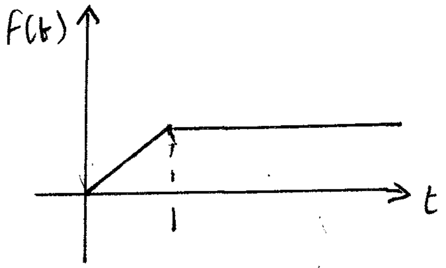
פתרונות לתרגילים על פרק ד

1. א. לפי ההגדרה,
$$f(t)=\begin{cases} t & 0<t<1 \\ 1 & t\geq 1\end{cases}$$

$$\Rightarrow \quad (\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$
$$= \int_0^1 t e^{-st}\,dt + \int_1^\infty e^{-st}\,dt$$
$$= \left[t\,\frac{e^{-st}}{-s} - \int \frac{e^{-st}}{-s}\,dt\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^\infty$$
$$= -e^{-s}/s - \left[e^{-st}/s^2\right]_0^1 + e^{-s}/s$$
$$= \underline{\underline{\frac{1-e^{-s}}{s^2}}}$$



דרך אחרת

$$f(t)=\begin{cases} t & 0<t<1 \\ 1 & t\geq 1\end{cases}$$
$$= t + \underbrace{(1-t)\,H(t-1)}_{\begin{cases} 0 & t<1 \\ 1-t & t\geq 1\end{cases}}$$

$$\Rightarrow \quad \mathcal{L}f = \mathcal{L}(t) + \mathcal{L}\big((1-t)\,H(t-1)\big)$$
$$= 1/s^2 - \mathcal{L}(t)\cdot e^{-s}$$
$$= \underline{\underline{1/s^2 - 1/s^2\, e^{-s}}}$$

תכונה של $\mathcal{L}$: $\mathcal{L}(f(t-a)H(t-a)) = e^{-as}\cdot\mathcal{L}f$

$\mathcal{L}(t)=1/s^2$

ב.
$$\mathcal{L}(4t^2-5\sin 3t) = 4\,\mathcal{L}(t^2) - 5\,\mathcal{L}(\sin 3t)$$

ליניאריות של $\mathcal{L}$

$$= 4\cdot(2/s^3) - 5\cdot(3/(s^2+9))$$

$\mathcal{L}(t^2)=2/s^3$
$\mathcal{L}(\sin 3t)=3/(s^2+9)$

$$= \underline{\underline{8/s^3 - 15/(s^2+9)}}$$

דף - 2

המשך של ① ...

ב.

$$\mathcal{L}(e^{-2t-5}+t^2e^{3t}) = e^{-5}\mathcal{L}(e^{-2t}) + \mathcal{L}(t^2e^{3t})$$

$$= e^{-5}\cdot\frac{1}{s+2} + \mathcal{L}(t^2)\Big|_{s\to s-3}$$

$$\mathcal{L}(f(t)e^{at}) = (\mathcal{L}f)\big|_{s\to s-a}$$
$$\mathcal{L}(e^{at}) = 1/(s-a)$$

$$= \underline{\underline{\frac{e^{-5}}{s+2} + \frac{2}{(s-3)^3}}}$$

$$\mathcal{L}(t^2) = 2/s^3$$

ג.

$$\mathcal{L}(t\,H(t-2)) = \mathcal{L}(f(t-2)\,H(t-2))$$

$f(t-2)=t$ כאשר

$f(t)=t+2$ $\Longleftrightarrow$

$$= \mathcal{L}(f(t))\cdot e^{-2s}$$

$$\mathcal{L}(f(t-a)H(t-a)) = e^{as}\mathcal{L}f$$

$$= e^{-2s}\cdot(\mathcal{L}(t) + 2\mathcal{L}(1))$$

$$= \underline{\underline{e^{-2s}\left(1/s^2 + 2/s\right)}}$$

דרך אחרת:

$$\mathcal{L}(t\,H(t-2)) = -d/ds\,(\mathcal{L}(H(t-2)))$$

$$\mathcal{L}(tf(t)) = -(\mathcal{L}f)'$$

$$= -d/ds\,(e^{-2s}/s)$$

$$\mathcal{L}(H(t-a)) = e^{-as}/s$$

$$= -\left(-2e^{-2s}/s + e^{-2s}(-1/s^2)\right)$$

$$= \underline{\underline{e^{-2s}\left(1/s^2 + 2/s\right)}}$$

א. 2.

$$\frac{s}{s^2+2s-3} = \frac{s}{(s+3)(s-1)}$$

$$= \frac{A}{s+3} + \frac{B}{s-1}$$

$$s = A(s-1) + B(s+3) \quad \text{לכל}$$

$$\left.\begin{array}{ll} s=1: & 1 = 4B \\ s=-3: & -3 = -4A \end{array}\right\} \quad \begin{array}{l} B = 1/4 \\ A = 3/4 \end{array}$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+2s-3}\right) = \mathcal{L}^{-1}\left(\frac{3/4}{s+3} + \frac{1/4}{s-1}\right)$$

$$= \underline{\underline{\tfrac{3}{4} e^{-3t} + \tfrac{1}{4} e^{t}}} \leftarrow \mathcal{L}(e^{at}) = 1/(s-a)$$

ב.

$$\frac{(s+1)^3}{s^4} = \frac{s^3+3s^2+3s+1}{s^4}$$

$$= 1/s + 3/s^2 + 3/s^3 + 1/s^4$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{(s+1)^3}{s^4}\right) = \underline{\underline{1 + 3t + \tfrac{3}{2}t^2 + \tfrac{1}{6}t^3}}$$

$$\mathcal{L}(t^n) = n!/s^{n+1}$$

ג.

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \tfrac{1}{4}\mathcal{L}^{-1}(1/s^2) - \tfrac{1}{8}\mathcal{L}^{-1}\left(2/(s^2+4)\right)$$

$$= \underline{\underline{\tfrac{1}{4}t - \tfrac{1}{8}\sin 2t}}$$

ד.

$$\mathcal{L}^{-1}(e^{-2s}/s^3) = \mathcal{L}^{-1}(1/s^3)\big|_{t\to t-2} H(t-2)$$

$$\mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}f$$

$$= \tfrac{1}{2}(t-2)^2 H(t-2)$$

$$= \underline{\underline{\begin{cases} 0 & t<2 \\ \tfrac{1}{2}(t-2)^2 & t\geq 2 \end{cases}}}$$

$$\begin{cases} y'' + 5y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \qquad .k \quad .3$$

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) = s\mathcal{L}(y) - 1$$
$$\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s^2\mathcal{L}(y) - s$$

$$\xrightarrow{\mathcal{L}}$$

$$(s^2\mathcal{L}(y) - s) + 5(s\mathcal{L}(y) - 1) + 4\mathcal{L}(y) = 0$$
$$\Rightarrow (s^2 + 5s + 4)\mathcal{L}(y) = s + 5$$
$$\Rightarrow \mathcal{L}(y) = \frac{s+5}{s^2+5s+4} = \frac{s+5}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$s + 5 = A(s+4) + B(s+1)$$
$$\left.\begin{aligned} s=-1:&\quad 4 = 3A \\ s=-4:&\quad 1 = -3B \end{aligned}\right\}$$

$$A = 4/3,\ B = -1/3, \qquad \mathcal{L}(y) = \frac{4/3}{s+1} - \frac{1/3}{s+4}$$

$$\Rightarrow \underline{\underline{y = 4/3\, e^{-t} - 1/3\, e^{-4t}}}$$

$$\mathcal{L}(e^{at}) = 1/(s-a)$$

$$\begin{cases} x'' + 16x = \cos 4t \\ x(0) = 0 \\ x'(0) = 1 \end{cases} \qquad .a$$

$$\mathcal{L}(x') = s\mathcal{L}(x) - x(0) = s\mathcal{L}(x)$$
$$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s^2\mathcal{L}(x) - 1$$

$$\xrightarrow{\mathcal{L}}$$

$$s^2\mathcal{L}(x) - 1 + 16\mathcal{L}(x) = \mathcal{L}(\cos 4t) = \frac{s}{s^2+16}$$
$$\Rightarrow (s^2+16)\mathcal{L}(x) = 1 + \frac{s}{s^2+16}$$
$$\Rightarrow \mathcal{L}(x) = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2}$$
$$\Rightarrow x = \mathcal{L}^{-1}\left(\frac{1}{s^2+16}\right) + \mathcal{L}^{-1}\left(\frac{s}{(s^2+16)^2}\right)$$
$$= 1/4 \sin 4t + \mathcal{L}^{-1}\left(\frac{s}{(s^2+16)^2}\right)$$

$$\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$$

המשך של 2ב...

דרך א':

$$\frac{s}{(s^2+16)^2} = -\tfrac{1}{2}\,\tfrac{d}{ds}\left(\frac{1}{s^2+16}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{s}{(s^2+16)^2}\right) = \tfrac{1}{2}\,t\,\mathcal{L}^{-1}\left(\frac{1}{s^2+16}\right) \leftarrow \mathcal{L}(tf(t)) = -(\mathcal{L}f)'$$

$$= \tfrac{1}{2}\,t \cdot \tfrac{1}{4}\sin 4t$$

דרך ב':

$$\frac{s}{(s^2+16)^2} = \frac{s}{s^2+16}\cdot\frac{1}{s^2+16}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{s}{(s^2+16)^2}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+16}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+16}\right) \quad \mathcal{L}(f*g) = \mathcal{L}f\cdot\mathcal{L}g$$

$$= \cos 4t \;*\; \tfrac{1}{4}\sin 4t$$

$$= \tfrac{1}{4}\int_0^t \cos 4u \sin 4(t-u)\,du$$

$$= \tfrac{1}{8}\int_0^t \sin 4t + \sin(4(t-u)-4u)\,du \leftarrow$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

$$= \tfrac{1}{8}\left[u\sin 4t + \tfrac{1}{8}\cos(4t-8u)\right]_{u=0}^{t}$$

$$= \tfrac{1}{8}\left[(t\sin 4t) + \tfrac{1}{8}\underbrace{\left(\cos(-4t) - \cos(4t)\right)}_{0}\right]$$

$$= \tfrac{1}{8}\,t\sin 4t$$

בסך הכל:

$$x = \tfrac{1}{4}\sin 4t + \tfrac{1}{8}\,t\sin 4t$$

$$\begin{cases} x''+16x = f(t) \\ x(0) = 0 \\ x'(0) = 1 \end{cases} \quad \text{?}$$

$$\mathcal{L}(x') = s\mathcal{L}(x) - x(0) = s\mathcal{L}(x)$$

$$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s^2\mathcal{L}(x) - 1$$

$$\xrightarrow{\mathcal{L}} s^2\mathcal{L}(x) - 1 + 16\mathcal{L}(x) = \mathcal{L}(f)$$

$$F(t) = \begin{cases} \cos 4t & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$$

$$= \cos 4t - \cos 4t \cdot H(t-\pi)$$

$$= \cos 4t - \cos 4(t-\pi)\, H(t-\pi)$$

$$\Rightarrow \mathcal{L}f = \frac{s}{s^2+16} - e^{-\pi s} \cdot \frac{s}{s^2+16}$$

$$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$$

$$\mathcal{L}(g(t-a)\, H(t-a)) = e^{-as}\mathcal{L}g$$

$$\mathcal{L}(x) = \frac{1+\mathcal{L}(f)}{s^2+16}$$

$$= \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - e^{-\pi s}\frac{s}{(s^2+16)^2}$$

$$\Rightarrow \quad x = \frac{1}{4}\sin 4t$$

$$\frac{s}{(s^2+16)^2} = -\frac{1}{2}\,{}^{d}/_{ds}\left(\frac{1}{s^2+16}\right)$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+16)^2}\right) = \frac{1}{2}\, t\, \mathcal{L}^{-1}\left(\frac{1}{s^2+16}\right)$$

$$+ \frac{1}{8}\, t \sin 4t$$

$$\mathcal{L}(g(t-\pi)\, H(t-\pi)) = e^{-\pi s}\mathcal{L}g$$

$$- \frac{1}{8}(t-\pi)\underbrace{\sin 4(t-\pi)}_{=\sin 4t} H(t-\pi)$$

$$= \begin{cases} \left(\frac{1}{4} + \frac{t}{8}\right)\sin 4t & 0 \le t < \pi \\ \left(\frac{1}{4} + \frac{\pi}{8}\right)\sin 4t & t > \pi \end{cases}$$

החלק של ③ ...

ב.

$$f(t)=\begin{cases}0 & 0\le t<1\\ 5 & t>1\end{cases}$$

$$f(t)=5H(t-1)$$

$$\left.\begin{aligned} y'+y&=f(t)\\ y(0)&=0\end{aligned}\right\}$$

$\mathcal{L}$: $\mathcal{L}(y')=s\mathcal{L}y-y(0)=s\mathcal{L}y$

$$s\mathcal{L}y+\mathcal{L}y=\mathcal{L}f=5e^{-s}/s$$

$$\Rightarrow\quad \mathcal{L}y=\frac{5}{s(s+1)}e^{-s}$$

$$\Rightarrow\quad y=\mathcal{L}^{-1}\left(\frac{5}{s(s+1)}\right)*\delta(t-1)$$ **דרך א'**

$$\frac{5}{s(s+1)}=\frac{5}{s}-\frac{5}{s+1}$$

$$=\left[5(1-e^{-t})\right]*\delta(t-1)$$

$$=\underline{\underline{5(1-e^{1-t})\ H(t-1)}}$$

$$=\underline{\underline{\begin{cases}0 & 0\le t<1\\ 5(1-e^{1-t}) & t\ge 1\end{cases}}}$$

$$\left[g(t)*\delta(t-1)\right]=\int_0^t g(s)\,\delta(t-s-1)\,ds$$

$$=\begin{cases}g(t-1) & t-1>0\\ 0 & t-1<0\end{cases}$$

$$=g(t-1)\,H(t-1)$$

$$\mathcal{L}y=\frac{5}{s(s+1)}e^{-s}$$ **דרך ב'**

$$\mathcal{L}(g(t-a)H(t-a))=e^{-as}\mathcal{L}g\ \Rightarrow$$

$$y=\mathcal{L}^{-1}\left(\frac{5}{s(s+1)}\right)\Big|_{t\to t-1}H(t-1)$$

$$=\mathcal{L}^{-1}\left(\frac{5}{s}-\frac{5}{s+1}\right)\Big|_{t\to t-1}H(t-1)$$

$$=5(1-e^{-t})\Big|_{t\to t-1}H(t-1)$$

$$=5(1-e^{1-t})\,H(t-1)$$