פתרונות לתרגילים על פרק 1

(1)

$$
\begin{aligned}
((\underline{\nabla}\wedge\underline{v})\wedge\underline{v})_i &= \varepsilon_{ijk}\,(\underline{\nabla}\wedge\underline{v})_j\,v_k\\
&= \varepsilon_{ijk}\,\varepsilon_{jlm}\,(\partial_l v_m)\,v_k\\
&= \varepsilon_{jki}\,\varepsilon_{jlm}\,(\partial_l v_m)\,v_k\\
&= (\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})\,(\partial_l v_m)\,v_k\\
&= (\partial_l v_i)\,v_l-(\partial_i v_k)\,v_k\\
&= v_l(\partial_l v_i)-\partial_i\left(\tfrac{1}{2}v_k v_k\right)\\
(\underline{\nabla}\wedge\underline{v})\wedge\underline{v} &= (\underline{v}\cdot\underline{\nabla})\underline{v}-\underline{\nabla}\left(\tfrac{1}{2}|\underline{v}|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_i(v_k v_k)\\
&= v_k\cdot\partial_i v_k+\partial_i v_k\cdot v_k\\
&= 2v_k(\partial_i v_k)
\end{aligned}
$$

$\square$

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(2) א.

$$
\begin{aligned}
(\underline{\nabla}\wedge(\underline{A}\wedge\underline{B}))_i &= \varepsilon_{ijk}\,\partial_j\,(\underline{A}\wedge\underline{B})_k\\
&= \varepsilon_{ijk}\,\partial_j\,(\varepsilon_{klm}A_l B_m)\\
&= \varepsilon_{kij}\,\varepsilon_{klm}\,\partial_j(A_l B_m)\\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\,\partial_j(A_l B_m)\\
&= \partial_j(A_i B_j)-\partial_l(A_l B_i)\\
&= (\partial_j A_i)B_j+A_i(\partial_j B_j)-(\partial_l A_l)B_i-A_l(\partial_l B_i)\\
\underline{\nabla}\wedge(\underline{A}\wedge\underline{B}) &= (\underline{B}\cdot\underline{\nabla})\underline{A}+(\underline{\nabla}\cdot\underline{B})\underline{A}-(\underline{\nabla}\cdot\underline{A})\underline{B}-(\underline{A}\cdot\underline{\nabla})\underline{B}
\end{aligned}
$$

$\square$

ב.

$$
\begin{aligned}
\underline{\nabla}\wedge(\underline{\omega}\wedge\underline{r}) &= (\underline{r}\cdot\underline{\nabla})\underline{\omega}+(\underline{\nabla}\cdot\underline{r})\underline{\omega}-(\underline{\nabla}\cdot\underline{\omega})\underline{r}-(\underline{\omega}\cdot\underline{\nabla})\underline{r}\\
&= 0+3\underline{\omega}-0-\omega_i\underbrace{\partial_i(\underline{r})}_{=\underline{e}_i}\\
&= 3\underline{\omega}-\underline{\omega}=\underline{\underline{2\underline{\omega}}}
\end{aligned}
$$

$\underline{A}=\underline{\omega}$

$\underline{B}=\underline{r}$

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(3) א.

אלה קואורדינטות של $\underline{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$v_i=\varepsilon_{ijk}\,\omega_j x_k \quad\Leftarrow\quad \underline{v}=\underline{\omega}\wedge\underline{r}$$

$$
\begin{aligned}
[(\underline{v}\cdot\underline{\nabla})\underline{v}]_i &= v_j\nabla_j v_i\\
&= (\varepsilon_{jlm}\,\omega_l x_m)\,\nabla_j\,(\varepsilon_{inp}\,\omega_n x_p)\\
&= \varepsilon_{jlm}\,\omega_l x_m\,\varepsilon_{inp}\,\omega_n\,\delta_{jp}\\
&= \varepsilon_{jlm}\,\varepsilon_{inj}\,\omega_l\omega_n x_m\\
&= \varepsilon_{jlm}\,\varepsilon_{jin}\,\omega_l\omega_n x_m\\
&= (\delta_{li}\delta_{mn}-\delta_{ln}\delta_{mi})\,\omega_l\omega_n x_m\\
&= \omega_i\omega_m x_m-\omega_l\omega_l x_i\\
\Rightarrow\quad (\underline{v}\cdot\underline{\nabla})\underline{v} &= \underline{\underline{(\underline{\omega}\cdot\underline{r})\underline{\omega}-|\underline{\omega}|^2\underline{r}}}
\end{aligned}
$$

המשך של ③ ...

דרך אחרת: משוואה ①:

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = (\underline{\nabla}\wedge\underline{v})\wedge\underline{v} + \underline{\nabla}\left(\tfrac{1}{2}|\underline{v}|^2\right)$$

(משוואה ② $\Leftarrow \underline{\nabla}\wedge\underline{v} = 2\underline{\omega}$)

$$= 2\underline{\omega}\wedge(\underline{\omega}\wedge\underline{r}) + \underline{\nabla}\left(\tfrac{1}{2}|\underline{\omega}\wedge\underline{r}|^2\right)$$

$$= 2\underbrace{\left((\underline{\omega}\cdot\underline{r})\underline{\omega} - (\underline{\omega}\cdot\underline{\omega})\underline{r}\right)}_{\underline{a}\wedge(\underline{b}\wedge\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}} + \tfrac{1}{2}\underline{\nabla}\Big(\underbrace{|\underline{\omega}|^2\cdot|\underline{r}|^2 - (\underline{\omega}\cdot\underline{r})^2}_{|\underline{a}|^2\cdot|\underline{b}|^2 = (\underline{a}\cdot\underline{b})^2 + |\underline{a}\wedge\underline{b}|^2}\Big)$$

$$= 2\left((\underline{\omega}\cdot\underline{r})\underline{\omega} - |\underline{\omega}|^2\underline{r}\right) + \tfrac{1}{2}\Big(|\omega|^2 \underbrace{\underline{\nabla}(|\underline{r}|^2)}_{= 2\underline{r}} - \underbrace{\underline{\nabla}(\underline{\omega}\cdot\underline{r})^2}_{\text{כלל השרשרת}}\Big)$$

$$\text{כלל השרשרת}: \quad \underline{\nabla}(\underline{\omega}\cdot\underline{r})^2 = 2(\underline{\omega}\cdot\underline{r})\underbrace{\underline{\nabla}(\underline{\omega}\cdot\underline{r})}_{= \underline{\omega}}$$

$$= 2\left((\underline{\omega}\cdot\underline{r})\underline{\omega} - |\underline{\omega}|^2\underline{r}\right) + |\underline{\omega}|^2\underline{r} - (\underline{\omega}\cdot\underline{r})\underline{\omega}$$

$$= \underline{\underline{(\underline{\omega}\cdot\underline{r})\underline{\omega} - |\underline{\omega}|^2\underline{r}}}$$

ב.

$$\left(\underline{\nabla}\wedge(f(r)\,\underline{k}\wedge\underline{r})\right)_i = \varepsilon_{ijk}\,\partial_j\left(f(r)(\underline{k}\wedge\underline{r})_k\right)$$

$$= \varepsilon_{ijk}\,\partial_j\Big(f(r)\,\varepsilon_{klm}\underbrace{\hat{k}_l}_{\delta_{3l}}x_m\Big)$$

$$= \varepsilon_{ijk}\,\varepsilon_{k3m}\,\partial_j\left(f(r)\,x_m\right)$$

$$= \varepsilon_{kij}\,\varepsilon_{k3m}\Big(f(r)\underbrace{(\partial_j x_m)}_{\delta_{jm}} + \underbrace{\partial_j(f(r))}_{f'(r)\,\partial_j(r) \,=\, f'(r)\cdot x_j/r}x_m\Big)$$

$$= (\delta_{i3}\delta_{jm} - \delta_{im}\delta_{j3})\left(\delta_{jm}f(r) + x_j x_m\cdot f'(r)/r\right)$$

$$= \left(\delta_{i3}f(r)(\delta_{jm}\delta_{jm}) + \delta_{i3}\,x_j x_j\cdot f'(r)/r\right)$$

$$\quad - \left(\delta_{i3}f(r) + x_3 x_i\cdot f'(r)/r\right)$$

$$\delta_{jm}\delta_{jm} = 3 \longrightarrow \quad = 2f(r)\,\delta_{i3} + \frac{f'(r)}{r}\cdot r^2\,\delta_{i3} - \frac{f'(r)}{r}x_3 x_i$$

$$\underline{\underline{\underline{\nabla}\wedge(f(r)\,\underline{k}\wedge\underline{r}) = 2f(r)\,\underline{\hat{k}} + rf'(r)\,\underline{\hat{k}} - f'(r)\,z/r\cdot\underline{r}}}$$

4.

א.

$$\underline{r} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix} \Rightarrow \frac{\partial\underline{r}}{\partial r} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad \frac{\partial\underline{r}}{\partial\theta} = \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{pmatrix}, \quad \frac{\partial\underline{r}}{\partial\phi} = \begin{pmatrix} -r\sin\theta\sin\phi \\ r\sin\theta\cos\phi \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$\underline{\underline{\underline{\hat{r}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad \underline{\hat{\theta}} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \quad \underline{\hat{\phi}} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}}}$$

$\underline{\hat{r}}\cdot\underline{\hat{\theta}} = 0,\ \underline{\hat{r}}\cdot\underline{\hat{\phi}} = 0,\ \underline{\hat{\theta}}\cdot\underline{\hat{\phi}} = 0 \Rightarrow$ ניצבים אחד לשני $\{\underline{\hat{r}}, \underline{\hat{\theta}}, \underline{\hat{\phi}}\}$

המשך של ④ ...

ב.

$$\partial f/\partial r = \partial f/\partial x \cdot \partial x/\partial r + \partial f/\partial y \cdot \partial y/\partial r + \partial f/\partial z \cdot \partial z/\partial r$$
$$= \partial f/\partial x \cdot \sin\theta\cos\phi + \partial f/\partial y \cdot \sin\theta\sin\phi + \partial f/\partial z \cdot \cos\theta$$
$$\partial f/\partial\theta = \partial f/\partial x \cdot r\cos\theta\cos\phi + \partial f/\partial y \cdot r\cos\theta\sin\phi - \partial f/\partial z \cdot r\sin\theta$$
$$\partial f/\partial\phi = \partial f/\partial x \cdot (-r\sin\theta\sin\phi) + \partial f/\partial y\,(r\sin\theta\cos\phi)$$

$$\Rightarrow \left.\begin{aligned}
&\underline{\underline{\partial/\partial r = \sin\theta\cos\phi\,\partial/\partial x + \sin\theta\sin\phi\,\partial/\partial y + \cos\theta\,\partial/\partial z}}\\
&\underline{\underline{\partial/\partial\theta = r(\cos\theta\cos\phi\,\partial/\partial x + \cos\theta\sin\phi\,\partial/\partial y - \sin\theta\,\partial/\partial z)}}\\
&\underline{\underline{\partial/\partial\phi = r\sin\theta(-\sin\phi\,\partial/\partial x + \cos\phi\,\partial/\partial y)}}
\end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned}
&\underline{\underline{\partial/\partial x = \sin\theta\cos\phi\,\partial/\partial r + \tfrac{1}{r}\cos\theta\cos\phi\,\partial/\partial\theta - \tfrac{1}{r\sin\theta}\sin\phi\,\partial/\partial\phi}}\\
&\underline{\underline{\partial/\partial y = \sin\theta\sin\phi\,\partial/\partial r + 1/r\,\cos\theta\sin\phi\,\partial/\partial\theta + \tfrac{1}{r\sin\theta}\cos\phi\,\partial/\partial\phi}}\\
&\underline{\underline{\partial/\partial z = \cos\theta\,\partial/\partial r - \tfrac{1}{r}\sin\theta\,\partial/\partial\theta}}
\end{aligned}\right\}$$

$$\Rightarrow \underline{\underline{\underline{\nabla} = \underline{\hat r}\,\partial/\partial r + 1/r\,\underline{\hat\theta}\,\partial/\partial\theta + \tfrac{1}{r\sin\theta}\,\underline{\hat\phi}\,\partial/\partial\phi}}$$

ג.

$$\underline{\hat r} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix} \Rightarrow \partial\underline{\hat r}/\partial r = 0$$

$$\underline{\hat\theta} = \begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix} \Rightarrow \partial\underline{\hat\theta}/\partial\theta = \begin{pmatrix}-\sin\theta\cos\phi\\ -\sin\theta\sin\phi\\ -\cos\theta\end{pmatrix} = -\underline{\hat r}$$

$$\underline{\hat\phi} = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix} \Rightarrow \partial\underline{\hat\phi}/\partial\phi = \begin{pmatrix}-\cos\phi\\ -\sin\phi\\ 0\end{pmatrix} = -\sin\theta\,\underline{\hat r} - \cos\theta\,\underline{\hat\theta}$$

$$\underline{a} = (\underline{a}\cdot\underline{\hat r})\underline{\hat r} + (\underline{a}\cdot\underline{\hat\theta})\underline{\hat\theta} + (\underline{a}\cdot\underline{\hat\phi})\underline{\hat\phi}$$
כי $\{\underline{\hat r},\underline{\hat\theta},\underline{\hat\phi}\}$ בסיס אורתונורמלי

ד.

$$(\underline{\hat r}\,\partial/\partial r)\cdot\underline{f} = \partial/\partial r\,(\underline{\hat r}\cdot\underline{f}) - \partial\underline{\hat r}/\partial r\cdot\underline{f}$$
$$= \partial f_r/\partial r$$

$$(\underline{\hat\theta}\,\partial/\partial\theta)\cdot\underline{f} = \partial/\partial\theta\,(\underline{\hat\theta}\cdot\underline{f}) \underbrace{-\partial\underline{\hat\theta}/\partial\theta}_{\underline{\hat r}}\cdot\underline{f}$$
$$= \partial f_\theta/\partial\theta + f_r$$

$$(\underline{\hat\phi}\,\partial/\partial\phi)\cdot\underline{f} = \partial/\partial\phi\,(\underline{\hat\phi}\cdot\underline{f}) - \partial\underline{\hat\phi}/\partial\phi\cdot\underline{f}$$
$$= \partial f_\phi/\partial\phi - (-\sin\theta\,\underline{\hat r} - \cos\theta\,\underline{\hat\theta})\cdot\underline{f}$$
$$= \partial f_\phi/\partial\phi + \sin\theta\cdot f_r + \cos\theta\cdot f_\theta.$$

□

המשך (3)(4)...

$$\underline{\nabla}\cdot\underline{f} = \left(\hat{\underline{r}}\,\partial/\partial r + 1/r\,\hat{\underline{\theta}}\,\partial/\partial\theta + \tfrac{1}{r\sin\theta}\,\hat{\underline{\phi}}\,\partial/\partial\phi\right)\cdot\underline{f}$$
$$= \partial f_r/\partial r + 1/r\left(\partial f_\theta/\partial\theta + f_r\right) + \tfrac{1}{r\sin\theta}\left(\partial f_\phi/\partial\phi + \sin\theta\cdot f_r + \cos\theta\cdot f_\theta\right)$$
$$= \frac{\partial f_r}{\partial r} + \frac{2f_r}{r} + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{1}{r}\cot\theta\cdot f_\theta + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$

ה.

$$\underline{\nabla}^2 V = \underline{\nabla}\cdot(\underline{\nabla}V)$$
$$\overset{(3)}{=} \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$

כאשר $\underline{f} = \underline{\nabla}V \overset{(2)}{=} \underbrace{\hat{\underline{r}}\frac{\partial V}{\partial r}}_{\Downarrow} + \underbrace{\frac{1}{r}\hat{\underline{\theta}}\frac{\partial V}{\partial\theta}}_{\Downarrow} + \frac{1}{r\sin\theta}\hat{\underline{\phi}}\frac{\partial V}{\partial\phi}$

$$f_r = \frac{\partial V}{\partial r} \qquad f_\theta = \frac{1}{r}\frac{\partial V}{\partial\theta} \qquad f_\phi = \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\right)$$
$$= \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$= 1/r^2\left(r^2\,\partial^2V/\partial r^2 + 2r\,\partial V/\partial r\right)$$
$$= \frac{1}{r}\left(r\,\partial^2V/\partial r^2 + 2\,\partial V/\partial r\right)$$
$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rV)$$

הערה: גם אפשר להשתמש בנוסחאות הנ"ל למצוא נוסחה ל- $\underline{\nabla}\wedge\underline{f}$ בקואורדינטות כדוריות (לפי הבסיס $(\hat{\underline{r}},\hat{\underline{\theta}},\hat{\underline{\phi}})$). [חומר רשות]

$$\underline{f} = f_r\hat{\underline{r}} + f_\theta\hat{\underline{\theta}} + f_\phi\hat{\underline{\phi}}, \quad \underline{\nabla} = \hat{\underline{r}}\,\partial/\partial r + 1/r\,\hat{\underline{\theta}}\,\partial/\partial\theta + \frac{1}{r\sin\theta}\hat{\underline{\phi}}\,\partial/\partial\phi$$

$\{\hat{\underline{r}},\hat{\underline{\theta}},\hat{\underline{\phi}}\}$ בסיס אורתונורמלי, ולכן $\hat{\underline{r}}\wedge\hat{\underline{\theta}} = \hat{\underline{\phi}}$, $\hat{\underline{\theta}}\wedge\hat{\underline{\phi}} = \hat{\underline{r}}$, $\hat{\underline{r}}\wedge\hat{\underline{\phi}} = -\hat{\underline{\theta}}$.

$$\Rightarrow (\hat{\underline{r}}\,\partial/\partial r)\wedge\underline{f} = \partial/\partial r(\hat{\underline{r}}\wedge\underline{f}) - \partial\hat{\underline{r}}/\partial r\wedge\underline{f} = \partial/\partial r(f_\theta\hat{\underline{\phi}} - f_\phi\hat{\underline{\theta}}) = \frac{\partial f_\theta}{\partial r}\hat{\underline{\phi}} - \frac{\partial f_\phi}{\partial r}\hat{\underline{\theta}}$$

$$(\hat{\underline{\theta}}\,\partial/\partial\theta)\wedge\underline{f} = \partial/\partial\theta(\hat{\underline{\theta}}\wedge\underline{f}) - \partial\hat{\underline{\theta}}/\partial\theta\wedge\underline{f} = \partial/\partial\theta(-f_r\hat{\underline{\phi}} + f_\phi\hat{\underline{r}}) + \hat{\underline{r}}\wedge\underline{f}$$
$$= -\frac{\partial f_r}{\partial\theta}\hat{\underline{\phi}} + \frac{\partial f_\phi}{\partial\theta}\hat{\underline{r}} - f_r\underbrace{\frac{\partial\hat{\underline{\phi}}}{\partial\theta}}_{0} + f_\phi\underbrace{\frac{\partial\hat{\underline{r}}}{\partial\theta}}_{\hat{\underline{\theta}}} + (f_\theta\hat{\underline{\phi}} - f_\phi\hat{\underline{\theta}})$$
$$= \left(f_\theta - \frac{\partial f_r}{\partial\theta}\right)\hat{\underline{\phi}} + \frac{\partial f_\phi}{\partial\theta}\hat{\underline{r}}$$

$$(\hat{\underline{\phi}}\,\partial/\partial\phi)\wedge\underline{f} = \partial/\partial\phi(\hat{\underline{\phi}}\wedge\underline{f}) - \partial\hat{\underline{\phi}}/\partial\phi\wedge\underline{f} = \partial/\partial\phi(f_r\hat{\underline{\theta}} - f_\theta\hat{\underline{r}}) - (-\sin\theta\,\hat{\underline{r}} - \cos\theta\,\hat{\underline{\theta}})\wedge\underline{f}$$
$$= \frac{\partial f_r}{\partial\phi}\hat{\underline{\theta}} - \frac{\partial f_\theta}{\partial\phi}\hat{\underline{r}} + f_r\underbrace{\frac{\partial\hat{\underline{\theta}}}{\partial\phi}}_{\cos\theta\,\hat{\underline{\phi}}} - f_\theta\underbrace{\frac{\partial\hat{\underline{r}}}{\partial\phi}}_{\sin\theta\,\hat{\underline{\phi}}} + \sin\theta\underbrace{\hat{\underline{r}}\wedge\underline{f}}_{f_\theta\hat{\underline{\phi}} - f_\phi\hat{\underline{\theta}}} + \cos\theta\underbrace{\hat{\underline{\theta}}\wedge\underline{f}}_{-f_r\hat{\underline{\phi}} + f_\phi\hat{\underline{r}}}$$
$$= \left(\frac{\partial f_r}{\partial\phi} - \sin\theta\cdot f_\phi\right)\hat{\underline{\theta}} - \left(\frac{\partial f_\theta}{\partial\phi} - \cos\theta\cdot f_\phi\right)\hat{\underline{r}}$$

המשך של ④ ב...

$$\Rightarrow \underline{\nabla}\wedge\underline{f} = \left(\hat{\underline{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\underline{\theta}}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\hat{\underline{\phi}}\frac{\partial}{\partial\phi}\right)\wedge\underline{f} = \left(\frac{\partial f_\theta}{\partial r}\hat{\underline{\phi}} - \frac{\partial f_\phi}{\partial r}\hat{\underline{\theta}}\right)$$

$$+ \frac{1}{r}\left(\left(f_\theta - \frac{\partial f_r}{\partial\theta}\right)\hat{\underline{\phi}} + \frac{\partial f_\phi}{\partial\theta}\hat{\underline{r}}\right)$$

$$+ \frac{1}{r\sin\theta}\left(\left(\frac{\partial f_r}{\partial\phi} - \sin\theta\cdot f_\phi\right)\hat{\underline{\theta}} + \left(\cos\theta\cdot f_\phi - \frac{\partial f_\theta}{\partial\phi}\right)\hat{\underline{r}}\right)$$

קואורדינטות לפי הבסיס $\{\hat{\underline{r}},\hat{\underline{\theta}},\hat{\underline{\phi}}\}$

$$= \begin{pmatrix} \frac{1}{r}\frac{\partial f_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial f_\theta}{\partial\phi} + \frac{1}{r}\cot\theta\cdot f_\phi \\ \frac{1}{r\sin\theta}\frac{\partial f_r}{\partial\phi} - \frac{\partial f_\phi}{\partial r} - \frac{1}{r}f_\phi \\ \frac{\partial f_\theta}{\partial r} - \frac{1}{r}\frac{\partial f_r}{\partial\theta} + \frac{1}{r}f_\theta \end{pmatrix}_{\{\hat{\underline{r}},\hat{\underline{\theta}},\hat{\underline{\phi}}\}}$$

$$= \begin{pmatrix} \partial/\partial r \\ \frac{1}{r}\partial/\partial\theta \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \end{pmatrix} \wedge \begin{pmatrix} f_r \\ f_\theta \\ f_\phi \end{pmatrix} + \begin{pmatrix} \frac{1}{r}\cot\theta\cdot f_\phi \\ -\frac{1}{r}f_\phi \\ \frac{1}{r}f_\theta \end{pmatrix}$$

5

א.

$$\underline{r} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix} \Rightarrow \frac{\partial \underline{r}}{\partial\rho} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix},\ \frac{\partial \underline{r}}{\partial\theta} = \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \\ 0 \end{pmatrix},\ \frac{\partial \underline{r}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\hat{\underline{\rho}} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad \hat{\underline{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \qquad \hat{\underline{z}} = \hat{\underline{k}}$$

ברור ש- $\{\hat{\underline{\rho}},\hat{\underline{\theta}},\hat{\underline{z}}\}$ בסיס אורתונורמלי

ב.

$$\partial/\partial\rho = \frac{\partial x}{\partial\rho}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\rho}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\rho}\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$\partial/\partial\theta = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z} = -\rho\sin\theta\frac{\partial}{\partial x} + \rho\cos\theta\frac{\partial}{\partial y}$$

$$\partial/\partial z = \partial/\partial z$$

$$\Rightarrow \partial/\partial x = \cos\theta\,\partial/\partial\rho - \frac{1}{\rho}\sin\theta\,\partial/\partial\theta$$

$$\partial/\partial y = \sin\theta\,\partial/\partial\rho + \frac{1}{\rho}\cos\theta\,\partial/\partial\theta$$

$$\partial/\partial z = \partial/\partial z$$

$$\Rightarrow \underline{\nabla} = \hat{\underline{i}}\frac{\partial}{\partial x} + \hat{\underline{j}}\frac{\partial}{\partial y} + \hat{\underline{k}}\frac{\partial}{\partial z} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}\partial/\partial\rho + \frac{1}{\rho}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}\partial/\partial\theta + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\partial/\partial z$$

$$= \hat{\underline{\rho}}\,\partial/\partial\rho + \frac{1}{\rho}\hat{\underline{\theta}}\,\partial/\partial\theta + \hat{\underline{z}}\,\partial/\partial z$$

<u>הערה</u>: המשמעות של האופרטור $x\,\partial/\partial x$ היא $f \longmapsto x\,\partial f/\partial x$

זה הרכבה של שני אופרטורים: $\partial/\partial x : f \longmapsto \partial f/\partial x$ ו- $x\cdot : f \longmapsto x\cdot f$ (כפל עם x)

אבל ההרכבה ההפוכה היא $\underbrace{f \longmapsto x\cdot f \longmapsto \frac{\partial}{\partial x}(x\cdot f) = x\frac{\partial f}{\partial x} + f}$

כותבים: $x\,\partial/\partial x f + I$

צבות

מכאן רואים שחשוב לכתוב $\sin\theta\,\partial/\partial\theta$ ולא בסדר ההפוך!

המשך של 5 ...

ג.

$$\hat{\underline{z}} = \begin{pmatrix}0\\0\\1\end{pmatrix},\quad \hat{\underline{\rho}} = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} \Rightarrow \partial\hat{\underline{\rho}}/\partial\rho = 0 = \partial\hat{\underline{z}}/\partial z$$

$$\hat{\underline{\theta}} = \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} \Rightarrow \partial\hat{\underline{\theta}}/\partial\theta = \begin{pmatrix}-\cos\theta\\ -\sin\theta\\ 0\end{pmatrix} = -\hat{\underline{\rho}}$$

ד.

$$(\hat{\underline{\rho}}\,\partial/\partial\rho)\cdot\underline{f} = \partial/\partial\rho\,(\hat{\underline{\rho}}\cdot\underline{f}) - \partial\hat{\underline{\rho}}/\partial\rho\cdot\underline{f} = \frac{\partial f_\rho}{\partial\rho} -$$

$$(\hat{\underline{\theta}}\,\partial/\partial\theta)\cdot\underline{f} = \partial/\partial\theta\,(\hat{\underline{\theta}}\cdot\underline{f}) - \partial\hat{\underline{\theta}}/\partial\theta\cdot\underline{f} = \frac{\partial f_\theta}{\partial\theta} + \hat{\underline{\rho}}\cdot\underline{f} = \frac{\partial f_\theta}{\partial\theta} + f_\rho$$

$$(\hat{\underline{z}}\,\partial/\partial z)\cdot\underline{f} = \partial/\partial z\,(\hat{\underline{z}}\cdot\underline{f}) - \partial\hat{\underline{z}}/\partial z\cdot\underline{f} = \frac{\partial f_z}{\partial z}$$

$$\underline{\nabla}\cdot\underline{f} = \left(\hat{\underline{\rho}}\,\partial/\partial\rho + 1/\rho\,\hat{\underline{\theta}}\,\partial/\partial\theta + \hat{\underline{z}}\,\partial/\partial z\right)\cdot\underline{f} = \frac{\partial f_\rho}{\partial\rho} + \frac{1}{\rho}\left(\frac{\partial f_\theta}{\partial\theta} + f_\rho\right) + \frac{\partial f_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\rho) + \frac{1}{\rho}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_z}{\partial z}$$

$$\underline{\nabla}^2 V = \underline{\nabla}\cdot(\underline{\nabla}V) \overset{ב}{=} \underline{\nabla}\cdot\left(\hat{\underline{\rho}}\frac{\partial V}{\partial\rho} + \frac{1}{\rho}\hat{\underline{\theta}}\frac{\partial V}{\partial\theta} + \hat{\underline{z}}\frac{\partial V}{\partial z}\right) = \underline{\nabla}\cdot\underline{f}$$

כאשר $\underline{f} = \hat{\underline{\rho}}\frac{\partial V}{\partial\rho} + \frac{1}{\rho}\hat{\underline{\theta}}\frac{\partial V}{\partial\theta} + \hat{\underline{z}}\frac{\partial V}{\partial z}$

$$\left(f_\rho = \frac{\partial V}{\partial\rho},\ f_\theta = \frac{1}{\rho}\frac{\partial V}{\partial\theta},\ f_z = \frac{\partial V}{\partial z}\right)$$

נוסחה ל- $\underline{\nabla}\cdot\underline{f}$ הנ"ל

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\rho) + \frac{1}{\rho}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2}$$

ה.

$$(\hat{\underline{\rho}}\,\partial/\partial\rho)\wedge\underline{f} = \partial/\partial\rho\,(\hat{\underline{\rho}}\wedge\underline{f}) - \underbrace{\partial\hat{\underline{\rho}}/\partial\rho}_{0}\wedge\underline{f} = \partial/\partial\rho\,(f_\theta\hat{\underline{z}} - f_z\hat{\underline{\theta}}) = \frac{\partial f_\theta}{\partial\rho}\hat{\underline{z}} - \frac{\partial f_z}{\partial\rho}\hat{\underline{\theta}} + f_\theta\underbrace{\frac{\partial\hat{\underline{z}}}{\partial\rho}}_{0} - f_z\underbrace{\frac{\partial\hat{\underline{\theta}}}{\partial\rho}}_{0}$$

$$= \frac{\partial f_\theta}{\partial\rho}\hat{\underline{z}} - \frac{\partial f_z}{\partial\rho}\hat{\underline{\theta}}$$

$$\underline{f} = f_\rho\hat{\underline{\rho}} + f_\theta\hat{\underline{\theta}} + f_z\hat{\underline{z}}$$
$$\Rightarrow \hat{\underline{\rho}}\wedge\underline{f} = f_\theta\hat{\underline{z}} - f_z\hat{\underline{\theta}}$$

$$(\hat{\underline{\theta}}\,\partial/\partial\theta)\wedge\underline{f} = \partial/\partial\theta\,(\hat{\underline{\theta}}\wedge\underline{f}) - \underbrace{\partial\hat{\underline{\theta}}/\partial\theta}_{-\hat{\underline{\rho}}}\wedge\underline{f} = \partial/\partial\theta\,(-f_\rho\hat{\underline{z}} + f_z\hat{\underline{\rho}}) + \hat{\underline{\rho}}\wedge\underline{f}$$

$$= -\frac{\partial f_\rho}{\partial\theta}\hat{\underline{z}} + \frac{\partial f_z}{\partial\theta}\hat{\underline{\rho}} + f_z\underbrace{\frac{\partial\hat{\underline{\rho}}}{\partial\theta}}_{\hat{\underline{\theta}}} + (f_\theta\hat{\underline{z}} - f_z\hat{\underline{\theta}})$$

$$\underline{f} = f_\rho\hat{\underline{\rho}} + f_\theta\hat{\underline{\theta}} + f_z\hat{\underline{z}}$$
$$\Rightarrow \hat{\underline{\theta}}\wedge\underline{f} = -f_\rho\hat{\underline{z}} + f_z\hat{\underline{\rho}}$$

$$= \left(f_\theta - \frac{\partial f_\rho}{\partial\theta}\right)\hat{\underline{z}} + \frac{\partial f_z}{\partial\theta}\hat{\underline{\rho}}$$

$$(\hat{\underline{z}}\,\partial/\partial z)\wedge\underline{f} = \partial/\partial z\,(\hat{\underline{z}}\wedge\underline{f}) - \underbrace{\partial\hat{\underline{z}}/\partial z}_{0}\wedge\underline{f} = \partial/\partial z\,(f_\rho\hat{\underline{\theta}} - f_\theta\hat{\underline{\rho}}) = \frac{\partial f_\rho}{\partial z}\hat{\underline{\theta}} - \frac{\partial f_\theta}{\partial z}\hat{\underline{\rho}} + f_\rho\underbrace{\frac{\partial\hat{\underline{\theta}}}{\partial z}}_{0} - f_\theta\underbrace{\frac{\partial\hat{\underline{\rho}}}{\partial z}}_{0}$$

$$\underline{f} = f_\rho\hat{\underline{\rho}} + f_\theta\hat{\underline{\theta}} + f_z\hat{\underline{z}}$$
$$\Rightarrow \hat{\underline{z}}\wedge\underline{f} = f_\rho\hat{\underline{\theta}} - f_\theta\hat{\underline{\rho}}$$

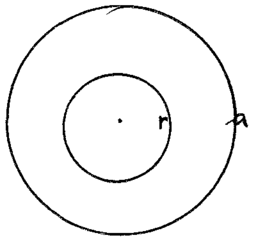
$$= \frac{\partial f_\rho}{\partial z}\hat{\underline{\theta}} - \frac{\partial f_\theta}{\partial z}\hat{\underline{\rho}}$$

$$\Rightarrow \underline{\nabla}\wedge\underline{f} = \left(\hat{\underline{\rho}}\frac{\partial}{\partial\rho} + \frac{1}{\rho}\hat{\underline{\theta}}\frac{\partial}{\partial\theta} + \hat{\underline{z}}\frac{\partial}{\partial z}\right)\wedge\underline{f} = \left(\frac{\partial f_\theta}{\partial\rho}\hat{\underline{z}} - \frac{\partial f_z}{\partial\rho}\hat{\underline{\theta}}\right) + \frac{1}{\rho}\left(\left(f_\theta - \frac{\partial f_\rho}{\partial\theta}\right)\hat{\underline{z}} + \frac{\partial f_z}{\partial\theta}\hat{\underline{\rho}}\right) + \left(\frac{\partial f_\rho}{\partial z}\hat{\underline{\theta}} - \frac{\partial f_\theta}{\partial z}\hat{\underline{\rho}}\right)$$

קואורדינטות לפי בבסיס $\{\hat{\underline{\rho}},\hat{\underline{\theta}},\hat{\underline{z}}\}$ $\longrightarrow$

$$= \begin{pmatrix} \frac{1}{\rho}\frac{\partial f_z}{\partial\theta} - \frac{\partial f_\theta}{\partial z} \\ -\frac{\partial f_z}{\partial\rho} + \frac{\partial f_\rho}{\partial z} \\ \frac{\partial f_\theta}{\partial\rho} + \frac{f_\theta}{\rho} - \frac{1}{\rho}\frac{\partial f_\rho}{\partial\theta} \end{pmatrix}_{\{\hat{\underline{\rho}},\hat{\underline{\theta}},\hat{\underline{z}}\}}$$

6. **א.** הצפיפות היא $\rho(r)=k(a-r)$ (k קבוע)



$$M = \text{מסה} = \iiint_{\text{כדור}} \rho\, dV = \int_0^a \rho(r)\cdot 4\pi r^2\, dr$$
$$= 4\pi k \int_0^a (a-r) r^2\, dr$$
$$= 4\pi k \left[\frac{ar^3}{3} - \frac{r^4}{4}\right]_0^a$$
$$= 4\pi k \left(\frac{a^4}{3} - \frac{a^4}{4}\right) = \frac{\pi k a^4}{3}$$

$$\underline{\underline{\rho(r) = \frac{3M}{\pi a^4}(a-r)}} \Leftarrow \quad k = 3M/\pi a^4 \Leftarrow$$

ב. אנחנו יודעים ש- $\nabla\cdot \underline{F} = -4\pi G\rho$

מסימטריה, הכיוון של $\underline{F}$ רדיאלי והאורך תלוי רק ב- r : $\underline{F}=F(r)\hat{\underline{r}}$

משפט Gauss $\Leftarrow$ $\oiint_{\partial R} \underline{F}\cdot d\underline{S} = \iiint_R (\nabla\cdot\underline{F})dV = \iiint_R -4\pi G\rho\, dV$

$R=$ כדור ברדיוס r $\Leftarrow$

$$\underbrace{\oiint_{\substack{\text{ספירה}\\ \text{ברדיוס } r}} \underline{F}\cdot d\underline{S}}_{} = -4\pi G \underbrace{\iiint_{\substack{\text{כדור}\\ \text{ברדיוס } r}} \rho\, dV}_{}$$

$$F(r)\oiint_{\text{ספירה}} F(r)\hat{\underline{r}}\cdot\hat{\underline{r}}\,dA \qquad\qquad \text{מסה בתוך כדור}$$

$$= F(r)\cdot\left(\text{שטח של ספירה}\right) = F(r)\cdot 4\pi r^2$$

$$= \begin{cases} M & r\geq a \\ \int_0^r \rho(u)\cdot 4\pi u^2\, du & r<a \end{cases}$$
$$= 4\pi k\int_0^r (a-u)u^2\, du$$
$$= 4\pi k\left[\frac{au^3}{3} - \frac{u^4}{4}\right]_0^r$$
$$= 4\pi k r^3 (a/3 - r/4)$$

$$F(r) = \begin{cases} \underline{\underline{-GM/r^2}} & \underline{\underline{r\geq a}} \\ -4\pi k G r(a/3 - r/4) & \underline{\underline{r<a}} \\ \quad = \underline{\underline{-\frac{GM}{a^4}(4ar-3r^2)}} & \end{cases} \Leftarrow$$

ג. פונקציה פוטנציאל תלויה רק ב- r : $V=V(r)$

$$(4ב) \Rightarrow \underline{F} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\underline{r}}$$

$$\Rightarrow \frac{\partial V}{\partial r} = \begin{cases} GM/r^2 & r\geq a \\ \frac{GM}{a^4}(4ar-3r^2) & r<a \end{cases} \Rightarrow V = \begin{cases} -GM/r + \text{קבוע} & r\geq a \\ \frac{GM}{a^4}(2ar^2-r^3) + \text{קבוע} & r<a \end{cases}$$

$$\underset{\substack{\text{רציפות}\\ r=a\text{-ב}}}{\Rightarrow} V = \underline{\underline{\begin{cases} -GM/r & r\geq a \\ \frac{GM}{a^4}(2ar^2-r^3-a^3) & r<a \end{cases}}} + \text{קבוע}$$

.7

$$\underline{B} = \begin{pmatrix} 2\cos\theta / r^3 \\ \sin\theta / r^3 \\ 0 \end{pmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

צריך לבדוק לקיום פוטנציאל וקטורי: צ"ל ש- $\nabla\cdot\underline{B}=0$

$$\nabla\cdot\underline{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \underbrace{B_r}_{2\cos\theta/r^3}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\underbrace{B_\theta}_{\sin\theta/r^3}\sin\theta) \quad \Leftarrow \text{(דף-4)}$$

$$= \tfrac{1}{r^2}\,\partial/\partial r\,(\tfrac{2}{r}\cos\theta) + \frac{1}{r\sin\theta}\,\partial/\partial\theta\,(\tfrac{1}{r^3}\sin^2\theta)$$

$$= \tfrac{1}{r^2}(-\tfrac{2}{r^2}\cos\theta) + \frac{1}{r\sin\theta}(\tfrac{1}{r^3}\cdot 2\sin\theta\cos\theta)$$

$$= -\tfrac{2}{r^4}\cos\theta + \tfrac{2}{r^4}\cos\theta = 0 \quad \checkmark$$

סימטריה ($\underline{B}$ לא תלוי ב- ϕ) $\Leftarrow$ הפוטנציאל $\underline{A}$ לא תלוי ב- ϕ

$$\begin{pmatrix} 2\cos\theta/r^3 \\ \sin\theta/r^3 \\ 0 \end{pmatrix} = \begin{pmatrix} \tfrac{1}{r}\,\partial f_\phi/\partial\theta + \tfrac{1}{r}\cot\theta\cdot f_\phi \\ -\partial f_\phi/\partial r - \tfrac{1}{r} f_\phi \\ \partial f_\theta/\partial r - \tfrac{1}{r}\,\partial f_r/\partial\theta + \tfrac{1}{r} f_\theta \end{pmatrix} \quad \Leftarrow \text{(דף-5)}$$

אפשר $f_r = f_\theta = 0$

רק צריכים למצוא f_ϕ כך ש-

$$\tfrac{2}{r^2}\cos\theta = \partial f_\phi/\partial\theta + \cot\theta\cdot f_\phi \quad (1)$$

$$-\tfrac{1}{r^2}\sin\theta = r\,\partial f_\phi/\partial r + f_\phi = \frac{\partial}{\partial r}(r f_\phi) \quad (2)$$

$$\tfrac{1}{r}\sin\theta + \begin{pmatrix}\text{לא תלוי} \\ \text{ב-}r\end{pmatrix} = r f_\phi \quad \Leftarrow (2) \qquad \Leftarrow (2)$$

$$\tfrac{1}{r^2}\sin\theta + \begin{pmatrix}\text{לא תלוי} \\ \text{ב-}r\end{pmatrix} = f_\phi \quad \Leftarrow$$

בדוק גם (1): $\partial/\partial\theta\,(\tfrac{1}{r^2}\sin\theta) + \cot\theta(\tfrac{1}{r^2}\sin\theta) = \tfrac{2}{r^2}\cos\theta$ ✓

קיבלנו תשובה: $\underline{A} = f_\phi\,\hat{\underline{\phi}} = \frac{1}{r^2}\sin\theta\,\hat{\underline{\phi}}$

__הערה__: הפתרון לא יחיד: $\nabla\wedge(\nabla V) = \underline{0}$ ולכן כל"ים שגם $\underline{A}+\nabla V$ פוטנציאל וקטורית לכל פונקציה V.

בדרך כלל בוחרים V כך ש- $\nabla\cdot\underline{A} = 0$

$$\nabla\cdot\left(\tfrac{1}{r^2}\sin\theta\,\hat{\underline{\phi}}\right) \underset{\substack{\text{(דף-4)} \\ f_r=f_\theta=0}}{=} \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\tfrac{1}{r^2}\sin\theta\right) = 0 \quad \checkmark$$

.8

$$\underline{E} = \begin{pmatrix} 2\cos\theta/r^3 \\ \sin\theta/r^3 \\ 0 \end{pmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

__הערה__ אפשר לבדוק קודם שקיים פונקציית פוטנציאל: $\nabla\wedge\underline{E} = \underline{0}$

$$\nabla\wedge\underline{E} = \begin{pmatrix} 0 \\ 0 \\ \partial E_\theta/\partial r - \tfrac{1}{r}\,\partial E_r/\partial\theta + \tfrac{1}{r}E_\theta \end{pmatrix} = \underline{0} \quad \Leftarrow \text{(דף-5)}$$

$E_\phi = 0$

$\frac{\partial E_r}{\partial\phi} = \frac{\partial E_\theta}{\partial\phi} = 0$

$$\tfrac{1}{r}\left(-\frac{\partial E_r}{\partial\theta} + \frac{\partial}{\partial r}(r E_\theta)\right) = \tfrac{1}{r}\left[\frac{\partial}{\partial\theta}\left(\tfrac{2}{r^3}\cos\theta\right) + \frac{\partial}{\partial r}\left(\tfrac{1}{r^2}\sin\theta\right)\right] = 0$$

המשך של (8) ...

רוצים למצוא פונקציה V כך ש- $\underline{E} = -\underline{\nabla} V$

מ-(פו-3) , $\underline{\nabla} V = \hat{r}\,\partial V/\partial r + 1/r\,\hat{\theta}\,\partial V/\partial \theta + \frac{1}{r\sin\theta}\,\hat{\varphi}\,\partial V/\partial \varphi$

מקומפוננטות לפי הבסיס $\{\hat{r},\hat{\theta},\hat{\varphi}\}$ מקבלים שלוש משוואות:

$$\hat{r}: \quad 2\cos\theta/r^3 = -\partial V/\partial r \quad\Rightarrow\quad \partial V/\partial r = -2/r^3 \cos\theta$$

$$\hat{\theta}: \quad \sin\theta/r^3 = -1/r\,\partial V/\partial\theta \quad\Rightarrow\quad \partial V/\partial\theta = -\frac{1}{r^2}\sin\theta$$

$$\Rightarrow V = \frac{1}{r^2}\cos\theta$$ + לא תלוי ב- r,θ

$$\hat{\varphi}: \quad 0 = -\frac{1}{r\sin\theta}\,\partial V/\partial\varphi \quad\Rightarrow$$ V לא תלוי ב- φ

קיבלנו: $V = \frac{1}{r^2}\cos\theta +$ קבוע

---

9. נניח ש- $\underline{r}(t)$ וקטור המיקום של נקודה הנעה עם הנוזל. כ"א:

$$\frac{d\underline{r}}{dt} = \left(\text{מהירות של נוזל ב- } \underline{r}(t)\right) = \underline{v}(\underline{r}(t), t)$$

$$\frac{d}{dt}\underbrace{\left(\rho(\underline{r}(t),t)\right)}_{\text{הרכבה של } \rho,\ t \mapsto (\underline{r}(t),t)} \overset{\text{כלל השרשרת}}{=} D\rho \underset{\text{כפל מטריצות}}{*} D(\underline{r}(t),t)$$

$$= \left(\frac{\partial\rho}{\partial x}\ \frac{\partial\rho}{\partial y}\ \frac{\partial\rho}{\partial z}\ \frac{\partial\rho}{\partial t}\right) * \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \\ 1 \end{pmatrix}$$

$$= \underline{\nabla}\rho \cdot \underline{v} + \partial\rho/\partial t \quad \square$$

---

10. א.

$$\varepsilon_{ijk}\varepsilon_{ikl} = (\delta_{jk}\delta_{kl} - \delta_{jl}\delta_{kk})$$
$$= \delta_{jl} - \delta_{jl}(3)$$

$\delta_{kk} = \sum_k 1 = 3$ (מימד 3)

$$= \underline{-2\delta_{jl}}$$

ב.

$$\varepsilon_{ijk}\varepsilon_{klm}\varepsilon_{mni} = (\varepsilon_{kij}\varepsilon_{klm})\varepsilon_{mni}$$

i,k,m פעמיים
j,l,n פעם אחת

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\varepsilon_{mni}$$
$$= \underbrace{\delta_{il}\delta_{jm}\varepsilon_{mni}}_{i=l,\ j=m} - \underbrace{\delta_{im}\delta_{jl}\varepsilon_{mni}}_{i=m,\ j=l}$$
$$= \varepsilon_{jnl} - \delta_{jl}\underbrace{\varepsilon_{ini}}_{0}$$
$$= \varepsilon_{jnl}$$

המשך של 10 ...

ג. $\underbrace{\varepsilon_{ijk}}_{\substack{\text{אנטי - סימטרית} \\ i\leftrightarrow k \text{ -ב}}} \underbrace{a_i a_k}_{\substack{\text{סימטרית} \\ i\leftrightarrow k \text{ ב}}} = 0$

ד. $\varepsilon_{ijk}\,\delta_{jk} = \varepsilon_{ijj} = 0$

---

11 א. תחת החלפת בסיס $f^j = P_i^{\ j} e^i \underset{Q=P^{-1}}{\Leftarrow}$

$$(\delta')_i^{\ j} = Q_i^{\ k}\,\delta_k^{\ l}\,P_l^{\ j}$$
$$= Q_i^{\ l} P_l^{\ j}$$
$$= (QP)_i^{\ j}$$
$$= (I)_i^{\ j} = \delta_i^{\ j} \quad \square$$

ב. $\varepsilon_i^{\ jk}$ הוא הטנזור של ההעתקה $T(a,b) = c$

כאשר $c_i = \varepsilon_i^{\ jk} a_j b_k$

כלומר: $T(a,b) = a \wedge b$

לבסיס אורתונורמלי עם אוריינטציה חיובית, $\{f^1, f^2, f^3\}$

$$f^1\wedge f^2 = f^3,\quad f^1\wedge f^3 = -f^2,\quad f^2\wedge f^3 = f^1$$

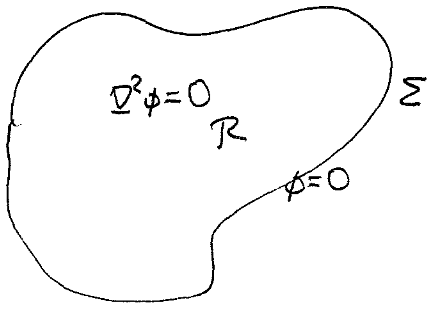
ומכאן רואים שהקומפוננטות של T לפי $\{f^1,f^2,f^3\}$ הן עדיין $\varepsilon_i^{\ jk}$. $\square$

---

12 א.

$$\nabla\cdot(\phi\nabla\phi) = \phi\,\nabla\cdot(\nabla\phi) + (\nabla\phi)\cdot(\nabla\phi)$$
$$= \phi\,\nabla^2\phi + |\nabla\phi|^2$$

ב.



$$\iiint_R \nabla\cdot(\phi\nabla\phi)\,dV \overset{\text{Gen}}{\underset{\text{Gauss}}{=}} \oiint_\Sigma \phi\nabla\phi\cdot d\Sigma$$

$\| $ (א) $\qquad\qquad \|$ ($\phi=0$,ב-Σ)

$$\iiint_R \phi\nabla^2\phi + |\nabla\phi|^2\,dV \qquad\qquad 0$$

$\|$ ($\nabla^2\phi=0$)

$$\iiint_R |\nabla\phi|^2\,dV$$

$$\Leftarrow \quad \iiint_R |\nabla\phi|^2\,dV = 0$$

אבל $|\nabla\phi|^2 \geq 0$ ומכאן רואים שאם $\nabla\phi$ רציפה), $|\nabla\phi|^2 = 0$ בכל R.

כלומר $\nabla\phi = 0$ בכל R

$\Leftarrow$ ϕ פונקציה קבועה ב- R

בגלל ש- $\phi=0$ ב- Σ, גם $\phi=0$ בכל R. $\square$

המשך של ⑫ ...

ג: אם $\phi=\psi$ ב- Σ , $\nabla^2\phi=0=\nabla^2\psi$ ב- R $\Leftarrow$ $\begin{cases}\phi-\psi=0 & \text{ב-}\ \Sigma\\ \nabla^2(\phi-\psi)=0 & \text{ב-}\ R\end{cases}$

מסעיף ב, מקבלים ש- $\phi-\psi=0$ בכל R

$\Leftarrow$ $\phi=\psi$ בכל R

<u>הערה</u> [חומר רשות] עם ϕ,ψ שתי פונקציות שהן שוות ל-0 מוגדרות מחוץ לתחום חסום R

וגם $\nabla^2\phi=f$, $\nabla^2\psi=f$ בכל נקודה

$\Leftarrow$ הפונקציה $\phi-\psi$ מקיימת: $\nabla^2(\phi-\psi)=0$ בכל נקודה

$\phi-\psi=0$ על ∂R

$\overset{ב}{\Leftarrow}$ $\phi-\psi=0$ בכל R.

משמעות פיזית: לצפיפות נתון ρ, קיים פוטנציאל הכובד V והוא הפתרון יחיד (עד כדי + קבוע) של המשוואה

$\nabla\cdot \underline{F}=-4\pi G\rho$, $\underline{F}=-\underline{\nabla}V$ $\longrightarrow$ $\underline{\nabla}^2 V = 4\pi G\rho$

---

⑬ א:

$$\underset{\text{ספירה ברדיוס } a}{\oiint}\frac{\underline{r}}{\|\underline{r}\|^3}\cdot d\underline{S} = \underset{\text{ספירה ברדיוס } a}{\iint}\frac{a\hat{r}}{a^3}\cdot \underset{\hat{N}}{\hat{r}}\, dA$$

$$= \frac{1}{a^2}\underset{\text{ספירה ברדיוס } a}{\iint} dA = \frac{1}{a^2}\underbrace{(4\pi a^2)}_{\text{שטח של ספירה}} = 4\pi$$

ב:

$$\underline{\nabla}\cdot(\underline{r}/\|\underline{r}\|^3) = \underline{\nabla}\cdot(\hat{r}/r^2)$$

$$\overset{\text{פנ-4}}{=} \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) = 0$$

$f_r = 1/r^2$, $f_\theta = f_\phi = 0$

או:

$$\underline{\nabla}\cdot(\underline{r}/\|\underline{r}\|^3) = \partial_i(x_i/r^3)$$

$$= (\partial_i x_i)\cdot\frac{1}{r^3} + x_i\,\partial_i(1/r^3)$$

$$= 3\cdot 1/r^3 + x_i(-3/r^4)(\partial_i r)$$

$$= 3/r^3 + x_i(-3/r^4)(x_i/r) \underset{x_i x_i = r^2}{=} 0$$

או:

$$\underline{\nabla}\cdot(\underline{r}/\|\underline{r}\|^3) = \underline{\nabla}\cdot(\tfrac{1}{r^3}\underline{r})$$

$$= 1/r^3(\underline{\nabla}\cdot\underline{r}) + \underline{\nabla}(1/r^3)\cdot\underline{r}$$

$$= 1/r^3(3) - 3/r^4\,\underbrace{\underline{\nabla}(r)}_{\hat{r}}\cdot\underline{r} = 3/r^3 - 3/r^4\cdot r = 0$$

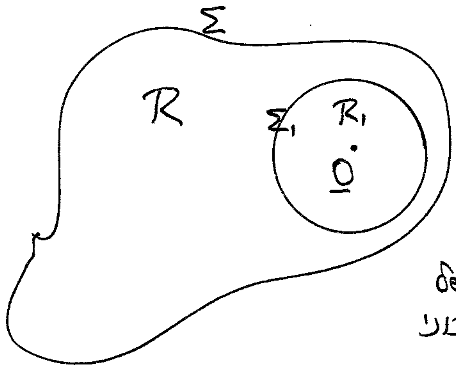
המשך של (13) ...

ג. נניח ש- $\Sigma=\partial R$ כאשר R תחום ב- $\mathbb{R}^3$.

אם $\underline{0}\notin R$ אז $\underline{r}/\|\underline{r}\|^3$ שדה וקטורי מוגדר על R,

$$(*)\quad \oiint_{\partial R} \frac{\underline{r}}{\|\underline{r}\|^3}\cdot d\underline{S} \overset{\text{Gauss}}{=} \iiint_R \underline{\nabla}\cdot\left(\frac{\underline{r}}{\|\underline{r}\|^3}\right) dV \overset{(ב)}{=} \iiint_R 0\, dV = 0$$

אם $\underline{0}\in R$, נניח ש- Σ_1 הוא ספירה ברדיוס a (מרכזו ב-$\underline{0}$) כאשר a מספיק קטן ש- $\Sigma_1\subset R$. נסמן ב- R_1 הכדור ברדיוס a מרכזו ב-$\underline{0}$, אז $\partial R_1=\Sigma_1$.



נגדיר תחום $R' = R\setminus R_1$

(כלומר התחום R חוץ מהכדור R_1)

השפה של R' היא $\partial R' = \Sigma \cup (-\Sigma_1)$

כיוון של $\underline{n}$ חיצוני

כיוון של $\underline{n}$ חיצוני ל-R' ולכן פנימי לכדור R_1

$$\oiint_{\partial R'} \frac{\underline{r}}{\|\underline{r}\|^3}\cdot d\underline{S} \overset{(*)}{=} 0 \quad \Leftarrow \quad \underline{0}\notin R'$$

$$\| $$

$$\oiint_{\Sigma} \frac{\underline{r}}{\|\underline{r}\|^3}\cdot d\underline{S} - \oiint_{\Sigma_1} \frac{\underline{r}}{\|\underline{r}\|^3}\cdot d\underline{S} = 0 \quad \Leftarrow$$

$$\oiint_{\Sigma} \frac{\underline{r}}{\|\underline{r}\|^3}\cdot d\underline{S} = \oiint_{\Sigma_1} \frac{\underline{r}}{\|\underline{r}\|^3}\cdot d\underline{S} \overset{(א)}{=} 4\pi \quad \Leftarrow$$

□

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(14) א. $\phi(x,y)=\tan^{-1}(y/x) \quad \Leftarrow \quad \frac{\partial\phi}{\partial x} = -\frac{y}{x^2}\cdot\frac{1}{1+y^2/x^2}$

$$= -\frac{y}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{x}\cdot\frac{1}{1+y^2/x^2}$$

$$= \frac{x}{x^2+y^2}$$

$$\underline{\nabla}\phi = \begin{pmatrix} -y/(x^2+y^2) \\ x/(x^2+y^2) \end{pmatrix}$$

תחום ההגדרה של ϕ הוא $\{(x,y): x\neq 0\} = \mathbb{R}^2 \setminus$ (ציר-y)

המשך של (14) ...

ב'

$$\oint_{\partial\Sigma} \frac{y\,dx - x\,dy}{x^2+y^2} = \oint_{\partial\Sigma} P\,dx + Q\,dy$$

$$P = \frac{y}{x^2+y^2}, \quad Q = -\frac{x}{x^2+y^2}$$

$$\underset{\text{Green}}{\overset{\text{משפט}}{=}} \iint_\Sigma \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$\frac{\partial Q}{\partial x} = \partial/\partial x\left(-\frac{x}{x^2+y^2}\right)$$

$$= -\frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \partial/\partial y\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{1\cdot(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \iint_\Sigma 0\,dx\,dy = 0$$

לכל תחום Σ לא כולל $(0,0)$ (כאן P,Q מוגדרות כפונקציות דיפרנציאביליות וכו')

__הערה__ מסעיף (א), אם Σ לא כולל $\mathbb{R}^2\setminus(y$ ציר$)$, אפשר לומר

$$\oint_{\partial\Sigma} \frac{y\,dx - x\,dy}{x^2+y^2} = \oint_{\partial\Sigma} -\nabla\phi\cdot d\underline{r} = 0$$

אבל ההוכחה משפט Green יותר כללי.

ג' כאשר C מעגל ברדיוס a, מרכזו ב- $(0,0)$

$$\oint_C \frac{y\,dx - x\,dy}{x^2+y^2} = \oint_C \begin{pmatrix} y/(x^2+y^2) \\ -x/(x^2+y^2)\end{pmatrix}\cdot d\underline{r}$$

$C:\ x = a\cos\theta$
$\quad y = a\sin\theta$
$\quad 0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \underbrace{\begin{pmatrix} a\sin\theta/a^2 \\ -a\cos\theta/a^2\end{pmatrix}}_{\underline{F}} \cdot \underbrace{\underbrace{\begin{pmatrix} -a\sin\theta \\ a\cos\theta\end{pmatrix}}_{\dot{\underline{r}}}\,d\theta}_{d\underline{r}}$$

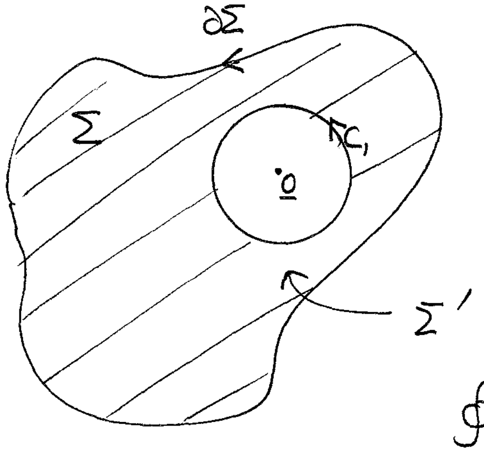
$$= \int_0^{2\pi} -1\,d\theta = -2\pi$$

כאשר $\underline{0}\notin\Sigma$, מסעיף (ב),

$$\oint_{\partial\Sigma} \frac{y\,dx - x\,dy}{x^2+y^2} = 0$$

המשך של 14ג ...

לתחום Σ שכולל $\underline{0}$, נבחר מעגל קטן C_1 ברדיוס a, ממורכז ב-$\underline{0}$
כך ש-$C_1 \subset \Sigma$. נסמן ב-Σ_1 הדיסק
ברדיוס a, ממורכז ב-$\underline{0}$ $(\partial\Sigma_1 = C_1)$



לתחום $\Sigma' = \Sigma \setminus \Sigma_1$

השפה: $\partial\Sigma' = \partial\Sigma \cup (-C_1)$

$$\oint_{\partial\Sigma'} \frac{y\,dx - x\,dy}{x^2+y^2} = 0 \overset{\text{ב}}{\Longleftarrow} \underline{0} \notin \Sigma'$$

$$\parallel$$

$$\oint_{\partial\Sigma} \frac{y\,dx - x\,dy}{x^2+y^2} - \oint_{C_1} \frac{y\,dx - x\,dy}{x^2+y^2}$$

$$\Longleftarrow \quad \oint_{\partial\Sigma} \frac{y\,dx - x\,dy}{x^2+y^2} = \oint_{C_1} \frac{y\,dx - x\,dy}{x^2+y^2} = -2\pi$$

$$\oint_{\partial\Sigma} \frac{y\,dx - x\,dy}{x^2+y^2} = \begin{cases} 0 & \underline{0} \notin \Sigma \\ -2\pi & \underline{0} \in \Sigma \end{cases}$$

ד. על חלק קטן של מסילה, (או מסילה שמוכל ב-$(\mathbb{R}^2 \setminus (\text{ציר } y))$ -

$$\int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \int_C -\nabla\phi \cdot d\underline{r} = -(\text{שינוי ב-}\phi)$$

$$= -(\text{שינוי בזווית קוטבית על } C)$$

מכאן רואים שעל מסילה סגורה C, הערך של $-\oint_C \frac{y\,dx - x\,dy}{x^2+y^2}$
הוא השינוי בזווית קוטבית כאשר נעים על C (בכיוון של C).
ז"א $2\pi \cdot \underbrace{(\text{מספר שלם})}_{\text{winding number of } C \text{ around } \underline{0}}$

I: -2π II 2π III 0 IV -4π

הערה: ϕ = פונקציה פוטנציאל מקומי של השדה $\begin{pmatrix} y/(x^2+y^2) \\ -x/(x^2+y^2) \end{pmatrix}$