A Note on Two-row Hecke Algebra Representations [1]

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Abstract. In this note, two forms for the representations of the Iwahori-Hecke algebra associated with 2-row Young diagrams will be given. One of the forms appears naturally from the topological construction of such representations given elsewhere, while the other form comes from a specialisation of Wenzl's general construction of irreducible Hecke algebra representations. In both constructions the bases used may be naturally indexed by Young tableaux. The combinatorial structure of the transformation between these bases is the subject of this note, involving q-numbers.

1: The two constructions

1.1 Wenzl's construction

The Iwahori-Hecke algebra $H_n(q)$ (type $A^{(1)}_{n-1}$) is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with the following relations,

$$\left.\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i=1,2,\ldots,n-2 \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for } \left|i-j\right|>1 \\ (\sigma_i-1)(\sigma_i+q) &= 0 && \text{for } 1\le i\le n-1\,. \end{aligned}\right\} \tag{1.1}$$

When $q=1$, these relations reduce to those existing between the generators $(i\,i{+}1)$ of $\mathbf{C}S_n$. When q is not a root of unity, the representation theory of $H_n(q)$ is similar to that of S_n, with irreducible representations being defined for each Young diagram with n squares. Such representations may be constructed using similar techniques to those used for symmetric group representations (i.e. Young symmetrisers), and this was carried out in [We]. In this note we restrict our attention to representations associated with two-row Young diagrams, Λ_m, with $n-m$ and m squares in the two rows ($0\le m\le [n/2]$). A *tableau* t of shape Λ_m is an assignment of the integers $1,2,\ldots,n$ to the squares of Λ_m. By a *standard tableau* is meant a tableau whose labels increase along any row (left to right) or down any column. Such a standard tableau of shape Λ_m is thus specified by the labels $1\le a_1<\cdots<a_m\le n$ appearing (left to right) in the second row, subject to the condition,

$$a_i \ge 2i \qquad \forall i\,. \tag{1.2}$$

Let I_m denote the set of such standard tableaux; it has order,

$$\binom{n}{m} - \binom{n}{m-1}\,, \tag{1.3}$$

where $0\le m\le [n/2]$.

Let $e_i=(\sigma_i+q)/(1+q)$. This defines generators for $H_n(q)$ which are all projections. In [We], Wenzl constructs a representation π_1 of $H_n(q)$, on the vector space $U=\langle \mathbf{v}_t \mid t\in I_m\rangle$ indexed by I_m, by setting,

$$\pi_1(e_i)\mathbf{v}_t = a_d\mathbf{v}_t + \left(a_d a_{-d}\right)^{1/2}\mathbf{v}_{\sigma_i(t)}\,, \tag{1.4}$$

where $\sigma_i(t)$ is the tableau obtained from t by interchanging the labels i and $i{+}1$. The number a_d is defined by,

$$a_d = \frac{(1-q^{d+1})}{(1-q^d)(1+q)}\,.$$

Here the integer d is defined to be $r_i - r_{i+1} + c_{i+1} - c_i$, where the square labelled i in the tableau t, lies in the ${r_i}^{\text{th}}$ row and ${c_i}^{\text{th}}$ column. Note that although it is possible for $\sigma_i(t)\notin I_m$ even though $t\in I_m$, in such cases $d=\pm1$ and so $a_d a_{-d}=0$. Hence (1.4) gives a well-defined action of e_i upon U.

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1.2 A topological construction

Let X_n denote the configuration space of n distinct ordered points in $\mathbf{C}$. Then there is a natural fibration of X_{m+n} over X_n with projection map given by forgetting the first m points. The fibre over $\mathbf{w} \in X_n$, denoted $Y_{\mathbf{w},m}$, is the configuration space of m distinct points in $\mathbf{C}\backslash\mathbf{w}$. For any local system χ on X_{m+n}, a natural flat connection (the Gauss-Manin connection) exists on the vector bundle over X_n with fibre $H^m(Y_{\mathbf{w},m},\chi)$. The monodromy of this connection provides a representation of B_n on the middle cohomology of $Y_{\mathbf{w},m}$. For more details, we refer to [L 2]. It is possible to pick out of the monodromy representation a natural subrepresentation which factors through $H_n(q)$ and may be defined as follows.

Let $\mathcal{T}_m$ denote the set of all subsets of $\{1,2,\ldots,n\}$ of order m, and V be a vector space with basis $\{f_{\underline{\alpha}}\}$ indexed by $\underline{\alpha} \in \mathcal{T}_m$. Consider the subspace W of V generated by,

$$g_{\underline{\beta}} = \sum_{i \notin \underline{\beta}} q^{s_i} f_{\underline{\beta}\cup\{i\}} \tag{1.5}$$

where $\underline{\beta}$ runs over elements of $\mathcal{T}_{m-1}$, and s_i denotes the number of elements of $\underline{\beta}$ less than i. The representation π_2 is defined on V/W by,

$$[\pi_2(\sigma_i)]f_{\underline{\alpha}} = \begin{cases} (1-q)f_{\underline{\alpha}} + f_{\sigma_i(\underline{\alpha})} & \text{if } i \notin \underline{\alpha},\, i+1 \in \underline{\alpha}, \\ q f_{\sigma_i(\underline{\alpha})} & \text{if } i \in \underline{\alpha},\, i+1 \notin \underline{\alpha}, \\ f_{\underline{\alpha}} & \text{otherwise,} \end{cases} \tag{1.6}$$

where $\sigma_i(\underline{\alpha})$ denotes the m-tuple obtained from $\underline{\alpha}$ by interchanging i and $i+1$.

Theorem 1 *When q is a root of unity, (1.4) and (1.6) define isomorphic irreducible representations of $H_n(q)$, associated with the two-row Young diagram Λ_m.*

A basis for V/W is given by those $f_{\underline{\alpha}}$ for $\underline{\alpha} \in \mathcal{S}_w$ where,

$$\mathcal{S}_m = \big\{\{\text{labels on second row of squares of } \Lambda_m \text{ in } t\} \mid t \in I_m\big\} \subset \mathcal{T}_m\,.$$

This statement is equivalent to showing that a certain matrix has maximal rank. For, let $\mathbf{A}$ be a matrix whose rows are labelled by the $\binom{n}{m-1}$ elements of $\mathcal{T}_m\backslash\mathcal{S}_m$ and whose columns are indexed by elements of $\mathcal{T}_{m-1}$. The entries in a row associated with label $\underline{\alpha} \in \mathcal{T}_m\backslash\mathcal{S}_m$ are all either powers of q or zero. Indeed, if $\underline{\beta}$ is an $(m-1)$-set,

$$A_{\underline{\alpha}\underline{\beta}} = \begin{cases} 0 & \text{if } \underline{\beta} \not\subset \underline{\alpha} \\ q^{i-1} & \text{if } \underline{\beta} = \underline{\alpha}_{(i)} \end{cases} \tag{1.7}$$

where $\underline{\alpha}_{(i)}$ denotes the set obtained from $\underline{\alpha}$ by removing the i^{th} entry when arranged in ascending order.

Example 1 For $n=4$ and $m=2$ we have $|\mathcal{T}_2| = 6$, $|\mathcal{T}_1| = 4$ and $|\mathcal{S}_2| = 2$. Indeed, $\mathcal{T}_1 = \big\{\{1\},\{2\},\{3\},\{4\}\big\}$ and $\mathcal{T}_2\backslash\mathcal{S}_2 = \big\{\{1,2\},\{1,3\},\{1,4\},\{2,3\}\big\}$ and the matrix $\mathbf{A}$ is shown below.

$$\begin{pmatrix} q & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ q & 0 & 0 & 1 \\ 0 & q & 1 & 0 \end{pmatrix}$$

It has determinant $q(q+1)$.

Example 2 For $n=6$ and $m=3$, $\mathbf{A}$ is a 15×15 matrix and its determinant is $-q^8(1+q)^5(1+q+q^2)$. In general, each row in $\mathbf{A}$ contains $1, q, \ldots, q^{m-1}$ exactly once, and its determinant is non-zero whenever $q \in \mathbf{C}^*$ is not a root of unity.

Theorem 2 *The determinant of the matrix* $\mathbf{A}$ *defined by (1.7) is given by,*

$$\pm \prod_{r=1}^{m} \left(q^{1/2 r(r-1)} (1 + q + \cdots + q^{r-1}) \right)^{\alpha_{n,m-r}}$$

where $\alpha_{n,k} = \binom{n}{k} - \binom{n}{k-1}$ *is the dimension of* π_{Λ_k}.

This may be verified inductively. Note that when q is a root of unity, $H_n(q)$ may not be semi-simple. Indeed, (smaller) irreducible representations of $H_n(q)$ may be picked out, at such roots of unity (see [We]). However, in every construction it is seen that if q is a primitive k^{th} root of unity,

(i) for $k > n$, $H_n(q)$ is semi-simple;

(ii) the representation π_1 constructed in §1.1 is irreducible if $k > m$.

Thus if $k > m$, q behaves, for the purposes of the representation theory, like a non-root-of-unity. It may be seen that the singular behaviour of $\det \mathbf{A}$ appears only at roots of unity of order at most m.

Hence for q away from roots of unity, V/W has a natural basis indexed by $\mathcal{S}_m$, which may be naturally corresponded with I_m.

2: Inner products

In this section we shall consider inner products defined on the spaces of §1 on which representations π_1 and π_2 are defined. These inner products will be identified by the condition that they are invariant under the involutions,

$$\tau_i = 2e_i - 1, \qquad \forall i \, .$$

This condition is equivalent to requiring the 1 and $-q$ eigenspaces of the action of σ_i to be orthogonal, $\forall i$.

2.1 Inner product for π_1

It may easily be seen that the 0 and 1 eigenspaces of e_i under the representation π_1 are spanned by,

$$\{a_{-d}^{1/2}\mathbf{v}_t - a_d^{1/2}\mathbf{v}_{\sigma_i(t)} \mid t \in I_m \text{ with } d \neq \pm 1\} \cup \{\mathbf{v}_t \mid t \in I_m \text{ with } d = -1\}$$
$$\{a_d^{1/2}\mathbf{v}_t + a_{-d}^{1/2}\mathbf{v}_{\sigma_i(t)} \mid t \in I_m \text{ with } d \neq \pm 1\} \cup \{\mathbf{v}_t \mid t \in I_m \text{ with } d = 0\}$$

respectively (see (1.4)). Choosing the standard inner product on $\langle\{\mathbf{v}_t \mid t \in I_m\}\rangle$ under which $\{\mathbf{v}_t\}$ are orthonormal, it may be seen that this makes τ_i preserve the inner product $\forall i$. It is also clear that this inner product is essentially unique, and thus, by Theorem 1, the isomorphism between U of π_1 and V/W of π_2 must transform the natural inner product on U to one on V/W, also preserved under τ_i, $\forall i$.

2.2 Inner product for π_2

In this section we proceed to construct an inner product on V/W invariant under τ_i, $\forall i$. This is equivalent to a degenerate inner product $\langle\,|\,\rangle$ on V for which,

(i) $\langle \underline{\alpha} \mid \underline{\beta} \rangle = 0$ whenever $\underline{\alpha}$ is a 1 eigenvector and $\underline{\beta}$ is a $-q$ eigenvector of $\pi_1(\sigma_i)$ on V, $\forall i$;

(ii) $\langle f_{\underline{\alpha}} \mid g_{\underline{\beta}} \rangle = 0$, $\forall \underline{\beta} \in \mathcal{T}_{m-1}$, $\underline{\alpha} \in \mathcal{T}_m$.

It is easily seen that (i) is equivalent to,

$$\left.\begin{aligned} \langle f_{\underline{\alpha}\cup\{i\}} \mid f_{\underline{\beta}\cup\{i+1\}} \rangle &= \langle f_{\underline{\alpha}\cup\{i+1\}} \mid f_{\underline{\beta}\cup\{i\}} \rangle \\ \langle f_{\underline{\alpha}\cup\{i\}} \mid f_{\underline{\beta}} \rangle &= \langle f_{\underline{\alpha}\cup\{i+1\}} \mid f_{\underline{\beta}} \rangle \\ \langle f_{\underline{\alpha}\cup\{i\}} \mid f_{\underline{\beta}\cup\{i\}} \rangle - q\langle f_{\underline{\alpha}\cup\{i+1\}} \mid f_{\underline{\beta}\cup\{i+1\}} \rangle &= (q-1)\langle f_{\underline{\alpha}\cup\{i\}} \mid f_{\underline{\beta}\cup\{i+1\}} \rangle \end{aligned}\right\} \qquad (2.1)$$

whenever $\underline{\alpha}, \underline{\beta} \in \mathcal{T}_{m-1}$ with $i, i{+}1 \notin \underline{\alpha}, \underline{\beta}$. Denote by $s_{\underline{\alpha}}$, the sum of the elements of $\underline{\alpha}$, for any subset $\underline{\alpha}$ of $\{1, 2, \ldots, n\}$. Set,

$$(\underline{\alpha}, \underline{\beta}) = q^{s_{\underline{\alpha}}+s_{\underline{\beta}}} \langle f_{\underline{\alpha}} \mid f_{\underline{\beta}} \rangle \tag{2.2}$$

Then (2.1) reduces to,

$$\left.\begin{aligned}
(\underline{\alpha} \cup \{i\}, \underline{\beta} \cup \{i{+}1\}) &= (\underline{\alpha} \cup \{i{+}1\}, \underline{\beta} \cup \{i\}) \\
(\underline{\alpha} \cup \{i\}, \underline{\beta}) &= (\underline{\alpha} \cup \{i{+}1\}, \underline{\beta}) \\
(\underline{\alpha} \cup \{i\}, \underline{\beta} \cup \{i\}) - q^{-1}(\underline{\alpha} \cup \{i{+}1\}, \underline{\beta} \cup \{i{+}1\}) &= (1 - q^{-1})(\underline{\alpha} \cup \{i\}, \underline{\beta} \cup \{i{+}1\})
\end{aligned}\right\} \tag{2.3}$$

Moreover (ii) reduces to,

$$\sum_{i \notin \underline{\beta}} q^{s_i - i} (\underline{\alpha}, \underline{\beta} \cup \{i\}) = 0 \tag{2.4}$$

for all $\underline{\alpha} \in \mathcal{T}$ and $\underline{\beta} \in \mathcal{T}_{m-1}$.

We say that $(\underline{\alpha}, \underline{\beta})$ is in *standard form* if $\underline{\alpha} \cup \underline{\beta} = \{1, 2, \ldots, k\}$ some k with $m \le k \le 2m$. Suppose that such inner products, in standard form, depend only on $|\underline{\alpha} \cup \underline{\beta}|$, say $(\underline{\alpha}, \underline{\beta}) = A_d$ where $d = |\underline{\alpha} \cap \underline{\beta}|$. Equation (2.3) may be used to reduce any $(\underline{\alpha}, \underline{\beta})$ to combinations of inner products in standard form. Suppose $(\underline{\alpha}, \underline{\beta})$ is general, with $|\underline{\alpha} \cup \underline{\beta}| = d$, say $\underline{\alpha} \cup \underline{\beta} = \{x_1, \ldots, x_d\}$ arranged in increasing order. In the segment between x_{i-1} and x_i ($x_0 \equiv 0$), there may be a_i integers which are in neither $\underline{\alpha}$ or $\underline{\beta}$. Then,

$$(\underline{\alpha}, \underline{\beta}) = \sum_{r=0}^{d} \left\{ A_{d-r} \sum_{|S|=r} \left(\prod_{i=1}^{d} P_{i,S}(q^{a_1 + \cdots + a_i - (\#S<i)}) \right) \right\} \tag{2.5}$$

where the second summation is over all subsets S of $\{1, 2, \ldots, d\}$, of order r, and,

$$P_{i,S} = \begin{cases} x & \text{if } i \notin S \\ 1 - x & \text{if } i \in S\,. \end{cases}$$

Applying (2.4) with $\underline{\alpha} = \{1, 2, \ldots, m\}$ and $\underline{\beta} = \{1, 2, \ldots, d, m{+}1, \ldots, 2m - d - 1\}$, gives,

$$A_{d+1}(1 + q^{-1} + \cdots + q^{d+1-m}) + A_d(q^{d-m} + \cdots + q^{m-n}) = 0\,.$$

Thus up to scale,

$$A_d = (-1)^d q^{-\,{}^1\!/\!{}_2 d(d+1) + d(2m-n)} \frac{(n - 2m + d)!_q (m-d)!_q}{(n-2m)!_q m!_q} A_0 \tag{2.6}$$

where $[m]_q \equiv (1 - q^m)/(1-q)$ is a q-number and $m!_q \equiv \prod_{k=1}^{m} [k]_q$ is a q-factorial.

It may be verified that (2.6) provides a solution to (2.3) and (2.4). Since the inner product on U is uniquely determined by its invariance under the action of $\pi_1(\tau_i)$, this must also be true of the corresponding inner product on $V\!/W$, with respect to the action of $\pi_2(\tau_i)$. Hence the isomorphism of Theorem 1 transforms the standard inner product on U of §2.1, to one on $V\!/W$ induced by the inner product,

$$\begin{aligned}\langle f_{\underline{\alpha}} \mid f_{\underline{\beta}} \rangle = q^{-s_{\underline{\alpha}} - s_{\underline{\beta}}} \sum_{r=0}^{d} \Bigg\{ &(-1)^r q^{-\,{}^1\!/\!{}_2 r(r+1) + r(2m-n)} (n - 2m + r)!_q (m - r)!_q \\ &\sum_{|S|=d-r} \left(\prod_{i=1}^{d} P_{i,S}(q^{a_1 + \cdots + a_i - (\#S<i)}) \right) \Bigg\}\end{aligned} \tag{2.7}$$

on V (see (2.2) and (2.5)). By Theorem 2, a basis for $V\!/W$ may be obtained as $\{f_{\underline{\alpha}} + W \mid \underline{\alpha} \in \mathcal{S}_m\}$. The determinant of that part of (2.7) associated with $\underline{\alpha}, \underline{\beta} \in \mathcal{S}_m$ is thus essentially Δ^2, where Δ is the determinant of the transformation matrix between bases in U and $V\!/W$ given by the isomorphism of Theorem 1.

3: Further remarks

In §1.2, the origins of the representation π_2 of (1.6) were briefly outlined. The basis $f_{\underline{\alpha}}$ is essentially dual to a natural basis for homology in terms of iterated loops. An alternative basis for cohomology comes from using de Rahm cohomology. Let $z_1, \ldots, z_m$ be local coordinates in $Y_{\mathbf{w},m}$. Then, since $Y_{\mathbf{w},m}$ is a Stein manifold, $H^m(Y_{\mathbf{w},m}, \chi)$ may be computed in terms of holomorphic differentials, $f\, dz_1 \ldots dz_m$ where f is a (multi-valued) holomorphic function on $Y_{\mathbf{w},m}$ which twists according to χ. The appropriate functions f, for such a basis, turn out to be,

$$\left\{ f_0 \sum_{\sigma \in S_m} \prod_{i=1}^{m} (z_i - w_{\alpha_{\sigma(i)}})^{-1} = h_{\underline{\alpha}} \;\middle|\; \underline{\alpha} \in \mathcal{T}_m \right\}$$

where $f_0 = \prod_{\substack{i,j=1 \\ i<j}}^{m} (z_i - z_j)^{-2a} \prod_{i=1}^{m} \prod_{j=1}^{n} (z_i - w_j)^a$ and $q = \exp(2\pi i a)$ (see [L 2]).

One may consider combinations $\sum_{\underline{\alpha} \in \mathcal{T}_m} A_{\underline{\alpha}}(\mathbf{w}) h_{\underline{\alpha}} = f$, such that $\frac{\partial f}{\partial w_i} \sim 0$ in the sense of cohomology, $\forall i$. This reduces to a set of differential equations for the functions $A_{\underline{\alpha}}(\mathbf{w})$, which in terms of $\mathbf{A}(\mathbf{w}) = (\mathbf{A}_{\underline{\alpha}}(\mathbf{w}))_{\underline{\alpha} \in \mathcal{S}_m}$ are,

$$\frac{\partial \mathbf{A}}{\partial w_i} = \sum_{j \neq i} \frac{\mathbf{C}_{ij} \mathbf{A}}{w_i - w_j} \tag{3.1}$$

in which $\mathbf{C}_{ij}$ are constant matrices. The associated monodromy action is π_2 and once again exists on a quotient of V, since components of $\mathbf{A}$ are indexed by $\underline{\alpha} \in \mathcal{S}_m$. However, with respect to $\sim$, relations also exist amongst the $h_{\underline{\alpha}}$. This gives rise to a representation on V/W' where,

$$\begin{aligned} W' &= \langle g'_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{T}_{m-1} \rangle \\ g'_{\underline{\alpha}} &= \sum_{i \notin \underline{\alpha}} (h_{\underline{\alpha}}) \,. \end{aligned} \tag{3.2}$$

The system (3.1) is reminiscent of the Knizhnik-Zamolodchikov equations of conformal field theory [KZ], and indeed it was shown in [L 1] and [L 2] to be isomorphic to,

$$\frac{\partial f}{\partial w_i} = a \sum_{j \neq i} \frac{\mathbf{\Omega}_{ij}}{w_i - w_j} f \tag{3.3}$$

up to a shift, where $f \colon X_n \to (V_{1/2}^{\otimes n} \otimes \check{V}_t)_0 = X$ is a holomorphic vector valued function of $\mathbf{w} \in X_n$. Here V_t is the spin-t representation of $\mathfrak{sl}_2$ and $t = n/2 - m$, while $\mathbf{\Omega}_{ij} \in \mathrm{End}(X)$ is the polarised Casimir operator acting on i and j alone.

Theorem 3 [L 1] *An isomorphism $\alpha \colon X \to V/W'$ exists under which,*

$$\alpha \circ (\mathbf{\Omega}_{ij} - 1/2\mathbf{I}) a = \mathbf{C}_{ij} \circ \alpha \,.$$

It can be shown that the matrices $\mathbf{C}_{ij}$ are just $a(\mathbf{P}_{ij} - \mathbf{I})$ where $\mathbf{P}_{ij}$ interchanges the elements i and j, as an action on the basis $f_{\underline{\alpha}}$ for V. Let us now investigate the inner product correspondence induced by α. A natural inner product exists on $V_{1/2}$, which therefore carries over to one on $X = (V_{1/2}^{\otimes n} \otimes \check{V}_t)_0$. Now, $\mathbf{\Omega}_{ij} + 1/2\mathbf{I}$ acts on $V_{1/2}^{\otimes n}$ by transposing the i^{th} and j^{th} factors. The natural inner product is invariant under this action. However, under α, by Theorem 3, the induced inner product on V/W' will be invariant under

$a^{-1}\mathbf{C}_{ij} + \mathbf{I} = \mathbf{P}_{ij}$. Hence, as an inner product on V, $\langle f_{\underline{\alpha}} \mid f_{\underline{\beta}} \rangle$ can only depend upon $d = |\underline{\alpha} \cap \underline{\beta}|$, say A'_d. Since the inner product is well-defined on $V/_{W'}$,

$$\langle f_{\underline{\alpha}} \mid g'_{\underline{\beta}} \rangle = 0 \qquad \forall \underline{\alpha} \in \mathcal{T}_m, \underline{\beta} \in \mathcal{T}_{m-1}$$

Hence it may be deduced that,

$$\langle f_{\underline{\alpha}} \mid f_{\underline{\beta}} \rangle = A'_d = (-1)^d (n-2m+d)!(m-d)!.C \tag{3.4}$$

for some constant C. This defines the inner product on $V/_{W'}$.

The results of this section should be compared with those of §2. In particular, the structure of the representation on $V/_{W'}$, as contrasted with that in §2 which is on $V/_W$, can be seen to differ only in that the relations W' of (3.2), are those of W at $q = 1$ (see (1.5)). The inner product (2.7) which is natural, in that it is invariant under some *global* action (the involutions associated with the monodromy action), specialises at $q = 1$ to (3.4), which is obtained by imposing a local invariance constraint (invariance under the *local* action matrices $\mathbf{C}_{ij}$).

The two bases of Theorem 1 (§1) should be thought of as 'global', as opposed to the 'local' bases of the present section. An example of a local basis coming from conformal field theory, is given in [TK]. Not only does the transformation between 'local' and 'global' bases inevitably involve hypergeometric functions, but it also depends upon the basepoint chosen, on the local side of the picture, with respect to which to relativise. Of course, transforming between the two local bases, or between the two global bases discussed in this note, will involve only algebraic functions of q.

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