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A formula for topology/deformations and its significance

by

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Abstract. The formula is

$$\partial e = (\mathrm{ad}_e)b + \sum_{i=0}^{\infty} \frac{B_i}{i!}(\mathrm{ad}_e)^i(b-a),$$

with $\partial a + \frac{1}{2}[a,a] = 0$ and $\partial b + \frac{1}{2}[b,b] = 0$, where a, b and e in degrees -1, -1 and 0 are the free generators of a completed free graded Lie algebra $L[a,b,e]$. The coefficients are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n.$$

The theorem is that

- this formula for ∂ on generators extends to a derivation of square zero on $L[a,b,e]$;
- the formula for ∂e is unique satisfying the first property, once given the formulae for ∂a and ∂b, along with the condition that the "flow" generated by e moves a to b in unit time.

The immediate significance of this formula is that it computes the infinity cocommutative coalgebra structure on the chains of the closed interval. It may be derived and proved using the geometrical idea of flat connections and one-parameter groups or flows of gauge transformations. The deeper significance of such general DGLAs which want to combine deformation theory and rational homotopy theory is proposed as a research problem.

1. Introduction. This paper fits into the general framework of constructions of algebraic models of cell complexes using differential graded Lie algebras and, conversely, topological and algebraic interpretations of DGLAs. We give some background.

Let L be a free Lie algebra on a graded vector space over $\mathbb{Q}$ provided with a derivation of degree plus one (or of degree minus one).

The choice of a differential of degree -1 with L concentrated in nonnegative degrees is natural for the interpretation of these objects in topology. If X is a cell complex with one 0-cell and only 2-, 3-, 4-, ... cells, the rational

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homotopy theory of Quillen [Q] assigns a free differential Lie algebra L with one generator in degree k for each $k+1$-cell ($k>0$). The homology of L is the Whitehead Lie algebra of homotopy groups tensor $\mathbb{Q}$ shifted down by one degree. The homology of $L/[L,L]$ is the ordinary reduced homology ($\mathbb{Q}$ coefficients) of the space shifted down by one degree. One imagines that enlarging this discussion to allow cells in degree 1 would be related to some Lie algebras associated to non-trivial fundamental groups, but little is known here, to our knowledge.

The choice of a differential of degree $+1$ with L concentrated in non-negative degrees is natural for the interpretation of such differential Lie algebras controlling the deformation theory of some mathematical structure. In this case one considers elements in degree 1 satisfying $dx+\frac{1}{2}[x,x]=0$. One also makes sense of the expression $x'=dy.y^{-1}+yxy^{-1}$ (gauge transformation) for y in (some completion of) degree 0 and declares y to be an equivalence between x and x'. The set of equivalence classes is a formal version of the moduli space of the structure whose deformations are controlled by L. One imagines that enlarging the discussion with elements in degrees $-1,-2,\ldots$ would involve degree -1 elements acting as equivalences between equivalences, degree -2 elements as equivalences between equivalences between equivalences etc., but little is known, to our knowledge.

The geometric or topological interpretation of a general differential Lie algebra is a mixed object which combines both of the above discussions: homotopy theory of spaces and moduli spaces of deformations of some structure.

One knows that free Lie algebras arise from a standard ("bar") construction starting from any differential cocommutative and coassociative coalgebra over $\mathbb{Q}$. In this instance the generators are those of the coalgebra shifted down by one. The differential is the original differential extended to a derivation plus the original comultiplication extended to a derivation. The square of this derivation being zero is equivalent to coassociativity. Then one may extend the notion of a cocommutative coassociative coalgebra on a graded space to be any derivation of square zero on the free Lie algebra (possibly completed) generated by the graded space shifted down by one. This defines the notion of an infinity differential graded cocommutative, coassociative coalgebra. The higher terms beyond quadratic of the differential are chain homotopies restoring coassociativity.

Now one also knows that for any chain complex the cellular approximations to the diagonal are homotopic, any two homotopies between two of them are themselves homotopic, etc. It follows that there is on the chains an infinity cocommutative, coassociative coalgebra structure.

PROBLEM. *Study this free differential Lie algebra attached to a cell complex, determine its topological and geometric meaning as an intrinsic object.*

Give closed form formulae for the differential and for the induced maps associated to subdivisions.

In this note we will say something about the interval, the circle and the real line. We use the Maurer–Cartan idea to find an explicit formula involving Bernoulli numbers. We will see that the subdivision map corresponding to splitting an interval into two by adding a midpoint is described by the Baker–Campbell–Hausdorff formula.

Previous work. The abstract picture about the diagonal goes back to Steenrod's construction of cohomology operations mod p. The infinity coalgebra story was known for a long time by Hinich et al. In the appendix to [TZ] there is a cell-by-cell canonical construction which is not explicit. In [CG], there is an explicit sum-over-trees construction based on Chen's iterated integrals and Whitney forms but not a closed form expression.

2. Preliminaries on flows on pre-DGLAs

General DGLAs. Recall that a *DGLA* is a vector space A over a field k with grading $A = \bigoplus_{n\in\mathbb{Z}} A_n$ along with a bilinear map $[\cdot,\cdot]\colon A\times A\to A$ (bracket) and a linear map $\partial\colon A\to A$ (differential) for which $\partial^2 = 0$ while

(1) (symmetry of bracket) $[b,a] = -(-1)^{|a||b|}[a,b]$,
(2) (Jacobi identity) $[[a,b],c] = [a,[b,c]] - (-1)^{|a||b|}[b,[a,c]]$,
(3) (Leibniz rule) $\partial[a,b] = [\partial a, b] + (-1)^{|a|}[a,\partial b]$.

Note that the three properties above are valid only for homogeneous elements a, b and c of A (that is an element of $\bigcup_n A_n$), and $|a|\in\mathbb{Z}$ denotes the grading. The bracket and differential are required to respect the grading, in that for homogeneous elements, $|[a,b]| = |a|+|b|$ while $|\partial a| = |a|-1$. The adjoint action of A on itself is given by $\mathrm{ad}_e\, a = [e,a]$ and acts on the grading by $\mathrm{ad}_e : A_n\to A_{n+|e|}$. In this notation (2) and (3) can be rewritten as

(2′) (Jacobi identity) $\mathrm{ad}_{[a,b]} = [\mathrm{ad}_a, \mathrm{ad}_b]$,
(3′) (Leibniz rule) $[\partial, \mathrm{ad}_a] = \mathrm{ad}_{\partial a}$,

in which the brackets on the right-hand side refer to the (signed) commutator of operators defined by $[x,y] = xy - (-1)^{|x||y|}yx$ where the product is composition of operators, and the grading $|x|$ of a (homogeneous) operator is the shift in grading which x induces. Thus the gradings of ad_e and ∂ are $|e|$ and -1, respectively.

When the condition $\partial^2 = 0$ is removed, the resulting algebraic structure will be called a *pre-DGLA*.

Auxiliary spaces over $\mathbb{Q}[t]$. For simplicity we will work over $k=\mathbb{Q}$, though the discussion also holds for any field of characteristic 0. Assume now that A is a free Lie algebra on a finite-dimensional graded vector space V, so it has generators $x_1,\ldots,x_k$ where x_i label a basis for V. In order to deal with convergence issues which otherwise would arise, we will need to work in certain finite-dimensional quotients of A. There is an additional grading on A by the number of Lie brackets,

$$A=\bigoplus_{n=0}^{\infty}A^{(n)},$$

in which $A^{(n)}$ is the finite-dimensional vector space generated by expressions involving exactly n Lie brackets in elements of V, so that it is spanned by all words of the form

$$[x_{i_0},[x_{i_1},\ldots,[x_{i_{n-1}},x_{i_n}]\ldots]]$$

where $i_0,\ldots,i_n\in\{1,\ldots,k\}$ label (not necessarily distinct) basis elements of V, and there may be relations between them induced by (1), (2). The grading is well-defined since it is respected by (1), (2) and A is free as a Lie algebra.

For non-negative integers N, set $B^{(N)}=\bigoplus_{n=0}^{N}A^{(n)}$; it has a natural Lie algebra structure induced from A, and as such can be identified as the (graded) Lie algebra quotient in which the vanishing of all expressions involving exactly $N+1$ Lie brackets is imposed as relations (as a consequence, all expressions with more than N brackets must also vanish). Then we have a tower of Lie algebra homomorphisms

$$A\to B^{(N)}\to B^{(N-1)}\to\cdots\to B^{(0)}=A^{(0)}=V,$$

with $B^{(N)}\to B^{(N-1)}$ mapping all elements of $A^{(N)}$ to zero.

Define $U^{(N)}=\bigoplus_{n=0}^{N}(A^{(n)}\otimes\mathbb{Q}[t])$. Picking a basis $\{\mathbf{e}_{n,r}\mid 1\le r\le\dim A^{(n)}\}$ for each $A^{(n)}$, a typical element $x\in U^{(N)}$ can be written as

$$x=\sum_{n=0}^{N}\sum_{r=1}^{\dim A^{(n)}}p_{n,r}(t)\mathbf{e}_{n,r}$$

for some polynomials $p_{n,r}(t)\in\mathbb{Q}[t]$. Since such an element involves only a finite number of such polynomials, one can equivalently think of elements of $U^{(N)}$ as

$$x=\sum_{m=0}^{\infty}t^m\mathbf{x}_m$$

where only a finite number of the vectors $\mathbf{x}_m\in B^{(N)}$ are non-zero. That is, an element of $U^{(N)}$ is a formal polynomial in t with coefficients in $B^{(N)}$.

There is an obvious linear operator of differentiation by t defined on $U^{(N)}$ by

$$\frac{d}{dt}\Big(\sum_{m=0}^{M} t^m.\mathbf{x}_m\Big) = \sum_{m=0}^{M-1} t^m.(m+1)\mathbf{x}_{m+1}.$$

Differential structures over a free Lie algebra. For arbitrary elements $v_1, \dots, v_k \in A$, there is defined a unique linear map $\partial\colon A \to A$ satisfying $\partial(x_i) = v_i$ for all i, along with the Leibniz rule, that is, giving A the structure of a pre-DGLA. The condition that this defines the structure of a DGLA on A is that $\partial \circ \partial = 0$, that is, that $\partial^2(x) = 0$ for all $x \in A$. Applying the Leibniz rule twice gives $\partial^2[u,v] = [\partial^2 u, v] + [u, \partial^2 v]$, from which inductively it follows that a sufficient condition on v_i for them to generate a DGLA structure on A is that $\partial^2 x_i = 0$ for all i, that is, $\partial v_i = 0$ for all i.

From freeness of A, it can inductively be deduced that any such pre-DGLA structure has $\partial(A^{(n)}) \subset \bigoplus_{m\geq n} A^{(m)}$, so that it also induces a well-defined pre-DGLA structure on $B^{(N)}$ which will be a full DGLA structure so long as ∂^2 vanishes on A to order at least N in the Lie bracket grading.

Flatness and flows in a pre-DGLA. For any $v \in A_0$, consider the "formal" ordinary differential equation

$$\frac{du}{dt} = \partial v - \mathrm{ad}_v(u)$$

for $u \in U^{(N)}$, where both sides are considered as elements of $U^{(N)}$. Writing $u(t)$ in the form $\sum_{n=0}^{\infty}(t^n.x_n)$ where $x_n \in B^{(N)}$, we break the differential equation into the recurrence relation

$$(n+1)x_{n+1} = -\,\mathrm{ad}_v(x_n), \quad n > 0, \qquad x_1 = \partial v - \mathrm{ad}_v(x_0)$$

from which we see that ($n \geq 1$)

$$x_n = \frac{(-\mathrm{ad}_v)^{n-1}}{n!}x_1 = \frac{(-\mathrm{ad}_v)^{n-1}}{n!}(\partial v) + \frac{(-\mathrm{ad}_v)^{n}}{n!}x_0,$$

giving a unique solution for $u(t) \in U^{(N)}$ once the initial condition $x_0 = u(0) \in B^{(N)}$ is fixed. Note that $x_n = 0 \in B^{(N)}$ for all $n > N+1$, so that indeed $u(t)$ is *polynomial* in t, for every choice of initial condition.

The differential equation, and hence also the solution space, is invariant under time-translation and so is said to define the *flow* on $U^{(N)}$ generated by v. By evaluation at a given $t = t_0 \in \mathbb{Q}$, this flow defines an action of $(\mathbb{Q}, +)$ on $U^{(N)}$ by

$$t_0.x_0 \equiv u(t_0),$$

namely the action of t_0 on an element $x_0 \in U^{(N)}$ is given by "flowing according to the flow generated by v for time t_0". Observe that despite the intuition based on a continuous model and derivatives, the formal definitions are only valid at *rational* "times" t, and that this works because of the rationality of coefficients in all the expansions. Explicitly,

$$t_0.x_0 = x_0 + \sum_{n=1}^{N+1} t_0^n \left(\frac{(-\operatorname{ad}_v)^{n-1}}{n!}(\partial v) + \frac{(-\operatorname{ad}_v)^n}{n!} x_0 \right).$$

An element $x \in A_{-1}$ is said to be *flat* if $\partial x + \frac{1}{2}[x,x] = 0$. Similarly, if this equality holds up to order N brackets, then x is a flat element of $B_{-1}^{(N)}$.

3. The DGLA model for the interval. The interval consists of two points and a single 1-cell. Its model should therefore have two generators in degree -1 (corresponding to its endpoints) and a single generator in degree 0.

THEOREM 1. *There is a unique completed free differential graded Lie algebra, A, with generating elements a, b and e, in degrees -1, -1 and 0 respectively, for which a and b are flat while the flow generated by e moves from a to b in unit time. The differential is specified by*

$$\partial e = \operatorname{ad}_e b + \sum_{i=0}^{\infty} \frac{B_i}{i!} (\operatorname{ad}_e)^i (b-a),$$

where B_i denotes the ith Bernoulli number defined as coefficients in the expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Proof. Let A be the free graded Lie algebra generated by a, b and e. For any non-negative integer N, define the derived spaces $B^{(N)}$ and $U^{(N)}$ as in the previous section. We prove the result of the theorem on the truncated free Lie algebra $B^{(N)}$ for all N and see that the corresponding differentials are compatible for all N. For any $x \in B_{-1}^{(N)}$, define a map $\partial_x \colon A \to A$ by its action on the generators,

$$\partial_x a = -\tfrac{1}{2}[a,a], \qquad \partial_x b = -\tfrac{1}{2}[b,b], \qquad \partial_x e = x,$$

extended to the whole of A via linearity and the Leibniz rule. (Here ∂_x is well-defined since derivation by the Leibniz rule preserves (1) and (2).) This defines a pre-DGLA structure on $B^{(N)}$.

The flow on $B^{(N)}$ generated by $e \in U_{-1}^{(N)}$ has $du/dt = x - \operatorname{ad}_e(u)$. For the particular solution with $u(0) = a$, the solution is given as in the previous section by

$$u(t) = a + \sum_{n=1}^{N+1} t^n \left(\frac{(-\operatorname{ad}_e)^{n-1}}{n!} x + \frac{(-\operatorname{ad}_e)^n}{n!} a \right) = e^{-t.\operatorname{ad}_e} a + \frac{e^{-t.\operatorname{ad}_e} - 1}{(-\operatorname{ad}_e)} x$$

where as always the operator exponential is defined by its series expansion $e^X = \sum_{n=0}^{\infty} X^n/n!$ so that the operator acting on x is defined by its series expansion

$$\sum_{n=1}^{\infty} \frac{t^n}{n!}(-\mathrm{ad}_e)^{n+1}.$$

Observe that although there are formally infinite series in the expression for $u(t)$, as elements of $U^{(N)}$ they are finite sums. The condition that $u(1) = b$ is precisely that

$$x = \frac{(-\mathrm{ad}_e)}{e^{-\mathrm{ad}_e} - 1}(b - e^{-\mathrm{ad}_e} a) = (\mathrm{ad}_e)b + \frac{\mathrm{ad}_e}{e^{\mathrm{ad}_e} - 1}(b - a),$$

that is, the value of ∂e given in the theorem (compatible elements of $B^{(N)}$ for different N). This proves uniqueness.

It only remains to verify existence, in other words to show that the pre-DGLA structure is in fact a full DGLA structure, that is, $\partial_x^2 = 0$ for this particular value of x. From the Leibniz rule for ∂_x, it follows that for all $p, q \in B^{(N)}$, $\partial_x^2[p,q] = [\partial_x^2 p, q] + [p, \partial_x^2 q]$, so that it is only necessary to check that $\partial_x^2 = 0$ on the generators. For the generator a, we have

$$\partial_x^2(a) = \partial_x\big(-\tfrac{1}{2}[a,a]\big) = [a, \partial_x a] = \big[a, -\tfrac{1}{2}[a,a]\big] = 0,$$

the final equality following from the Jacobi identity. Similarly $\partial_x^2(b) = 0$.

To prove that $\partial_x^2(e) = 0$, consider the flow u generated by e for which $u(0) = a$ as above. By our choice of x, this flow also has $u(1) = b$. Consider the function $f(t) = \partial_x u + \frac{1}{2}[u,u]$ (the curvature), taking values in $B_{-2}^{(N)}$ at rational t; equivalently, f defines an element of $U_{-2}^{(N)}$. Its derivative is

$$\begin{aligned}\frac{df}{dt} &= \partial_x \frac{du}{dt} + \left[u, \frac{du}{dt}\right] = \partial_x(x - \mathrm{ad}_e\, u) + [u, x - \mathrm{ad}_e\, u] \\ &= \partial_x^2(e) - (\mathrm{ad}_{\partial_x e}(u) + \mathrm{ad}_e\, \partial_x u) + ([u,x] - [u, \mathrm{ad}_e\, u]) \\ &= \partial_x^2(e) - \mathrm{ad}_e(f(t))\end{aligned}$$

where we have used the Leibniz rule and the fact that $[u, \mathrm{ad}_e\, u] = \frac{1}{2}\,\mathrm{ad}_e[u,u]$ from the Jacobi identity. Thus f satisfies a first-order linear differential equation with constant (operator) coefficients of the same form as that satisfied by u where now x is replaced by $\partial_x^2 e$, while $f(0) = f(1) = 0$ (since a and b are flat). It follows that $\partial_x^2 e = 0$ in $U_{-2}^{(N)}$ (for all N), as required. ∎

Remark. From the last calculation in the proof, it can be seen that in the DGLA of Theorem 1, a flow u defined by an arbitrary element $v \in B_0^{(N)}$ on $B_{-1}^{(N)}$ has curvature $f \in U_{-2}^{(N)}$ satisfying $df/dt = -\,\mathrm{ad}_v\, f$, so that if $u(0)$ is flat (hence $f(0) = 0$) then f is identically zero. That is, the flow on A_{-1} defined by an arbitrary element of A_0 preserves flatness.

Remark. The vanishing of odd Bernoulli numbers after the first is exactly the condition to make the formula for ∂e in terms of a, b invariant under interchange of a, b accompanied by a sign change of e,

$$\partial e = (b-a) + \frac{1}{2}\,\mathrm{ad}_e(a+b) + \frac{1}{12}(\mathrm{ad}_e)^2(b-a) - \frac{1}{720}(\mathrm{ad}_e)^4(b-a) + \cdots .$$

In other words, if a flow in direction e moves a to b in unit time, then $-e$ moves b to a in unit time.

Differential geometric interpretation. We would like to push the language of 'flatness' and 'flows' used in the above formal proof a little further. The real differential geometric meaning of these constructions is however yet to be understood.

The proof of Theorem 1 used the language of curvature and flatness of connections, alluding to the interpretation of the Maurer–Cartan equation

$$\partial a = -\tfrac{1}{2}[a,a]$$

as the condition for a connection a (as given by its associated 1-form) to be flat. However, there is also a deformation theory interpretation, namely that the differential structure in a DGLA (A,∂) can be deformed by replacing ∂ by $\partial_a \equiv \partial + \mathrm{ad}_a$ on condition that the Maurer–Cartan condition is satisfied. Indeed, the deformed structure defines a DGLA so long as $\mathrm{ad}_{\partial_a x} = [\partial_a, \mathrm{ad}_x]$ and $\partial_a^2 = 0$; we calculate

$$\begin{aligned}
&\mathrm{ad}_{\partial x + [a,x]} = \mathrm{ad}_{\partial x} + \mathrm{ad}_{[a,x]} = [\partial, \mathrm{ad}_x] + [\mathrm{ad}_a, \mathrm{ad}_x] = [\partial + \mathrm{ad}_a, \mathrm{ad}_x],\\
&(\partial + \mathrm{ad}_a)^2 = \partial^2 + \partial.\mathrm{ad}_a + \mathrm{ad}_a.\partial + (\mathrm{ad}_a)^2 = [\partial, \mathrm{ad}_a] + \mathrm{ad}_{\frac{1}{2}[a,a]} = \mathrm{ad}_{\partial a + \frac{1}{2}[a,a]}
\end{aligned}$$

(by repeated applications of the Jacobi identity and the Leibniz rule), so that the first condition is automatic, while the vanishing of ∂_a^2 is guaranteed by the Maurer–Cartan equation.

So we will think of any $a \in A_{-1}$ as defining a "connection" $\partial_a = \partial + \mathrm{ad}_a$. Furthermore, any $e \in A_0$ generates a flow on $u \in A_{-1}$, which we consider as an "infinitesimal gauge transformation" flowing connections by $du/dt = \partial e - \mathrm{ad}_e\, u$; as we saw from the proof of Theorem 1, this flow preserves flatness of connections. In addition, $e \in A_0$ also defines a flow on $v \in A_0$ by $dv/dt = -\,\mathrm{ad}_e\, v$, for which

$$\begin{aligned}
\frac{d}{dt}(\partial_u v) &= \frac{d}{dt}(\partial v + \mathrm{ad}_u v) = \partial\frac{dv}{dt} + \mathrm{ad}_u\frac{dv}{dt} + \mathrm{ad}_{du/dt} v\\
&= \partial(-\mathrm{ad}_e v) + \mathrm{ad}_u(-\mathrm{ad}_e v) + \mathrm{ad}_{\partial e} v - \mathrm{ad}_{\mathrm{ad}_e u} v\\
&= -\mathrm{ad}_e \partial v - \mathrm{ad}_e \mathrm{ad}_u v = -\mathrm{ad}_e(\partial_u v),
\end{aligned}$$

so that the condition $\partial_u v = 0$ is preserved by the flow, that is, the flow e defines a "parallel transport" between fibres $\ker \partial_u \subset A_0$ over each "point" $u(t)$.

4. The DGLA model of the circle. To obtain a circle from an interval we need only identify the endpoints. So we obtain a single 0-cell a and a single 1-cell e. In our algebraic model, we impose the condition $a = b$ and immediately the differential of Theorem 1 collapses. The resulting model of the circle is a free Lie algebra with two generators a and e in degrees -1 and 0, respectively, and differential ∂:

$$\partial a = -\tfrac{1}{2}[a, a], \qquad \partial e = \operatorname{ad}_e a.$$

The twisted differential over the point a would then be just $\partial_a = \partial + \operatorname{ad}_a$ and here $\partial_a e = 0$ so that the localization to a point (namely to a) would be just generated by e in grading 0; this corresponds to the single generating loop e.

So our model now relates points to flat connections; in particular, the endpoints of the interval correspond to a and b, while interior rational points t_0 give other flat elements $u(t_0)$. On the other hand, the 1-cell is represented by e, which defines an infinitesimal gauge transformation flowing between all these (rational) points (flat connections). Furthermore, as will be discussed in a further paper, this model can be extended to higher dimensions, and then it will become apparent that the algebraic analogue of localization to a point a (corresponding to consideration of the loop space based at that point) is the replacement of the whole complex with differential ∂ by the complex truncated to non-negative degrees with degree 0 part restricted to $\ker \partial_a$ and with differential ∂_a. A path connecting points then induces a flow preserving the complexes.

5. Gluing intervals. Suppose X is any 1-complex. Using Theorem 1, we can construct its DGLA model as a free Lie algebra on generators a_i in degree -1 for each 0-cell, and e_i in degree 0 for each 1-cell, along with a differential ∂ which is uniquely defined by its action on the generators making a_i flat, and giving ∂e_i by the corresponding formula from Theorem 1 in which a and b are replaced by the algebra elements associated with the endpoints of the interval labelled by e_i.

For example, a 1-complex consisting of two adjoining intervals, $[a_0, a_1]$ and $[a_1, a_2]$ with corresponding 1-cells e_1 and e_2, is modelled by the DGLA B, which, as a Lie algebra, is free on generators a_0, a_1, a_2 in degree -1 and e_1, e_2 in degree 0. The differential ∂ has

$$\partial a_0 = -\tfrac{1}{2}[a_0, a_0], \qquad \partial a_1 = -\tfrac{1}{2}[a_1, a_1], \qquad \partial a_2 = -\tfrac{1}{2}[a_2, a_2],$$

$$\partial e_1 = \operatorname{ad}_{e_1} a_1 + \sum_{i=0}^{\infty} \frac{B_i}{i!} (\operatorname{ad}_{e_1})^i (a_1 - a_0),$$

$$\partial e_2 = \operatorname{ad}_{e_2} a_2 + \sum_{i=0}^{\infty} \frac{B_i}{i!} (\operatorname{ad}_{e_2})^i (a_2 - a_1).$$

The geometric model is a subdivided interval. The previous discussion defines a flow on B_{-1}; the flow according to e_1 flows from a_0 to a_1 in unit time, and then the flow according to e_2 flows from a_1 to a_2 in unit time. That is, to any rational point p on either interval there corresponds a flat element $u_p \in A_{-1}$.

Removing the interior point a_1 would give a single interval $[a_0, a_2]$ with 1-cell e whose DGLA model, A, is a free Lie algebra on generators a_0, a_2 in degree -1 and e in degree 0, along with the differential ∂ defined by

$$\partial a_0 = -\tfrac{1}{2}[a_0, a_0], \qquad \partial a_2 = -\tfrac{1}{2}[a_2, a_2],$$
$$\partial e = \mathrm{ad}_e\, a_2 + \sum_{i=0}^{\infty} \frac{B_i}{i!}(\mathrm{ad}_e)^i (a_2 - a_0).$$

To parallel the geometric fact that the glued pair of intervals is just a subdivision of a single interval, we have the following theorem.

Theorem 2. *There is homomorphism $p\colon A \to B$ respecting the DGLA structure, for which $p(a_0) = a_0$, $p(a_2) = a_2$ while $p(e)$ is given by the Baker–Campbell–Hausdorff formula on e_1 and e_2,*

$$p(e) = e_1 + e_2 + \tfrac{1}{2}[e_1, e_2] + \tfrac{1}{12}[e_1, [e_1, e_2]] - \tfrac{1}{12}[e_2, [e_1, e_2]] + \cdots .$$

Proof. Denote by $\mathrm{BCH}(e_1, e_2)$ the Baker–Campbell–Hausdorff formula on e_1 and e_2. By the Jacobi identity, it follows that as operators, $\mathrm{ad}_{\mathrm{BCH}(e_1,e_2)} = \mathrm{BCH}(\mathrm{ad}_{e_1}, \mathrm{ad}_{e_2})$ and hence that (again as operators) $e^{\mathrm{ad}_{\mathrm{BCH}(e_1,e_2)}} = e^{\mathrm{ad}_{e_1}} e^{\mathrm{ad}_{e_2}}$.

By the Leibniz rule, to prove that p is a DGLA homomorphism, it is enough to check compatibility of the action of the differential on generators, that is, $p(\partial e) = \partial(p(e))$. As we saw in the proof of Theorem 1, ∂e_1, ∂e_2 satisfy

$$a_1 = e^{-\mathrm{ad}_{e_1}} a_0 + \frac{e^{-\mathrm{ad}_{e_1}} - 1}{-\mathrm{ad}_{e_1}}(\partial e_1), \qquad a_2 = e^{-\mathrm{ad}_{e_2}} a_1 + \frac{e^{-\mathrm{ad}_{e_2}} - 1}{-\mathrm{ad}_{e_2}}(\partial e_2).$$

Substituting the first equation into the second gives the identity in B:

$$a_2 = e^{-\mathrm{ad}_{e_2}} e^{-\mathrm{ad}_{e_1}} a_0 + e^{-\mathrm{ad}_{e_2}} \frac{e^{-\mathrm{ad}_{e_1}} - 1}{-\mathrm{ad}_{e_1}}(\partial e_1) + \frac{e^{-\mathrm{ad}_{e_2}} - 1}{-\mathrm{ad}_{e_2}}(\partial e_2).$$

On comparison with the identity $a_2 = e^{-\mathrm{ad}_e} a_0 + \frac{e^{-\mathrm{ad}_e} - 1}{-\mathrm{ad}_e}(\partial e)$ in A, and recalling from above that $e^{-\mathrm{ad}_{p(e)}} = e^{-\mathrm{ad}_{e_2}} e^{-\mathrm{ad}_{e_1}}$, the theorem now follows from Lemma 3 in the Appendix. ∎

Remark. According to the previous discussion, e_1 induces a flow on B_{-1} (points/connections) and also on B_0. If we think of B_0 as being in the fibre over the corresponding point, the flow on $v \in B_0$ is defined by the differential equation $dv/dt = -\mathrm{ad}_{e_1} v$. This is a homogeneous linear differential equation, and so its solution is $v(t) = e^{-t.\,\mathrm{ad}_{e_1}}v(0)$, so that the twisting in the fibre as we move along the edge e_1 from a_0 to a_1 is $e^{-\,\mathrm{ad}_{e_1}}$ and a 'flat section' has values v_0, v_1 over the endpoints of the interval $[a_0, a_1]$ related by $v_1 = e^{-\,\mathrm{ad}_{e_1}}v_0$. Similarly e_2 induces a flow from a_1 to a_2 as well as on the corresponding "fibres" and in composition

$$v_2 = e^{-\,\mathrm{ad}_{e_2}}v_1 = e^{-\,\mathrm{ad}_{e_2}}e^{-\,\mathrm{ad}_{e_1}}v_0 = e^{-\mathrm{ad}_{p(e)}}v_0.$$

Thus Theorem 2 implies that not only is the map $A \to B$ a DGLA homomorphism, but it is also compatible with the induced "flow" structures discussed previously.

6. Appendix: Some algebraic lemmas. We here give the proofs for three lemmas used in the previous section.

LEMMA 1. *Let e and v be arbitrary elements of a DGLA with e of degree* 0. *Then the following formula holds, where E denotes the operator* ad_e*:*

$$\partial(e^{-E}v) = e^{-E}(\partial v) + (-1)^{|v|}e^{-E}\,\mathrm{ad}_v\,\frac{e^E - 1}{E}(\partial e).$$

Proof. By the definition of e^{-E},

$$\partial(e^{-\,\mathrm{ad}_e}v) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial\big((\mathrm{ad}_e)^n v\big).$$

However

$$\begin{aligned}\partial((\mathrm{ad}_e)^n v) &= \partial([e,[e,\cdots[e,v]\cdots]])\\ &= [\partial e,[e,\cdots[e,v]\cdots]] + [e,[\partial e,\cdots[e,v]\cdots]] + \cdots\\ &\qquad + [e,[e,\cdots[\partial e,v]\cdots]] + [e,[e,\cdots[e,\partial v]\cdots]]\\ &= (-1)^{|v|+1}\big(\,\mathrm{ad}_{(\mathrm{ad}_e)^{n-1}v}(\partial e) + E.\,\mathrm{ad}_{(\mathrm{ad}_e)^{n-2}v}(\partial e) + \cdots\\ &\qquad\qquad + E^{n-1}\,\mathrm{ad}_v(\partial e)\big) + E^n(\partial v).\end{aligned}$$

By the Jacobi identity, the operator $\mathrm{ad}_{(\mathrm{ad}_e)^m v} = \mathrm{ad}_{[e,\cdots[e,v]\cdots]}$ can be equivalently written as the repeated commutator of operators $[\mathrm{ad}_e,\cdots[\mathrm{ad}_e,\mathrm{ad}_v]\cdots]$ $= [E,\cdots[E,V]\cdots]$ where $V \equiv \mathrm{ad}_v$. Since e is of degree zero, the grading on the operator E is also zero and so these graded operator commutators are all with standard sign, $[A,B] = AB - BA$. Inductively one obtains

$$\mathrm{ad}_{E^m v} = (\mathrm{ad}_E)^m V = \sum_{k=0}^{m}(-1)^k\binom{m}{k}E^{m-k}VE^k.$$

Substituting into the above expression we get

$$\partial((\mathrm{ad}_e)^n v) = (-1)^{|v|+1}\Big(\sum_{m=0}^{n-1}E^{n-m-1}\,\mathrm{ad}_{E^m v}\Big)(\partial e) + E^n(\partial v)$$

$$= (-1)^{|v|+1}\Big(\sum_{m=0}^{n-1}\sum_{k=0}^{m}(-1)^k\binom{m}{k}E^{n-k-1}VE^k\Big)(\partial e) + E^n(\partial v)$$
$$= (-1)^{|v|+1}\Big(\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}E^{n-k-1}VE^k\Big)(\partial e) + E^n(\partial v).$$

Combining over all n leads to

$$\partial(e^{-\operatorname{ad}_e}v)$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big((-1)^{|v|+1}\Big(\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}E^{n-k-1}VE^k\Big)(\partial e) + E^n(\partial v)\Big)$$
$$= (-1)^{|v|}\Big(\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}}{n!}\binom{n}{k+1}E^{n-k-1}VE^k\Big)(\partial e) + e^{-E}(\partial v)$$
$$= (-1)^{|v|}e^{-E}V\frac{e^E-1}{E}(\partial e) + e^{-E}(\partial v). \ \blacksquare$$

LEMMA 2. *Let e and v be arbitrary elements of a DGLA with e of degree 0. Then* $\operatorname{ad}_{e^{-E}v} = e^{-E}Ve^E$ *where* $E \equiv \operatorname{ad}_e$, $V \equiv \operatorname{ad}_v$.

Proof. From the expression for $\operatorname{ad}_{E^m v}$ obtained in the proof of Lemma 1,

$$\operatorname{ad}_{e^{-E}v} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\operatorname{ad}_{E^n v}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n+k}}{n!}\binom{n}{k}E^{n-k}VE^k = e^{-E}Ve^E. \ \blacksquare$$

LEMMA 3. *If e_1 and e_2 are elements of an arbitrary DGLA with* $f = \mathrm{BCH}(e_1,e_2)$, *then*

$$\frac{e^{-\operatorname{ad}_f}-1}{-\operatorname{ad}_f}(\partial f) = e^{-\operatorname{ad}_{e_2}}\frac{e^{-\operatorname{ad}_{e_1}}-1}{-\operatorname{ad}_{e_1}}(\partial e_1) + \frac{e^{-\operatorname{ad}_{e_2}}-1}{-\operatorname{ad}_{e_2}}(\partial e_2).$$

Proof. We will derive this identity as a compatibility condition. It follows from Lemma 1, directly from the definition of the exponential and the Leibniz rule, that for arbitrary elements e, v of a DGLA in which e has degree 0,

$$\partial(e^{-E}v) = e^{-E}(\partial v) + (-1)^{|v|}e^{-E}\operatorname{ad}_v\frac{e^E-1}{E}(\partial e),$$

where E denotes the operator ad_e. Applying this to evaluate $\partial(e^{-\operatorname{ad}_f}v)$ in two different ways, one directly from this formula, and the other as $\partial(e^{-\operatorname{ad}_{e_2}}e^{-\operatorname{ad}_{e_1}}v)$ by applying the formula twice, and equating the results, shows that for all Lie algebra elements v, there is an identity

$$e^{-F}\operatorname{ad}_v \frac{e^F-1}{F}(\partial f) = e^{-E_2}e^{-E_1}\operatorname{ad}_v \frac{e^{E_1}-1}{E_1}(\partial e_1) + e^{-E_2}\operatorname{ad}_{e^{-E_1}v}\frac{e^{E_2}-1}{E_2}(\partial e_2),$$

where again we have denoted the operators ad_{e_1}, ad_{e_2} and ad_f by E_1, E_2 and F, respectively. Furthermore, by Lemma 2, $\operatorname{ad}_{e^{-E_1}v} = e^{-E_1}\operatorname{ad}_v e^{E_1}$, while $e^{-F} = e^{-E_2}e^{-E_1}$ is an invertible operator, so that the relation becomes

$$\operatorname{ad}_v \frac{e^F-1}{F}(\partial f) = \operatorname{ad}_v \frac{e^{E_1}-1}{E_1}(\partial e_1) + \operatorname{ad}_v e^{E_1}\frac{e^{E_2}-1}{E_2}(\partial e_2).$$

Since this holds in all DGLAs, it holds in particular for the free DGLA on three generators e_1, e_2, v (all in degree 0), and we can therefore remove the ad_v operator from all terms, leaving an equality, which when multiplied by $e^{-F} = e^{-E_2}e^{-E_1}$ on the left, is the required identity. ∎

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The authors worked out in Israel (2001) the proof that the formula for ∂e that takes the flat element a to the flat element b by the flow associated to the infinitesimal gauge group action leads to a derivation of square zero, but its publication was somewhat delayed by the addition to our respective families of (three, one) wonderful children! The first author would like to thank Stony Brook for its hospitality during her visit in 2010, where the continuation of the story from an interval through 1-complexes to more general complexes was pursued. The authors would like to thank the referees for useful comments.

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