Appendix A

Topology

In this chapter we review basic concepts in topology that are used in this course; needless to say, this does not replace a comprehensive course in topology.

Definition A.1 A ***topological space*** *(מרחב טופולוגי) is a pair* $(\mathbb{X}, \tau)$*, where* $\mathbb{X}$ *is a non-empty set and* $\tau \subset \mathscr{P}(\mathbb{X})$ *is a collection of subsets, closed under* arbitrary *union, under pairwise intersection and containing both* $\mathbb{X}$ *and the empty set. The elements of* τ *are called* ***open sets*** *(קבוצות פתוחות).*

Example: For every non-empty set $\mathbb{X}$, the power set $\mathscr{P}(\mathbb{X})$ is a topology called the **discrete topology**. In this case, every subset of $\mathbb{X}$ is open. ▲▲▲

Definition A.2 A set $A \subset \mathbb{X}$ *in a topological space* $(\mathbb{X}, \tau)$ *is called* ***closed*** *(סגור) if its complement is open.*

Definition A.3 Let $(\mathbb{X}, \tau)$ *be a topological space and let* $x \in \mathbb{X}$*. A set* $U \subset \mathbb{X}$ *is called an* ***open neighborhood*** *(סביבה פתוחה) of* x *if*

$$x \in U \in \tau.$$

Let $(\mathbb{X}, d)$ be a **metric space**. For every $x \in \mathbb{X}$ and $r > 0$, denote the **open ball** (כדור פתוח)

$$B(x, r) = \{y \in \mathbb{X} \,:\, d(x, y) < r\}.$$

The metric d induces a topology τ on $\mathbb{X}$ as follows:

Definition A.4 In a metric space $(\mathbb{X}, d)$, a set $A \subset \mathbb{X}$ is defined as open if every $x \in A$ has open ball around it contained in A; that is

$$\forall x \in A \quad \exists r > 0, \quad B(x, r) \subset A.$$

In the particular case of $\mathbb{X} = \mathbb{R}$ and $d(x, y) = |x - y|$, a set $A \subset \mathbb{R}$ is open if every $x \in A$ has an open segment centered at x and contained in A.

Definition A.5 Let $(\mathbb{X}, \tau)$ be a topological space. A sequence $(x_n) \subset \mathbb{X}$ is said to ***converge*** *to $x \in \mathbb{X}$ if for every open neighborhood U of x there exists an $N \in \mathbb{N}$, such that for all $n > N$, $x_n \in U$. In other words, the sequence is eventually in every open neighborhood of x.*

Proposition A.6 In a metric space $(\mathbb{X}, d)$, the topological notion of convergence coincides with the metric notion of convergence.

Proof: Check it. ■

Definition A.7 Let $(\mathbb{X}, \tau)$ be a topological space. A collection $\mathscr{C} \subset \tau$ of open sets is called a ***base*** *(בסיס) for τ, if every open set can be represented as a union of elements of $\mathscr{C}$.*

Proposition A.8 In a metric space, the collection of all open balls forms a basis for the topology.

Proof: Let $(\mathbb{X}, d)$ be a metric space and let $A \subset \mathbb{X}$ be open. By definition of A being open, there exists for every $x \in A$ an $r(x) > 0$ such that

$$B(x, r(x)) \subset A.$$

Consider the set

$$U = \bigcup_{x \in A} B(x, r(x)).$$

Since U is a union of subsets of A, it follows that $U \subset A$. On the other hand, U contains all the elements of A, i.e., $A \subset U$, which implies that $U = A$. ■

Definition A.9 A topological space is called ***second countable*** *(ממניה שנייה) if it has a countable base.*

Proposition A.10 The space $\mathbb{R}$ endowed with its metric topology is second-countable.

Proof: Consider the countable collection of open intervals,

$$\mathscr{C} = \{(a,b) \ : \ a < b, \quad a,b \in \mathbb{Q}\}.$$

Since every open interval contains an element of $\mathscr{C}$, it follows that in every open set A, every element $x \in A$ has an element $I(x) \in \mathscr{C}$, such that

$$x \in I(x) \subset A.$$

Then,

$$A = \bigcup_{x \in A} I(x),$$

proving that $\mathscr{C}$ is a (countable) base for τ. ■

Definition A.11 Let $(\mathbb{X}, \tau)$ be a topological space. A subset $A \subset \mathbb{X}$ is called ***dense*** *(צפוף) if every $x \in \mathbb{X}$ is the limit of some sequence in A.*

Definition A.12 A topological space $(\mathbb{X}, \tau)$ is called ***separable*** *(ספרבילי) if it contains a countable dense set.*

Example: $\mathbb{R}$ is separable because $\mathbb{Q}$ is a countable dense set. ▲▲▲

Proposition A.13 Every second-countable space $(\mathbb{X}, \tau)$ is separable.

Proof: Let $\mathscr{C}$ be a countable base for τ. From each $U \in \mathscr{C}$ take a representative $x_U \in \mathscr{C}$. Then,

$$\{x_U \ : \ U \in \mathscr{C}\}$$

is a countable set; by construction, it is a dense set. ■