Appendix A

Topology

In this chapter we review basic concepts in topology that are used in this course; needless to say, this does not replace a comprehensive course in topology.

Definition A.1 A topological space (מר緩 ופאפתחים) is a pair \((X, \tau)\), where \(X\) is a non-empty set and \(\tau \subset \mathcal{P}(X)\) is a collection of subsets, closed under arbitrary union, under pairwise intersection and containing both \(X\) and the empty set. The elements of \(\tau\) are called open sets (.dtdאควנים).

Example: For every non-empty set \(X\), the power set \(\mathcal{P}(X)\) is a topology called the discrete topology. In this case, every subset of \(X\) is open. ▲ ▲ ▲

Definition A.2 A set \(A \subset X\) in a topological space \((X, \tau)\) is called closed (מערור) if its complement is open.

Definition A.3 Let \((X, \tau)\) be a topological space and let \(x \in X\). A set \(U \subset X\) is called an open neighborhood (סיבוב פתוח) of \(x\) if

\[x \in U \in \tau.\]

Let \((X, d)\) be a metric space. For every \(x \in X\) and \(r > 0\), denote the open ball (כדור פתוח)

\[B(x, r) = \{y \in X : d(x, y) < r\}.\]

The metric \(d\) induces a topology \(\tau\) on \(X\) as follows:
Definition A.4 In a metric space \((\mathbb{X}, d)\), a set \(A \subset \mathbb{X}\) is defined as open if every \(x \in A\) has open ball around it contained in \(A\); that is
\[
\forall x \in A \exists r > 0, \quad B(x, r) \subset A.
\]
In the particular case of \(\mathbb{X} = \mathbb{R}\) and \(d(x, y) = |x - y|\), a set \(A \subset \mathbb{R}\) is open if every \(x \in A\) has an open segment centered at \(x\) and contained in \(A\).

Definition A.5 Let \((\mathbb{X}, \tau)\) be a topological space. A sequence \((x_n) \subset \mathbb{X}\) is said to converge to \(x \in \mathbb{X}\) if for every open neighborhood \(U\) of \(x\) there exists an \(N \in \mathbb{N}\), such that for all \(n > N\), \(x_n \in U\). In other words, the sequence is eventually in every open neighborhood of \(x\).

Proposition A.6 In a metric space \((\mathbb{X}, d)\), the topological notion of convergence coincides with the metric notion of convergence.

Proof: Check it.

Definition A.7 Let \((\mathbb{X}, \tau)\) be a topological space. A collection \(\mathcal{C} \subset \tau\) of open sets is called a base for \(\tau\), if every open set can be represented as a union of elements of \(\mathcal{C}\).

Proposition A.8 In a metric space, the collection of all open balls forms a basis for the topology.

Proof: Let \((\mathbb{X}, d)\) be a metric space and let \(A \subset \mathbb{X}\) be open. By definition of \(A\) being open, there exists for every \(x \in A\) an \(r(x) > 0\) such that
\[
B(x, r(x)) \subset A.
\]
Consider the set
\[
U = \bigcup_{x \in A} B(x, r(x)).
\]
Since \(U\) is a union of subsets of \(A\), it follows that \(U \subset A\). On the other hand, \(U\) contains all the elements of \(A\), i.e., \(A \subset U\), which implies that \(U = A\).
Definition A.9 A topological space is called **second countable** if it has a countable base.

Proposition A.10 The space \( \mathbb{R} \) endowed with its metric topology is second-countable.

**Proof:** Consider the countable collection of open intervals,
\[ C = \{ (a, b) : a < b, \ a, b \in \mathbb{Q} \}. \]

Since every open interval contains an element of \( C \), it follows that in every open set \( A \), every element \( x \in A \) has an element \( I(x) \in C \), such that
\[ x \in I(x) \subset A. \]

Then,
\[ A = \bigcup_{x \in A} I(x), \]
proving that \( C \) is a (countable) base for \( \tau \).

**Definition A.11** Let \( (X, \tau) \) be a topological space. A subset \( A \subset X \) is called **dense** if every \( x \in X \) is the limit of some sequence in \( A \).

**Definition A.12** A topological space \( (X, \tau) \) is called **separable** if it contains a countable dense set.

**Example:** \( \mathbb{R} \) is separable because \( \mathbb{Q} \) is a countable dense set.

Proposition A.13 Every second-countable space \( (X, \tau) \) is separable.

**Proof:** Let \( C \) be a countable base for \( \tau \). From each \( U \in C \) take a representative \( x_U \in C \). Then,
\[ \{ x_U : U \in C \} \]

is a countable set; by construction, it is a dense set.