A Presentation for Manin and Schechtman's Higher Braid Groups [1]

R.J. Lawrence[2]

Department of Mathematics
Harvard University
Cambridge, Massachusetts

Abstract. Manin and Schechtman's higher braid groups $B(n,k)$ are a generalisation of the ordinary pure braid groups $P_n = B(n,1)$, in which the role that the symmetric group S_n plays in P_n is replaced by a weak Bruhat order. In this paper a family of concrete presentations of $B(n,k)$ will be given, which generalise Artin's presentation of P_n.

1: Introduction

Let $\pi_1^0, \ldots, \pi_n^0$ denote n hyperplanes in $\mathbf{C}^k$, in general position. That is, it is assumed that for any $S \subseteq \{1, 2, \ldots, n\}$,

$$\bigcap_{i \in S} \pi_i^0$$

has codimension $|S|$ in $\mathbf{C}^k$ if $|S| \leq k$, and is empty if $|S| \geq k+1$. Let $U(n,k)$ denote the family of all sets $\{\pi_1, \ldots, \pi_n\}$ of n hyperplanes in $\mathbf{C}^k$, in general position and such that π_i is parallel to π_i^0 for all $1 \leq i \leq n$. Denote by $B(n,k)$, the fundamental group of $U(n,k)$. This is the *higher braid group* in the sense of Manin and Schechtman [MS 2]. Note that although $U(n,k)$ depends on the initial choice of $\{\pi_i^0\}$, the structure of the group $B(n,k)$ and the topology of $U(n,k)$ depend only upon n and k.

In this discussion, n and k are arbitrary positive integers with $n \geq k$. When $k = 1$, $U(n,1)$ reduces to X_n, the configuration space of n distinct, ordered points in $\mathbf{C}$, whose fundamental group is the pure braid group $B(n,1) = P_n$ on n strings. When $k = n$, $U(n,n)$ is just $\mathbf{C}$, so that $B(n,n)$ is trivial, while $B(k+1,k) \cong \mathbf{Z}$. Choose hyperplanes specified by,

$$\mathbf{r}.\mathbf{n}_i = \alpha_i \; , \tag{1.1}$$

for suitable $\mathbf{n}_i \in \mathbf{C}^k$ and $\alpha_i \in \mathbf{C}$, so as to put $\{\pi_i^0\}$ in general position. Then elements of $U(n,k)$ may be specified by points $\mathbf{x} = (x_i) \in \mathbf{C}^n$ with,

$$\pi_i \colon \mathbf{r}.\mathbf{n}_i = \alpha_i - x_i \; . \tag{1.2}$$

For every $(k+1)$-set $J \subseteq I = \{1, 2, \ldots, n\}$, a hyperplane π_J may be defined in $\mathbf{C}^n$, by the condition that $\{\pi_i \mid i \in J\}$ have a common point of intersection. The set of hyperplanes π_i will be in general position so long as $\mathbf{x}$ lies in the complement of the $\binom{n}{k+1}$ hyperplanes $\{\pi_J\}$ in $\mathbf{C}^n$. Thus $B(n,k)$ is the fundamental group of a complement of hyperplanes. It is therefore possible to obtain a presentation of $B(n,k)$ along the lines of [R].

[1] This work is supported in part by NSF Grant No. 9013738.

[2] The author is a Junior Fellow of the Society of Fellows.

In this paper we will explicitly construct such presentations of $B(n,k)$. Using a suitable set of initial hyperplanes π_i^0, generators for $B(n,k)$ are defined in §2. In §3 the associated relations are obtained by investigating the structure of the codimension-two subsets of $\mathbf{C}$ associated with $\{\pi_i\}$. Some special cases are discussed in §4, while some connections with the combinatorial structures of [MS 1] and those of the ordinary braid groups are considered in §§5 and 6.

It is well known that the complement of a set of complex hyperplanes possesses a strong combinatorial structure (see [OS]). It is therefore not surprising to discover that the formulae obtained for the presentation of $B(n,k)$ are complex, involving inequalities.

2: Construction of generators

The hyperplanes (1.1), will be in general position for suitable α_i, so long as, for all $J \subseteq I$ with $|J| = k$,

$$\det(\mathbf{n}_i)_{i\in J} \neq 0 \,,$$

where $(\mathbf{n}_i)_{i\in J}$ denotes the $k \times k$ matrix with columns $\mathbf{n}_i$ indexed by $i \in J$. A suitable choice for $\mathbf{n}_i$ is thus,

$$\mathbf{n}_i = (1, a_i, \ldots, a_i^{k-1}) \,, \tag{2.1}$$

where $a_i \in \mathbf{R}^+$ are distinct, say $0 < a_1 < a_2 < \cdots < a_n$. Throughout §§2 and 3, this choice of $\mathbf{n}_i$ will be assumed, with a_i fixed. Suppose $J \subset I$ with $|J| = k+1$. Let $j = \max(J)$. The condition for $\{\mathbf{r}.\mathbf{n}_i = \alpha_i \mid i \in J\}$ to not have a common point of intersection is that,

$$\det \mathbf{A}_J \neq 0 \,,$$

where $\mathbf{A}_J$ is a $(k+1) \times (k+1)$ matrix with rows $(\alpha_i, \mathbf{n}_i)$ indexed by $i \in J$. That is,

$$\sum_{i\in J} \Big\{ \alpha_i (-1)^{(\# J < i)} . \prod_{\substack{j<k \\ j,k \in J\setminus\{i\}}} (a_k - a_j) \Big\} \neq 0 \,. \tag{2.2}$$

The ratio between the coefficients of α_i and α_l $(i, l \in J)$ on the l.h.s. here has modulus $|\Pi^{l,i}_{J\setminus\{i,l\}}|$ where,

$$\Pi^{i,j}_K \equiv \prod_{\mu\in K} \left(\frac{a_\mu - a_i}{a_\mu - a_j} \right) \,. \tag{2.3}$$

This shows that if α_j is chosen sufficiently large in comparison with $\{\alpha_i \mid i \in J\setminus\{j\}\}$, then (2.2) will automatically be satisfied.

To be more precise, let π denote $\max\{|\Pi^{i,j}_K|\}$ over all $(k-1)$ sets $K \subseteq I$ and $i, j \notin K$. Consider also the set of all sums of all triples of expressions of the form $\Pi^{i_1,j_1}_{K_1} \Pi^{i_2,j_2}_{K_2}$. Let π' denote the minimum difference between distinct elements of this set. Put $M = {}^{6n\pi^2}/_{\pi'}$.

Theorem 1 *Suppose that $\mathbf{n}_i$ are defined by (2.1) and $\alpha_1, \ldots, \alpha_n$ are chosen to be real numbers with ${}^{\alpha_i}/_{\alpha_{i-1}} > M$ for $1 \le i \le n$ where $\alpha_0 \equiv 1$. Then the hyperplanes π_i^0 of (1.1) are in general position.*

The precise value of M is immaterial. It is only important to note that once $\{a_i\}$ are fixed, it is possible to choose M sufficiently large so that when $\{\alpha_i\}$ satisfies the conditions in Theorem 1, the sign of any of the expressions $\sum f_r \alpha_r$ we will wish to compute will be determined by that of f_m, where m is the largest r for which $f_r \neq 0$. Here f_r will be functions of the a's only, given in terms of $\{\Pi_{i,j}^K\}$. From now on, $\alpha_1, \alpha_2, \ldots, \alpha_n$ will be thought of as defining increasing orders of magnitude, in the sense just described.

Consider the hyperplane π_J in $\mathbf{C}^n$, given by the condition that $\{\pi_i \mid i \in J\}$ have a common point of intersection, where $J \subset I$ with $|J| = k+1$. This hyperplane will intersect the x_j-axis $\iff j \in J$, and the point of intesection will then be at,

$$x_j = x_j^J \equiv \alpha_j - \sum_{r \in J \backslash j} \left(\alpha_r \Pi_{J \backslash \{j,r\}}^{j,r} \right). \tag{2.4}$$

Assume $\{a_i\}$ are chosen such that $\left\{\Pi_K^{i,j} \mid K \subset I \backslash \{i,j\}\right\}$ are distinct $\forall i > j$. Let $j = \max(J)$. Define the generators β_J of $\pi_1(U_{n,k})$ to be given by a loop in $U_{n,k}$, based at $\mathbf{0}$ and lying in the copy of $\mathbf{C}$ on which $x_i = 0$, $\forall i \neq j$. The loop is given by x_j following a path around x_j^J based at 0, in a clockwise direction, defined with $\Im(x_j) > 0$ along the whole loop except that part 'close' to x_j^J (see Fig 1). Note that in this situation, the values of x_j^J in (2.4) are all positive real numbers. The $\binom{n}{k+1}$ generators β_J of $\pi_1(U_{n,k})$ have now been defined.

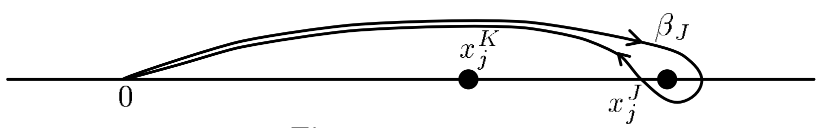

Figure 1

3: Analysis of relations

As discussed in [R], the relations in $B_{n,k}$ arise from the consideration of all codimension two subsets of $\mathbf{C}^n$ associated with the arrangement $\mathcal{C} = \{\pi_J \mid J \subset I,\ |J| = k+1\}$. Such subsets, Δ, come in two main types,

(i) $\bigcap\{\pi_K \mid K \subset J,\ \ |K| = k+1\}$ for $J \subset I$ with $|J| = k+2$;

(ii) $J \cap L$ for $J, L \subset I$ with $|J| = |L| = k+1$ and $|J \cap L| < k$.

For later convenience, we choose to subdivide case (ii) into two parts,

(ii)$'$ $\max(J) > \max(L)$;

(ii)$''$ $\max(J) = \max(L)$.

Clearly, since J and L may be interchanged, $\max(J) < \max(L)$ need not be considered. In each case, define $j, l \in I$ with $j > l$, as follows,

(i) $j = \max(J)$, $l = \max(J \backslash \{j\})$;

(ii)$'$ $j = \max(J)$, $l = \max(L)$;

(ii)$''$ $j = \max(J)$, $l = \max\big((J \cup L) \backslash \{j\}\big)$.

Without loss of generality, it may be assumed in (ii)$''$ that $l \in J$.

The relations R_Δ, associated with Δ, may clearly be discussed by only considering the arrangement $\mathcal{C}_{j,l}$ induced by $\mathcal{C}$ upon the two-dimensional subspace of $\mathbf{C}^n$ on which $x_i = 0$ for all $i \neq j, l$. The arrangement $\mathcal{C}_{j,l}$ on $\mathbf{C}^2$ (axes x_j and x_l) consists of lines,

$$\pi_{M \cup \{j\}}: x_j = \alpha_j - \sum_{r \in M} \alpha_r \Pi^{j,r}_{M \backslash \{r\}} \tag{3.1}$$

$$\pi_{M \cup \{l\}}: x_l = \alpha_l - \sum_{r \in M} \alpha_r \Pi^{l,r}_{M \backslash \{r\}} \tag{3.2}$$

for $M \subset I \backslash \{j, l\}$, $|M| = k$, together with lines $\pi_{M \cup \{j,l\}}$ for $|M| = k-1$, $M \subset I \backslash \{j, l\}$ whose slopes are $\Pi^{l,j}_M$,

$$\pi_{M \cup \{j,l\}}: x_j - \Pi^{j,l}_M x_l = \alpha_j - \sum_{r \in M \cup \{l\}} \alpha_r \Pi^{j,r}_{\{l\} \cup M \backslash \{r\}} = x_j^{M \cup \{j,l\}} . \tag{3.3}$$

The relations associated with Δ are now seen to be associated with a point of intesection $x_\Delta = (x_j^0, x_l^0)$ of n_Δ lines in $\mathcal{C}_{j,l}$ where,

$$n_\Delta = \begin{cases} k+2 & \text{if } \Delta \text{ is of type (i)}, \\ 2 & \text{if } \Delta \text{ is of type (ii)}' \text{ or (ii)}''. \end{cases}$$

These n_Δ lines are said to be the lines in $\mathcal{C}_{j,l}$ *associated with* Δ.

The generators β_J associated with lines π_J in $\mathcal{C}_{j,l}$, were defined in §2 to be given by loops based at $\mathbf{0}$ in the (complex) x_j-axis, unless $j \notin J$, in which case they are loops in the (complex) x_l-axis. Choose $x_\Delta^\varepsilon \equiv (x_j^0 - \varepsilon, x_l^0 - \varepsilon) = (x_j^\varepsilon, x_l^\varepsilon)$ with $\varepsilon > 0$ sufficiently small, so that the only members of $\mathcal{C}_{j,l}$ crossing the square $[x_j^0 - \varepsilon, x_j^0] \times [x_l^0 - \varepsilon, x_l^0]$ are the lines associated with Δ, while no line of $\mathcal{C}_{j,l}$ cuts the x_l-axis in the interval (x_l^ε, x_l^0). Let β_J^ε be the generators of,

$$\pi_1(\mathbf{C}^2 \backslash \mathcal{C}_{j,l}, x_\Delta^\varepsilon)$$

defined in the same way as β_J with basepoint x_Δ^ε replacing $\mathbf{0}$.

Definition *Suppose $g_1, \ldots, g_r$ are elements of a group G. It is said that the relation $\mathcal{R}\{g_1, \ldots, g_r\}$ holds if, and only if, the product $g_1, \ldots, g_r$ is unchanged by cyclic rotations of the g_i's.*

Lemma 1 [R] *If $\pi_{J_1}, \ldots, \pi_{J_{n_\Delta}}$ are the n_Δ lines associated with Δ in an anti-clockwise order, then the relation in $B_{n,k}$ associated with Δ is,*

$$R_\Delta\colon \mathcal{R}\{\beta^\varepsilon_{J_1}, \ldots, \beta^\varepsilon_{J_{n_\Delta}}\}\,.$$

To complete the evaluation of R_Δ it is only necessary to move the basepoint from x^ε_Δ to $\mathbf{0}$ and determine the transformed β^ε_J in terms of $\{\beta_J\}$. Let us note first that under the conditions of Theorem 1, in all cases $x^0_j, x^0_l \in \mathbf{R}^+$ while all lines in $\mathcal{C}_{j,l}$ associated with Δ are either parallel to the x_j-axis or cut it at a positive real point.

We shall define the curve followed by the basepoint in two parts. First, move the basepoint from x_Delta^ε to $(0, x^\varepsilon_l)$ along a path with x_l fixed, and $\Im(x_j) > 0$ at all intermediate positions. This transforms all $\beta^\varepsilon_{J_i}$ associated with Δ to similarly defined curves, based at $(0, x^\varepsilon_l)$. The fact that no line in $\mathcal{C}_{j,l}$ cuts the x_l-axis between x^ε_l and x^0_l has been used here. Finally, move the base point from $(0, x^\varepsilon_l)$ to $\mathbf{0}$ along a path in the (complex) x_l-axis with $\Im(x_l) > 0$ at all intermediate points. Any generator $\beta^\varepsilon_{J_i}$ associated with Δ, for which the lines π_{J_i} in $\mathcal{C}_{j,l}$ is parallel to the x_j-axis, will transform to β_{J_i}. The remaining generators transform according to the following lemma, see Fig 2.

Lemma 2 *Under the shift of basepoint described above, a generator β^ε_M $(j \in M)$ associated with Δ, transforms to,*

$$\xi^{-1}\eta\beta_M\eta^{-1}\xi$$

where ξ and η are the products of the generators β_K associated with those lines π_K in $\mathcal{C}_{j,l}$ satisfying,

$$\xi\colon x^K_j < 0, \quad 0 < x^K < x^0_j$$
$$\eta\colon 0 < x^K_j < x^M_j, \quad x^K > x^0_j$$

respectively, in order of increasing x^K_j. Here x^K denotes the value of x_j at which π_K crosses $x_l = x^0_l$.

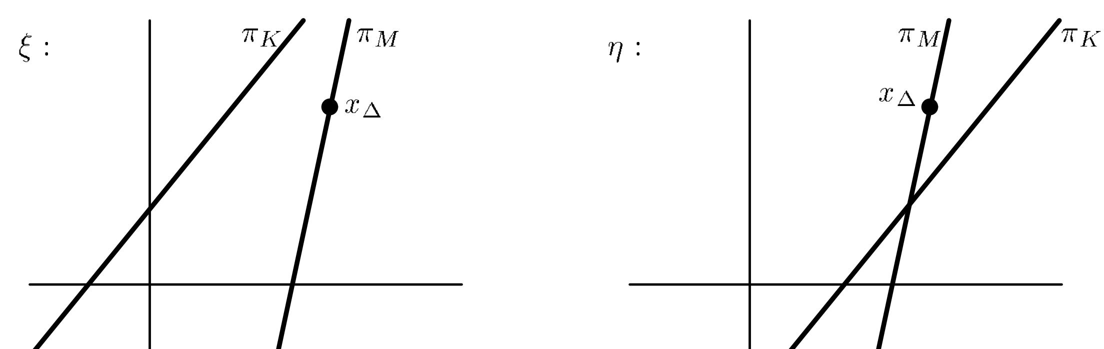

Figure 2

By (2.4), x_j^K can only be negative if $\max(K) > j$. Since x_l^0 has order at most that of α_j, it is impossible for x^K to be positive at the same time as $x_j^K < 0$ (see (3.3)). Hence $\xi = 1$ in Lemma 2, reducing the computation to that of determining the sets K satisfying the condition for η. Note that if K_m denotes $\max(K)$, then the sign of $x_j^{K\cup\{j\}}$ is, by (2.4), determined by that of,

$$\alpha_j - \alpha_{K_m}\Pi^{j,K_m}_{K\backslash\{K_m\}}\,.$$

This is positive if $K_m < j$, while if $K_m > j$ it is only positive if,

$$\Pi^{j,K_m}_{K\backslash\{K_m\}} < 0\,.$$

That is, $x_j^{K\cup\{j\}} > 0$ if, and only if, the number of elements of K greater than j, denoted $\#(K > j)$, is even. Using similar arguments it may be seen that the following lemma holds. Note that in all of cases (i), (ii)$'$ and (ii)$''$, all the generators β_M associated with Δ, for which $j \in M$, have $\max(M) = j$.

Lemma 3 *The condition $0 < x_j^K < x_j^M$ where $\max(M) = j$ is equivalent to,*

(a) $\#(K > j) = 0$,

and (b) $m \equiv \max\big((K \cup M)\backslash\{j\}\big) \in K$,

and (c) $\Pi^{j,m}_{K\backslash\{j,m\}} > \delta_{m\in M}\Pi^{j,m}_{M\backslash\{j,m\}}$, *where $\delta_{m\in M}$ is 0 or 1 as $m \notin M$ or $m \in M$.*

The consideration of the remaining part of the condition for β_K to be a term in η of Lemma 2, namely $x^K > x_j^0$, must be broken up into three cases.

Case (i): In this case the lines in $\mathcal{C}_{j,l}$ associated with Δ are $\pi_{J\backslash\{j\}}$ and $\pi_{J\backslash\{l\}}$ (parallel to the x_j and x_l axes, respectively) and $\big\{\pi_{J\backslash\{p\}} \;\big|\; p \in J\backslash\{j,l\}\big\}$ which have positive slopes. The condition η in Lemma 2 thus requires both j and l to be elements of K. Also $x_j^0 = x_j^{J\backslash\{l\}}$ and so $x^K > x_j^0$ requires,

$$\Pi^{j,l}_{K\backslash\{j,l\}}x_l^{J\backslash\{j\}} > x_j^{J\backslash\{l\}} - x_j^K$$

by (3.3). Applying (2.4) reduces this to,

$$-\delta_{m\in J}\Pi^{j,l}_{K\backslash\{j,l\}}\Pi^{l,m}_{J\backslash\{j,l,m\}} > \delta_{m\in K}\Pi^{j,m}_{K\backslash\{j,m\}} - \delta_{m\in J}\Pi^{j,m}_{J\backslash\{j,l,m\}}\,,$$

where $m = \max((J\cup K)\backslash\{j,l\})$. If $m \notin J$, this condition reduces to $\Pi^{j,m}_{K\backslash\{j,m\}} < 0$ which reduces to $m < l$, since by Lemma 3(a), $m < j$. In all cases, $m < l$, and condition η of Lemma 2 applied to $M = J\backslash\{p\}$ $(p \neq j)$ gives K of the form $\overline{K}\cup\{j,l\}$ with $\max \overline{K} < l$ and,

$$\left.\begin{aligned} \Pi^{j,l}_{\overline{K}} &> \delta_{l\neq p}\Pi^{j,l}_{J\backslash\{j,l,p\}} \\ \delta_{m\in J}\Pi^{l,m}_{J\backslash\{j,l,m\}}(\Pi^{j,l}_{\overline{K}} - \Pi^{j,l}_{J\backslash\{j,l,m\}}) &< \delta_{m\in K}\Pi^{j,m}_{K\backslash\{j,m\}}\,. \end{aligned}\right\} \quad (3.4)$$

As was mentioned above, the second condition is automatically satisfied if $m \notin J$.

Case (ii)′: In this case, the two lines associated with Δ are π_L (parallel to the x_j-axis) and π_J. Hence $x_l^0 = x_l^L$ and $x^K > x_j^0$ requires,

$$(\delta_{l\in K}\Pi^{j,l}_{K\backslash\{j,l\}} - \delta_{l\in J}\Pi^{j,l}_{J\backslash\{j,l\}})x_l^L > x_j^J - x_j^K \,.$$

This reduces to,

$$(\Pi^{j,l}_{K\backslash\{j,l\}}\delta_{l\in k} - \Pi^{j,l}_{J\backslash\{j,l\}}\delta_{l\in J})\Pi^{l,m}_{L\backslash\{l,m\}}\delta_{m\in L\backslash\{l\}} < (\Pi^{j,m}_{J\backslash\{j,m\}}\delta_{m\in J} - \Pi^{j,m}_{K\backslash\{j,m\}}\delta_{m\in K})\,, \quad (3.5)$$

where $m = \max\Big(\big((J\cup K)\backslash\{j,l\}\big)\cup\big(L\backslash\{l\}\big)\Big)$. Thus condition η of Lemma 2 applied to $M = J$ gives K of the form $\overline{K}\cup\{j,l\}$ with $\max(\overline{K}) < l$, so long as $\max(J\backslash\{j\}) \leq l$ and (3.5) together with,

$$\Pi^{j,l}_{K\backslash\{j,l\}} > \delta_{l\in J}\Pi^{j,l}_{J\backslash\{j,l\}}\,. \quad (3.6)$$

Indeed, if $\max(J\backslash\{j\}) > l$, no K satisfies condition η of Lemma 2 for $M = J$.

Case (ii)″ : In this case the two lines associated with Δ are π_J and π_L. Using a similar argument to those in the last two cases, it can be seen that condition η of Lemma 2 is satisfied by those K of the form $\overline{K}\cup\{j,l\}$ for which $\max(\overline{K}) < l$ and,

$$\Pi^{j,m}_{K\backslash\{j,m\}}\delta_{m\in K} < \Big[(\Pi^{j,l}_{K\backslash\{j,l\}} - \Pi^{j,l}_{L\backslash\{j,l\}}\delta_{l\in L})\Pi^{j,m}_{J\backslash\{j,m\}}\delta_{m\in J} + (\Pi^{j,l}_{J\backslash\{j,l\}} - \Pi^{j,l}_{K\backslash\{j,l\}})\Pi^{j,m}_{L\backslash\{j,m\}}\delta_{m\in L}\Big](\Pi^{j,l}_{J\backslash\{j,l\}} - \delta_{l\in L}\Pi^{j,l}_{L\backslash\{j,l\}})^{-1}\,, \quad (3.7)$$

where $m = \max\big((J\cup K\cup L)\backslash\{j,l\}\big)$. It is also required that,

$$\Pi^{j,l}_{K\backslash\{j,l\}} > \Pi^{j,l}_{M\backslash\{j,l\}}\delta_{l\in M}\,. \quad (3.8)$$

These give the possible K's corresponding to both $M = L$ and $M = J$.

Theorem 2 *$B(n,k)$ has a presentation with generators $\{\beta_J \mid J\subset I,\ \ |J| = k+1\}$ and relations,*

- *(i)* $\mathcal{R}\{\beta^*_{J\backslash\{p\}},\ldots\beta^*_{J\backslash\{l\}},\beta^*_{J\backslash\{j\}}\}$ *for* $|J| = k+2$, $j = \max(J)$, $l = \max(J\backslash\{j\})$, *where* p *ranges over* $J\backslash\{j\}$ *in increasing order;*
- *(ii)′* $\mathcal{R}\{\beta^*_J,\beta_L\}$ *for* $|J| = |L| = k+1$, $|J\cap L| < k$, $\max(J) = j > l = \max(L)$*;*
- *(ii)″* $\mathcal{R}\{\beta^*_J,\beta^*_L\}$ *for* $|J| = |L| = k+1$, $|J\cap L| < k$, $\max(J) = \max(L) = j$ *with* $\max(J\backslash\{j\}) = l \geq \max(L\backslash\{j\})$.

Here $\beta^*_M = \eta\beta_M\eta^{-1}$ *where* η *is the product of generators* β_K *over* K *whose maximal elements are* j,l *and which satisfy (3.4) in (i); (3.5), (3.6) in (ii)′; (3.7), (3.8) in (ii)″. The product is taken in order of decreasing* $\Pi^{j,l}_{K\backslash\{j,l\}}$. *In case (ii)′, if* $\max(J\backslash\{j\}) > l$ *then* $\beta^*_J \equiv \beta_J$.

This presentation depends on the choice of positive real numbers $a_1 < a_2 < \cdots < a_n$ such that,

$$\Pi_J^{i,j} = \prod_{k\in J} \frac{a_k - a_i}{a_k - a_j}$$

are distinct over all $J \subset I\backslash\{i,j\}$ of order $(k-1)$.

4: Special cases

In this section we shall illustrate the use of Theorem 2 in some special cases. If $n = k+1$, there is only one generator, so that $B(k+1,k) \cong \mathbf{Z}$. If $n = k+2$, there are n generators labelled by the $k+1$-sets $I\backslash\{j\}$. There is just one relation, which is of type (i),

$$\{\beta_{I\backslash\{1\}}, \ldots, \beta_{I\backslash\{n\}}\}\ .$$

In other words, the only relation states that $\beta_{I\backslash\{1\}}, \ldots, \beta_{I\backslash\{n\}}$ is central.

The generators for $B(n,1)$ are labelled by pairs $\{i,j\} \subset I$. There are no relations of type (ii)$''$. The relations of type (i) give,

$$\{\beta_{jk}, \beta_{ik}, \beta_{ij}\} \qquad i < j < k\ .$$

The relations of type (ii)$'$ give,

$$\begin{aligned} &\{\beta_{ij}, \beta_{kl}\} && \text{for } j > i > l > k \text{ or } j > l > k > i \\ &\{\beta_{jl}\beta_{ij}\beta_{jl}^{-1}, \beta_{kl}\} && \text{for } j > l > i > k\ . \end{aligned}$$

These are the standard relations existing between generators of the pure braid group P_n. Here all Π's are 1.

In the case of $B(n,2)$, the generators are labelled by triples $\{i,j,k\} \subset I$. The relations of type (i) are,

$$\{\beta_{jkl}, \beta_{ikl}, \beta_{ijl}^*, \beta_{ijk}\} \qquad \text{for } i < j < k < l\ ,$$

where $\beta_{ijl}^* = \eta\beta_{ijl}\eta^{-1}$ and η is the product of generators β_{qkl} with $q < i$, in decreasing order. The relations of type (ii)$'$ give,

$$\begin{aligned} &\{\beta_{ijk}, \beta_{lms}\} && \text{for } l < m < s < j < k,\ i < j\ ; \\ &\{\beta_{ijk}^*, \beta_{lms}\} && \text{for } i < j \le s < k,\ l < m < s\ . \end{aligned}$$

In the latter case, $\beta_{ijk}^* = \eta\beta_{ijk}\eta^{-1}$, where η is a product of generators β_{qsk} with q decreasing and such that $\max\{i,m\} < q < j$, for $j = s$, while for $j < s$, q must satisfy,

$$\Pi_q^{ks}\Pi_l^{sm}\delta_{\alpha m} < \Pi_i^{kj}\delta_{\alpha j} - \Pi_s^{kq}\delta_{\alpha q} \tag{4.1}$$

with $\alpha = \max\{m,q,j\}$. Finally the relations of type (ii)$''$ are given by,

$$\{\beta_{ilj}^*, \beta_{kmj}^*\} \qquad \text{for } k < m < l < j,\ k \ne i < l$$

and $\beta^*_{ilj} = \xi\beta_{ilj}\xi^{-1}$, $\beta^*_{kmj} = \eta\beta_{kmj}\eta^{-1}$ in which ξ and η are suitable products of generators β_{plj}, in decreasing order of p. The p's involved in ξ are those with $i < p < m$ (equality only if $k < i$), while the condition required for η is that,

$$\Pi^{jl}_i \Pi^{ip}_l \delta_{\alpha p} > \Pi^{ip}_l \Pi^{jl}_p \delta_{\alpha i} - \Pi^{jm}_k \Pi^{il}_p \delta_{\alpha m} \, ,$$

where $\alpha = \max\{i, m, p\}$. This latter condition reduces to,

$$\left.\begin{array}{ll} & l > p > i \text{ and } p \geq m \text{ (equality only if } k < i) \\ \text{or} & p < i < m \end{array}\right\}$$

In a similar way, (4.1) (type (ii)') may be reduced to,

$$\left.\begin{array}{ll} & m < q < s \\ \text{or} & m < j, \quad q < s \\ \text{or} & m = j > q \text{ and } \Pi^{ks}_q \Pi^{sj}_l < \Pi^{kj}_i \\ \text{or} & m = j = q \text{ and } l < i \end{array}\right\} \tag{4.2}$$

It is apparent that the values of $\{a_i\}$ only enter into the presentation given above, in those relations associated with case (ii)$'$, for which $i, l < j = m < s < k$ $(i \neq l)$ and then the allowable values of q are such that $q < j$ with $\Pi^{ks}_q \Pi^{sj}_l < \Pi^{kj}_i$.

5: Relation to ordinary braid group

Let $\mathcal{C}$ be an arrangement of $n+1$ hyperplanes in $\mathbf{C}^k$, constructed as in as in §2, with $n+1$ replacing n. Then, by construction, $\{\pi_{J\cup\{n+1\}} \bigm| J \subset I, |J| = k\}$ cut the x_{n+1}-axis in $\binom{n}{k}$ distinct points. Any element of $B(n,k)$ is associated with a loop in $\mathbf{C}^n$, based at $\mathbf{0}$, and as $\mathbf{x} = (x_1, \ldots, x_n)$ moves, the $\binom{n}{k}$ values $a_J(\mathbf{x})$ of x_{n+1}, for which π_{n+1} passes through the intersection of k other hyperplanes $\{\pi_i \bigm| i \in J\}$, will follow paths in $\mathbf{C}$. It is possible that some of the $a_J(\mathbf{x})$ may coincide, for some values of $\mathbf{x}$.

For any $J, K \subset I$, $|J| = |K| = k$, $|J \cap K| < k-1$, a path in $U(n,k)$ exists in which $a_J(\mathbf{x}) = a_K(\mathbf{x})$ at some point $\mathbf{x}$ on the path, while all other $a_L(\mathbf{x})$ remain distinct along the entire path. For every such pair, J, K, pick such a path, and deform it slightly so that $a_J(\mathbf{x})$ and $a_K(\mathbf{x})$ never coincide, but do wind around each other. Denote the corresponding element of $P_{\binom{n}{k}}$ by $h_{J,K}$. This will be conjugate to $\beta_{J,K}$ in $P_{\binom{n}{k}}$, using the standard notation for generators of $B(*,1)$ (see §4). For every $J \subset I$ of order $k+1$, choose an element $g_J \in P_{\binom{n}{k}}$, corrsponding to a path in $U(n,k)$ based at $\mathbf{0}$, associated with the generator β_J of $B(n,k)$. Such a path may need to be slightly deformed so as to ensure $\{a_J(\mathbf{x})\}$ distinct throughout the path.

Let G be the group generated by $\{g_J \bigm| |J| = k+1\}$ and $\{h_{J,K} \bigm| |J| = |K| = k, \ |J \cap K| < k-1\}$. Let H be the subgroup of G generated by $\{h_{J,K}\}$, and K be its normal closure in G. Then the choice of g_J is arbitrary up to composition with elements of K.

The analysis above shows that,

$$B(n,k) \cong G/K \;, \tag{5.1}$$

a quotient of a subgroup of $P_{\binom{n}{k}}$. The generators g_J and $h_{J,K}$ used to construct G and K will be conjugates of,

$$\prod_{j\in J}\Big(\prod_{\substack{i\in J\\ i<j}} \beta_{J\setminus\{i\},J\setminus\{j\}}\Big) \quad \text{and} \quad \beta_{J,K}\;,$$

respectively, in $P_{\binom{n}{k}}$.

The construction described here is analogous to viewing $B(n,1)$ as P_n. That is, consider $X_{n+1,1}$ fibred over $X_{n,1}$ with projection map given by forgetting the last point. The fibre will be $\mathbf{C}$ with n points removed, and has fundamental group F_n. In the case of general k, one can still consider the map from $X_{n+1,k}$ to $X_{n,k}$ obtained by forgetting the last hyperplane. However, this map is not a fibration; this corresponds to the non-triviality of K.

It should be observed that when $k=1$, the higher braid group reduces to P_n, the pure braid group, and not to B_n, the full braid group. A natural action of S_n exists on X_n, by permuting the points, and B_n appears as the fundamental group of of ${X_n}/{S_n}$. There is no obvious action of S_n on $U(n,k)$ for $k>1$. Let $V(n,k)$ denote the space of all $(\mathbf{a},\mathbf{x})$ with $\mathbf{a}\in X_n$ and $\mathbf{x}\in\mathbf{C}$, for which the hyperplanes $\{\mathbf{r}.\mathbf{n}_i = x_i\}$ with $\mathbf{n}_i$ defined by (2.1), are in general position. Clearly, the projection to X_n, given by $(\mathbf{a}.\mathbf{x})\mapsto\mathbf{a}$ defines a fibration with fibre $U(n,k)$. A free action of S_n exists on $V(n,k)$, given by,

$$\sigma(\mathbf{a}.\mathbf{x}) = \big(\sigma(\mathbf{a}),\,\sigma(\mathbf{x})\big)$$

for $\sigma\in S_n$. The resulting quotient fibration,

$$\begin{array}{c} {V(n,k)}/{S_n} \\ \downarrow \\ {X_n}/{S_n} \end{array}$$

gives rise to a homomorphism,

$$B_n \longrightarrow \mathrm{Aut}\big(B(n,k)\big)$$

in much the same way that a homomorphism $B_n\to\mathrm{Aut}(F_n)$ exists (see for example [B]). This provides yet another construction for braid group representations, distinct from, but similar to [L].

6: Further remarks

In [MS 1], it was shown how the space of orders on k-sets satisfying a certain condition, generalises, in some sense, the structure of the symmetric group S_n. These orders may be viewed as the order of $\{a_J(\mathbf{0}) \mid |J| = k\}$, which clearly depends on the choice of $a_1, \ldots, a_{n+1}$.

It is clear from the analysis of this paper, that $B(n,k)$, for $k > 1$, has no unique natural presentation, but rather possesses a family of such, indexed by possible real $\{a_i\}$. In other words, a different presentation may be defined for each component of the space of allowed $\{a_i\}$, a complement of hypersurfaces in $\mathbf{R}^n$. This corresponds to the fact that for $k > 1$, $B(n,k)$ is associated with only a *weak* Bruhat order. The results of §5 (see (5.1)) may also be seen as the formulation in terms of groups, analogous to the constructions of partially ordered sets in [MS 1].

Acknowledgements This work was carried out while the author was at the Mathematical Sciences Research Institute, Berkeley, and is supported in part by NSF Grant No. 8505550. The author also wishes to thank MSRI and the University of California at Berkeley for their hospitality. The author wishes to thank P. Orlik for useful comments.

References

[Ar] E. Artin, 'Theorie der Zöpfe', *Abh. Math. Sem. Univ. Hamburg* **4** (1925) p.47–72.

[B] J.S. Birman, *Braids, Links and Mapping Class Groups* Princeton University Press (1974).

[L] R.J. Lawrence, 'Homology representations of braid groups', *D.Phil. Thesis, Oxford* (June 1989).

[MS 1] Yu.I. Manin, V.V. Schechtman, 'Higher Bruhat Orders, Related to the Symmetric Group', *Funct. Analysis and its Applications* **20** (1987) p.148–150.

[MS 2] Yu.I. Manin, V.V. Schechtman, 'Arrangements of Hyperplanes, Higher Braid Groups and Higher Bruhat Orders', *Adv. Studies in Pure Maths.* **17** (1989) p.289–308.

[OS] P. Orlik, L. Solomon, 'Combinatorics and Topology of Complements of Hyperplanes', *Invent. Math.* **56** (1980) p.167–189.

[R] R. Randell, 'The Fundamental Group of the Complement of a Union of Complex Hyperplanes', *Invent. Math.* **80** (1985) p.467–468.

Berkeley, California
March 1991.
Amended February 1992.