$\boxed{16}$

③ $f,g\in\mathcal{R}[a,b]$
$k\in\mathbb{R}$
$\Downarrow$
$f+g,\ k\cdot f\in\mathcal{R}[a,b]$

וכן:

$\int_a^b (f+g)=\int_a^b f+\int_a^b g$

$\int_a^b k\cdot f = k\int_a^b f$

<u>הוכחה</u>

לפי א', תכונה ג' $\Rightarrow$ $\left.\begin{array}{l}\overline{\int_a^b}(f+g)\le\overline{\int_a^b}f+\overline{\int_a^b}g\\ \underline{\int_a^b}(f+g)\ge\underline{\int_a^b}f+\underline{\int_a^b}g\end{array}\right\}$

$\Rightarrow\quad \overline{\int_a^b}(f+g)\le\overline{\int_a^b}f+\overline{\int_a^b}g\underset{f,g\in\mathcal{R}}{=}\underline{\int_a^b}f+\underline{\int_a^b}g\le\underline{\int_a^b}(f+g)$

(ה) $\overline{\int}\ge\underline{\int}$ $\Rightarrow$ $\int_a^b f+\int_a^b g=\overline{\int_a^b}(f+g)=\underline{\int_a^b}(f+g)$ $\square$

תכונה ד' $\Rightarrow$ $\overline{\int_a^b}k\cdot f=k\overline{\int_a^b}f=k\underline{\int_a^b}f=\underline{\int_a^b}k\cdot f\quad(k\ge 0)$

$\overline{\int_a^b}k\cdot f=k\underline{\int_a^b}f=k\overline{\int_a^b}f=\underline{\int_a^b}k\cdot f\quad(k<0)$ $\square$

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④ * $f\in\mathcal{R}[a,b]$

* g פונקציה מוגדרת על $[a,b]$ כך ש- $g(x)=f(x)$ לכל $x\in[a,b]$ חוץ ממספר סופי של נקודות

$\Downarrow$

$g\in\mathcal{R}[a,b]$

<u>הוכחה</u>

נניח ש- $y_1,\dots,y_n$ הנקודות ב- $[a,b]$ כך ש- $f(x)\neq g(x)$.

$\left\{\begin{array}{l}f\in\mathcal{R}[a,b]\\ \varepsilon>0\end{array}\right.\Rightarrow\inf_T(\bar S_T(f)-\underline S_T(f))=0\Rightarrow\exists T:\ \bar S_T(f)-\underline S_T(f)<\varepsilon$

נגדיר חלוקה U ע"י $(\overset{a}{\overset{\shortparallel}{x_0}}<x_1<\dots<\overset{b}{\overset{\shortparallel}{x_N}})\equiv U=T\cup\{y_1,\dots,y_n\}$

לכן $\bar S_U(f)-\underline S_U(f)\le\bar S_T(f)-\underline S_T(f)<\varepsilon$

כי $\bar S_U(f)\le\bar S_T(f)$, $\underline S_U(f)\ge\underline S_T(f)$, $U\supseteq T$

$y_1\cdots y_n\notin(x_{i-1},x_i)$

$\Downarrow$

על (x_{i-1},x_i) $f=g$,

$\Downarrow$

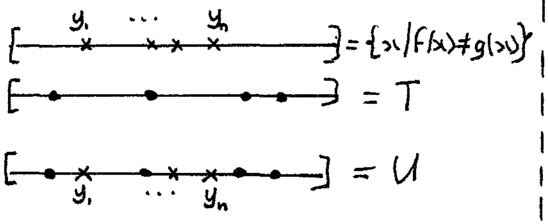
$\sup_{(x_{i-1},x_i)}f=\sup_{(x_{i-1},x_i)}g$

$\inf_{(x_{i-1},x_i)}f=\inf_{(x_{i-1},x_i)}g$

$\bar S_U(g)-\underline S_U(g)<\varepsilon$

$\Downarrow$

$\inf_U(\bar S_U(g)-\underline S_U(g))=0\Rightarrow g\in\mathcal{R}[a,b]$



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⑤ * $f\in\mathcal{B}[a,b]$

* $\forall c,d:\ a<c<d<b$
$f\in\mathcal{R}[c,d]$

$f\in\mathcal{R}[a,b]$

<u>הוכחה</u>

$\begin{array}{l}f\in\mathcal{B}[a,b]\\ \varepsilon>0\end{array}\Rightarrow(\exists m,M:\ \forall x\in[a,b]\ \ m\le f(x)\le M)$

נסמן את $\frac{\varepsilon}{4(M-m)}$ ב- δ.

$f\in\mathcal{R}[a+\delta,b-\delta]\Rightarrow\exists T=(\overset{a+\delta}{\overset{\shortparallel}{x_0}}<x_1<\dots<\overset{b-\delta}{\overset{\shortparallel}{x_n}}):\ \bar S_T(f)-\underline S_T(f)<\frac{\varepsilon}{2}$

$U=T\cup\{a,b\}=(a<x_0<\dots<x_n<b)\Rightarrow$

$$\bar S_U(f)-\underline S_U(f)=(x_0-a)\Big(\sup_{(a,x_0)}f-\inf_{(a,x_0)}f\Big)+(\bar S_T(f)-\underline S_T(f))+(b-x_n)\Big(\sup_{(x_n,b)}f-\inf_{(x_n,b)}f\Big)$$

$$<\underset{x_0-a}{\underset{\shortparallel}{\delta}}\cdot(M-m)+\frac{\varepsilon}{2}+\underset{b-x_n}{\underset{\shortparallel}{\delta}}\cdot(M-m)=\varepsilon$$

על כל תחום $X\subset[a,b]$, $m\le f(x)\le M$

$\Rightarrow\inf_X f\ge m,\ \sup_X f\le M\Rightarrow\sup_X f-\inf_X f\le(M-m)$

⑥
* $f\in\mathcal{R}[a,b]$
* $m\le f\le M$
* $g\in\mathcal{C}[m,M]$

$$\Downarrow$$

$$g\circ f\in\mathcal{R}[a,b]$$

נניח ש- $\varepsilon>0$

g רציפה $\Leftarrow$ $\left(\exists\delta>0:\ |s-t|<\delta\Rightarrow|g(s)-g(t)|<\varepsilon\right)$

$f\in\mathcal{R}[a,b]$ $\Leftarrow$ $\left(\exists T:\ \overline{S}_T(f)-\underline{S}_T(f)<\delta\cdot\varepsilon\right)$

נסמן $\{1,2,\dots,n\}\supseteq A=\{i:1\le i\le n,\ \sup\limits_{(x_{i-1},x_i)}f-\inf\limits_{(x_{i-1},x_i)}f\ge\delta\}$

<u>למה 1</u> $\sum\limits_{i\in A}(x_i-x_{i-1})\le\varepsilon$

<u>הוכחה:</u>

$$\delta\varepsilon>\overline{S}_T(f)-\underline{S}_T(f)=\sum_{i=1}^{n}(x_i-x_{i-1})\Big(\sup_{(x_{i-1},x_i)}f-\inf_{(x_{i-1},x_i)}f\Big)$$
$$\ge\sum_{i\in A}(x_i-x_{i-1})\Big(\sup_{(x_{i-1},x_i)}f-\inf_{(x_{i-1},x_i)}f\Big)$$
$$\ge\delta\sum_{i\in A}(x_i-x_{i-1})$$

□

<u>למה 2</u> $i\notin A\Rightarrow\sup\limits_{(x_{i-1},x_i)}(g\circ f)-\inf\limits_{(x_{i-1},x_i)}(g\circ f)\le\varepsilon$

<u>הוכחה</u> $i\notin A$ $\Leftarrow$ $\sup\limits_{(x_{i-1},x_i)}f-\inf\limits_{(x_{i-1},x_i)}f<\delta$

נסמן את $\inf\limits_{(x_{i-1},x_i)}f,\ \sup\limits_{(x_{i-1},x_i)}f$ ב- c,d. אזי $(g\circ f)(x_{i-1},x_i)=g(f(x_{i-1},x_i))\subseteq g([c,d])$

$d-c<\delta\ \Rightarrow\ |g(s)-g(t)|<\varepsilon\ \forall s,t\in[c,d]\ \Rightarrow\ \sup\limits_{[c,d]}g-\inf\limits_{[c,d]}g\le\varepsilon\ \Rightarrow\ \sup\limits_{(x_{i-1},x_i)}(g\circ f)-\inf\limits_{(x_{i-1},x_i)}(g\circ f)\le\varepsilon$

□

$$\overline{S}_T(g\circ f)-\underline{S}_T(g\circ f)=\sum_{i=1}^{n}(x_i-x_{i-1})\Big(\sup_{(x_{i-1},x_i)}(g\circ f)-\inf_{(x_{i-1},x_i)}(g\circ f)\Big)$$

$$=\sum_{i\in A}(x_i-x_{i-1})\underbrace{\Big(\sup_{(x_{i-1},x_i)}(g\circ f)-\inf_{(x_{i-1},x_i)}(g\circ f)\Big)}_{\le(\sup_{[m,M]}g-\inf_{[m,M]}g)}+\sum_{i\notin A}(x_i-x_{i-1})\underbrace{\Big(\sup_{(x_{i-1},x_i)}(g\circ f)-\inf_{(x_{i-1},x_i)}(g\circ f)\Big)}_{\le\varepsilon\ (\text{למה }2)}$$

$$\le\underset{(\text{למה }1)}{\varepsilon}\cdot\Big(\sup_{[m,M]}g-\inf_{[m,M]}g\Big)+(b-a)\cdot\varepsilon$$

$$\Rightarrow\ \inf_T\left(\overline{S}_T(g\circ f)-\underline{S}_T(g\circ f)\right)=0$$

$$\Rightarrow\ g\circ f\in\mathcal{R}[a,b]$$

<u>מסקנה</u> $f,g\in\mathcal{R}[a,b]\Rightarrow f\cdot g\in\mathcal{R}[a,b]$

$\left[f\cdot g=\frac{1}{2}\left((f+g)^2-f^2-g^2\right)\right]$

<u>מסקנה</u> $f\in\mathcal{R}[a,b]\ \Downarrow\ |f|\in\mathcal{R}[a,b]$

<u>מסקנה</u> $f\in\mathcal{R}[a,b]$, $\inf\limits_{[a,b]}f>0$ $\Downarrow$ $1/f\in\mathcal{R}[a,b]$

<u>מסקנה</u> $f\in\mathcal{R}[a,b]\ \Downarrow_{\alpha>0}\ f^{\alpha}\in\mathcal{R}[a,b]$