pm 2\pi/3$;

the identity in Figure 6(ii) to be satisfied.

In order for w_1 and w_2 to determine well-defined weights on cells of type 1 and type 2, it is necessary for,

w_1 to remain unchanged when all labels are permuted as a result of a rotation through π;

w_2 to be unchanged under rotation through $\pm 2\pi/3$ of a type-2 cell and its labels

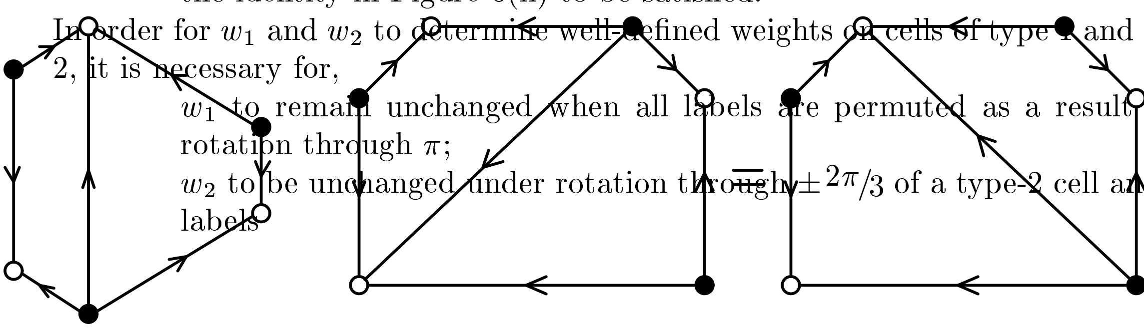

Figure 6: 2

0,20

The purely combinatorial data above may be alternatively expressed in terms of vector spaces and 2-vector spaces in the following way. Let n_+ and n_- be $|I_+|$

and $|I_-|$, respectively. Consider two 2-vector spaces, V_+ and V_- of dimensions n_+ and n_-, respectively. The data may be viewed as three pairings,

$$V_+ \otimes V_- \longrightarrow \mathbb{1}_2$$

under which the basis vector $e_a \otimes f_b$ maps to the element of $\mathbb{1}_2$, given by the vector space whose basis is labelled by $I_{ij}(a,b)$. Denote these vector spaces $V_{ij}(a,b)$. The weights in may now be considered as maps between the vector spaces V_{ij}. Indeed, the relative orientations of the boundary edges on a cell may be used to determine which vector spaces appear in the tensor product giving the domain, and which appear in the image space. For example, the weights associated with the two right hand cells of Figure 3 may be viewed as maps,

$$V_{12}(a',b) \otimes V_{12}(a,b') \longrightarrow V_{01}(a,b) \otimes V_{01}(a',b')$$

$$V_{02}(b,a') \otimes V_{02}(c,b') \otimes V_{02}(a,c') \longrightarrow V_{12}(a,a') \otimes V_{12}(b,b') \otimes V_{12}(c,c')$$

or equivalently as tensors with each index appropriately raised or lowered. The vertex labels may appear an arbitrary number of times in any allowed composition of weights. However, edge labels, those coming from bases for $V_{ij}(a,b)$, may only appear twice, once as upper indices and once as lower indices.

Starting from a polyhedral data, one can attempt to construct a 2-ETFT, that is, to define the appropriate operations Z and of §3. To do this, first note that, to any triangulated M^2 with boundary, there is associated a contracted product of weights which gives rise to an element of the vector space,

$$\Big(\bigotimes \check{V}_{12}(a_{.}, a_{.})\Big) \otimes \Big(\bigotimes V_{02}(a_{.}, a_{.})\Big) \tag{4.1}$$

for each assignment of labels $a_{.}$ to vertices of $\partial(\mathcal{T}_*) \simeq (\partial\mathcal{T})^*$; here ˇ denotes the dual. The tensor products above are over all edges of types 12 or 02 in $\partial(\mathcal{T}_*)$, or equivalently of types 1 or 0 in $(\partial\mathcal{T})^*$. The direct sum of the vector spaces (4.1) over all vertex labels $a_{.}$ is the vector space $(\partial M, \partial\mathcal{T})$ to be associated with ∂M, while the direct sum of the vectors in (4.1) associated with M, defines a vector $Z(M,\mathcal{T}) \in (\partial M, \partial\mathcal{T})$. Indeed, given any closed 1-dimensional triangulated manifold, $(\Sigma, \mathcal{U})$, the construction above provides a family of vector spaces, depending upon the local vertex labels, and this structure can be contracted to give the two important parts,

(i) the direct sum of these vector spaces, $(\Sigma, \mathcal{U})$;

(ii) a pairing $(\Sigma,\mathcal{U}) \otimes (\Sigma^*,\mathcal{U}) \longrightarrow K$, via the contraction of a product of tensors with types $0b$ and 1 along common type-01 edges.

To a not necessarily closed manifold Σ^1 similar constructions associate a family of vector spaces dependent upon the vertex labels on the boundary points. This family of vector spaces defines an element $Z(\Sigma, \mathcal{U})$ of the 2-vector space denoted $(\partial\Sigma, \partial\mathcal{U})$, obtained as a tensor product of V^+ and V^- spaces, one for each point in $\partial\Sigma$. The gluing operation between the families associated with two such 1-dimensional manifolds sharing a common boundary point, is given by the vector spaces associated with type-02 edges. Our conclusions are summarised below.

Proposition 1 *Given a polyhedral data for 2-ETFT, an associated 2-ETFT on 2-dimensional, oriented, triangulated manifolds may be constructed, so long as the data satisfies – above.*

However, in order that the theory extends to one on all oriented 2-dimensional manifolds, without chosen triangulations, it is necessary that the data satisfy, additionally, the relations given by Figures 7 and 8.

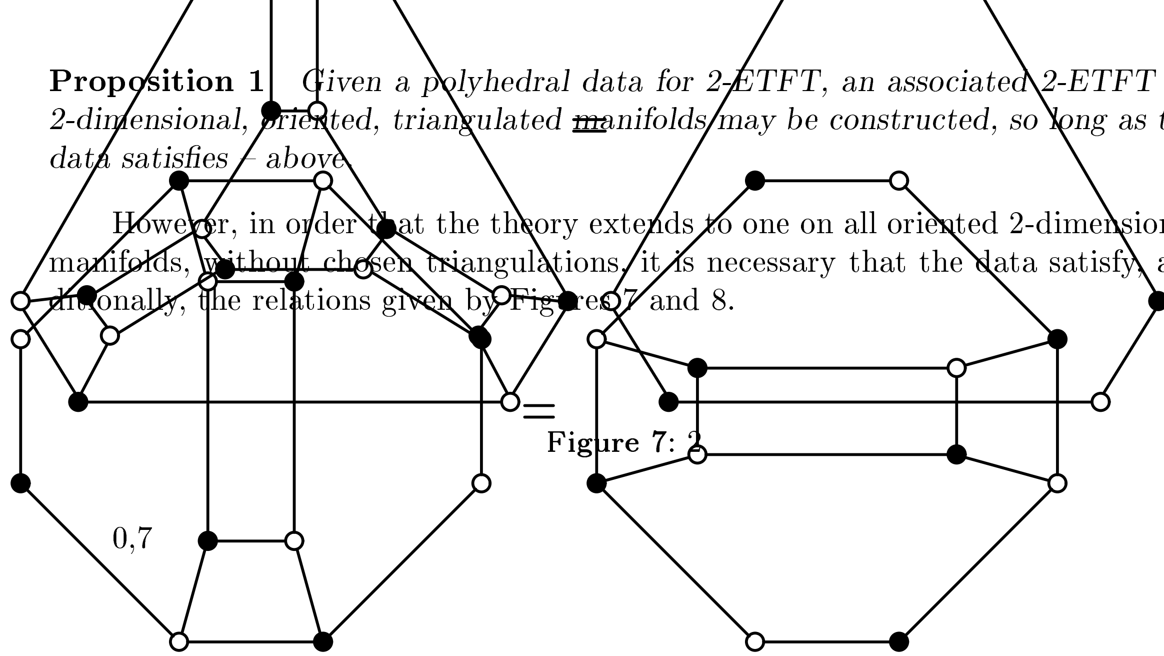

Figure 7: 2

0,7

Figure 8: 3

8,11

To see this, note first that any two triangulations of a closed manifold, M^2, may be obtained from each other by a sequence of moves under each of which only a local piece of the triangulation is affected, via a change as illustrated in either of the two diagrams of Figure 9. Two such triangulations, $\mathcal{T}_1$ and $\mathcal{T}_2$, representing the same manifold, will give rise to identical invariants, $Z(M, \mathcal{T}_1)$ and $Z(M, \mathcal{T}_2)$, since the local moves on triangulations in Figure 9 give rise to the local changes in the blow-up illustrated in Figures 7 and 8. For manifolds with boundary, two different triangulations matching on ∂M may be transformed into each other by the same types of move, so that $Z(M, \mathcal{T})$ is a vector in $(\partial M, \partial \mathcal{T})$, independent of the choice of triangulation, $\mathcal{T}$ of M, once the restricted triangulation on ∂M is fixed. Different subdivisions $\partial \mathcal{T}$ of ∂M may give rise to different vector spaces.

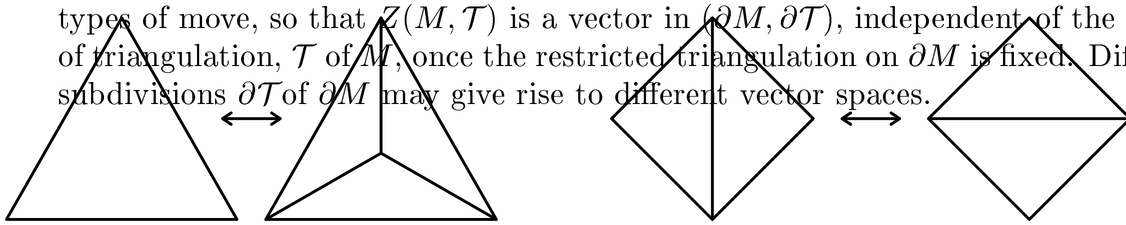

Figure 9: 3

,18

Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are two triangulations of a manifold Σ^1. These give rise in general to different families of vector spaces $[Z(\Sigma, \mathcal{U}_i)]^a$, $i = 1, 2$, indexed by an allowed labelling a of $\partial\Sigma$. However, a change between two such subdivisions may be accomplished via moves in which an interval is subdivided into two, or conversely, an internal vertex is removed. These moves on triangulations $\mathcal{U}$ of Σ translate into moves on $\mathcal{U}_*$, and for each such move there is a natural transformation between the corresponding vector spaces. This is given by contractions of the tensors in Figures 3 and 4. Figure 10 is such an example in the case of the addition of an internal vertex. By this procedure, for each sequence, μ, of local moves bringing $\mathcal{U}_1$ to $\mathcal{U}_2$, a map $\theta^a_\mu(\Sigma;\mathcal{U}_1,\mathcal{U}_2)\colon \big(Z(\Sigma,\mathcal{U}_1)\big)^a \longrightarrow \big(Z(\Sigma,\mathcal{U}_2)\big)^a$ is defined. It turns out that the identity in Figure 7 along with – ensure that θ^a_μ is independent of μ. Now we define $Z(\Sigma)^a$ to be the inverse limit of $\big(Z(\Sigma,\mathcal{U})\big)^a$ over triangulations $\mathcal{U}$ of Σ.

Proposition 2 *Given a polyhedral data for 2-ETFT satisfying conditions – above, along with the equalities of tensor contractions depicted in Figures 7 and 8, an associated 2-ETFT on 2-dimensional oriented manifolds may be constructed.*

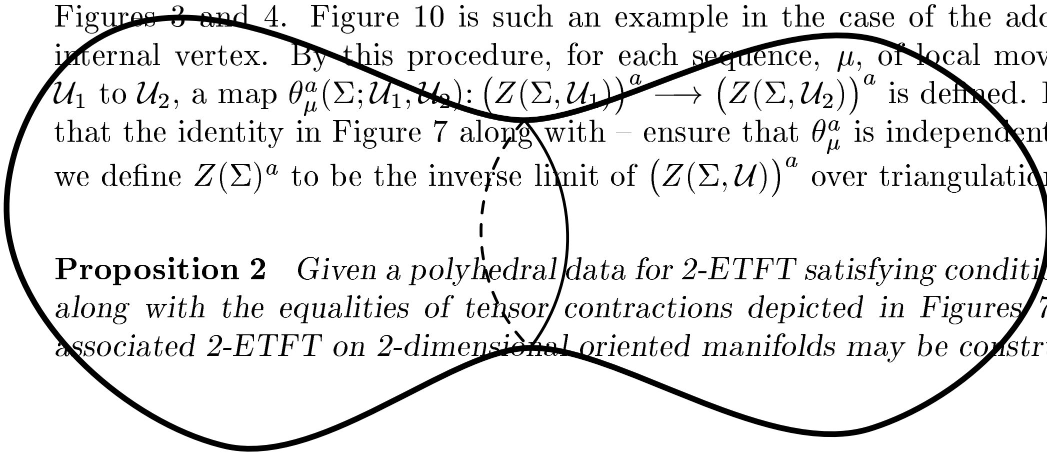

Figure 1: 0

105,31

6:. Examples

In this section some examples of ETFT's will be given.

Example 1 Consider a polyhedral data for a 2-ETFT, that is , and . Suppose that in , $|I_+| = |I_-| = 1$, so that supplies three sets, I_{01}, I_{02} and I_{12}. Suppose that $|I_{01}| = |I_{02}| = 1$. Then gives rise to weights,

$$\begin{aligned} w_{0a} &\text{ and } w_{0b} \in K \\ w_1&\colon I_{12} \times I_{12} \longrightarrow K \\ w_2&\colon I_{12} \times I_{12} \times I_{12} \longrightarrow K \end{aligned}$$

Conditions , and are automatically satisfied, while and require w_1 to be symmetric and w_2 to be invariant under cyclic permutation of the three indices. Put $I = I_{12}$. The identities depicted in Figures 7 and 8 give rise to,

$$\sum_{\lambda,\mu} w_2(a_1,a_2,\lambda)w_2(\mu,a_3,a_4)w_1(\lambda,\mu) = \sum_{\lambda,\mu} w_2(a_4,a_1,\lambda)w_2(\mu,a_2,a_3)w_1(\lambda,\mu) \tag{5.1}$$

$$w_{0a}^{-4} w_{0b}^{-1} w_2(a_1,a_2,a_3) = \sum_{\lambda_i,\mu_i} w_2(a_1,\nu_1,\mu_2)w_2(a_2,\lambda_1,\nu_2)w_2(a_3,\mu_1,\lambda_2) \\ w_1(\lambda_1,\lambda_2)w_1(\mu_1,\mu_2)w_1(\nu_1,\nu_2) \tag{5.2}$$

for all $a_i \in I$. Whenever w_1 and w_2 satisfy (5.1) and (5.2), a 2-dimensional ETFT is obtained.

A specific solution of (5.1) and (5.2) is obtained from any finite group G, by setting $I = G$ and

$$w_1(g,h) = c_1\delta(gh=1)$$
$$w_2(g,h,k) = c_2\delta(ghk=1)$$

where (5.1) is automatically satisfied and (5.2) gives the constraint,

$$c_1^3c_2^2w_{0a}^4w_{0b}|G| = 1\ . \tag{5.3}$$

For a closed triangulated 2-dimensional manifold $(M,\mathcal{T})$, the associated invariant is,

$$Z(M) = c_1^{n_1}c_2^{n_2}|G|^{n_1-n_2+c}\prod_v(w_{0a}^{d_v-2}w_{0b})$$

where n_i is the number of i-simplices in $\mathcal{T}$, and c is the number of components of M. Here d_v denotes the degree of the vertex v, the product being over all vertices v of $\mathcal{T}$. By (5.3), since $3n_2 = 2n_1$, $\sum(d_v-2) = 2n_1 - 2n_0$ and $\chi = n_0 - n_1 + n_2$, thus,

$$Z(M) = (w_{0a}^{-2}w_{0b})^{\chi}|G|^c$$

giving rise to the only two global invariants present, namely χ and c.

For an arbitrary solution of (5.1) and (5.2), let V be a vector space with basis indexed by I. Then w_1 gives a symmetric pairing $V\otimes V \to K$, and using it to identify V^* with V, it is seen that w_2 gives maps $m: V\otimes V \to V$ and $\Delta: V \to V\otimes V$. By (5.1), m is associative and Δ is coassociative. It is therefore not surprising to see that solutions of (5.1) and (5.2) are indexed by algebraic structures. The general solution is indexed by ambialgebras, c.f. [Q].

Example 2 The detailed analysis of §4 can be similarly carried out for 3-ETFT's. A polyhedral data for 3-ETFT will associate a class of allowed labellings to 0-, 1- and 2-dimensional cells in a blow-up $\mathcal{T}_*$ of an arbitrary 3-dimensional triangulation, while allowing weights to be associated to already labeled 3-dimensional cells. Just as in §4, there are an infinite number of cells of types 0 and 1, but they can be decomposed into a finite number of elemental cells. For the purposes of defining a specific 3-ETFT, it suffices to define all the allowed cell labellings and weights on *all* the possible cell types, rather than just for the elemental ones.

There are two forms of type-0123 cell, namely, points with either of two orientations. Associate to both cells the 3-vector space $\mathbb{1}_3$, that is, only one allowed label exists on points. The 1-dimensional cells are of type-012, -013, -023 and -123, to which there are associated sets giving the allowed labels, or equivalently 2-vector spaces. Suppose that the first three sets are of order 1, the last being I. The 2-dimensional cells come in 6 types, there being only one geometric form to each type, except type-01. Suppose that there is only one allowed label on all type-01, -02 and -03 cells. Two elements of I suffice to label the vertices and edges of either type-12 or type-13 cells, three being necessary for type-23 cells. The allowed sets of labels on these cells will be denoted by X_{ij}, Y_{ij} and A_{ijk}, respectively, $(i,j,k\in I)$. Both this and the next example use the data as given so far.

For a finite group G use $I = G$, while X_{ij}, Y_{ij} and A_{ijk} are sets of order $\delta(ij=1)$, $\delta(ij=1)$ and $\delta(ijk=1)$, respectively. The polyhedral data is completed by specifying weights on all type-0, -1, -2 and -3 cells. In the case when these weights are all independent of the allowed labelling of the vertices, edges and faces, the invariant obtained of a closed 3-manifold, M, depends only on the number of components, the Euler characteristic of M, and $|\operatorname{Hom}(\pi_1(M), G)|$. A more general system of weights can be defined in terms of a choice of 3-cocycle, and the resulting 3-ETFT gives rise to TFT's of the form investigated in [FQ].

Example 3 This example differs from the last, in that here I is chosen to index a set of irreducible representations of a quantum group, A. In the case $A = U_q\mathfrak{sl}_2$, at a root of unity $q = \exp(\pi i/r)$, I is chosen to be $\{0, 1/2, \ldots, (r-2)/2\}$, labelling the generators of the semi-simple part of the category of representations. The sets X_{ij} and Y_{ij} are both chosen to be of order 1 precisely when i and j represent dual representations, being of order 0 otherwise. The set A_{ijk} is chosen to specify the multiplicity of the trivial representation in the tensor product of the representations labelled by i, j and k.

The associated weights to be placed on labelled 3-dimensional cells of types 0,1,2 and 3 are now all given in terms of structure constants of A. In particular, the geometric form of a type-3 cell is that of a tetrakaidekahedron, which has twenty-four vertices, eight hexagonal and six square faces. A complete labelling of the vertices, edges and faces of such a cell is given by six elements of I, and four multiplicity labels. The weight to be associated with this cell is now a generalised quantum $6j$-symbol. For the case of $U_q\mathfrak{sl}_2$, these symbols were investigated in [KR]. The conditions on the weights, which in the case of a 3-ETFT are equivalent to the constraints of Figures 7 and 8 in the case of a 2-ETFT, are satisfied due to relations amongst the quantum $6j$-symbols, known as the orthogonality and the Elliot-Biedenharn relations.

As for any ETFT derived from polyhedral data, the associated (scalar) invariant of closed, top-dimensional manifolds has the form of a sum, over allowed labellings, of a product of local weights. Thus,

$$Z(M) = \sum_{\sigma} \prod_{C \subset \mathcal{T}^*} w_C(\sigma|_C) \,, \tag{5.4}$$

where $\mathcal{T}$ is a triangulation of M, σ ranges over allowed labellings of the cells and subcells of $\mathcal{T}^*$ of codimension at least one, and $w_C(\tau)$ is the weight associated with cell C when labelled according to τ. The product is over all top-dimensional cells in $\mathcal{T}^*$. For top-dimensional manifolds with boundary, the contraction of weights given by the right hand side of (5.4) provides a tensor, indexed by the boundary labelling $\sigma|_{\partial M}$.

The sum (5.4), in the case of this example, is that found in [TV] for $U_q\mathfrak{sl}_2$.

7:. Further remarks

In §4 it was seen in detail how the combinatorics of triangulations of manifolds of dimension 2 translates directly into rules for a structure, which was termed a polyhedral data for 2-ETFT, of weights and allowed labellings for the cells in blow-ups of arbitrary such triangulations. From such data, a 2-dimensional ETFT could be constructed, and it was apparent that such data is indexed by algebraic structures of a suitable type. The same combinatorics may be interpreted as giving the basic elements and axioms of a d-dimensional 2-ETFT, for any $d > 1$.

The translation from the structure which we have defined as a polyhedral data for 2-ETFT, to that of a d-dimensional 2-ETFT, is accomplished by introducing an additional dependence into the weights and sets of allowed labels on different cell types, upon a choice of suitable manifolds. Thus for any particular cell, C, in the blow-up of a 2-dimensional triangulation, $\mathcal{T}$, the dependence introduced will be upon $(d-2)$-, $(d-1)$- and d-dimensional manifolds, one manifold being given for each 0-, 1- and 2-simplex in $\mathcal{T}$ related to C. The manifolds will satisfy boundary constraints given by the incidence relations existing amongst the associated simplices in $\mathcal{T}$. Those structures attached to cells C, whose type is other than 2, 12 or 012, are traditionally thought of as gluing rules.

In an arbitrary dimension d, s-ETFT's can similarly be defined along with polyhedral data for s-ETFT's. Just as in the case of $s = 2$, the number of cell types in a blow-up of an s-dimensional triangulation is infinite, but they can be decomposed into a finite number of elemental cells. Thus, a polyhedral data may be given by specifying a (finite) set of allowed labellings and a finite number of weights (tensors). The analogue of Proposition 1 will then hold. The analogue of Proposition 2 relies on there being a finite simple set of generators for moves on (singular) s-dimensional triangulations of a manifold ([M], see also [P]). Hence there is a family of s-ETFT's coming from polyhedral data, indexed by finite collections of tensors constrained by a finite number of polynomial relations, arising from invariance under local moves on triangulations. Since for $s = 2$, a polyhedral data is equivalent to an ambialgebra, it is not unreasonable to also view such a polyhedral data for $s > 2$, as an *algebraic structure*. These structures will not be algebras in the usual sense, and will possess many, not necessarily binary, operations, rather than just a multiplication; similar structures were studied in [L 2] and [L 3]. The reader is referred to [L 1] for more details.

It may be remarked that the polynomial relations to be satisfied by a polyhedral data are not unlike a generalisation of the Moore-Seiberg polynomial relations [MoSe] of rational CFT. The form of the invariant for s-dimensional manifolds will always be that of (5.4). It may be thought of as a discretised version of the functional integrals arising in other approaches, see [W] and [F].

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