The Homological Approach Applied to Higher Representations

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Abstract. It was seen elsewhere how representations of the Iwahori-Hecke algebra associated with two-row Young diagrams could be constructed using elementary topology. Such representations also arise in the context of conformal field theory, where they are associated with the spin-½ representations of $\mathfrak{sl}_2$. In this paper, the analogue of this topological construction will be produced, giving rise to those braid group representations associated with higher representations of $\mathfrak{sl}_2$. The associated problem of the construction of link invariants from such representations is also discussed.

1: Introduction

It has been seen in [L 1] how monodromy representations of the braid group B_n may be obtained on a natural flat line bundle whose fibres have the form of a homology space. These homology spaces are evaluated on the space obtained from the configuration space of points in the complex plane with a number of punctures, by placing a suitable twisted local coefficient system on them. The representations obtained, turn out to factor through the Hecke algebras, and can be used in order to obtain the one-variable Jones polynomial, evaluated on the link obtained by closing a braid. The procedure can be identified with the Lie group $\mathfrak{sl}_2$ and the standard vector representation.

In this paper the analogue of this procedure will be developed for the case of the higher representations of $\mathfrak{sl}_2$. Starting with the homology space used in [L 1] it will be seen that, from a slightly more general local coefficient system, one derives those representations of the braid groups associated with higher representations of $\mathfrak{sl}_2$, by taking a suitable subrepresentation of the monodromy representation obtained on the twisted homology space. These representations no longer factor through the Iwahori-Hecke algebras; however they do factor through a quotient of the braid group, B_n, given by imposing a polynomial relation on the standard generators $\{\sigma_i\}$. In §2, it will be shown how such representations are derived from the techniques of Tsuchiya & Kanie [TK] generalised to higher representations. In §3, the details of the construction using homology, will be given, and the isomorphism which exists between these two approaches is discussed in §4.

Just as $\mathfrak{sl}_2$ with the standard (vector) representation gives rise to those Hecke algebra representations occurring in the one-variable Jones polynomial [J], it is reasonable to suppose that the braid group representations constructed here, although not factoring through the (Iwahori-) Hecke algebra, can also be used to construct knot and link invariants. Such questions are briefly discussed in §5.

2: Representations obtained using CFT

Let X_n denote the configuration space of n distinct (ordered) points in the complex plane $\mathbf{C}$. In this section we will recall the construction of Tsuchiya & Kanie [TK], based on conformal field theory on $\mathbf{P}^1$. Suppose that $\Phi_i(u;z)$ is a vertex operator of spin-j_i, considered as a map $\mathcal{H} \to \widehat{\mathcal{H}}$, in the sense of [BPZ], where $\mathcal{H}$ is a suitable Hilbert space with completion $\widehat{\mathcal{H}}$, $z \in \mathbf{C}\backslash\{0\}$ and u is an element of V_{j_i}, the spin-j_i representation of $\mathfrak{sl}_2$. One can then calculate an n-point function:

$$\left\langle v \;\middle|\; \Phi_n(u_n;z_n)\cdots\Phi_2(u_2;z_2)\Phi_1(u_1;z_1) \;\middle|\; v\right\rangle \tag{2.1}$$

so long as $\Phi_1,\ldots,\Phi_n$ are 'compatible', in the sense of [TK]. Here $u_i \in V_{j_i}$ for $1 \le i \le n$, while $|\mathrm{vac}\rangle \in V_0$ and $\langle v| \in \check{V}_t$ for some *target spin*, t. The complex valued function of the u_i's and z_i's defined by (2.1) may alternatively be viewed as a function of $z_1,\ldots,z_n$ whose values are linear functionals in the u_i's and v. It was shown in [TK] that this is well-defined in the region $0 < |z_1| < |z_2| < \cdots < |z_n|$, and that such an n-point function, considered as a map:

$$f\colon X_n \longrightarrow (V_{j_1} \otimes \cdots \otimes V_{j_n} \otimes \check{V}_t)_0 \tag{2.2}$$

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satisfies a system of differential equations of the form:

$$\kappa \frac{\partial f}{\partial z_i} = \sum_{k \neq i} \frac{\mathbf{\Omega}_{ik}}{z_i - z_k} f \tag{2.3}$$

(the equations of Knizhnik and Zamolodchikov). Here $^\vee$ denotes the dual representation of $\mathfrak{sl}_2$ and $(\ )_0$ denotes the $\mathfrak{sl}_2$-invariant part of the space within the brackets. As a (single-valued) holomorphic function, f is only well-defined by the expected value of an operator product in the region $0 < |z_1| < |z_2| < \cdots < |z_n|$, but it can be analytically continued to give a multivalued meromorphic function on the whole of X_n. In (2.3), $\mathbf{\Omega}_{ij}$ are matrices defined for all $i \neq j$, by taking the action of the polarisation of the Casimir operator, $\mathbf{\Omega}$, on the i^{th} and j^{th} factors of the tensor product $V_{j_1} \otimes \cdots \otimes V_{j_n} \otimes V_t$, with the identity action on remaining factors.

Equation (2.3) is equivalent to stating that f defines a covariant constant section of a flat line bundle over X_n, specified by $\{\mathbf{\Omega}_{ij}\}$. The collection of such covariant constant sections is spanned by all possible n-point functions of primary fields (vertex operators) of spins $\{j_i\}$. The monodromy gives rise to a representation of $\pi_1(X_n) = P_n$, the pure braid group of n strings, on the space of n-point functions. When $j_i = j$, independent of i, the action of P_n can be extended to one of the full braid group $B_n = \pi_1(\widetilde{X}_n)$ where $\widetilde{X}_n$ is the configuration space of n (unordered) distinct points in $\mathbf{C}$. The case of $j = 1/2$ is the one which was discussed in detail in [TK], and led to the Hecke algebra representations associated with two-row Young diagrams.

Let H, E and F be the usual generators for $\mathfrak{sl}_2$ with $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$. Then the polarisation of the Casimir operator $\mathbf{\Omega}$ is given by:

$$1/2 H \otimes H + E \otimes F + F \otimes E \, .$$

However, in the spin-j representation of $\mathfrak{sl}_2$, the full Casimir operator, $\mathbf{\Omega} = 1/2 H^2 + EF + FE$, acts by multiplication by $2j(j+1)$. Thus the polarisation of $\mathbf{\Omega}$ acts on $V_{j_1} \otimes V_{j_2}$ as:

$$j(j+1) - j_1(j_1+1) - j_2(j_2+1)$$

on those parts of the decomposition of $V_{j_1} \otimes V_{j_2}$ which transform according to V_j under the action of $\mathfrak{sl}_2$. When $j_i = j \, \forall i$, the system of differential equations (2.3) for vector-valued functions:

$$f \colon X_n \longrightarrow \left(V_j^{\otimes n} \otimes \check{V}_t\right)_0^\vee \tag{2.4}$$

has a solution set consisting of multi-valued meromorphic functions. The action of $\mathbf{\Omega}_{ik}$ on $\left(V_j^{\otimes n} \otimes \check{V}_t\right)_0^\vee$, for $1 \leq i, k \leq n$, is defined to be the action of the polarisation of $\mathbf{\Omega}$ on the i^{th} and k^{th} V_j factors. Since,as an $\mathfrak{sl}_2$-module, $V_j \otimes V_j$ has a decomposition into irreducible modules $V_j \otimes V_j = V_0 \oplus V_1 \oplus \cdots \oplus V_{2j}$ the eigenvalues of $\mathbf{\Omega}_{ik}$ on $V_j \otimes V_j$, are given by:

$$j'(j'+1) - 2j(j+1) \tag{2.5}$$

where $j' \in \{0, 1, \ldots, 2j\}$ with multiplicity $2j'+1$.

However, the dimension of $\left(V_j^{\otimes n} \otimes \check{V}_t\right)_0^\vee$ is the multiplicity of V_t in the decomposition of $V_j^{\otimes n}$ into irreducible modules under the action of $\mathfrak{sl}_2$. For fixed j, consider the array $b_{n,t}$ given by this multiplicity. Extend this to negative t by $b_{n,-t} = -b_{n,t-1}$ for $t \geq 1$. It is easily deduced from the decomposition:

$$V_{j_1} \otimes V_{j_2} = V_{|j_1 - j_2|} \oplus V_{|j_1 - j_2|+1} \oplus \cdots \oplus V_{j_1 + j_2}$$

that:

$$\sum_t b_{n,t} z^t = (z^{-j} + z^{1-j} + \cdots + z^j) \sum_t b_{n-1,t} z^t \, .$$

Thus $\sum_t b_{n,t} z^t = (z^{-j} + z^{1-j} + \cdots + z^j)^n (1 - z^{-1})$. It is now apparent that, for $t \geq 0$, $b_{n,t}$ may be expressed as the difference between the coefficients of z^{nj-t} and z^{nj-t-1} in $(1 + z + \cdots + z^{2j})^n$. Throughout this

paper $\binom{n}{m}_j$ will denote the coefficient of z^m in $(1+z+\cdots+z^{2j})^n$, so that $\binom{n}{m}_{1/2}$ is the usual binomial coefficient. The monodromy representations of B_n obtained by using the spin-j representation of $\mathfrak{sl}_2$ thus have dimensions:

$$\binom{n}{nj-t}_j - \binom{n}{nj-t-1}_j \tag{2.6}$$

where $0 \le t \le [nj]$.

The matrices $\mathbf{\Omega}_{ik}$ have eigenvalues as given by (2.5), with multiplicities given by that of V_t in the $\mathfrak{sl}_2$-module $\left(V_j^{\otimes n-2} \otimes V_{j'}\right)$, namely the coefficient of $z^{(n-2)+j+j'-t}$ in $(1+z+\cdots+z^{2j})^{n-2}(1+z+\cdots+z^{2j'})(1-z)$. That is:

$$\binom{n-2}{(n-2)j+j'-t}_j - \binom{n-2}{(n-2)j-j'-t-1}_j \tag{2.7}.$$

Example Take the case of $j=1$. There are three possible eigenvalues for $\mathbf{\Omega}_{ik}$, corresponding to the cases $j'=0,1,2$. These are -2, -1 and 1, respectively. However when $t=n-1$, $n-2$ and $n-3$, the dimensions of the monodromy representations obtained, together with the associated multiplicities of the eigenvalues of $\mathbf{\Omega}_{ik}$ are given below.

$t=n-1$	dim= $n-1$	0 ,	1 ,	$n-2$
$t=n-2$	dim= $\frac{1}{2}n(n-1)$	1 ,	$n-2$,	$\frac{1}{2}(n-1)(n-2)$
$t=n-3$	dim= $\frac{1}{6}n(n^2-7)$	$n-3$,	$\frac{1}{2}(n-1)(n-2)$,	$\frac{1}{6}(n+2)(n-2)(n-3)$

Thus, in the first case, $\mathbf{\Omega}_{ik}$ has only two eigenvalues, and the associated monodromy representation of B_n, factors through the Hecke algebra.

The braid group representations obtained, depend on the parameter κ. For large κ, it is easily seen that the monodromy obtained when z_i and z_k interchange is $\exp(\pi i \kappa^{-1} \mathbf{\Omega}_{ik})$, up to first order in κ^{-1}, and thus has eigenvalues:

$$\pm \exp\left(\pi \kappa^{-1}\left(j'(j'+1) - 2j(j+1)\right)\right).$$

Let $q = \exp\left(2\pi i/\kappa\right)$. Then, in the example above, it may be seen that, to first order in κ^{-1}, the eigenvalues of the generators of B_n in the monodromy representation are $\pm q^{-1}$, $\pm q^{-1/2}$ and $\pm q^{1/2}$.

Theorem 1 *Suppose that n, $2j$ and $2t$ are non-negative integers with $0 \le t \le [nj]$. Then the monodromy representation obtained from (2.3) for functions f as in (2.4), has dimension given in (2.6), and the generators σ_i of B_n, have eigenvalues $(-1)^{j'} q^{\frac{1}{2}j'(j'+1)-j(j+1)}$ with multiplicities as given by (2.7), for $0 \le j' \le 2j$.*

In the next section the structure of the monodromy representations produced via homology will be determined, and the comparison with the representation of B_n discussed in this section will be made by using the associated differential equations.

3: Topological construction of representation

For any $\mathbf{w} = (w_1, \ldots, w_n) \in X_n$, define $Y_{\mathbf{w},m}$ to be the configuration space of m ordered points in the punctured complex plane $\mathbf{C} \backslash \{w_1, \ldots, w_n\}$. Define a twisted local coefficient system, χ, on $Y_{\mathbf{w},m}$ based on $\mathbf{C}^*$, with twists of q and α when z_i goes clockwise around w_k and z_j respectively, for all $j \ne i$ and $k \in \{1, 2, \ldots, n\}$. Then $H^m(Y_{\mathbf{w},m}, \chi)$ defines a vector bundle over X_n with a natural flat connection, induced from homotopy equivalence of homology, and hence there is naturally defined a representation of B_n on the homology space. See [L 1] for further details. The particular values of the parameters q and α which we shall use are:

$$q = \exp(2\pi i a), \qquad \alpha = q^{-1/j}$$

where $2j \in \mathbf{N} \cup \{0\}$ and a is a real parameter. A suitable (multi-valued) analytic function on $Y_{\mathbf{w},m}$ which twists according to χ is:

$$f = \prod_{\substack{1 \le i \le m \\ 1 \le k \le n}} (z_i - w_k)^{-a} \prod_{1 \le i < l \le m} (z_i - z_l)^{a/j}$$

The cohomology space $H^m(Y_{\mathbf{w},m})$ may be computed by using analytic functions on $Y_{\mathbf{w},m}$, since $Y_{\mathbf{w},m}$ is a Stein manifold. It is easily seen that the only functions that need to be considered are of the form $f.g$ where g is a single-valued analytic function on $Y_{\mathbf{w},m}$ with trivial local coefficient system; and indeed that a suitable spanning set is obtained from g's of the form:

$$\prod_{i=1}^{m} (z_i - \alpha_i)^{-1}$$

where $\alpha_i \in \{w_i, \ldots, w_n\}$. For more details on the derivation of this, see [L 1].

Let $g_{\underline{\alpha}}$ denote the sum of the $m!$ terms obtained by symmetrising the product $\prod_{i=1}^{m} (z_i - \alpha_i)^{-1}$ in $z_1, \ldots, z_m$, where $\alpha_1, \ldots, \alpha_m \in \{w_1, \ldots, w_n\}$. Let $\underline{\beta}$ denote an $(m-1)$-tuple $\beta_1, \ldots, \beta_{m-1}$ of elements of $\{w_1, \ldots, w_n\}$, not necessarily distinct. Define g_k to be the sum of $(m-1)!$ terms of the form:

$$\prod_{\substack{i=1 \\ i \ne k}} (z_i - \beta_{\sigma(i)})^{-1}$$

over $\sigma \in S_m$ such that $\sigma(k) = m$. Then g_k is independent of z_k. The functions $f.g_k$ are all holomorphic functions of $z_1, \ldots, z_m$ on $Y_{\mathbf{w},m}$ with twisting χ, and hence in $H^m(Y_{\mathbf{w},m}, \chi)$,

$$\sum_k \frac{\partial}{\partial z_k}(f g_k) = 0 \,.$$

However,
$$\frac{1}{f} \sum_k \frac{\partial}{\partial z_k}(f g_k) = \sum_k \frac{\partial g_k}{\partial z_k} + \frac{a}{j} \sum_{i<l} \frac{g_i - g_l}{z_i - z_l} - a \sum_{i,l} \frac{g_i}{z_i - w_l}$$
$$= \frac{a}{2j} \sum_{i=1}^{m-1} g_{\underline{\beta}\beta_i} - a \sum_{l=1}^{n} g_{\underline{\beta} w_l}$$

and thus for all $\underline{\beta}$, there is a relation, existing at the level of cohomology:

$$\sum_{i=1}^{m-1} f.g_{\underline{\beta}\beta_i} - 2j \sum_{i=1}^{n} f.g_{\underline{\beta} w_l} = 0 \,. \tag{3.1}$$

Suppose that $\{\beta_1, \ldots, \beta_{m-1}\}$ contains λ copies of w_r. Then, in (3.1), the first term contains an expression $\lambda g_{\underline{\beta} w_r}$ while the second term contains $-2j g_{\underline{\beta} w_r}$. That is, so long as there are at most $2j$ occurrences of any w_r in $\{\beta_1, \ldots, \beta_{m-1}\}$, (3.1) will involve only $g_{\underline{\alpha}}$'s in which each $\underline{\alpha}$ contains at most $2j$ copies of any w_r. It is therefore meaningful to consider the subspace, H, of the homology space $H^m(Y_{\mathbf{w},m})$ spanned by $f.g_{\underline{\alpha}}$'s for $\underline{\alpha} \in \mathcal{S}$, where $\mathcal{S}$ is the set of m-tuples of w's consisting of the elements of $\{w_1, \ldots, w_n\}$ which contain at most $2j$ occurrences of each w_r. There is a natural action of S_m on this cohomology space given by permuting the z_i's, and this action commutes with that of B_n. One may therefore, consider the action of B_n on the symmetric part of the cohomology space, the space H being a subspace of this symmetric part.

The number of $\underline{\alpha}$'s, up to permutation, with $\alpha_1, \ldots, \alpha_m \in \{w_1, \ldots, w_n\}$ containing at most $2j$ occurrences of each w is $\binom{n}{m}_j$. There are relations between the associated $g_{\underline{\alpha}}$'s given by (3.1) for all $\underline{\beta}$ containing at most $2j$ occurrences of any w. The number of relations is $\binom{n}{m-1}_j$, so that the dimension of H is:

$$\binom{n}{m}_j - \binom{n}{m-1}_j \,. \tag{3.2}$$

This is to be compared with the dimension of the representation obtained in the last section using conformal field theory. It can be seen that the evaluation of (2.6), when $m = nj - t$, gives $\dim H$. However before such a comparison can properly be made, it is necessary to verify that H is an invariant subspace of the symmetric part of the cohomology space $H^m(Y_{\mathbf{w},m}, \chi)$, under the action of the braid group B_n.

It may be easily verified that, in $H^m(Y_{\mathbf{w},m}, \chi)$,

$$\begin{aligned}\frac{\partial}{\partial w_i}(fg_{\underline{\alpha}}) = \sum_{\substack{w_k \in \underline{\alpha} \\ k \neq i}} \Bigg\{ as\Big(1 - \frac{u_k}{2j}\Big)\frac{fg_{\underline{\alpha} w_k - w_i}}{w_i - w_k} + au_k\Big(1 - \frac{s}{2j}\Big)\frac{fg_{\underline{\alpha} w_i - w_k}}{w_i - w_k} \\ + \Big[as\Big(\frac{u_k}{2j} - 1\Big) + au_k\Big(\frac{s}{2j} - 1\Big)\Big]\frac{fg_{\underline{\alpha}}}{w_i - w_k}\Bigg\} + as \sum_{w_k \notin \underline{\alpha}} \frac{fg_{\underline{\alpha} w_k - w_i} - fg_{\underline{\alpha}}}{w_i - w_k}\end{aligned} \tag{3.3}$$

where s and u_k denote the number of occurrences of w_i and w_k in $\underline{\alpha}$, respectively. Thus if the multiplicity of any w_k in $\underline{\alpha}$ is not greater than $2j$, then the same is true of all the terms $fg_{\underline{\beta}}$ occurring in $\frac{\partial}{\partial w_i}(fg_{\underline{\alpha}})$. For, in (3.3) the last two terms clearly satisfy this condition. The first term contains $fg_{\underline{\alpha} w_k - w_i}$ in which the multiplicity of any w_i $(i \neq k)$ must be at most $2j$. It is possible, however, that w_k occurs $2j+1$ times, namely when $u_k = 2j$; but in this case the coefficient of this term vanishes. A similar argument applies to the second term. Hence the subspace $H \subseteq H^m(Y_{\mathbf{w},m}, \chi)$ is invariant under the action of $\frac{\partial}{\partial w_i}$, for any $i = 1, 2, \ldots, n$. Let $\mathbf{C}_{ik}$ denote the matrix whose $(\alpha, \beta)^{\text{th}}$ element is given by the coefficient of $\frac{fg_{\underline{\beta}}}{w_i - w_k}$ in $\frac{\partial}{\partial w_i}(fg_{\underline{\alpha}})$. Thus, at the level of cohomology:

$$\frac{\partial}{\partial w_i}(fg_{\underline{\alpha}}) = \sum_{k \neq i} \frac{(\mathbf{C}_{ik})_{\underline{\alpha}\underline{\beta}} fg_{\underline{\beta}}}{w_i - w_k} \tag{3.4}$$

A flat section of the bundle over X_n whose fibre over $\mathbf{w}$ is the cohomology space $H^m(Y_{\mathbf{w},m}, \chi)$ is thus given by $\sum_{\underline{\alpha}} A_{\underline{\alpha}}(fg_{\underline{\alpha}})$ where $A_{\underline{\alpha}}$ is a function of $w_1, \ldots, w_n$ such that:

$$\sum_{\underline{\alpha}} (fg_{\underline{\alpha}}) \frac{\partial A_{\underline{\alpha}}}{\partial w_i} + \sum_{\underline{\beta}} A_{\underline{\beta}} \frac{\partial}{\partial w_i}(fg_{\underline{\beta}}) \sim 0\,.$$

That is, $\frac{\partial A_{\underline{\alpha}}}{\partial w_i} + \sum_{k \neq i} \frac{(\mathbf{C}_{ik}{}^{\mathrm{T}}\mathbf{A})_{\underline{\alpha}}}{w_i - w_k} = 0$, by (3.4). This system of differential equations describes the evolution of the functions $(A_{\underline{\alpha}})$ and hence also the monodromy representation of B_n. It is to be compared with the monodromy representations obtained from conformal field theory, which are associated with the Knizhnik-Zamolodchikov equations, (2.3), in which $-\kappa^{-1}\mathbf{\Omega}_{ik}$ is to be compared with $\mathbf{C}_{ik}$. Since $(\mathbf{C}_{ik})_{\underline{\alpha}\underline{\beta}} = 0$ whenever $\underline{\alpha} \in \mathcal{S}$ and $\underline{\beta} \notin \mathcal{S}$, thus the evolution of the functions $(A_{\underline{\alpha}})$ preserves the subspace in which $A_{\underline{\alpha}} = 0$ for all $\underline{\alpha} \notin \mathcal{S}$. That is, B_n has a well-defined action upon H. In the next section it will be shown that the two representations obtained are isomorphic up to scaling when κ and a, and m and t are suitably compared.

Theorem 2 *There is a natural flat connection on the vector bundle with fibre $H \subseteq \big(H^m(Y_{\mathbf{w},m}, \chi)\big)^{S_m}$ over $\widetilde{X}_n$, and the associated monodromy representation of $B_n = \pi_1(\widetilde{X}_n)$ has dimension as in (3.2).*

For the rest of this paper we shall denote this representation by $\pi^j_{n,m}$ and the space H on which it acts by $W^j_{n,m}$. The constructions of this section should also be compared with those in [SV], where functions of the form $fg_{\underline{\alpha}}$ are shown to satisfy the Knizhnik-Zamolodchikov equations, and thus generate, as monodromy representations, the same representations as those obtained from n-point correlation functions.

4: The correspondence between the two approaches

As was remarked in the last section, if a correspondence exists between the approaches outlined in §§2,3 for producing representations of B_n, then t should be compared with $nj - m$ (see equations (2.6) and (3.2)

for the dimensions of the respective representations). Both representations are monodromy representations associated with flat connections of the following form (see Kohno [K]):

$$\sum_{i<j} \mathbf{A}_{ij}\, d\ln(z_i - z_j)$$

where the appropriate constant matrices, $\mathbf{A}_{ik}$, are $-\kappa^{-1}\mathbf{\Omega}_{ik}$ in the case of §2, and $\mathbf{C}_{ik}$ in the homology approach. The former matrices are generated from the polarisation of the Casimir operator on $V_j \otimes V_j$. The latter matrices were given in §3. Using (3.3) it is apparent that:

$$(\mathbf{C}_{ik})_{\underline{\alpha}\underline{\beta}} = \begin{cases} a(\,{}^{u_i u_k}/_{j} - u_i - u_k) & \text{if } \underline{\beta} = \underline{\alpha} \\ a u_i(1 - {}^{u_k}/_{2j}) & \text{if } \underline{\beta} = \underline{\alpha} w_k - w_i \\ a u_k(1 - {}^{u_i}/_{2j}) & \text{if } \underline{\beta} = \underline{\alpha} w_i - w_k \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Note that the notation used here requires that $\underline{\beta} = \underline{\alpha} w_i - w_k$, if and only if, the $(m+1)$-tuples $\underline{\beta} w_k$ and $\underline{\alpha} w_i$, of elements of $\{w_1, \ldots, w_n\}$ are identical up to permutations.

Consider one particular matrix $\mathbf{C}_{ik}$, for fixed i and k. For any s, t and $\underline{\beta}$ with $0 \le s, t \le 2j$; $\underline{\beta}$ being an $(m-s-t)$-tuple of elements of $\{w_1, \ldots, w_n\} \backslash \{w_i, w_k\}$ with each w_l occurring with multiplicity at most $2j$, let $v_{st}^{\underline{\beta}}$ denote the unordered m-tuple of w's consisting of the elements of $\underline{\beta}$ together with s occurrences of u_i and t occurrences of u_k. It is clear from the above description of $\mathbf{C}_{ik}$, that the space H on which it acts may be generated by subspaces, H_{st}, spanned by those elementary basis elements $\underline{\alpha} \equiv v_{st}^{\underline{\beta}}$, as $\underline{\beta}$ varies over all allowed $(m-s-t)$-tuples, with s and t fixed. Then for fixed $\underline{\beta}$, ${\mathbf{C}_{ik}}^{\mathrm{T}}$ preserves the subspace generated by the vectors $v_{st}^{\underline{\beta}}$ over values of s and t with sum d, for any fixed d with $0 \le d \le 4j$. The matrix ${\mathbf{C}_{ik}}^{\mathrm{T}}$ acts on $H_d \equiv \langle \{ H_{s,d-s} \mid 0 \le s, d-s \le 2j \} \rangle$ by:

$$H_{s,d-s} \longrightarrow H_{s-1,d-s+1} \oplus H_{s,d-s} \oplus H_{s+1,d-s-1}$$

The three spaces on the right hand side have identical dimensions and there is a natural correspondence between them, under which the vectors in $\left\{ v_{st}^{\underline{\beta}} \mid s+t=d \right\}$ with fixed d and $\underline{\beta}$, are identified with each other. Given this, the action of ${\mathbf{C}_{ik}}^{\mathrm{T}}$ can be written as an action on a subspace of $\mathbf{C}(\mathbf{Z} \times \mathbf{Z})$ specified by:

$$(s,t) \longmapsto as(1 - {}^{t}/_{2j})(s-1, t+1) \oplus at(1 - {}^{s}/_{2j})(s+1, t-1) \oplus a({}^{st}/_{j} - s - t)(s,t) \tag{4.3}$$

where $0 \le s, t \le 2j$. The multiplicities of the eigenvalues of $\mathbf{C}_{ik}$ are given by those in the above action, multiplied by a factor $\dim(H_{s,d-s})$ where d specifies the space H_d in which the associated eigenvector lies. The action of (4.3) may be more neatly rewritten as an action on homogeneous polynomials in x and y of degree d, $(0 \le d \le 4j)$ as:

$$\mathcal{L} = a(y-x)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) - \frac{a}{2j}(x-y)^2 \frac{\partial^2}{\partial x\, \partial y}$$

where (s,t) has been identified with $x^s y^t$. A change of variables, $u = x - y$ and $v = x + y$ simplifies this to:

$$\mathcal{L} = \frac{au^2}{2j}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 2au\frac{\partial}{\partial u}$$

acting on polynomials in u and v, of homogeneous degree d. It is now obvious that the eigenvalues of this action are:

$$a\nu(\nu - 1)/2j - 2a\nu \qquad \text{for } \nu = 0, 1, \ldots, \min(d, 4j-d)\,. \tag{4.4}$$

The dimension of $H_{s,d-s}$ is the number of unordered $(m-d)$-tuples, $\underline{\beta}$ taken from a set of $(n-2)$ w's, which are such that no w occurs more than $2j$ times, namely:

$$\binom{n-2}{m-d}_j - \binom{n-2}{m-d-1}_j \tag{4.5}$$

(the second term is subtracted to take into account the relations existing in H). The multiplicity of the eigenvalues of $\mathbf{C}_{ik}$ given in (4.4), for some fixed value of ν, is obtained by summing (4.5) over the allowed values of $d = \nu, \nu+1, \ldots, 4j-\nu$.

Theorem 3 *The natural action, $\pi^j_{n,m}$, of B_n on the subspace H of the S_m-invariant part of $H^m(Y_{\mathbf{w},m},\chi)$ provides a braid group representation of dimension $\binom{n}{m}_j - \binom{n}{m-1}_j$. The images $\pi^j_{n,m}(\sigma_i)$, of the standard generators have eigenvalues given in (4.4) with associated multiplicities $\binom{n-2}{m-\nu}_j - \binom{n-2}{m-4j+\nu-1}_j$.*

Put $j' = 2j-\nu$ and $t = nj-m$. We may now compare the results of Theorem 3 with those of Theorem 1. The above multiplicities can be rewritten as $\binom{n-2}{(n-2)j-t+j'}_j - \binom{n-2}{(n-2)j-t-j'-1}_j$ which should be compared with (2.7). The associated eigenvalues reduce to ${}^{a}\!/_{2}j\big(j'(j'+1) - 2j(j+1)\big) - aj$, which identify, up to suitable shifting, with the eigenvalues obtained in §2 using conformal field theory (see (2.5)).

We deduce that a correspondence does indeed exist for the construction of braid group representations, between the two approaches outlined in the preceding two sections. The comparison is given in the table below.

Homological approach	**CFT approach**
$\alpha = q^{-\,1/j}$	Spin-j
$nj - m$	Target spin t
$\nu = 2j - j'$ eigenvalue	j' component of $V_j \otimes V_j$
Eigenvalues $a\nu(\nu-1)/2j - 2a\nu$	Eigenvalues $\kappa^{-1}\big(j'(j'+1) - 2j(j+1)\big)$
Multiplicities $\binom{n-2}{m-\nu}_j - \binom{n-2}{m-4j+\nu-1}_j$	Multiplicities $\binom{n-2}{(n-2)j-t+j'}_j - \binom{n-2}{(n-2)j-t-j'-1}_j$
Natural connection on cohomology	Kohno connection
$\mathbf{C}_{ik}$	$\kappa^{-1}\mathbf{\Omega}_{ik}$
${}^{2j}\!/_{a}$ where $q = \mathbf{e}^{2\pi i a}$	κ

5: Further remarks

It was shown in §4 how a correspondence could be established between representations of B_n constructed in §2 using conformal field theory, and the homological approach. The representations produced do not factor through the Iwahori-Hecke algebra $H_n(q)$. Since the connection matrices $\kappa^{-1}\mathbf{\Omega}_{ik}$ and $\mathbf{C}_{ik}$ are conjugate, the associated representations must also be. The eigenvalues of the operators associated with the standard generators, σ_i, of the braid group B_n are thus given by:

$$\exp\big({}^{\pi}\!/_{\kappa} j'(j'+1)\big)(-1)^{j'} \tag{5.1}$$

for $j' = 0, 1, \ldots, 2j$, up to a constant factor. Hence the braid group representations obtained factor through a generalisation of the Hecke algebra. It would be interesting to investigate the representation theory of these new algebras; they are deformations of $\mathbf{C}S_n$ as q moves away from 1, although being much larger, as linear spaces.

The representation of B_n obtained from the conformal field theory approach of §2, on $V_j^{\otimes n}$, is that obtained from the solution of the Yang-Baxter equation associated with $\mathfrak{sl}_2$ and the spin-j representation. It therefore gives rise to a link invariant in the usual way, which, up to normalisation, has the form:

$$\mathrm{tr}\big(\mu^{\otimes n} \circ \pi(\beta)\big) \tag{5.2}$$

where $\beta \in B_n$ is a braid whose closure is the link concerned; and $\mu \in \mathrm{End}(V_j)$ is an *enhanced* Yang-Baxter operator (see [T]). From the discussion of §2, it is clear that π decomposes into braid group actions on $\big(V_j^{\otimes n} \otimes \check{V}_t\big)_0$, and that these subrepresentations may alternatively be obtained using the homology picture. The link invariant of (5.2) can thus be written as the linear combination of traces of the representations of §3 over all possible values of m, that is, $0 \le m \le [jn]$. It is easily seen that:

$$\sum_{m=0}^{[jn]} \left[\binom{n}{m}_j - \binom{n}{m-1}_j \right]^2 = \binom{2n}{2jn}_j - \binom{2n}{2jn-1}_j .$$

Indeed, the representation $\pi^j_{2n,2nj}$ of B_{2n} induces a representation of $B_n \times B_n \leq B_{2n}$, and as such may be decomposed as:

$$\bigoplus_{m=0}^{[jn]} \pi^j_{n,m} \otimes \pi^j_{n,m}$$

where the first factor acts on the first B_n, and similarly for the second factor. From this it is apparent that the link invariant of (5.2) can be written in the form:

$$\langle v_n \mid A \mid v_n \rangle$$

where A gives the action of $\beta \otimes \mathrm{id} \in B_{2n}$ under $\pi^j_{2n,2nj}$, and v_n defines the vector in the cohomology $H^{2nj}(Y_{2n,2nj}, \chi)$ associated with the closure operation on the braid β (see Fig 1).

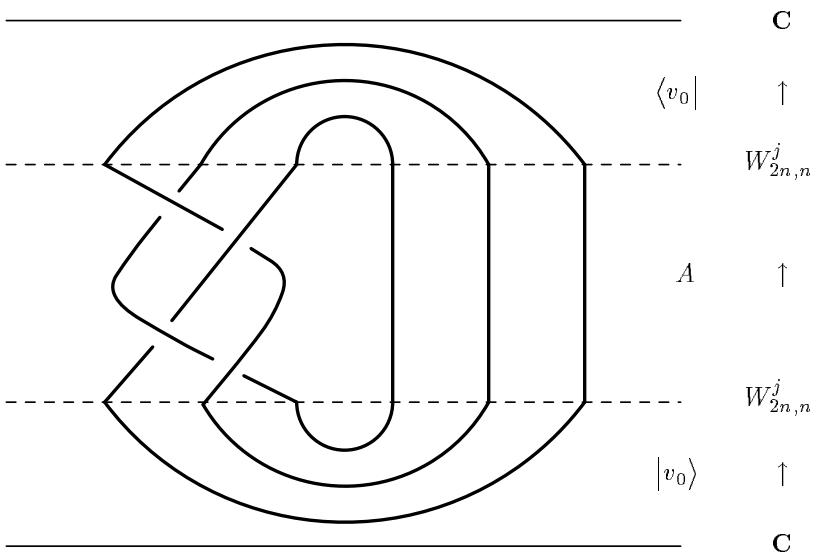

Figure 1

That is, one can produce a functorial approach to the invariant in which, for each plane with marked points $w_1, \ldots, w_{2n}$, a vector space $H^{2nj}(Y_{\mathbf{w},2nj}, \chi)$ is associated. The morphisms are defined by the natural connection, induced by homotopy invariance of homology, between objects with the same n. When n increases or decreases by one, a map:

$$H^{2nj}(Y_{\mathbf{w},2nj}, \chi) \longrightarrow H^{2(n+1)j}(Y_{\mathbf{w}w_{2n+1}w_{2n+2},2(n+1)j}, \chi')$$

is introduced in such a way that the standard relations for the algebra of tangles are satisfied. This is a generalised version of the approach of [L 2].

It is now obvious how the link invariant associated with $\mathfrak{sl}_2$ and higher representations is related to the Jones polynomial. Suppose L is a link expressed as a braid closure $\widehat{\beta}$ with $\beta \in B_n$. The one-variable Jones polynomial associated with the $2j^{\text{th}}$ cabling of β may be evaluated using the induced action of the braid in B_{4nj}, given by the $2j^{\text{th}}$-cabling of $\beta \otimes 1 \in B_{2n}$, on:

$$H^{2nj}(Y_{\mathbf{w},2nj}, \chi_1) \tag{5.3}$$

where $\mathbf{w}$ consists of $2n$ collections of clusters of $2j$ points (see [L 2]). The local coefficient system χ_1 is here defined with twistings of q and q^{-2} when z_i goes around w_j and z_k, respectively (using the usual notation). Transforming this picture so that each clump of $2j$ w-points is transformed into a single point, leaving $2n$ w's

and $2nj$ z's. Of course, this is only meaningful, if we restrict the action of B_{4nj} to that subgroup preserving the integrity of the clusters of points. The equivalent twisting, when z_i goes around one of the new w's, is clearly q^{2j}. Moreover the subspace of (5.3) used to select out the Jones polynomial is given by repeated loops, with no w mentioned more than once. In the transformed picture this gives no w mentioned more than $2j$ times (one for each point in a cluster). The invariant obtained is now in the form of that of the link invariant of L evaluated with $\mathfrak{sl}_2$ and the spin-j representation, with new κ given by $2j/a'$. Here $q' = \mathbf{e}^{2\pi i a'}$ is the q associated with the transformed model, that is, q^{2j}, and so $a' = 2ja$. Thus the two values of κ, for the spin-j evaluation on L ($\kappa' = 2j/a'$), and the spin-$1/2$ evaluation on the $2j$-cabled link $L^{(2j)}$ ($\kappa = 1/a$), are identical.

It has now been verified that:

Theorem 4 *The spin-j representation of $\mathfrak{sl}_2$ gives rise to an invariant on links, L, which may be identified with the Jones polynomial on $L^{(2j)}$, the $2j$-cabled link of L.*

This result is already well known (see [R]), although the above ties it in nicely with the homological approach. It may also be noted that $|v_n\rangle$ is naturally obtained from the vector v_{2nj} associated with the closure operation on $2nj$-braids, in the notation of [L 2], under the clumping procedure discussed above, whereby blocks of $2j$ consecutive strands are transformed into single strands.

In this paper, the case of $\mathfrak{sl}_2$ and general representations, has been discussed in detail. A similar procedure may be carried out for $\mathfrak{sl}_m$; see [L 3].

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