

On Habiro's cyclotomic expansions of the Ohtsuki invariant

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Abstract

We give a self-contained treatment of Le and Habiro's approach to the Jones function of a knot and Habiro's cyclotomic form of the Ohtsuki invariant for manifolds obtained by surgery around a knot. On the way we reproduce a state sum formula of Garoufalidis and Le for the colored Jones function of a knot. As a corollary, we obtain bounds on the growth of coefficients in the Ohtsuki series for manifolds obtained by surgery around a knot, which support the slope conjecture of Jacoby and the first author.

1 Introduction

Suppose that M is a compact oriented 3-manifold without boundary. For any Lie algebra, g , and integral level, k , there is defined an invariant, $Z_{k+\check{c}_g}(M, L)$, of embeddings of links L in M , known as the Witten-Reshetikhin-Turaev invariant (see [27], [24]). It is known that for links in S^3 , $Z_K(S^3, L)$ is a polynomial in $q = \exp \frac{2\pi i}{K}$, namely the generalized Jones polynomial of the link L . In this paper we consider only the case $g = su(2)$.

Now assume that M is a rational homology sphere, with $H = |H_1(M, \mathbf{Z})|$. In the normalization for which the invariant for S^3 is 1, denote the invariant for the pair (M, \emptyset) , as an algebraic function of q at K^{th} roots of unity, by $Z_K(M)$. For a rational homology sphere M and odd prime K , [21] showed that $Z_K(M) \in \mathbf{Z}[q]$, so that for some $a_{m,K}(M) \in \mathbf{Z}$, one has

$$Z_K(M) = \sum_{m=0}^{\infty} a_{m,K}(M)(q-1)^m,$$

the representation being determined up to the addition of elements of the ideal generated by the K^{th} cyclotomic polynomial, $\phi_K(q)$. Although the $a_{m,K}$ are not uniquely determined, it is known from [22] that there exist rational numbers $\lambda_m(M)$ such that,

$$a_{m,K}(M) \equiv \lambda_m(M)$$

as elements of $\mathbf{Z}/K\mathbf{Z}$ for all sufficiently large primes K . For integer homology spheres, $\lambda_0(M) = 1$ and $\lambda_1(M) = 6\lambda(M)$ where $\lambda(M)$ denotes the Casson–Walker invariant of M in Casson’s normalization. As a result, one may define a formal power series

$$Z_\infty(M) = \sum_{m=0}^{\infty} \lambda_m(q-1)^m,$$

with rational coefficients, which is an invariant of rational homology 3–spheres, M , known as the *Ohtsuki series* of M . This is the asymptotic expansion of the trivial connection contribution to $Z_K(M)$ (see [25]). Work of Habiro (see [6], [7]) showed that $Z_\infty(M)$ can always be written in the form

$$Z_\infty(M) = \sum_{n=0}^{\infty} a_n(q)(q-1)^n \{n\}!$$

where $a_n(q) \in \mathbf{Z}[q, q^{-1}]$ and $\{n\}!$ is a q -factorial (see §2.1). This expression has the advantage that at a root of unity it reduces to a finite sum (polynomial in $q^{\pm 1}$), while it also can be considered as an element of $\mathbf{Z}[[q-1]]$.

In this paper we review the technique used by Le [17] and Habiro [7], using the R -matrix presentation of link and manifold invariants from [23], [24], to compute the colored Jones function for a knot and hence the Ohtsuki series for integer homology spheres obtained by surgery around a knot. We carry out the quantum group calculation in general, obtaining the colored Jones polynomial of a knot as a state sum directly in terms of q -numbers as in [3] (though here it is written for arbitrary knot diagrams, not just for braid closures). For arbitrary integer homology spheres the result would be similar, but the knot case is simpler, since then the Gauss diagram involves only one circle.

An outline of the present paper is as follows. In §2, basic notation is defined relating to q -numbers along with a summary of facts needed concerning the quantum group $U_q sl_2$. In §3, we follow [24] and Le to give a description of the sl_2 colored Jones polynomial of a knot, first in terms of R -matrices, then using the universal invariant as an algebra element and finally evaluating to get the state sum formula (Theorem 1) for the colored Jones function based on a Gauss diagram description of the knot. In §4, following Le and Habiro, the transformation from the colored Jones function of a knot to the Ohtsuki series of the 3-manifolds obtained by surgery around the knot, is

discussed and combining with Theorem 1 gives Habiro's cyclotomic form for the Ohtsuki series. This formulation is used to give bounds on the growth of coefficients in the Ohtsuki series, which support the slope conjecture of [9]. Finally in §5, the application of the formulae to the trefoil and figure-8 knots are demonstrated.

2 q -numbers and $U_q sl_2$

2.1 q -numbers

Let $q = v^2 = e^{\hbar}$ be a formal parameter. Define q -numbers, q -factorials and q -binomial coefficients according to

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]! = \prod_{i=1}^n [i], \quad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]}.$$

The dependence on q will be omitted from the notation. We will also define asymmetric q -numbers, q -factorials and q -binomial coefficients according to

$$\{n\} = \frac{q^n - 1}{q - 1}, \quad \{n\}! = \prod_{i=1}^n \{i\}, \quad \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{\{n\}!}{\{m\}!\{n-m\}}.$$

There is an inductive relation as for ordinary binomial coefficients

$$\begin{bmatrix} n+1 \\ r+1 \end{bmatrix} = v^{r-n} \begin{bmatrix} n \\ r \end{bmatrix} + v^{r+1} \begin{bmatrix} n \\ r+1 \end{bmatrix}, \quad \left\{ \begin{matrix} n+1 \\ r+1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ r \end{matrix} \right\} + q^{r+1} \left\{ \begin{matrix} n \\ r+1 \end{matrix} \right\}$$

from which it follows that the asymmetric q -numbers, q -factorials and q -binomial coefficients all lie in $\mathbf{Z}[q]$. Also, even when x is not an integer, say an element of an algebra, we can make sense of quantum binomial coefficients of the form

$$\begin{bmatrix} x \\ m \end{bmatrix} = \prod_{i=1}^m \frac{[x-i+1]}{[i]}, \quad \left\{ \begin{matrix} x \\ m \end{matrix} \right\} = \prod_{i=1}^m \frac{\{x-i+1\}}{\{i\}},$$

where $m \in \mathbf{N}$. The two sets of q -numbers are related by

$$\{n\} = v^{n-1} [n], \quad \{n\}! = v^{\frac{1}{2}n(n-1)} [n]!, \quad \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = v^{m(n-m)} \begin{bmatrix} n \\ m \end{bmatrix}.$$

We will also denote by $\{n\}_-$, the asymmetric q -number obtained from $\{n\}$ by replacing q by q^{-1} , and by $\{n\}_-!$ the corresponding q -factorial. Thus $\{n\}_- = q^{1-n} \{n\}$ and $\{n\}_-! = q^{-\frac{1}{2}n(n-1)} \{n\}!$.

2.2 The quantum group $U_q sl(2)$

The quantum group $A = U_q sl(2)$ is defined by generators H , X and Y with relations

$$[X, Y] = [H], \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

Set $K = e^{\hbar H/4} = v^{H/2}$. The comultiplication $\Delta: A \rightarrow A \otimes A$ is given by

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X) = X \otimes K + K^{-1} \otimes X, \quad \Delta(Y) = Y \otimes K + K^{-1} \otimes Y,$$

with antipode $S: A \rightarrow A$ being an antihomomorphism acting on the generators by

$$S(H) = -H, \quad S(X) = -vX, \quad S(Y) = -v^{-1}Y.$$

Additionally A is a quasi-triangular Hopf algebra with universal R -matrix, $R \in A \otimes A$ satisfying the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. A formula for R (see [2]) is

$$R = \sum_{l=0}^{\infty} \frac{(1 - q^{-1})^l}{\{l\}!} q^{\frac{1}{4}H \otimes H} K^l X^l \otimes K^{-l} Y^l = \sum_{n,l=0}^{\infty} \frac{(1 - q^{-1})^l (\hbar/4)^n}{\{l\}! n!} H^n K^l X^l \otimes H^n K^{-l} Y^l$$

Write this as $R = \sum_i \alpha_i \otimes \beta_i$ (thus the suffix i refers to a pair (n, l) of non-negative integers). As in any quasi-triangular Hopf algebra, $R^{-1} = (\text{Id} \otimes S^{-1})R$,

$$R^{-1} = \sum_{n,l=0}^{\infty} \frac{(1 - q) q^{\frac{1}{2}l(l-1)} (-\hbar/4)^n}{\{l\}! n!} H^n K^{-l} X^l \otimes H^n K^l Y^l.$$

We will write these two formulae together for $\sigma = \pm 1$ as

$$R^\sigma = \sum_i \alpha_i^{(\sigma)} \otimes \beta_i^{(\sigma)} = \sum_{n,l=0}^{\infty} \frac{(1 - q^{-\sigma})^l (\sigma \hbar/4)^n}{\{l\}_\sigma! n!} H^n K^{\sigma l} X^l \otimes H^n K^{-\sigma l} Y^l$$

The square of the antipode is then given by

$$S^2(a) = uau^{-1}, \quad \text{for all } a \in A,$$

where $u = m(S \otimes \text{Id})R_{21}$ and $uS(u)$ is central. Furthermore, A is a ribbon Hopf algebra, that is there is a central (ribbon) element v such that

$$v^2 = uS(u), \quad S(v) = v, \quad \epsilon(v) = 1, \quad \Delta(v) = (R_{12}R_{21})^{-1}(v \otimes v).$$

In our case, $K^2 = v^{-1}u$. This special (charmed) element satisfies the property $S^2(a) = K^2 a K^{-2} \quad \forall a \in A$, while

$$f \equiv \sum_i \alpha_i K^{-2} \beta_i = \sum_i \beta_i K^2 \alpha_i = \sum_{l=0}^{\infty} \frac{(q-1)^l}{\{l\}!} q^{l^2/2} e^{hH^2/4} K^{-2-2l} X^l Y^l .$$

lies in the center of A , and in fact $f = v^{-1}$.

2.3 Finite dimensional $U_q sl(2)$ -modules

For each $\mu \in \mathbf{N}$, there is a μ -dimensional module Λ_μ with basis $\{v_i^{(\mu)}\}_{i=0}^{\mu-1}$. The quantum group action is described by the lowest weight vector $v_0^{(\mu)}$ for which (we omit the superscripts μ)

$$H(v_0) = (1 - \mu)v_0, \quad Y(v_0) = 0, \quad X^i(v_0) = v_i .$$

Since $[X^i, Y] = [i][H - i + 1]X^{i-1}$, the whole action is given by

$$H(v_i) = (2i + 1 - \mu)v_i, \quad X(v_i) = v_{i+1}, \quad Y(v_i) = [i][\mu - i]v_{i-1} .$$

Since $f \in Z(A)$, it acts as multiplication by a scalar in any irreducible representation. In particular, in Λ_μ , f acts by scaling by

$$f_\mu = e^{h(\mu-1)^2/4} v^{\mu-1} = v^{\frac{\mu^2-1}{2}},$$

as can be verified by direct evaluation on v_0 , where H and K act as $1 - \mu$ and $v^{\frac{1-\mu}{2}}$ respectively.

3 The Jones function of a knot

3.1 Functorial description

Suppose L is an oriented framed link in S^3 . The *generalized colored Jones polynomial* [23] of L is defined when each component L_i is colored by a representation V_i of A and will be denoted by $J_L(V_1, \dots, V_c)$. We will assume for simplicity that V_i are irreducible. Indeed, according to [23], there is a functor, \mathcal{F} , from the category of (colored) ribbon tangles to the category of vector spaces, under which links (closed tangles) map to scalars, namely J_L . A slice

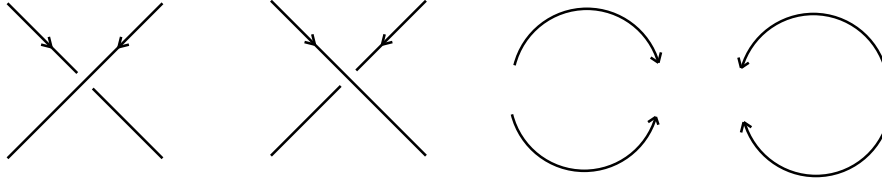


Figure 1: Generators of the category of tangles

of a colored (oriented) ribbon tangle (object in the category) is an ordered list of colors V_1, \dots, V_r with orientations $\epsilon_1, \dots, \epsilon_r$ and is mapped under the functor \mathcal{F} to $V_1^{\epsilon_1} \otimes \dots \otimes V_r^{\epsilon_r}$, where V^- denotes the dual representation V^* . This functor is defined by its images on the generators shown in Figure 1. In our conventions, downward oriented strands are counted positively in objects, while tangle morphisms are composed considering them from bottom to top.

The two orientations of crossing, in which the strands are labeled with representations V and W as shown, have as images the maps $V \otimes W \rightarrow W \otimes V$ given by $P \circ R$ and $R^{-1} \circ P$, respectively, where P is the permutation of the factors. The four cup and cap sections transform to maps

$$\begin{array}{ll} V^* \otimes V & \longrightarrow \mathbf{C} \\ (x, y) & \longmapsto x(y) \end{array} \qquad \begin{array}{ll} V \otimes V^* & \longrightarrow \mathbf{C} \\ (y, x) & \longmapsto x(K^2 y) \end{array}$$

and

$$\begin{array}{ll} \mathbf{C} & \longrightarrow V \otimes V^* \\ 1 & \longmapsto \sum_i e_i \otimes e^i \end{array} \qquad \begin{array}{ll} \mathbf{C} & \longrightarrow V^* \otimes V \\ 1 & \longmapsto \sum_i K^{-2} e^i \otimes e_i \end{array}$$

respectively. Here $\{e^i\}$ is the dual basis for V^* to the basis $\{e_i\}$ for V .

Suppose that L has one component (knot) and T is a 1-tangle presentation of L , that is $T \in \text{Morph}(a, a)$ (where a is the object consisting of one downward oriented point) such that its closure with blackboard framing is ambient isotopic to L . Then the above prescription can be used to compute $\mathcal{F}(T)$ (when a and T are colored by V) as a map $V \rightarrow V$, for any representation V (the color of the one open strand). This map commutes with

the action of A , so that if V is irreducible it is given by multiplication by a scalar, namely

$$J'_L(\mu) = \frac{J_L(\Lambda_\mu)}{J_U(\Lambda_\mu)},$$

where U is the unknot with framing zero. For U we have

$$J_U(\Lambda_\mu) = \text{tr}_{\Lambda_\mu}(K^2) = \text{tr}_{\Lambda_\mu}(K^{-2}) = v^{\mu-1} + v^{\mu-3} + \dots + v^{1-\mu} = [\mu].$$

By construction, J'_L is multiplicative under connect sum on L (corresponding to composition of tangles T). In particular, the algebra element corresponding to a trivial 1-tangle with framing 1 is f , so that J'_L changes by a factor $(f_\mu)^t$ under a framing change of t in L .

3.2 Universal $sl(2)$ formulation of Jones polynomial of a knot

The above functorial prescription for J'_L when L is a knot expressed as the closure of a 1-tangle T , may be rewritten algebraically as follows. See [14], [15] and [12] for very similar presentations. First assume that T is presented as a tangle diagram in generic position, in which the crossings are between downward oriented strands (this is always possible). Recall that $R^\sigma = \sum_i \alpha_i^{(\sigma)} \otimes \beta_i^{(\sigma)}$ where the sum is over $i \in I$ an indexing set (in this case, pairs of non-negative integers (n, l)). Place an element of I at each crossing. In the neighborhood of each crossing decorate the crossing strands by elements of A according to Figure 2; α_i^σ on the overcrossing arc and β_i^σ on the undercrossing arc, where σ denotes the sign of the crossing (+ for a slash crossing and $-$ for a backslash crossing). At local maxima/minima oriented leftwards decorate the strand by K^2 , K^{-2} respectively; rightward oriented cups and caps receive no decoration.

Then $\mathcal{F}(T)$ is the evaluation in the chosen representation of the algebra element read off the diagram by tracing the tangle strand according to its orientation, and writing down the decorations in the form of a product from left to right, and then summing over all labels i at crossings. In fact, in the correct quotient of A , the algebra element itself is an invariant (this is the universal $U_h sl(2)$ invariant); however we do not need this here.

In a general tangle diagram, crossings need not always occur only between downward oriented strands. However, other crossings can always be redrawn in terms of such downward oriented crossings, at the cost of introducing extra

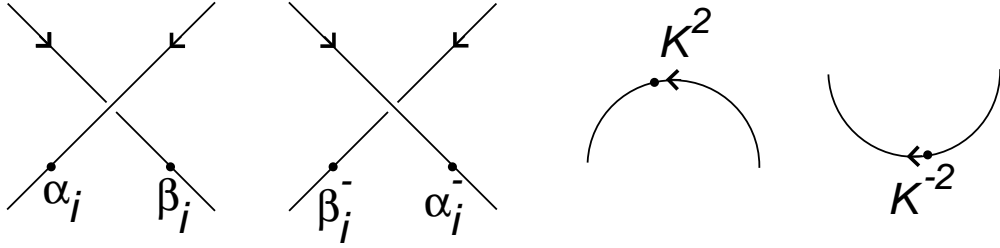


Figure 2: Decorations on 1-tangle elements

caps and cups, by “internally rotating” the crossing. For example a crossing of two upward oriented strands can be obtained by “internally rotating” a downward oriented crossing through π , either clockwise or anticlockwise and consistency of the results follows from the fact that $(S^2 \otimes S^2)R = R$ along with $S^2(a) = K^2 a K^{-2} \forall a \in A$.

There are altogether eight crossing types given by the orientations on the two strands as well the sign of the crossing itself. The result is that for an arbitrarily oriented crossing, the undercrossing arc is decorated with $\beta_i^{(\sigma)}$ while the overcrossing arc is decorated with $\alpha_i^{(\sigma)}$ *except* in the case of a left pointing crossing in which case it is decorated by $S^{2\sigma}(\alpha_i^{(\sigma)})$.

3.3 Gauss diagram formulation of Jones function of a knot

We will now proceed to carry out the above prescription at the quantum group level for an arbitrary 1-tangle, producing a prescription for $J'_L(\mu)$ as a state sum of scalar quantities.

The combinatorial data involved in specifying a knot L , presented as a 1-tangle closure can be contained in its Gauss diagram. That is, consider the knot as parametrised by an (oriented) circle. Each crossing in L corresponds to exactly two points on the circle, one for the underpass and the other for the overpass. Encode this by joining the points on the circle by an oriented chord from overpass to underpass, marked by a sign to denote the sign of the crossing. The resulting diagram is known as the *Gauss diagram* of L ; it is an

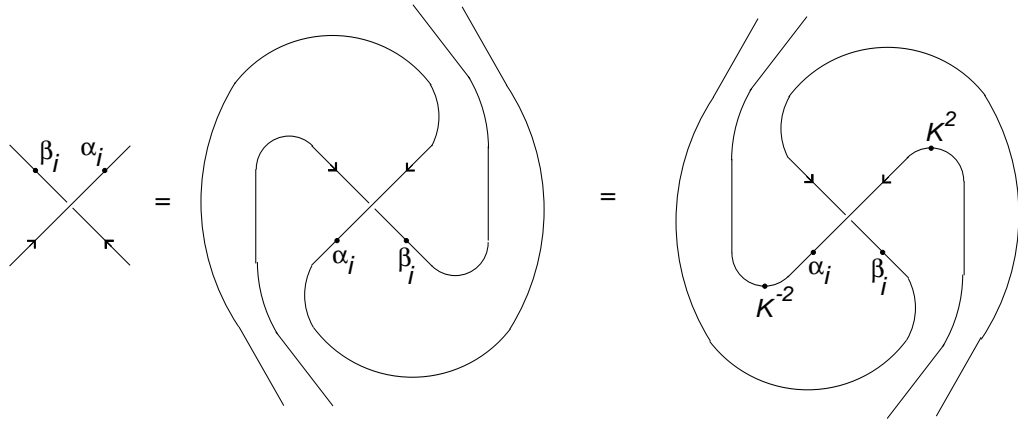


Figure 3: Internal rotation of a crossing

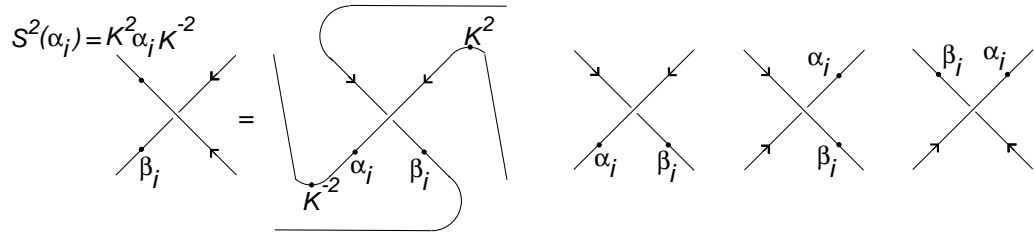


Figure 4: Decorations on all orientations of a positive crossing

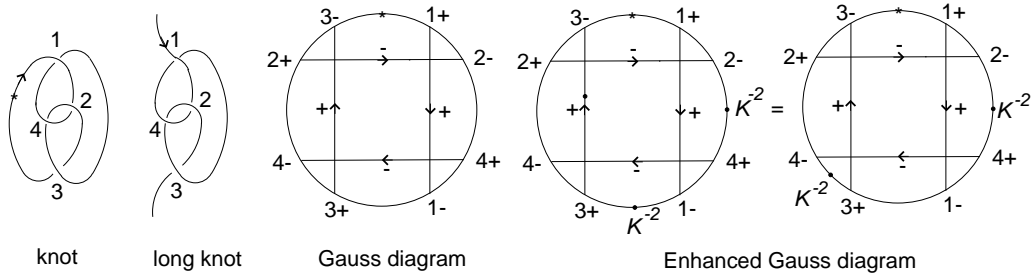


Figure 5: Gauss diagrams and enhanced Gauss diagrams

oriented signed chord diagram on an oriented circle. This is now sufficient data to encode the knot.

Unfortunately the above description of the algorithm for $J'_L(\mu)$ requires also a tangle presentation with left pointing crossings/cups/caps marked. This requires that the circle be marked with a basepoint as well as blobs $K^{\pm 2}$ at left pointing cups/caps with additionally left pointing crossings marked. (Note that only left pointing cups/caps/crossings in the long knot are marked; these may differ from those in the closure). Since $S^{2\sigma}(a) = K^{2\sigma}aK^{-2\sigma}$, instead of marking a chord as “left pointing” (whose corresponding crossing was left pointing), one may equivalently place $K^{2\sigma}$ and $K^{-2\sigma}$ blobs on the outer circle just before and after the outward pointing (corresponding to the overcrossing arc) end of that chord. The result is an *enhanced Gauss diagram* in which additional marks have been placed on the outer circle, namely the basepoint and blobs $K^{\pm 2}$. Figure 5 shows an example for the figure 8 knot.

Let D be such an enhanced Gauss diagram for a knot K with c crossings. In order to write formulae more easily, we give complete combinatorial data for D . Number the crossings (chords of D) by $j = 1, \dots, c$. Let σ_j denote the sign (orientation) of the crossing, positive for a slash crossing pointing up/down. As we go around the circle, we encounter $2c$ contributions from crossings which we will index by $i = 1, \dots, 2c$; for each such i there is a pair $(j(i), \epsilon_i)$ giving the crossing number $j = j(i)$ and the sign with which it is encountered, ϵ_i , which is positive for an overcrossing and negative for an undercrossing. Finally let i_1, \dots, i_b denote the positions of the blobs $K^{\pm 2}$, the a^{th} blob appearing between the i_a^{th} and $(i_a + 1)^{\text{th}}$ positions on the circle, with contribution $K^{2\delta_a}$. Throughout the remainder of this paper, we will

always use j as an index for crossings and i, k as indices for vertices on the Gauss diagram.

The evaluation of $\mathcal{F}(T)$ in the representation Λ_λ may be carried out by computing the action on the lowest weight vector $v_0^{(\lambda)}$ (equivalently it is the 00 matrix element), and then

$$J'_L(\mu) = \sum_{\substack{n_j, l_j=0 \\ j=1, \dots, c}}^{\infty} \left(\prod_{j=1}^c \frac{(1 - q^{-\sigma_j})^{l_j} (\sigma_j \hbar / 4)^{n_j}}{\{l_j\}_{\sigma_j!} n_j!} \right) W_D^\mu(\mathbf{n}, \mathbf{l}). \quad (1)$$

Here $W_D^\mu(\mathbf{n}, \mathbf{l})$ is the 00 matrix element in the evaluation on Λ_μ of the quantum group word $W_D(\mathbf{n}, \mathbf{l})$ obtained by tracing round the outer circle of D starting at the basepoint, with contributions of $H^{n_j} K^{\sigma_j l_j} X^{l_j}$ (outward arrow/overcrossing), $H^{n_j} K^{-\sigma_j l_j} Y^{l_j}$ (inward arrow/undercrossing) and $K^{\pm 2}$ (blobs on the circle), written in order from left to right. It is convenient to use the notation $X^+ = X$ and $X^- = Y$. Using

$$(X^\epsilon)^l H = (H - 2\epsilon l)(X^\epsilon)^l, \quad (X^\epsilon)^l K = v^{-\epsilon l} K (X^\epsilon)^l,$$

we push all powers of X and Y to the end of the word, obtaining $W_D(\mathbf{n}, \mathbf{l}) = A(\mathbf{n}, \mathbf{l})B(\mathbf{l})$ where

$$B(\mathbf{l}) = \prod_{i=1}^{2c} (X^{\epsilon_i})^{l_{j(i)}}, \quad (2)$$

the product being written from left to right. (Note that $B(\mathbf{l})$ is independent of n_1, \dots, n_c .) The term $A(\mathbf{n}, \mathbf{l})$ is the product of H 's and K 's remaining after all the powers of X and Y have been pushed to the right, namely

$$A(\mathbf{n}, \mathbf{l}) = \left[\prod_{i=1}^{2c} (H - 2 \sum_{k < i} \epsilon_k l_k)^{n_i} \left(\prod_{k < i} v^{-\epsilon_k l_k \cdot \epsilon_i \sigma_i l_i} \right) K^{\epsilon_i \sigma_i l_i} \right] \prod_{a=1}^b \left(\prod_{k < i_a} v^{-\epsilon_k l_k \cdot 2\delta_a} \right) K^{2\delta_a}.$$

where by abuse of notation we have written l_i and n_i in place of $l_{j(i)}$ and $n_{j(i)}$ respectively. Collecting terms and noting that $\sum_i \epsilon_i \sigma_i l_i = 0$,

$$A(\mathbf{n}, \mathbf{l}) = v^{-\sum_i \sum_{k < i} \epsilon_i \epsilon_k \sigma_i l_i l_k - 2 \sum_a \sum_{k < i_a} \delta_a \epsilon_k l_k} K^{2 \sum_a \delta_a} \prod_{i=1}^{2c} (H - 2 \sum_{k < i} \epsilon_k l_k)^{n_i}.$$

Since both $A(\mathbf{n}, \mathbf{l})$ and $B(\mathbf{l})$ preserve the weight, $W_D^\mu(\mathbf{n}, \mathbf{l}) = A^\mu(\mathbf{n}, \mathbf{l})B^\mu(\mathbf{l})$ where the superscripts μ denote the 00 matrix element in the evaluation on

Λ_μ . Thus $A^\mu(\mathbf{n}, \mathbf{l})$ is obtained from the above expression by replacing all occurrences of H and K by $1 - \mu$ and $v^{\frac{1-\mu}{2}}$, respectively. We may now carry out the sum over n_1, \dots, n_c in (1) for $J'_L(\mu)$, since the sum on n_j may be taken out to give a factor (where $\lambda = \mu - 1$)

$$\left[\sum_{n_j=0}^{\infty} \frac{(\sigma_j \hbar / 4)^{n_j}}{n_j!} (H - 2 \sum_{k < j+} \epsilon_k l_k)^{n_j} (H - 2 \sum_{k < j-} \epsilon_k l_k)^{n_j} \right]_{00} = v^{\frac{\sigma_j}{2}(\lambda+2 \sum_{k < j+} \epsilon_k l_k)(\lambda+2 \sum_{k < j-} \epsilon_k l_k)}$$

Moving to the zero framed knot introduces a framing correction of $f_\mu^{-\sum_j \sigma_j}$, that is $\prod_j v^{-\frac{\sigma_j}{2}(\mu^2-1)}$. The contribution of the term $A^\mu(\mathbf{n}, \mathbf{l})$ to $J'_{L_0}(\mu)$ becomes simply a power of v . Substituting into (1), we get the zero framed colored Jones polynomial as

$$J'_{L_0}(\mu) = \sum_{\substack{l_j=0 \\ j=1, \dots, c}}^{\infty} \left(\prod_{j=1}^c \frac{(1 - q^{-\sigma_j})^{l_j}}{\{l_j\}_{\sigma_j}!} \right) v^{a^\mu(\mathbf{l})} B^\mu(\mathbf{l}), \quad (3)$$

where (again $\lambda = \mu - 1$)

$$\begin{aligned} a^\mu(\mathbf{l}) = & - \sum_i \sum_{k < i} \epsilon_i \epsilon_k \sigma_i l_i l_k - 2 \sum_a \sum_{k < i_a} \delta_a \epsilon_k l_k - \lambda \sum_a \delta_a \\ & + \sum_j \sigma_j \left(-\lambda + \lambda \sum_{k < j-} \epsilon_k l_k + \lambda \sum_{k < j+} \epsilon_k l_k + 2 \left(\sum_{k < j-} \epsilon_k l_k \right) \left(\sum_{k < j+} \epsilon_k l_k \right) \right) \end{aligned}$$

The function $a^\mu(\mathbf{l})$ is a linear function of μ , and depends quadratically on the parameters l_j . Looking at only the parity of terms (that is, mapping onto $\mathbf{Z}/2\mathbf{Z}[\mu]$),

$$a^\mu(\mathbf{l}) \equiv \sum_i \sum_{k < i} \epsilon_i \epsilon_k l_i l_k + (b + c + \sum_i \sum_{k < i} l_k) (\mu - 1)$$

Since $\sum_i \epsilon_i l_i = 0$ (each term l_j appears twice with opposite signs), the first term is $-\sum_j l_j^2$. Reversing the order of summation in the last term, $\sum_i \sum_{k < i} l_k \equiv \sum_i i l_i \equiv \sum_j p_j l_j$, where p_j is the number of segments on the Gauss diagram between the two occurrences of the crossing j , that is between $j+$ and $j-$. Observing that $b + c$ is always even and that p_j is always odd (as follow from changing the signs of crossings in the knot diagram so as

to obtain an unknot and then applying Reidemeister moves to untie it and observing that in the process these parities are preserved), it follows that

$$a^\mu(\mathbf{1}) \equiv \mu \sum_j l_j \pmod{2}.$$

The final evaluation required in order to remove all traces of the quantum group A from our formula for $J'_{L_0}(\mu)$ is that of $B^\mu(\mathbf{1})$ in (2), namely the 00 matrix element in the representation Λ_μ of the product of terms X^{l_i} and Y^{l_i} described by the Gauss diagram. Using the explicit form for the basis for Λ_μ , one can read the product $B(\mathbf{1})$ from right to left, each term X pushes up the level, while each Y pushes it down with a factor $[i][\mu - i]$ starting at level i . Reading instead from left to right, X decreases the level while Y raises the level and introduces the factor $[i + 1][\mu - 1 - i]$ when starting from level i . Thus we obtain

$$B^\mu(\mathbf{1}) = \prod_{j=1}^c \prod_{i=s(j)}^{s(j)+l_j-1} [i + 1][\mu - 1 - i]$$

where $s(j) = -\sum_{i < j^-} l_i \epsilon_i$ is the level at the start of the term Y^{l_j} contributed by j^- in the Gauss diagram (if any $s(j)$ is negative, the expression vanishes). Writing $[i + 1][\mu - 1 - i] = v^{-\sigma\mu} \{i + 1\}_\sigma \{\mu - 1 - i\}_\sigma$, we obtain

$$B^\mu(\mathbf{1}) = \prod_{j=1}^c v^{-\mu l_j \sigma_j} \frac{\{s(j) + l_j\}_{\sigma_j}!}{\{s(j)\}_{\sigma_j}!} \{\mu - s(j) - 1\}_{\sigma_j} \cdots \{\mu - s(j) - l_j\}_{\sigma_j}.$$

Substituting into (3) and putting $c^\mu(\mathbf{1}) = \frac{1}{2}a^\mu(\mathbf{1}) - \frac{1}{2}\mu \sum_j \sigma_j l_j \in \mathbf{Z}[\mu, l_j]$, we obtain the following result.

Theorem 1 [3] *The colored Jones function of a zero framed knot is given by*

$$J'_{L_0}(\mu) = \sum_{\substack{l_j=0 \\ j=1, \dots, c}}^{\infty} q^{c^\mu(\mathbf{1})} \prod_{j=1}^c \left\{ \begin{matrix} s(j) + l_j \\ l_j \end{matrix} \right\}_{\sigma_j} (q^{\sigma_j(\mu - s(j) - 1)} - 1) \cdots (q^{\sigma_j(\mu - s(j) - l_j)} - 1),$$

where $s(j) = -\sum_{i < j^-} \epsilon_i l_{j(i)}$ and $c^\mu(\mathbf{1})$ is a function dependent on the Gauss diagram, linear in μ and quadratic in l_j , with integer coefficients, given explicitly by (4) and (5) below.

Several remarks and standard facts about the colored Jones function follow immediately from this theorem.

- In [3], this theorem is only stated explicitly for braid closures. A very similar (though inequivalent) state sum for the colored Jones function was recently obtained in Theorem 7, [4].
- Considering $q = 1 + h$, each term can be thought of as a power series in h whose coefficients are polynomials in μ . The contribution from a particular set of c non-negative integers l_j is divisible by $h^{\sum_j l_j}$. That is, the coefficient of h^N comes only from l_j 's with $\sum_j l_j \leq N$, a finite number of contributions. So $J'_{L_0}(\mu)$ is meaningful as a formal power series in h whose coefficients are polynomials in μ (even when μ is not an integer); this is the *Jones function*.
- If $\mu \in \mathbf{N}$, then the contribution from any term for which $s(j) \geq \mu - 1$ will vanish. Thus only a finite number of $\mathbf{1}$'s will contribute, giving a polynomial; this is the *colored Jones polynomial* (μ fixed integer and q variable).
- For each l , the contribution is a polynomial in q and q^λ , so that $J'_{L_0}(\lambda)$ can be written as a power series in h whose coefficients are polynomials in q^λ . (See the Melvin-Morton-Rozansky conjecture of [20], [26] proved in [1].)
- Since $\begin{Bmatrix} n \\ m \end{Bmatrix} \in \mathbf{Z}[q]$ for $m, n \in \mathbf{Z}^+$, it follows that for $\mu \in \mathbf{N}$ the contribution from l is divisible by $h^{\sum_j l_j} \prod_j \{l_j\}!$. At a root of unity of order K , only terms with $l_j < K \forall j$ will contribute. Again there are a finite number of contributions, of number independent of μ (for $\mu \in \mathbf{N}$) so that $J'_{L_0}(\mu)$ is a polynomial in q and q^μ (K fixed, $\mu \in \mathbf{N}$ variable). In particular it follows, under these conditions, that $J'_{L_0}(\mu)$ is a periodic function of μ of period K .
- The coefficient of μ in $c^\mu(\mathbf{1})$ simplifies to

$$d(\mathbf{1}) = \sum_j q_j l_j - \frac{1}{2} \sum_a \delta_a - \frac{1}{2} \sum_j \sigma_j \quad (4)$$

where $q_j = \frac{1}{2} \sum_{i>j+} \sigma_i - \frac{1}{2} \sum_{i>j-} \sigma_i - \frac{1}{2} \sigma_j \in \mathbf{Z}$. When the knot is presented in a blackboard framing which is also zero framed, then

$\sum_j \sigma_j = 0$, so that

$$d(\mathbf{1}) = \sum_j q_j l_j - \frac{1}{2} \sum_a \delta_a$$

where now q_j has the meaning of half the sum of the signs of crossings encountered when tracing around the knot from the overcrossing arc of the j^{th} crossing to its undercrossing arc; equivalently, it is the linking number of the two parts of the knot which the j^{th} crossing, if split so as to give a link, separates.

- In the limit $q \rightarrow 1$ with q^λ (or $q^{\lambda+1}$) constant, we obtain a formal power series in $h = q - 1$. In particular, the Kashaev invariant obtained by looking at the colored Jones polynomial in color K at the K^{th} root of unity $q = e^{\frac{2\pi i}{K}}$ is obtained by setting $\mu = K$, $q^\mu = 1$,

$$K_L(q) = \sum_{\substack{l_j=0 \\ j=1,\dots,c}}^{\infty} q^{c^0(\mathbf{1})} \prod_{j=1}^c (q^{-\sigma_j} - 1) \dots (q^{-\sigma_j l_j} - 1) \left\{ \begin{matrix} s(j) + l_j \\ l_j \end{matrix} \right\}_+ \left\{ \begin{matrix} s(j) + l_j \\ l_j \end{matrix} \right\}_-$$

where $c^0(\mathbf{1})$ is the constant term in $c^\mu(\mathbf{1}) = d(\mathbf{1})\mu + c^0(\mathbf{1})$, namely

$$c^0(\mathbf{1}) = -d(\mathbf{1}) - \sum_{\substack{a \\ k < i_a}} \delta_a \epsilon_k l_k - \frac{1}{2} \sum_j \sigma_j l_j - \frac{1}{2} \sum_{\substack{i \\ k < i}} \sigma_i \epsilon_i \epsilon_k l_i l_k + \sum_{\substack{j \\ i < j-, k < j+}} \sigma_j \epsilon_i \epsilon_k l_i l_k \quad (5)$$

- When $\mu = 2$, the colored Jones polynomial reduces to the one-variable Jones polynomial [10]. It can be checked that the only non-zero contributions to the sum in Theorem 1 in this case come from $l_j \in \{0, 1\}$ for which $-\sum_{k < i} \epsilon_k l_k \in \{0, 1\}$ for all i . This sum is similar to (though different from) the beautifully simple state model given by Kauffman for the bracket polynomial in [11].

Put concisely, the colored Jones function depends on two parameters, q and μ , and one can make sense of it if at least one of the following holds: (i) q is a root of unity of fixed order, as a polynomial in q and q^μ (μ integer variable); (ii) $\mu \in \mathbf{N}$ fixed, as a polynomial in q ; (iii) $q = 1 + h$ is considered as a formal parameter with expressions as formal power series in h and μ is fixed; (iv) q is considered as a formal parameter with expressions as formal power series in h and q^μ is fixed.

4 The WRT and Ohtsuki invariants

4.1 The $SU(2)$ and $SO(3)$ 3-manifold invariants

Suppose that q is a K^{th} root of unity. The prescription of [24] defines for a link L with c components and framing $f_i = \pm 1$ on the i^{th} component,

$$\langle L \rangle = \sum_{\mu_1, \dots, \mu_c=1}^{K-1} J'_{L_0}(\mu_1, \dots, \mu_c) \prod_{i=1}^c [\mu_i]^2 q^{\sum_i f_i \Omega(\mu_i)},$$

where $q^{\Omega(\mu)} = f_\mu$ is the framing normalization, that is $\Omega(\mu) = \frac{\mu^2-1}{4}$.

The $SU(2)$ invariant of a 3-manifold M obtained from S^3 by surgery around a link L is then

$$Z_K(M) = G_+^{-\sigma_+} G_-^{-\sigma_-} \langle L \rangle,$$

where σ_\pm are the numbers of positive/negative eigenvalues of the linking matrix of L and G_\pm are the bracket values for the unknot with framings ± 1 . When L is a knot ($c = 1$) with framing $f = \pm 1$, so that M is an integer homology 3-sphere, this reduces to

$$Z_K(M) = \frac{\sum_{\mu=1}^{K-1} q^{f\mu^2/4} [\mu]^2 J'_{L_0}(\mu)}{\sum_{\mu=1}^{K-1} q^{f\mu^2/4} [\mu]^2}.$$

The $(PSU(2) =)SO(3)$ invariant is obtained by the same procedure except that the sums are over a period $2K$ with μ_i restricted to be even; see [13].

4.2 The $SO(3)$ perturbative invariant (Ohtsuki series)

Following Le and Habiro, we now show how to obtain a cyclotomic form (similar to Habiro's form, though without the balancing) for the Ohtsuki series from Theorem 1. Assume that M is obtained by $f = \pm 1$ -surgery around a knot L , so that M is an integer homology sphere.

The expression for $J'_{L_0}(\mu)$ in Theorem 1 is a sum over l of terms which are polynomial in q^μ and q (and their inverses). A term $q^{m\mu}$ in $J'_{L_0}(\mu)$ contributes

$$\frac{\sum_{\mu=1}^{2K-1} q^{f\mu^2/4} [\mu]^2 q^{m\mu}}{\sum_{\mu=1}^{2K-1} q^{f\mu^2/4} [\mu]^2}$$

to $Z_K(M)$. Completing the square, $\sum q^{f\mu^2/4} q^{m\mu} = q^{-fm^2} \sum q^{f\mu^2/4}$, thus the contribution to $Z_K(M)$ of a term $q^{m\mu}$ in $V'_{L_0}(\mu)$ is

$$\frac{q^{-f(m+1)^2} + q^{-f(m-1)^2} - 2q^{-fm^2}}{2q^{-f} - 2} \in \frac{1}{2} \mathbf{Z}[q^{-f}].$$

Let B be the ring of formal power series in $h = q - 1$ whose coefficients are polynomials in $q^{\pm\mu}$ with integer coefficients. For $f = \pm 1$, define a map $\Phi_f: B \rightarrow \mathbf{Z}[[h]]$ linear over $\mathbf{Z}[[h]]$ by

$$\Phi_f(q^{m\mu}) = \frac{q^{-f(m+1)^2} + q^{-f(m-1)^2} - 2q^{-fm^2}}{q^{-f} - 1},$$

so that $Z_K(M) = \frac{1}{2} \Phi_f(J'_{L_0}(\mu))$.

Lemma 1 *For any $a, b \in \mathbf{Z}$, $l \in \mathbf{N}$, $\Phi_f(q^{a\mu}(q^{\mu-b-1} - 1) \dots (q^{\mu-b-l} - 1))$ is divisible by the l^{th} cyclotomic polynomial $\phi_l(q)$ in $\mathbf{Z}[q^{\pm 1}]$.*

Proof Suppose ζ is a root of unity of order l . It is sufficient to show that the expression in the lemma vanishes when evaluated at any such $q = \zeta$. Expanding out the product,

$$\Phi_f(q^{a\mu}(q^{\mu-b-1} - 1) \dots (q^{\mu-b-l} - 1)) = \sum_{S \subset \{1, \dots, l\}} (-1)^{l-|S|} q^{\Sigma S - b|S|} \Phi_f(q^{(a+|S|)\mu})$$

where ΣS and $|S|$ denote the sum of elements and the number of elements in the set S , respectively. Under the map $S \mapsto S + 1$, which adds 1 (modulo l) to every element of S , the number ΣS increments by $|S|$ while $|S|$ stays unchanged. Under repeated application of this map, we get orbits of size l (for $0 < |S| < l$) and two orbits with one element, namely the empty set and the whole set. The contribution from any orbit of size l contains a factor

$\sum_{i=0}^{l-1} q^{i|S|}$ which vanishes when evaluated at $q = \zeta$. The contribution from the two size one orbits is

$$(-1)^l \Phi_f(q^{a\mu}) + q^{\frac{1}{2}l(l+1)-bl} \Phi_f(q^{(a+l)\mu}).$$

When evaluated at $q = \zeta$, $\Phi_f(q^{a\mu}) = \Phi_f(q^{(a+l)\mu})$ while $\zeta^l = 1$ and $\zeta^{\frac{1}{2}l(l+1)} = (-1)^{l+1}$, and again the contribution vanishes. \square

Now for any $p \leq l$, $q^{a\mu}(q^{\mu-b-1} - 1) \dots (q^{\mu-b-l} - 1)$ can be written as a linear combination (over $\mathbf{Z}[q^{\pm 1}]$) of expressions of a similar form with chains of length p , $q^{a'\mu}(q^{\mu-b'-1} - 1) \dots (q^{\mu-b'-p} - 1)$. As a corollary, the expression in Lemma 1 is divisible by $\phi_p(q)$ for any $p \leq l$ and therefore also (since they are coprime) by $\langle l \rangle! \equiv \prod_{p=1}^l \phi_p(q)$. Here $\langle l \rangle! \in \mathbf{Z}[q]$ is the minimal monic polynomial in q which vanishes at all roots of unity of order $\leq l$ (but not at $q = 1$).

Corollary For any $a, b \in \mathbf{Z}$, $l \in \mathbf{N}$, $\Phi_f(q^{a\mu}(q^{\mu-b-1} - 1) \dots (q^{\mu-b-l} - 1))$ is divisible by $\langle l \rangle!$ in $\mathbf{Z}[q^{\pm 1}]$.

Lemma 2 For any integers a, b_j , $\Phi_f\left(q^{a\mu} \prod_{j=1}^l (q^{\mu-b_j} - 1)\right)$ is divisible by $h^{\lceil \frac{l}{2} \rceil}$.

Proof Let $\psi: B \rightarrow \mathbf{Z}[[h]]$ be defined by $\psi(q^{m\mu}) = q^{m^2}$ for $m \in \mathbf{Z}$, extended linearly over $\mathbf{Z}[[h]]$. Write $q = e^{\hbar}$ and extend ψ linearly to formal power series with rational coefficients. Then $\psi(e^{m\mu\hbar}) = e^{m^2\hbar}$. Comparing coefficients of powers of m , $\mu^{2n} \mapsto \frac{(2n)!}{n!\hbar^n}$ while $\mu^{2n+1} \mapsto 0$. In particular, for $s \leq r$, $\psi(\hbar^r \mu^s)$ is divisible by $h^{\lceil r/2 \rceil}$.

Now Φ_{-1} and ψ are related by

$$\Phi_{-1}(f) = \frac{1}{h} \psi((q^\mu + q^{-\mu} - 2)f) = \frac{1}{h} \psi((q^\mu - 1)(q^{-\mu} - 1)f)$$

Expanding as a power series in h , whose coefficients are polynomials in μ , $q^{a\mu} \prod_{j=1}^l (q^{\mu-b_j} - 1)(q^\mu - 1)(q^{-\mu} - 1)$ is divisible by h^{l+2} , and such that the coefficient of h^r ($r \geq l+2$) is a polynomial in μ of degree at most r . Replacing h by \hbar preserves these properties (though coefficients of powers of \hbar may now be polynomials with non-integral, but rational, coefficients). Hence applying ψ we obtain an element of $\mathbf{Q}[[\hbar]]$ divisible by $\hbar^{\lceil \frac{l+2}{2} \rceil}$. The lemma now follows. \square

Lemma 3 For any integers a, b_j, l_j ,

$$\Phi_f \left(q^{a\mu} \prod_j (q^{\mu-b_j-1} - 1) \cdots (q^{\mu-b_j-l_j} - 1) \right)$$

is divisible by $h^{\lceil \frac{1}{2} \sum_j l_j \rceil} < \max l_j >!$ in $\mathbf{Z}[[h]]$ and in $\mathbf{Z}[q^{\pm 1}]$.

Proof Note that Φ_1 and Φ_{-1} interchange when q and q^{-1} are interchanged, so it is sufficient to prove the lemma for $\Phi \equiv \Phi_{-1}$. Next observe that the expression in the lemma is actually a polynomial in $q^{\pm 1}$ and so divisibility in $\mathbf{Z}[[h]]$ and in $\mathbf{Z}[q^{\pm 1}]$ are equivalent. Since h and $< n >!$ are coprime in $\mathbf{Z}[[h]]$, it is sufficient to prove divisibility by $h^{\lceil \frac{1}{2} \sum_j l_j \rceil}$ and by $< \max l_j >!$ separately; for the first we work in $\mathbf{Z}[[h]]$ (Lemma 2) and the second in $\mathbf{Z}[q^{\pm 1}]$ (Corollary). \square

Putting Lemma 3 together with Theorem 1 from the last section gives the following.

Theorem 2 The WRT invariant of a manifold, M , obtained by $f = \pm 1$ surgery around a knot, L , is given by

$$Z_K(M) = \sum_{\substack{l_j=0 \\ j=1, \dots, c}}^{\infty} \frac{1}{2} q^{c^0(\mathbf{1})} \Phi_f \left(q^{d(\mathbf{1})\mu} \prod_{j=1}^c \prod_{p=1}^{l_j} (q^{\sigma_j(\mu-s(j)-p)} - 1) \right) \prod_{j=1}^c \left\{ \begin{matrix} s(j) + l_j \\ l_j \end{matrix} \right\}_{\sigma_j}$$

in which the contribution from $\mathbf{1}$ is divisible by $h^{\lceil \frac{1}{2} \sum_j l_j \rceil} < \max l_j >!$. Here $s(j) = -\sum_{i < j} \epsilon_i l_i$ while $c^0(\mathbf{1})$ and $d(\mathbf{1})$ are given by (5) and (4) respectively. This is meaningful when q is a root of unity (then the sum is finite as the only non-vanishing contributions are when $l_j < K$ for all j ; the sum is the WRT invariant) and as a formal power series in h (the sum is the Ohtsuki series, $Z_{\infty}(M)$).

In [7], Habiro obtains a more symmetric cyclotomic form for Ohtsuki series which is needed to prove integrality in general. For the case of surgery around knots, the above simple-minded approach is sufficient. (To remove the possibility of a single factor of two in the denominator, one needs to use the fact that the Jones function is invariant under a change of sign of μ , which is not apparent from the above formulations.)

4.3 Growth rates in Ohtsuki series

Suppose that M is an integral homology sphere obtained by surgery around a knot. Then the Ohtsuki series $Z_\infty(M)$ is a formal power series $\sum_{n=0}^{\infty} \lambda_n(M)h^n$ in $h = q - 1$. In this section we use the formula given in Theorem 2 to determine a bound on the growth rate of coefficients $\lambda_n(M)$ with n . By the theorem, the only terms \mathbf{l} which contribute to $\lambda_N(M)$ are those for which $\lceil \frac{1}{2} \sum_j l_j \rceil \leq N$, that is $\sum_j l_j \leq 2N$; there are $\binom{c+2N-1}{c-1}$ such \mathbf{l} 's.

The double product in the summand in Theorem 2 is a product of $\sum_j l_j$ factors of the form $q^{\pm\mu+a} - 1$ where $a \in \mathbf{Z}$. Thus the contribution from \mathbf{l} to $Z_\infty(M)$ can be expanded out to give a sum of $2^{\sum_j l_j} \leq 4^N$ terms of the form

$$\pm \frac{1}{2} \Phi_f(q^{\alpha\mu+\beta}) \cdot \prod_j \left\{ \begin{matrix} s(j) + l_j \\ l_j \end{matrix} \right\}_{\sigma_j} \quad (6)$$

where $\alpha = d(\mathbf{l}) + \sum_j \sigma_j |P_j|$ and $\beta = c^0(\mathbf{l}) - \sum_j \sigma_j (s(j)|P_j| + \Sigma(P_j))$, while $P_j \subset \{1, \dots, l_j\}$. Using (4) and (5), along with $\max |q_j| \leq \frac{c}{2}$ and $b \leq 4c$, one finds estimates

$$|\alpha| \leq (c+2)N + 2c, \quad |\beta| \leq (4c+7)N^2 + (c+4)N + 2c$$

Also the coefficient of h^n ($n \leq N$) in $\Phi_f(q^{\alpha\mu+\beta})$ (for $f = -1$) is

$$\binom{\beta + (\alpha + 1)^2}{n+1} + \binom{\beta + (\alpha - 1)^2}{n+1} - 2 \binom{\beta + \alpha^2}{n+1}$$

which is bounded by $4 \binom{f(N, c)}{n+1}$ where $f(N, c) = (c^2 + 8c + 11)N^2 + (4c^2 + 11c + 8)N + (4c^2 + 6c + 1)$.

The final term to estimate is of the coefficient of h^{N-n} in the product of c q -binomial coefficients on the right hand side of (6). Each of these q -binomial coefficients is of the form $\left\{ \begin{matrix} a \\ b \end{matrix} \right\}$ with $b \leq a \leq 2N$; such a coefficient is a sum of $\binom{a}{b}$ powers of q (with exponents at most $b(a-b)$) and so the coefficient of a power h^r is at most

$$\binom{2N}{N} \binom{N^2}{r} \leq 4^N \binom{N^2}{r}$$

Taking a product of c such terms can accumulate in the coefficient of h^r at most $4^{cN} \binom{cN^2}{r}$ and thus the coefficient of h^N in (6) is at most

$$\sum_{n=0}^N 2 \binom{f(N, c)}{n+1} \cdot 4^{cN} \binom{cN^2}{N-n} = 2 \cdot 4^{cN} \binom{f(N, c) + cN^2}{N+1}$$

Putting all the above estimates together gives a bound on $\lambda_N(M)$ of

$$\binom{2N+c-1}{c-1} \cdot 4^N \cdot 2 \cdot 4^{cN} \binom{f(N, c) + cN^2}{N+1} = x_N$$

for which $\frac{x_N}{x_{N-1}}$ asymptotically grows linearly with N , with slope $4^{c+1}e^2(c^2 + 9c + 11)$.

Theorem 3 *There exist bounds x_N dependent only on N and the number of crossings c in the knot K , for which $|\lambda_N(S_K^3)| < x_N$ for all $N \in \mathbf{N}$ while $\lim_{n \rightarrow \infty} \frac{x_{N+1}}{N x_N}$ exists and is a function of c which grows exponentially.*

This should be compared with the slope conjecture of [9] which states that $\frac{\lambda_N(M)}{\lambda_{N-1}(M)}$ is asymptotically linear in N , with coefficient, $\sigma(M)$, called the slope of M . Combining with the above result, it is reasonable to conjecture that $\sigma(M)$ has a bound $f(c)$, dependent only on the number of crossings c in a surgery presentation of M , which grows exponentially with c . This is consistent with known results ([9], [16]) on the slope for Seifert fibred manifolds, where the growth is quadratic in c .

As was pointed out to us by T. Le, a similar result should hold for arbitrary rational homology spheres (not necessarily obtained by surgery around knots), although the explicit formulation of the colored Jones function and Ohtsuki series is then a little more complex.

5 Examples

In the literature (see [3], [5], [8], [18], [19]) there are computations of the colored Jones function for the simplest knots, the trefoil, figure-8 knot and more generally, torus knots and twist knots. Applying Theorems 1 and 2, we reproduce the simplest of these results, although some are in different forms.

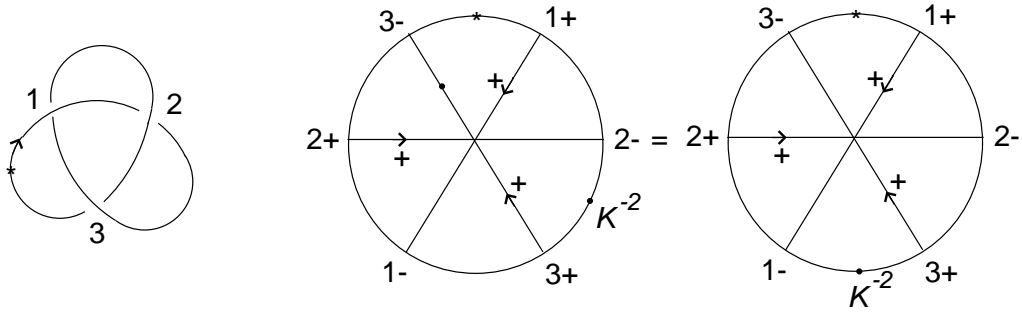


Figure 6: Trefoil knot and its enhanced Gauss diagram

5.1 The trefoil

Suppose that L is the right hand trefoil, presented as in Figure 6. In general, the formulae obtained will be simpler if the basepoint is chosen so that the first crossing encountered is an overcrossing (and the last before returning to the basepoint is an undercrossing). Crossing 3 is a left pointing crossing and the appropriate enhanced Gauss diagram is shown on the right in Figure 6.

In this case $c = 3$ and the six points (from crossings) on the outer circle of the Gauss diagram are marked $1+$, $2-$, $3+$, $1-$, $2+$ and $3-$, respectively. Also there is just one blob K^{-2} , so that $b = 1$ and $\delta_1 = -1$. The signs of the crossings σ_j are all $+1$. Hence

$$s(1) = l_2 - l_1 - l_3, \quad s(2) = -l_1, \quad s(3) = -l_3.$$

Non-vanishing contributions to $V'_{L_0}(\mu)$ appear only when $s(j)$ are all non-negative, so that $l_1 = l_3 = 0$. Then $s(1) = -l_2$ while $s(2) = s(3) = 0$. We calculate $q_2 = -2$ so that

$$d(l) = -1 - 2l, \quad c(l) = 1 + \frac{1}{2}l + \frac{1}{2}l^2,$$

and so $c^\mu(l) = -(1 + 2l)\mu + (1 + \frac{1}{2}l + \frac{1}{2}l^2)$. Theorem 1 now gives the Jones function of the trefoil as

$$J'_{L_0}(\mu) = \sum_{l=0}^{\infty} q^{-(2l+1)\mu + \frac{l(l+1)}{2} + 1} (q^{\mu-1} - 1) \dots (q^{\mu-l} - 1),$$

while the Kashaev invariant is

$$K_L(q) = \sum_{l=0}^{\infty} q^{\frac{l(l+1)}{2}+1} (q^{-1} - 1) \dots (q^{-l} - 1).$$

These formulae may be put in more standard form as

$$J'_{L_0}(\mu) = q \sum_{l=0}^{\infty} q^{-(l+1)\mu} (q^{1-\mu}; q)_l, \quad K_L(q) = q \sum_{l=0}^{\infty} (1 - q) \dots (1 - q^l),$$

where $(a; q)_k = (1 - a)(1 - aq) \dots (1 - aq^{k-1})$. This formula was obtained by Habiro[7] and Le, as well as another one,

$$J'_{L_0}(\mu) = \sum_{k=0}^{\infty} (-1)^k q^{\frac{-k(k+3)}{2}} q^{-k\mu} (q^{\mu+1}; q)_k (q^{\mu-1}; q^{-1})_k$$

5.2 The figure 8 knot

The figure 8 knot (4_1) is naturally presented in blackboard framing as a zero framed knot. Figure 5 shows its enhanced Gauss diagram along with the labeling of points on the circle. Here $c = 4$ with signs $\sigma_1 = \sigma_3 = 1$ and $\sigma_2 = \sigma_4 = -1$. There are two blobs K^{-2} so that $b = 2$, $\delta_a = -1$. Similar to the case of the trefoil, the only non-vanishing contributions come from $l_1 = l_3 = 0$. Set $l_2 = l$ and $l_4 = m$. Then $s(1) = s(4) = l - m$ and $s(2) = s(3) = 0$. Also $q_1 = q_3 = -1$, $q_2 = q_4 = 1$ so that from (4)

$$d(l, m) = l + m + 1.$$

The last term in (5) requires an enumeration of all triples (i, k, j) for which $i < j-$ and $k < j+$; the part of the list for which $i, k \in \{2\pm, 4\pm\}$ is $(2+, 2-, 3)$, $(2+, 4+, 3)$, $(2-, 2-, 3)$, $(2-, 2-, 4)$, $(2-, 4+, 3)$, $(4+, 2-, 3)$, $(4+, 2-, 4)$, $(4+, 4+, 3)$, $(4-, 2-, 3)$ and $(4-, 4+, 3)$. Substituting into (5) gives

$$c^0(l, m) = -(l + m + 1) - l + (-l + m) + \frac{1}{2}(l + m) + \frac{1}{2}(-l^2 - m^2) + (-l^2 + lm)$$

Theorem 1 supplies the Jones function of the knot 4_1 as

$$J'_L(\mu) = \sum_{l \geq m \geq 0} q^{c^\mu(l, m)} (q^{1-\mu} - 1) \dots (q^{l-\mu} - 1) \left\{ \begin{matrix} l \\ m \end{matrix} \right\}_- (q^{l-\mu-m+1} - 1) \dots (q^{l-\mu} - 1)$$

where $c^\mu(l, m) = (l + m + 1)\mu - \frac{3}{2}l^2 + lm - \frac{1}{2}m^2 - \frac{5}{2}l + \frac{1}{2}m - 1$. The Kashaev invariant is then

$$K_L(q) = \sum_{l \geq m \geq 0} q^{c^0(l, m)} (q-1) \dots (q^l-1)(q-1) \dots (q^m-1) \left\{ \begin{matrix} l \\ m \end{matrix} \right\}_+ \left\{ \begin{matrix} l \\ m \end{matrix} \right\}_-$$

This is to be compared with Habiro [7] and Le's formulae

$$J'_L(\mu) = \sum_{k=0}^{\infty} q^{k\mu} (q^{-\mu-1}; q^{-1})_k (q^{-\mu+1}; q)_k, \quad K_L(q) = \sum_{k=0}^{\infty} (q^{-1}; q^{-1})_k (q; q)_k$$

in which the symmetry $q \leftrightarrow q^{-1}$ due to the amphichirality of 4_1 is apparent.

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